Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Applied Mathematics 34 (2005) 123–137

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/yaama

Hypergeometric series and harmonic number identities

Wenchang Chu *, Livia De Donno

Dipartimento di Matematica, Università degli Studi di Lecce, Lecce-Arnesano P.O. Box 193, 73100 Lecce, Italy

Received 21 March 2004; accepted 27 May 2004

Available online 18 September 2004

Abstract

The classical hypergeometric summation theorems are exploited to derive several striking identities on harmonic numbers including those discovered recently by Paule and Schneider (2003).

Keywords: Binomial coefficient; Hypergeometric series; Harmonic number

1. Introduction and notation

Let x be an indeterminate. The generalized harmonic numbers are defined to be partial sums of the harmonic series:

$$H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k} \quad \text{for } n = 1, 2, \ldots . \tag{1.1}$$

For $x = 0$ in particular, they reduce to the classical harmonic numbers:

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n = 1, 2, \ldots . \tag{1.2}$$

* Corresponding author.
E-mail addresses: chu.wenchang@unile.it (W. Chu), liviadedonno@libero.it (L. De Donno).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2004.05.003

Given a differentiable function $f(x)$, denote two derivative operators by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x) \quad \text{and} \quad \mathcal{D}_0 f(x) = \frac{d}{dx} f(x)\bigg|_{x=0}.$$

Then it is an easy exercise to compute the derivative of binomial coefficients

$$\mathcal{D}_x \binom{x+n}{m} = \binom{x+n}{m} \sum_{\ell=1}^{m} \frac{1}{1+x+n-\ell}$$

which can be stated in terms of the generalized harmonic numbers as

$$\mathcal{D}_x \binom{x+n}{m} = \binom{x+n}{m} \big\{H_n(x) - H_{n-m}(x)\big\} \quad (m \leqslant n). \tag{1.3}$$

In this paper, we will frequently use its evaluation at $x = 0$:

$$\mathcal{D}_0 \binom{x+n}{m} = \binom{n}{m} \{H_n - H_{n-m}\} \quad (m \leqslant n). \tag{1.4}$$

For the inverse binomial coefficients, the analogous results read as

$$\mathcal{D}_x \binom{x+n}{m}^{-1} = \binom{x+n}{m}^{-1} \sum_{\ell=1}^{m} \frac{-1}{1+x+n-\ell}$$

and the explicit harmonic number expressions

$$\mathcal{D}_x \binom{x+n}{m}^{-1} = \binom{x+n}{m}^{-1} \big\{H_{n-m}(x) - H_n(x)\big\} \quad (m \leqslant n), \tag{1.5}$$

$$\mathcal{D}_0 \binom{x+n}{m}^{-1} = \binom{n}{m}^{-1} \{H_{n-m} - H_n\} \quad (m \leqslant n). \tag{1.6}$$

As pointed out by Richard Askey (cf. [1] and [3]), expressing harmonic numbers in terms of differentiation of binomial coefficients can be traced back to Issac Newton. Following the work of the two papers cited above, we will explore further the application of derivative operators to hypergeometric summation formulas. Several striking harmonic number identities discovered in [3] will be recovered and some new ones will be established.

Because hypergeometric series will play a central role in the present work, we reproduce its notation for those who are not familiar with it. Roughly speaking, a hypergeometric series is a series $\sum C_n$ where the term ratio C_{n+1}/C_n is a rational function in n. If the shifted factorial is defined by

$$(c)_0 = 1 \quad \text{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \text{for } n = 1, 2, \ldots \tag{1.7}$$

then the hypergeometric series (cf. [2]) reads explicitly as

$$ {}_{1+p}F_q\left[\begin{matrix} a_0, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{n!(b_1)_n \cdots (b_q)_n} z^n. \tag{1.8} $$

In order to illustrate how to discover harmonic number identities from hypergeometric series, we start with the Chu–Vandermonde–Gauss formula [2, §1.3]:

$$ {}_2F_1\left[\begin{matrix} -n, a \\ c \end{matrix}\,\middle|\, 1\right] = \frac{(c-a)_n}{(c)_n}. $$

Under parameter replacements $a \to -n-\mu n$ and $c \to 1+\lambda n+x$ with $\lambda, \mu \in \mathbb{N}_0$, it can equivalently be stated as the following binomial convolution identity

$$ \sum_{k=0}^{n} \binom{n+\mu n}{k} \binom{x+\lambda n+n}{n-k} = \binom{x+\lambda n+\mu n+2n}{n}. \tag{1.9} $$

In view of (1.4), we derive, by applying the derivative operator $\mathcal{D}_0$ to both sides of the last identity, the following relation:

$$ \begin{aligned} &\sum_{k=0}^{n} \binom{n+\mu n}{k} \binom{n+\lambda n}{n-k} \{H_{\lambda n+n} - H_{\lambda n+k}\} \\ &\quad = \binom{2n+\lambda n+\mu n}{n} \{H_{\lambda n+\mu n+2n} - H_{\lambda n+\mu n+n}\}. \end{aligned} $$

According to the factor inside the braces $\{\cdots\}$, splitting the left-hand side into two sums with respect to k and then evaluating the first one by (1.9), we get immediately the following simplified result.

