Lower bounds for blow-up time in parabolic problems under Dirichlet conditions

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Received 2 May 2006  
Available online 12 July 2006  
Submitted by Steven G. Krantz

Abstract

We consider an initial-boundary value problem for the semilinear heat equation whose solution may blow up in finite time. We use a differential inequality technique to determine a lower bound on blow-up time if blow-up occurs. A second method based on a comparison principle is also presented.

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Keywords: Blow-up; Lower bound; Parabolic problems

1. Introduction

There is an extensive literature on the global existence or nonexistence and the blow-up in finite time of solutions to semilinear parabolic and hyperbolic differential equations and systems. Much of this work has dealt with initial value problems or initial-boundary value problems for prototypical equations involving the heat and wave operator. The nonlinearity is often of the from $f(u) = u^p$, $p > 1$, and usually occurs in the differential equation although it may appear in a Neumann boundary condition. Sufficient conditions for blow-up or existence/nonexistence of solutions have been presented and bounds on blow-up rate or on global solutions have also been determined. The structure of the blow-up set or asymptotic behavior of solutions has been...
the focus of yet other investigations. The reader may consult [1–10,12–14] and the references cited therein in regard to these studies.

A variety of methods have been used in the study of the questions mentioned above. The list includes Fourier coefficient method, Green function method, weighted energy arguments, comparison method, and the concavity method. Levine records a number of references on these methods in [6] (see also [14]). The methods used to determine the blow-up of solutions often indicate an upper bound for the blow-up time. Lower bounds for the blow-up time are more difficult to obtain and little attention appears to have been given to this question. It is precisely this question which we consider in this paper.

In Section 2 we consider an initial-boundary value problem for the semilinear heat equation

$$u_t = \Delta u + f(u)$$

under homogeneous Dirichlet boundary conditions and appropriate constraints on the nonlinearity $f(u)$. We impose conditions which insure that a solution exists locally, but the solution may possibly blow up in a certain measure at some finite time $t^*$. By means of a first order differential inequality, we determine a lower bound for the blow up time $t^*$. We then propose a second method which yields a lower bound for a different measure under slightly altered conditions on $f$ in Section 3. This latter technique makes use of a comparison principle. In a forthcoming paper, we determine a lower bound for blow-up time when Neumann conditions are prescribed on the boundary rather than Dirichlet conditions.

2. First method

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial \Omega$. We consider the semilinear problem

$$
\begin{align*}
&u_t = \Delta u + f(u) \quad \text{in } \Omega \times (0, t^*), \\
&u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
&u(x, 0) = g(x) \geq 0 \quad \text{in } \Omega,
\end{align*}
$$

(2.1)

where $\Delta$ is the Laplace operator and $u_t$ denotes partial differentiation of $u$ with respect to $t$. We impose the following conditions on the nonlinearity $f$:

(i) $f(0) = 0$, $f(s) > 0$ for $s > 0$,

(ii) $\int_s^\infty \frac{dn}{f(\eta)}$ is bounded for $s > 0$,

and there exist positive constants $n > 2$ and $\beta$ such that

(iii) $f(s)(\int_s^\infty \frac{dn}{f(\eta)})^{n+1} \to \infty$ as $s \to 0^+$,

(iv) $f'(s) \int_s^\infty \frac{dn}{f(\eta)} \leq (n + 1) - \beta$.  

(2.2)

We note that $f(s) = s^p$, $p > 1$, and $f(s) = 2(\cosh \gamma s - 1)$, $\gamma > 0$, satisfy these requirements. Further, we suppose that $g$ satisfies the compatibility condition $g(x) = 0$ for $x \in \partial \Omega$.

We know from results of Ball [1] and Kielhöfer [5] that the solution to problems such as (2.1) can fail to exist globally only if they blow up at some finite time. They may or may not blow up depending on the form of $f(u)$, the initial data $g(x)$, and the geometry of the relevant domain $\Omega$. 
Moreover, we know by the maximum principle [11] that when the initial and boundary data are nonnegative, then the solution is nonnegative in its time interval of existence. Thus, a nonnegative classical solution will exist for a period of time, but it may become unbounded at some finite time $t^*$. Our aim is to determine a lower bound for the blow-up time if it occurs. We remark that whether the solution blows up or not, the bound will still be valid.