Theorem 1. *With $\lambda, \mu \in \mathbb{N}_0$, there holds the following harmonic number identity*:

$$ \begin{aligned} &\sum_{k=0}^{n} \binom{n+\mu n}{k} \binom{n+\lambda n}{n-k} H_{\lambda n+k} \\ &\quad = \binom{2n+\lambda n+\mu n}{n} \{H_{\lambda n+n} + H_{\lambda n+\mu n+n} - H_{\lambda n+\mu n+2n}\}. \end{aligned} $$

One interesting special case corresponding to $\mu = 0$ can be stated as

$$ \sum_{k=0}^{n} \binom{n}{k} \binom{n+\lambda n}{n-k} H_{\lambda n+k} = \binom{2n+\lambda n}{n} \{2H_{\lambda n+n} - H_{\lambda n+2n}\}. \tag{1.10} $$

It can be further specialized, with $\lambda = 0$, to

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n} \{2H_n - H_{2n}\}. \tag{1.11}$$

There exist numerous hypergeometric series identities. However we are not going to have a full coverage about how they can be used to find harmonic number identities. The authors will limit themselves to examine, by the derivative operator method, only the classical identities named after Pfaff–Saalschütz, Dougall–Dixon and the Whipple transformation in next three sections. As applications, we will tabulate 26 closed formulas and 21 transformations on harmonic numbers at the end of the paper.

Just like the demonstration of Theorem 1 and (1.10), we will examine the above-mentioned hypergeometric theorems in the three steps: reformulation in terms of binomial formulas, application of the derivative operator $\mathcal{D}_0$ and reduction to harmonic number identities by specifying parameters. Because all the computations involved in the paper are routine manipulations on finite series, we will therefore omit the details for the limit of space.

2. The Pfaff–Saalschütz theorem

Recall the Saalschütz theorem [2, §2.2]

$${}_3F_2\left[\begin{matrix} -n,\ a, & b \\ c,\ 1+a+b-c-n & \end{matrix}\,\middle|\, 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

Performing the parameter replacement

$$\left.\begin{aligned} a &\to -n - \mu n - \mu' x \\ b &\to 1 + \lambda n + \lambda' x \\ c &\to 1 + \nu n + \nu' x \end{aligned}\right\} \quad (\lambda, \mu, \nu \in \mathbb{N}_0)$$

we may express it as a binomial identity

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{k+\lambda n+\lambda' x}{k}\binom{n+\mu n+\mu' x}{k}}{\binom{k+\nu n+\nu' x}{k}\binom{k+(\lambda-\mu-\nu-2)n+(\lambda'-\mu'-\nu')x}{k}} = \frac{\binom{(\lambda-\nu)n+(\lambda'-\nu')x}{n}\binom{(\mu+\nu+2)n+(\mu'+\nu')x}{n}}{\binom{n+\nu n+\nu' x}{n}\binom{(\lambda-\mu-\nu-1)n+(\lambda'-\mu'-\nu')x}{n}}.$$

Applying $\mathcal{D}_0$ to the cases $\mu' = \nu' = 0$, $\lambda' = \nu' = 0$ and $\lambda' = \mu' = 0$ of the last identity, we get respectively the following harmonic number identities.

Theorem 2. *For $\lambda, \mu, \nu \in \mathbb{N}_0$ with $\lambda > 1 + \mu + \nu$, we have the harmonic number identity*:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{\lambda n+k}{k}\binom{\mu n+n}{k}}{\binom{\nu n+k}{k}\binom{(\lambda-\mu-\nu-2)n+k}{k}} \{H_{\lambda n+k} - H_{(\lambda-\mu-\nu-2)n+k}\}$$

$$= \frac{\binom{(\lambda-\nu)n}{n}\binom{(\mu+\nu+2)n}{n}}{\binom{\nu n+n}{n}\binom{(\lambda-\mu-\nu-1)n}{n}}\{H_{(\lambda-\nu)n} - H_{(\lambda-\nu-1)n} + H_{\lambda n} - H_{(\lambda-\mu-\nu-1)n}\}.$$

Theorem 3. *For $\lambda, \mu, \nu \in \mathbb{N}_0$ with $\lambda > 1 + \mu + \nu$, we have the harmonic number identity*:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{\lambda n+k}{k}\binom{\mu n+n}{k}}{\binom{\nu n+k}{k}\binom{(\lambda-\mu-\nu-2)n+k}{k}}\{H_{\mu n+n-k} - H_{(\lambda-\mu-\nu-2)n+k}\}$$
$$= \frac{\binom{(\lambda-\nu)n}{n}\binom{(\mu+\nu+2)n}{n}}{\binom{\nu n+n}{n}\binom{(\lambda-\mu-\nu-1)n}{n}}\{H_{(\mu+\nu+1)n} - H_{(\mu+\nu+2)n} + H_{\mu n+n} - H_{(\lambda-\mu-\nu-1)n}\}.$$

Theorem 4. *For $\lambda, \mu, \nu \in \mathbb{N}_0$ with $\lambda > 1 + \mu + \nu$, we have the harmonic number identity*:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{\lambda n+k}{k}\binom{\mu n+n}{k}}{\binom{\nu n+k}{k}\binom{(\lambda-\mu-\nu-2)n+k}{k}}\{H_{\nu n+k} - H_{(\lambda-\mu-\nu-2)n+k}\}$$
$$= \frac{\binom{(\lambda-\nu)n}{n}\binom{(\mu+\nu+2)n}{n}}{\binom{\nu n+n}{n}\binom{(\lambda-\mu-\nu-1)n}{n}}\left\{\begin{matrix} H_{(\mu+\nu+1)n} - H_{(\mu+\nu+2)n} + H_{(\lambda-\nu)n} - H_{(\lambda-\nu-1)n} \\ + H_{\nu n+n} - H_{(\lambda-\mu-\nu-1)n} \end{matrix}\right\}.$$

3. The Dougall–Dixon theorem

This section will explore the Dougall–Dixon theorem [2, §4.3]

$${}_5F_4\left[\begin{matrix} a, & 1+a/2, & b, & d, & -n \\ & a/2, & 1+a-b, & 1+a-d, & 1+a+n \end{matrix}\,\middle|\, 1\right] = \frac{(1+a)_n(1+a-b-d)_n}{(1+a-b)_n(1+a-d)_n}$$

to establish harmonic number identities.