We define the function

$$\varphi(t) = \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n} dx$$

and compute

$$\varphi'(t) = n \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n-1} \left[ f(u) \right]^{-1} u_t \, dx$$

$$= n \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n-1} \left[ \frac{\Delta u}{f(u)} + 1 \right] \, dx$$

$$= -n(n+1) \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n-2} \left[ f(u) \right]^{-2} |\nabla u|^2 \, dx$$

$$+ n \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n-1} \left[ f(u) \right]^{-2} f'(u) |\nabla u|^2 \, dx + n \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-n-1} \, dx,$$

where $\nabla$ is the gradient operator, and by (2.2) the surface integral vanishes on integration by parts. Further use of (2.2) results in

$$\varphi'(t) \leq -n\beta \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-(n+2)} \left[ f(u) \right]^{-2} |\nabla u|^2 \, dx + n \int_{\Omega} \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-(n+1)} \, dx. \quad (2.4)$$

We now seek a bound on the second term in (2.4) which will offset the first term on the right side.

In order to simplify our computations, we let

$$v = \left[ \int_{-\infty}^{\infty} \frac{d\eta}{f(\eta)} \right]^{-1} \quad \text{ (2.5)}$$

and rewrite (2.4) as

$$\varphi'(t) \leq -\frac{4\beta}{n} \int_{\Omega} |\nabla v^n|^2 \, dx + n \int_{\Omega} v^{n+1} \, dx. \quad (2.6)$$

Now by Hölder’s inequality, we have

$$\int_{\Omega} v^{n+1} \, dx \leq |\Omega|^{\frac{n+1}{2m}} \left( \int_{\Omega} v^{\frac{2m}{n+1}} \, dx \right)^{\frac{2(n+1)}{2m}},$$

$$\int_{\Omega} |\nabla v^n|^2 \, dx \leq |\Omega|^{\frac{n+1}{2m}} \left( \int_{\Omega} v^{\frac{2m}{n+1}} \, dx \right)^{\frac{2(n+1)}{2m}}.$$
where \(|\Omega|\) denotes the volume of \(\Omega\), and by the Schwarz inequality
\[
\int_{\Omega} v^{n+1} \, dx \leq |\Omega| \left\{ \int_{\Omega} v^{2n} \, dx \int_{\Omega} v^n \, dx \right\}^{\frac{n+1}{3n}}.
\] (2.7)
We again use the Schwarz inequality to bound
\[
\int_{\Omega} v^{2n} \, dx \leq \left\{ \int_{\Omega} (v^2)^{\frac{n}{2}} \, dx \int_{\Omega} v^n \, dx \right\}^{1/2}
\]
and make use of the Sobolev inequality [15],
\[
\left( \int_{\Omega} |w|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\Omega} |\nabla w|^p \, dx \right)^{\frac{1}{p}},
\] (2.8)
with \(q = 6\), \(p = 2\), \(w = v^{n/2}\), and best constant
\[
C = 4^{1/3} \cdot 3^{-1/2} \cdot \pi^{-2/3}.
\] (2.9)
It follows that
\[
\int_{\Omega} v^{2n} \, dx \int_{\Omega} v^n \, dx \leq C^3 \left\{ \int_{\Omega} |\nabla v|^2 \, dx \int_{\Omega} v^n \, dx \right\}^{3/2}.
\] (2.10)
Substituting into (2.7), we have
\[
\int_{\Omega} v^{n+1} \, dx \leq |\Omega|^{\frac{n-2}{3n}} C^{\frac{n+1}{n}} \left\{ \int_{\Omega} |\nabla v|^2 \, dx \int_{\Omega} v^n \, dx \right\}^{\frac{n+1}{2n}},
\] (2.11)
which we can write as
\[
\int_{\Omega} v^{n+1} \, dx \leq |\Omega|^{\frac{n-2}{3n}} C^{\frac{n+1}{n}} \left[ \alpha \int_{\Omega} |\nabla v|^2 \, dx \right]^{\frac{n+1}{2n}} \left[ \left( \alpha^{-1} \int_{\Omega} v^n \, dx \right)^{n+1} \right]^{\frac{n-1}{n}},
\] (2.12)
where \(\alpha\) is a positive constant to be determined. Using the inequality
\[
a^r b^q \leq ra + qb, \quad r + q = 1,
\]
for \(a, b > 0\) in (2.12) and combining it with (2.6), we obtain
\[
\varphi'(t) \leq -\left( \frac{4\beta}{n} - \frac{n+1}{2} |\Omega|^{\frac{n-2}{3n}} C^{\frac{n+1}{n}} \right) \int_{\Omega} |\nabla v|^2 \, dx
\]
\[
+ \left( \frac{n-1}{2} |\Omega|^{\frac{n-2}{3n}} C^{\frac{n+1}{n}} \alpha^{-\frac{n+1}{n-1}} \right) \left( \int_{\Omega} v^n \, dx \right)^{\frac{n+1}{n-1}}.
\]
Choosing
\[
\alpha = \frac{8\beta}{n(n+1)} |\Omega|^{\frac{n-2}{3n}} C^{-\frac{n+1}{n}},
\] (2.13)
we have the differential inequality
\[ \phi'(t) \leq K \left[ \phi(t) \right]^\frac{n+1}{n-1}, \tag{2.14} \]
where
\[ K = \frac{n-1}{2} |\Omega|^{\frac{n-2}{n}} C^{\frac{n+1}{2}} \alpha^{-\frac{n+1}{n-1}}. \tag{2.15} \]
It follows on integrating (2.14) from 0 to \( t \) that
\[ \left[ \phi(0) \right]^{-\frac{2}{n-1}} - \left[ \phi(t) \right]^{-\frac{2}{n-1}} \leq \frac{2K}{n-1} t, \]
so that letting \( t \to t^* \), we conclude that
\[ t^* \geq \frac{n-1}{2K} \left\{ \int_\Omega \left[ \int_0^\infty \frac{d\eta}{f(\eta)} \right] dx \right\}^{-\frac{2}{n-1}}. \tag{2.16} \]