3.1. Performing parameter replacement

$$\left.\begin{matrix} a \to -n-x \\ b \to 1+bn \\ d \to 1+dn \end{matrix}\right\} \quad (b, d \in \mathbb{N}_0)$$

we can reformulate the Dougall–Dixon theorem as the following binomial identity:

$$\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{k+dn}{k}}{\binom{k-x}{k}\binom{x+bn+n}{k}\binom{x+dn+n}{k}} = x\frac{\binom{x+n}{n}\binom{1+x+bn+dn+n}{n}}{\binom{x+bn+n}{n}\binom{x+dn+n}{n}}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 5. *With $b, d \in \mathbb{N}_0$, there holds the following harmonic number identity*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{k+dn}{k}}{\binom{n+bn}{k}\binom{n+dn}{k}}\big\{1+(n-2k)(2H_k-H_{bn+k}-H_{dn+k})\big\}=\frac{\binom{1+bn+dn+n}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}.$$

3.2. Performing parameter replacement

$$\left.\begin{array}{l} a\to -n-x\\ b\to 1+bn\\ d\to -n-dn\end{array}\right\}\quad (b,d\in\mathbb{N}_0)$$

we can reformulate the Dougall–Dixon theorem as the following binomial identity:

$$\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{n+dn}{k}}{\binom{k-x}{k}\binom{n+bn+x}{k}\binom{k+dn-x}{k}}=(-1)^n x\frac{\binom{x+n}{n}\binom{x+bn-dn}{n}}{\binom{n+bn+x}{n}\binom{n+dn-x}{n}}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 6. *With $b, d \in \mathbb{N}_0$, there holds the following harmonic number identity*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{n+dn}{k}}{\binom{n+bn}{k}\binom{k+dn}{k}}\big\{1+(n-2k)(2H_k-H_{bn+k}+H_{dn+k})\big\}=(-1)^n\frac{\binom{bn-dn}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}.$$

3.3. Performing parameter replacement

$$\left.\begin{array}{l} a\to -n-x\\ b\to -n-bn\\ d\to -n-dn\end{array}\right\}\quad (b,d\in\mathbb{N}_0)$$

we can reformulate the Dougall–Dixon theorem as the following binomial identity:

$$\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{n+bn}{k}\binom{n+dn}{k}}{\binom{k-x}{k}\binom{k+bn-x}{k}\binom{k+dn-x}{k}}=(-1)^n x\frac{\binom{x+n}{n}\binom{2n+bn+dn-x}{n}}{\binom{n+bn-x}{n}\binom{n+dn-x}{n}}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 7. *With $b, d \in \mathbb{N}_0$, there holds the following harmonic number identity*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{n+bn}{k}\binom{n+dn}{k}}{\binom{k+bn}{k}\binom{k+dn}{k}}\big\{1+(n-2k)(2H_k+H_{bn+k}+H_{dn+k})\big\}=(-1)^n\frac{\binom{2n+bn+dn}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}.$$

4. The Whipple transformation

In this section, the Whipple transformation [2, §4.3]

$$
{}_7F_6\left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & e, & -n \\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix}\,\middle|\,1\right]
$$
$$
=\frac{(1+a)_n(1+a-b-d)_n}{(1+a-b)_n(1+a-d)_n}\,{}_4F_3\left[\begin{matrix} -n, & b, & d, & 1+a-c-e \\ & 1+a-c, & 1+a-e, & b+d-a-n \end{matrix}\,\middle|\,1\right]
$$

will be used to derive harmonic number identities.

4.1. Performing parameter replacement

$$
\left.\begin{aligned} a &\to -x-n \\ b &\to 1+bn \\ c &\to 1+cn \\ d &\to 1+dn \\ e &\to 1+en \end{aligned}\right\} \quad (b,c,d,e\in\mathbb{N}_0)
$$

we can restate the Whipple transformation as

$$
\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{k+en}{k}}{\binom{k-x}{k}\binom{n+bn+x}{k}\binom{n+cn+x}{k}\binom{n+dn+x}{k}\binom{n+en+x}{k}}
$$
$$
=x\frac{\binom{x+n}{n}\binom{1+x+bn+dn+n}{n}}{\binom{x+bn+n}{n}\binom{x+dn+n}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{1+x+cn+en+n}{\ell}}{\binom{x+cn+n}{\ell}\binom{x+en+n}{\ell}\binom{1+x+bn+dn+\ell}{\ell}}
$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 8. *For four nonnegative integers* $\{b,c,d,e\}$*, there holds*:

$$
\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{k+en}{k}}{\binom{n+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{n+en}{k}}
$$
$$
\times\big\{1+(n-2k)(2H_k-H_{bn+k}-H_{cn+k}-H_{dn+k}-H_{en+k})\big\}
$$
$$
=\frac{\binom{1+bn+dn+n}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{1+cn+en+n}{\ell}}{\binom{n+cn}{\ell}\binom{n+en}{\ell}\binom{1+bn+dn+\ell}{\ell}}.
$$

4.2. Performing parameter replacement

$$\left.\begin{array}{l} a \to -x-n \\ b \to 1+bn \\ c \to 1+cn \\ d \to 1+dn \\ e \to -n-en \end{array}\right\} \quad (b,c,d,e \in \mathbb{N}_0)$$

we can restate the Whipple transformation as

$$\begin{aligned} &\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{n+en}{k}}{\binom{k-x}{k}\binom{n+bn+x}{k}\binom{n+cn+x}{k}\binom{n+dn+x}{k}\binom{k+en-x}{k}} \\ &\quad = x\frac{\binom{x+n}{n}\binom{1+x+bn+dn+n}{n}}{\binom{x+bn+n}{n}\binom{x+dn+n}{n}}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{x+cn-en}{\ell}}{\binom{n+cn+x}{\ell}\binom{\ell+en-x}{\ell}\binom{1+x+bn+dn+\ell}{\ell}} \end{aligned}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 9. *For four nonnegative integers* $\{b,c,d,e\}$*, there holds*:

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{n+en}{k}}{\binom{n+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{k+en}{k}} \\ &\qquad \times\big\{1+(n-2k)(2H_k-H_{bn+k}-H_{cn+k}-H_{dn+k}+H_{en+k})\big\} \\ &\quad = \frac{\binom{1+bn+dn+n}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{cn-en}{\ell}}{\binom{n+cn}{\ell}\binom{\ell+en}{\ell}\binom{1+bn+dn+\ell}{\ell}}. \end{aligned}$$

4.3. Performing parameter replacement

$$\left.\begin{array}{l} a \to -x-n \\ b \to 1+bn \\ c \to -n-cn \\ d \to 1+dn \\ e \to -n-en \end{array}\right\} \quad (b,c,d,e \in \mathbb{N}_0)$$

we can restate the Whipple transformation as

$$\begin{aligned} &\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{n+cn}{k}\binom{k+dn}{k}\binom{n+en}{k}}{\binom{k-x}{k}\binom{n+bn+x}{k}\binom{k+cn-x}{k}\binom{n+dn+x}{k}\binom{k+en-x}{k}} \\ &\quad = x\frac{\binom{x+n}{n}\binom{1+x+bn+dn+n}{n}}{\binom{x+bn+n}{n}\binom{x+dn+n}{n}}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{\ell+n+cn+en-x}{\ell}}{\binom{\ell+cn-x}{\ell}\binom{\ell+en-x}{\ell}\binom{1+x+bn+dn+\ell}{\ell}} \end{aligned}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 10. *For four nonnegative integers* $\{b,c,d,e\}$, *there holds*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{n+cn}{k}\binom{k+dn}{k}\binom{n+en}{k}}{\binom{n+bn}{k}\binom{k+cn}{k}\binom{n+dn}{k}\binom{k+en}{k}}$$
$$\times\big\{1+(n-2k)(2H_k-H_{bn+k}+H_{cn+k}-H_{dn+k}+H_{en+k})\big\}$$
$$=\frac{\binom{1+bn+dn+n}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{\ell+dn}{\ell}\binom{n+cn+en+\ell}{\ell}}{\binom{\ell+cn}{\ell}\binom{\ell+en}{\ell}\binom{1+bn+dn+\ell}{\ell}}.$$

4.4. Performing parameter replacement

$$\left.\begin{array}{l}a\to -x-n\\ b\to 1+bn\\ c\to -n-cn\\ d\to -n-dn\\ e\to -n-en\end{array}\right\}\quad (b,c,d,e\in\mathbb{N}_0)$$

we can restate the Whipple transformation as

$$\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{k+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{n+en}{k}}{\binom{k-x}{k}\binom{n+bn+x}{k}\binom{k+cn-x}{k}\binom{k+dn-x}{k}\binom{k+en-x}{k}}$$
$$=(-1)^n x\frac{\binom{x+n}{n}\binom{bn-dn+x}{n}}{\binom{x+bn+n}{n}\binom{n+dn-x}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{n+dn}{\ell}\binom{\ell+n+cn+en-x}{\ell}}{\binom{\ell+cn-x}{\ell}\binom{\ell+en-x}{\ell}\binom{\ell-n+bn-dn+x}{\ell}}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 11. *For four nonnegative integers* $\{b,c,d,e\}$, *there holds*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{k+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{n+en}{k}}{\binom{n+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{k+en}{k}}$$
$$\times\big\{1+(n-2k)(2H_k-H_{bn+k}+H_{cn+k}+H_{dn+k}+H_{en+k})\big\}$$
$$=(-1)^n\frac{\binom{bn-dn}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{\ell+bn}{\ell}\binom{n+dn}{\ell}\binom{\ell+cn+en+n}{\ell}}{\binom{\ell+cn}{\ell}\binom{\ell+en}{\ell}\binom{\ell+bn-dn-n}{\ell}}.$$

4.5. Performing parameter replacement

$$\left.\begin{array}{l}a\to -x-n\\ b\to -n-bn\\ c\to -n-cn\\ d\to -n-dn\\ e\to -n-en\end{array}\right\}\quad (b,c,d,e\in\mathbb{N}_0)$$

we can restate the Whipple transformation as

$$\sum_{k=0}^{n}\{x+n-2k\}\binom{n}{k}\frac{\binom{x+n}{k}\binom{n+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{n+en}{k}}{\binom{k-x}{k}\binom{k+bn-x}{k}\binom{k+cn-x}{k}\binom{k+dn-x}{k}\binom{k+en-x}{k}}$$
$$=(-1)^n x\frac{\binom{x+n}{n}\binom{2n+bn+dn-x}{n}}{\binom{n+bn-x}{n}\binom{n+dn-x}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{n+bn}{\ell}\binom{n+dn}{\ell}\binom{\ell+n+cn+en-x}{\ell}}{\binom{\ell+cn-x}{\ell}\binom{\ell+en-x}{\ell}\binom{2n+bn+dn-x}{\ell}}$$

which leads us, under the derivative operator $\mathcal{D}_0$, to the following result.