We summarize the foregoing in the following theorem.

**Theorem 2.1.** If \( u \) is a nonnegative solution of semilinear problem (2.1) which becomes unbounded in \( \phi \) measure at \( t = t^* \), then \( t^* \) is bounded below by (2.16), where \( K \) is given by (2.15).

In the particular case \( f(u) = u^p \), \( p > 1 \), we can take
\[ \phi(t) = \int_\Omega u^{n(p-1)} \, dx \]
and compute
\[ \phi'(t) = -(np-n)(np-n-1) \int_\Omega u^{n(p-1)-2} |\nabla u|^2 \, dx + (np-n) \int_\Omega u^{(n+1)(p-1)} \, dx. \]
Letting \( v = u^{p-1} \), we then bound
\[ \int_\Omega v^{n+1} \, dx \]
as in the argument above and obtain the lower bound
\[ t^* \geq \frac{n-1}{2K_1} \left( \int \frac{g(x)^{n(p-1)} \, dx}{|\Omega|} \right)^{-\frac{2}{n-1}}, \tag{2.17} \]
where
\[ K_1 = (p-1) \frac{(n-1)}{2} |\Omega|^{\frac{n-2}{2}} C^{\frac{n(p-1)}{2}} \alpha_1^{-\frac{n+1}{n-1}} \]
and
\[ \alpha_1 = \frac{2n(np-n-1)}{n+1} |\Omega|^{\frac{2-n}{n}} C^{\frac{n(p-1)}{2}} \alpha^{-\frac{n+1}{n}}. \]
We note that condition (2.2)(iii) can be replaced by a condition on \( f \) which is more easily checked and which implies (2.2)(iii), namely,

\[
\lim_{s \to 0^+} \frac{f'(s)}{[f(s)]^{\frac{1}{n+1}}} = 0.
\]