Theorem 12. *For four nonnegative integers* $\{b,c,d,e\}$*, there holds*:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{\binom{n+bn}{k}\binom{n+cn}{k}\binom{n+dn}{k}\binom{n+en}{k}}{\binom{k+bn}{k}\binom{k+cn}{k}\binom{k+dn}{k}\binom{k+en}{k}}$$
$$\times\big\{1+(n-2k)(2H_k+H_{bn+k}+H_{cn+k}+H_{dn+k}+H_{en+k})\big\}$$
$$=(-1)^n\frac{\binom{2n+bn+dn}{n}}{\binom{n+bn}{n}\binom{n+dn}{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\binom{n+bn}{\ell}\binom{n+dn}{\ell}\binom{n+cn+en+\ell}{\ell}}{\binom{\ell+cn}{\ell}\binom{\ell+en}{\ell}\binom{2n+bn+dn}{\ell}}.$$

5. Harmonic number identities and transformations

In order to facilitate computation of harmonic number sums, we present a useful limiting relation concerning harmonic numbers. Suppose that $\lambda,\nu,n,k\in\mathbb{N}_0$ (the set of nonnegative integers) with $k\leqslant n$ and $\{P_k(y),Q_k(y)\}$ are two families of monic polynomials with $P_k(y)$ and $Q_k(y)$ being of degree k in y, then there holds

$$\lim_{y\to\infty}\left\{\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}H_{ny+k}-\frac{P_{\lambda(n-k)+\nu}(y)}{Q_{\lambda(n-k)+\nu}(y)}H_{ny+n-k}\right\}=0. \tag{5.1}$$

In fact, it is not hard to see that $P_{\lambda k+\nu}(y)/Q_{\lambda k+\nu}(y)$ tends to one and $H_{ny+k}\approx\log(ny+k)$ as $y\to\infty$. Now rewrite the function in question into two terms

$$\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}H_{ny+k}-\frac{P_{\lambda(n-k)+\nu}(y)}{Q_{\lambda(n-k)+\nu}(y)}H_{ny+n-k}$$
$$=\{H_{ny+k}-H_{ny+n-k}\}\frac{P_{\lambda(n-k)+\nu}(y)}{Q_{\lambda(n-k)+\nu}(y)}+H_{ny+k}\left\{\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}-\frac{P_{\lambda(n-k)+\nu}(y)}{Q_{\lambda(n-k)+\nu}(y)}\right\}.$$

When $y\to\infty$, the former part in the last line tends to zero because the fraction is bounded and the difference in braces behaves like $\log((ny+k)/(ny+n-k))\to 0$; the latter part in the last line tends to zero too since the fractional difference is a fraction with numerator degree less than denominator degree in view of the fact that both $P(y)$ and $Q(y)$ are polynomials with the leading coefficients equal to one.

Theorem 13. *Let* $\{P_k(y), Q_k(y)\}$ *be two families of monic polynomials with* $P_k(y)$ *and* $Q_k(y)$ *being of degree* k *in* y. *If* $f_n(k)$ *is a function independent of* y *which satisfies the reflection property* $f_n(k) = -f_n(n-k)$, *then there holds the following limiting relation*:

$$\lim_{y\to\infty}\sum_{k=0}^{n} f_n(k)\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}H_{ny+k} = 0. \tag{5.2}$$

Proof. By means of the summation index involution $k \to n-k$, we can reformulate the finite sum stated in the theorem as

$$\sum_{k=0}^{n} f_n(k)\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}H_{ny+k} = \frac{1}{2}\sum_{k=0}^{n} f_n(k)\left\{\frac{P_{\lambda k+\nu}(y)}{Q_{\lambda k+\nu}(y)}H_{ny+k} - \frac{P_{\lambda(n-k)+\nu}(y)}{Q_{\lambda(n-k)+\nu}(y)}H_{ny+n-k}\right\}.$$

In view of (5.1), the differences in the braces on the right hand side tends to zero as $y \to \infty$. We therefore obtain the limiting relation about harmonic number sums stated in the theorem. □

There is a large class of functions satisfying the reflection property in the theorem, for example

$$f_n(k) = \binom{n}{k}^{\mu}\frac{\binom{n+k}{k}^{\nu}}{\binom{2n}{k}^{\nu}}(n-2k) \quad (\mu,\nu\in\mathbb{N}_0) \tag{5.3}$$

which come out frequently for the limiting process in the construction of Tables 1 and 2.

Table 1
The harmonic number identities of type $\sum_{k=0}^{n} A(n,k) = C(n)$

No	$A(n,k)$	$C(n)$	Note
1	$\binom{n}{k}^2\binom{2n+k}{k}\{H_{2n+k}-H_k\}$	$2\binom{2n}{n}^2\{H_{2n}-H_n\}$	Theorem 2: $\lambda=2$, $\mu=\nu=0$
2	$\binom{n}{k}^2\binom{2n+k}{k}\{H_k-H_{n-k}\}$	$\binom{2n}{n}^2\{H_{2n}-H_n\}$	Theorem 3: $\lambda=2$, $\mu=\nu=0$
3	$\binom{n}{k}\binom{2n}{k}\binom{3n+k}{k}\{H_{3n+k}-H_k\}$	$\binom{3n}{n}^2\{2H_{3n}-H_n-H_{2n}\}$	Theorem 2: $\lambda=3$, $\mu=1$ and $\nu=0$
4	$\binom{n}{k}\binom{2n}{k}\binom{3n+k}{k}\{H_{2n-k}-H_k\}$	$\binom{3n}{n}^2\{2H_{2n}-H_n-H_{3n}\}$	Theorem 3: $\lambda=3$, $\mu=1$ and $\nu=0$
5	$\binom{n}{k}^2\binom{3n+k}{2n}\{H_{3n+k}-H_k\}$	$\binom{3n}{n}\{H_{2n}+H_{3n}-2H_n\}$	Theorem 2: $\lambda=3$, $\mu=0$ and $\nu=1$
6	$\binom{n}{k}^2\binom{3n+k}{2n}\{H_k-H_{n-k}\}$	$\binom{3n}{n}\{H_{3n}-H_{2n}\}$	Theorem 3: $\lambda=3$, $\mu=0$ and $\nu=1$
7	$\binom{n}{k}^2\binom{3n+k}{2n}\{H_{n+k}-H_k\}$	$\binom{3n}{n}\{3H_{2n}-2H_n-H_{3n}\}$	Theorem 4: $\lambda=3$, $\mu=0$ and $\nu=1$

Table 1
(Continued)

No	$A(n,k)$	$C(n)$	Note
8	$\binom{n}{k}\{1+(n-2k)H_k\}$	1	Theorem 5: $b=0$, $d\to\infty$, cf. [3, Eq. 1]
9	$\binom{n}{k}^2\{1+2(n-2k)H_k\}$	0	Theorem 5: b, $d\to\infty$, cf. [3, Eq. 2]
10	$\binom{n+k}{k}\binom{2n-k}{n}\{1+(n-2k)(H_k-H_{n+k})\}$	$\binom{1+2n}{n}$	Theorem 5: $b=0$, $d=1$
11	$\binom{n+k}{k}^2\binom{2n-k}{n}^2\{1+2(n-2k)(H_k-H_{n+k})\}$	$\binom{1+3n}{n}$	Theorem 5: $b=d=1$
12	$\binom{2n}{k}\binom{2n}{n+k}\{1+(n-2k)(H_k+H_{n+k})\}$	$\binom{2n-1}{n}$	Theorem 6: $b=0$, $d=1$
13	$\binom{n}{k}\binom{2n}{k}\binom{2n}{n+k}\{1+(n-2k)(2H_k+H_{n+k})\}$	$(-1)^n$	Theorem 6: $b\to\infty$, $d=1$
14	$\binom{n}{k}\binom{n+k}{n}\binom{2n-k}{n}\{1+(n-2k)(2H_k-H_{n+k})\}$	1	Theorem 6: $b=1$, $d\to\infty$
15	$\binom{n}{k}^2\binom{n+k}{n}\binom{2n-k}{n}\{1+(n-2k)(3H_k-H_{n+k})\}$	$(-1)^n$	Theorem 6: $b=1$, $d=0$
16	$\binom{n}{k}^3\{1+3(n-2k)H_k\}$	$(-1)^n$	Theorem 7: $b=0$, $d\to\infty$, cf. [3, Eq. 3]
17	$\binom{n}{k}^4\{1+4(n-2k)H_k\}$	$(-1)^n\binom{2n}{n}$	Theorem 7: $b=d=0$, cf. [3, Eq. 4]
18	$\binom{n}{k}^2\binom{2n}{k}\binom{2n}{n+k}\{1+(n-2k)(3H_k+H_{n+k})\}$	$(-1)^n\binom{3n}{n}$	Theorem 7: $b=0$, $d=1$
19	$\binom{2n}{k}^2\binom{2n}{n+k}^2\{1+2(n-2k)(H_k+H_{n+k})\}$	$(-1)^n\binom{4n}{n}$	Theorem 7: $b=d=1$
20	$\binom{n}{k}^{-1}\{1-(n-2k)H_k\}$	$(1+n)H_{n+1}$	Theorem 8: $e\to\infty$, $b=c=d=0$
21	$\binom{n}{k}^{-2}\{1-2(n-2k)H_k\}$	$2\frac{(1+n)^2}{2+n}H_{n+1}$	Theorem 8: $b=c=d=e=0$
22	$\frac{1-(n-2k)(H_k+H_{n+k})}{\binom{2n}{k}\binom{2n}{n+k}}$	$\frac{1+2n}{2+2n}+(n+\frac{1}{2})H_{1+2n}$	Theorem 8: $e=1$, $b=c=d=0$
23	$\frac{\binom{n+k}{k}}{\binom{2n}{k}}\{1-(n-2k)H_{n+k}\}$	$(1+2n)\{H_{1+2n}-H_n\}$	Theorem 8: $b=d=0$, $c=1$ and $e\to\infty$
24	$\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\{1-2(n-2k)H_{n+k}\}$	$2\frac{(1+2n)^2}{2+3n}\{H_{1+2n}-H_n\}$	Theorem 8: $b=d=0$, $c=e=1$
25	$\frac{\binom{2n}{k}}{\binom{n}{k}\binom{n+k}{k}}\{1-(n-2k)(H_k-H_{n+k})\}$	$\frac{n(n+1)}{n-1}\{H_{n+1}+H_{n-1}-H_{2n}\}$	Theorem 9: $n>1$, $b=c=d=0$ and $e=1$
26	$\frac{\binom{2n}{k}^2}{\binom{n+k}{k}^2}\{1+2(n-2k)H_{n+k}\}$	$\frac{2n}{3}\{H_{2n}-H_{n-1}\}$	Theorem 10: $n>0$, $b=d=0$ and $c=e=1$

Table 2
The harmonic number transformations of type $\sum_{k=0}^{n} A(n,k) = \sum_{\ell=0}^{n} B(n,\ell)$

No	$A(n,k)$	$B(n,\ell)$	Note
1	$\binom{n}{k}\frac{\binom{n+k}{k}^3}{\binom{2n}{k}^3}\{1+(n-2k)(H_k-3H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times\frac{1+2n}{1+2n-\ell}\frac{\binom{n+\ell}{\ell}^2}{\binom{1+2n+\ell}{\ell}}$	Theorem 8: $e=0$, $b=c=d=1$
2	$\binom{n}{k}^2\frac{\binom{n+k}{k}^4}{\binom{2n}{k}^4}\{1+2(n-2k)(H_k-2H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times\binom{n}{\ell}\frac{\binom{n+\ell}{\ell}^2\binom{1+3n}{\ell}}{\binom{2n}{\ell}^2\binom{1+2n+\ell}{\ell}}$	Theorem 8: $b=c=d=e=1$
3	$\binom{n}{k}\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\{1+(n-2k)(H_k-2H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times\frac{\binom{n+\ell}{\ell}^2}{\binom{1+2n+\ell}{\ell}}$	Theorem 9: $b=d=1$, $c=0$ and $e\to\infty$
4	$\binom{n}{k}^3\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\{1+(n-2k)(3H_k-2H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2}{\binom{1+2n+\ell}{\ell}}$	Theorem 9: $b=d=1$, $c\to\infty$ and $e=0$
5	$\binom{n}{k}^2\frac{\binom{n+k}{k}^3}{\binom{2n}{k}^3}\{1+(n-2k)(2H_k-3H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2}{\binom{2n}{\ell}\binom{1+2n+\ell}{\ell}}$	Theorem 9: $e\to\infty$, $b=c=d=1$
6	$\binom{n}{k}^3\frac{\binom{n+k}{k}^3}{\binom{2n}{k}^3}\{1+3(n-2k)(H_k-H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times(-1)^\ell\frac{\binom{n}{\ell}^2\binom{n+\ell}{\ell}^2}{\binom{2n}{\ell}\binom{1+2n+\ell}{\ell}}$	Theorem 9: $e=0$, $b=c=d=1$
7	$\binom{n}{k}\frac{\binom{2n}{k}^2}{\binom{n+k}{k}^2}\{1+(n-2k)(H_k+2H_{n+k})\}$	$(-1)^\ell\binom{n}{\ell}\frac{\binom{3n+\ell}{\ell}}{\binom{n+\ell}{\ell}^2}$	Theorem 10: $b=0$, $c=e=1$ and $d\to\infty$
8	$\binom{n}{k}^4\frac{\binom{n+k}{k}}{\binom{2n}{k}}\{1+(n-2k)(4H_k-H_{n+k})\}$	$(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2}{\binom{2n}{n}}$	Theorem 10: $b=1$, $c=e=0$ and $d\to\infty$
9	$\binom{n}{k}^3\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\{1+(n-2k)(3H_k-2H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2}{\binom{1+2n+\ell}{\ell}}$	Theorem 10: $b=d=1$, $c=0$ and $e\to\infty$
10	$\binom{n}{k}^4\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\{1+2(n-2k)(2H_k-H_{n+k})\}$	$\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\times(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^3}{\binom{1+2n+\ell}{\ell}}$	Theorem 10: $b=d=1$, $c=e=0$
11	$\binom{n}{k}^4\frac{\binom{2n}{k}}{\binom{n+k}{k}}\{1+(n-2k)(4H_k+H_{n+k})\}$	$(-1)^n\times\binom{n}{\ell}^2\frac{\binom{2n+\ell}{\ell}}{\binom{n+\ell}{\ell}}$	Theorem 11: $b\to\infty$, $d=1$ and $c=e=0$
12	$\binom{n}{k}^5\frac{\binom{n+k}{k}}{\binom{2n}{k}}\{1+(n-2k)(5H_k-H_{n+k})\}$	$\frac{(-1)^n}{\binom{2n}{n}}\times\binom{n}{\ell}^2\binom{n+\ell}{\ell}^2$	Theorem 11: $b=1$, $c=d=e=0$
13	$\binom{n}{k}^3\frac{\binom{2n}{k}^2}{\binom{n+k}{k}^2}\{1+(n-2k)(3H_k+2H_{n+k})\}$	$(-1)^n\times\frac{\binom{n}{\ell}^2\binom{3n+\ell}{\ell}}{\binom{n+\ell}{\ell}^2}$	Theorem 11: $b\to\infty$, $d=0$ and $c=e=1$
14	$\binom{n}{k}\frac{\binom{2n}{k}^3}{\binom{n+k}{k}^3}\{1+(n-2k)(H_k+3H_{n+k})\}$	$n\times(-1)^\ell\frac{\binom{n}{\ell}\binom{3n+\ell}{\ell}}{(2n-\ell)\binom{n+\ell}{\ell}^2}$	Theorem 11: $n>0$, $b=0$ and $c=d=e=1$
15	$\binom{n}{k}^2\frac{\binom{2n}{k}^3}{\binom{n+k}{k}^3}\{1+(n-2k)(2H_k+3H_{n+k})\}$	$\frac{(-1)^n}{\binom{2n}{n}}\times\binom{n}{\ell}\frac{\binom{2n}{\ell}\binom{3n+\ell}{\ell}}{\binom{n+\ell}{\ell}^2}$	Theorem 11: $b\to\infty$, $c=d=e=1$

Table 2
(Continued)

No	$A(n,k)$	$B(n,\ell)$	Note
16	$\binom{n}{k}^5\{1+5(n-2k)H_k\}$	$(-1)^n\binom{n}{\ell}^2\binom{n+\ell}{n}$	Theorem 12: $b=c=d=0$, $e\to\infty$, cf. [3, Eq. 5]
17	$\binom{n}{k}^6\{1+6(n-2k)H_k\}$	$(-1)^n\binom{n}{\ell}^2\binom{n+\ell}{n}\binom{2n-\ell}{n}$	Theorem 12: $b=c=d=e=0$
18	$\binom{n}{k}^5\frac{\binom{2n}{k}}{\binom{n+k}{k}}\{1+(n-2k)(5H_k+H_{n+k})\}$	$(-1)^n\binom{2n}{n}\times\binom{n}{\ell}^3\frac{\binom{2n+\ell}{\ell}}{\binom{n+\ell}{\ell}\binom{2n}{\ell}}$	Theorem 12: $e=1$, $b=c=d=0$
19	$\binom{n}{k}^4\frac{\binom{2n}{k}^2}{\binom{n+k}{k}^2}\{1+2(n-2k)(2H_k+H_{n+k})\}$	$(-1)^n\binom{2n}{n}\times\frac{\binom{n}{\ell}^3\binom{3n+\ell}{\ell}}{\binom{n+\ell}{\ell}^2\binom{2n}{\ell}}$	Theorem 12: $b=d=0$, $c=e=1$
20	$\binom{n}{k}^3\frac{\binom{2n}{k}^3}{\binom{n+k}{k}^3}\{1+3(n-2k)(H_k+H_{n+k})\}$	$(-1)^n\frac{\binom{3n}{n}}{\binom{2n}{n}}\times\binom{n}{\ell}^2\frac{\binom{2n}{\ell}\binom{3n+\ell}{\ell}}{\binom{n+\ell}{\ell}^2\binom{3n}{\ell}}$	Theorem 12: $b=0$, $c=d=e=1$
21	$\binom{n}{k}^2\frac{\binom{2n}{k}^4}{\binom{n+k}{k}^4}\{1+2(n-2k)(H_k+2H_{n+k})\}$	$(-1)^n\frac{\binom{4n}{n}}{\binom{2n}{n}^2}\times\binom{n}{\ell}\frac{\binom{2n}{\ell}^2\binom{3n+\ell}{\ell}}{\binom{4n}{\ell}\binom{n+\ell}{\ell}^2}$	Theorem 12: $b=c=d=e=1$

Now we take entry 4 from Table 2 to exemplify how to derive harmonic number identities from the theorems established in this paper.

Specifying with $b=d=1$ and $e=0$, we can state the transformation in Theorem 9 as

$$\sum_{k=0}^{n}\binom{n}{k}^3\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\frac{\binom{k+cn}{k}}{\binom{n+cn}{k}}\big\{1+(n-2k)(3H_k-2H_{n+k}-H_{cn+k})\big\}$$
$$=\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\sum_{\ell=0}^{n}(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2\binom{cn}{\ell}}{\binom{1+2n+\ell}{\ell}\binom{n+cn}{\ell}}.$$

It is easy to see that the coefficient corresponding to H_{cn+k} is given by (5.3) with $\mu=3$ and $\nu=2$. In view of Theorem 13, the limit $c\to\infty$ of the last equation reads as

$$\sum_{k=0}^{n}\binom{n}{k}^3\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\big\{1+(n-2k)(3H_k-2H_{n+k})\big\}=\frac{\binom{1+3n}{n}}{\binom{2n}{n}^2}\sum_{\ell=0}^{n}(-1)^\ell\frac{\binom{n}{\ell}\binom{n+\ell}{\ell}^2}{\binom{1+2n+\ell}{\ell}}\tag{5.4}$$

which is exactly the fourth identity displayed in Table 2.

We remark that the right-hand side of this last identity can further be evaluated by Dixon's formula and we therefore get the following closed formula:

$$\sum_{k=0}^{n}\binom{n}{k}^3\frac{\binom{n+k}{k}^2}{\binom{2n}{k}^2}\big\{1+(n-2k)(3H_k-2H_{n+k})\big\}$$
$$=\begin{cases}0, & n-\text{odd},\\ (-1)^m\binom{3m}{m,m,m}\Big/\binom{4m}{2m}^2, & n=2m.\end{cases}\tag{5.5}$$

Specifying free parameters in Theorems 2–12 and then applying Theorem 13, we can similarly establish 26 closed summation formulas and 21 transformations on harmonic numbers, which are displayed respectively in Tables 1 and 2.

As a partial answer to the question posed at the end of the paper by Paule and Schneider [3], the examples 8, 9, 16, 17 numbered with in Table 1 and 16, 17 in Table 2 confirm that the sum

$$\Xi_\lambda(n) := \sum_{k=0}^{n} \binom{n}{k}^{\lambda} \big\{1+\lambda(n-2k)H_k\big\} \quad (\lambda, n \in \mathbb{N})$$

are representable in terms of terminating hypergeometric series for $1 \leqslant \lambda \leqslant 6$. In addition, the hypergeometric method presented in this paper shows that these binomial-harmonic number sums trace back to the same origin, the very well poised terminating hypergeometric series. In fact, if we define

$$\Omega_\lambda(n,x) := {}_{1+\lambda}F_\lambda\left[\begin{matrix} -x-n, & 1-(x+n)/2, & \langle -n\rangle_{\lambda-1} \\ & -(x+n)/2, & \langle 1-x\rangle_{\lambda-1} \end{matrix}\,\middle|\, (-1)^{\lambda}\right],$$

where $\langle w\rangle_\lambda$ stands for λ copies of w. Then it is not difficult to check that

$$\Xi_\lambda(n) = \mathcal{D}_0\big\{(x+n)\Omega_\lambda(n,x)\big\}.$$

However, the problem posed by Paule and Schneider [3] remains open for $\lambda > 6$, i.e., whether $\Xi_\lambda(n)$ can be expressed as a single terminating hypergeometric series.

Acknowledgments

In a recent preprint "Hypergéométrie et fonction zêta de Riemann" by Christian Krattenthaler and Tanguy Rivoal, a multisum expression for $\Xi_\lambda(n)$ has been derived, but as pointed out by Krattenthaler to the authors, that it is (*most likely*) not possible to express these sums as single hypergeometric sums. They make also the same observation, namely that the identities proved in the paper by Paule and Schneider come from applying differentiation to known hypergeometric summation or transformation theorems. In this sense, their work has some common background with ours, but they have different aims. The authors thank to Krattenthaler for the information.

References

[1] G.E. Andrews, K. Uchimura, Identities in combinatorics IV: differentiation and harmonic numbers, Util. Math. 28 (1985) 265–269.

[2] W.N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935.

[3] P. Paule, C. Schneider, Computer proofs of a new family of harmonic number identities, Adv. in Appl. Math. 31 (2003) 359–378.