We also remark that a condition that implies global existence rather than blow-up can be obtained from the computations here. From (2.6) and (2.11), we have

\[
\varphi'(t) \leq -\frac{4\beta}{n} \int_{\Omega} \nabla v^{\frac{n}{2}} \, dx + n |\Omega| \frac{n-2}{2n} C \frac{n+1}{n} \left\{ \int_{\Omega} |\nabla v^{\frac{n}{2}} |^2 \, dx \int v^n \, dx \right\}^{\frac{n+1}{2n}}
\]

\[
= - \left( \int_{\Omega} |\nabla v^{\frac{n}{2}} |^2 \, dx \right)^{\frac{n+1}{2n}} \left( \int_{\Omega} v^n \, dx \right)^{\frac{n-1}{2n}} \left\{ \frac{4\beta}{n} \lambda^{\frac{n-1}{2n}} |\Omega|^{\frac{n-2}{2n}} C \frac{n+1}{n} \left( \int_{\Omega} v^n \, dx \right)^{\frac{1}{n}} \right\}.
\]

Using the property

\[
\lambda \int_{\Omega} w^2 \, dx \leq \int_{\Omega} |\nabla w|^2 \, dx,
\]

where \( \lambda \) is the positive first eigenvalue of the fixed membrane problem, we obtain

\[
\varphi'(t) \leq - \left( \int_{\Omega} |\nabla v^{\frac{n}{2}} |^2 \, dx \right)^{\frac{n+1}{2n}} \left( \int_{\Omega} v^n \, dx \right)^{\frac{n-1}{2n}} \left\{ \frac{4\beta}{n} \lambda^{\frac{n-1}{2n}} |\Omega|^{\frac{n-2}{2n}} C \frac{n+1}{n} \left( \int_{\Omega} v^n \, dx \right)^{\frac{1}{n}} \right\}.
\]

Consequently, if

\[
|\Omega|^{\frac{n-2}{2n}} C \frac{n+1}{n} \left( \int_{\Omega} \left[ \int_{\Omega} \frac{d\eta}{f(\eta)} \right] d\xi \right)^{\frac{n}{\frac{n+1}{n}}} \leq \frac{4\beta}{n^2} \lambda^{\frac{n-1}{2n}},
\]

then \( \varphi(t) \) is decreasing in \( t \) and hence does not blow up.

In fact, using (2.18) in (2.19), we have

\[
\varphi'(t) \leq - \lambda^{\frac{n-1}{2n}} \varphi(t) \left\{ C_1 - C_2 \left[ \varphi(t) \right]^\frac{1}{\frac{n}{2}} \right\}
\]

for computable constants \( C_1 \) and \( C_2 \). This inequality is integrable and leads to

\[
\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\eta(C_1 - C_2 \eta^{\frac{1}{n}})} \leq -\lambda^{\frac{n+1}{2n}} t,
\]

or by means of \( \varphi = \psi^n \),

\[
n \int_{\varphi(0)}^{[\varphi(t)]^\frac{1}{\frac{n}{2}}} \frac{d\psi}{\psi(C_1 - C_2 \psi)} \leq -\lambda^{\frac{n+1}{2n}} t.
\]

The latter integration then implies the exponential decay of \( \varphi(t) \).
3. Second method

As an alternative method to determine a lower bound for the blow-up time \( t^* \), we introduce an auxiliary function which leads to a comparison procedure. We again consider problem (2.1) under the conditions (2.2) except that we replace (2.2)(iii) by

\[
\int_s^\infty \frac{d\eta}{f(\eta)} \to \infty \quad \text{as} \quad s \to 0^+.
\]

We define the function

\[
\psi(x,t) = \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-n}
\]

and compute

\[
\psi_t = n \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-n-1} \left[ f(u) \right]^{-1} u_t
\]

and

\[
\Delta \psi = n(n+1) \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-n-2} \left[ f(u) \right]^{-2} |\nabla u|^2
\]

\[
- n \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-n-1} \left[ f(u) \right]^{-2} f'(u) |\nabla u|^2
\]

\[
+ n \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-n-1} \left[ f(u) \right]^{-1} \Delta u.
\]

It follows by the differential equation in (2.1) and condition (2.2)(iv) that

\[
\psi_t - \Delta \psi \leq n \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-(n+1)} - n\beta \left( \int_u^\infty \frac{d\eta}{f(\eta)} \right)^{-(n+2)} \left[ f(u) \right]^{-2} |\nabla u|^2
\]

\[
\leq n \psi^{\frac{n+1}{n}}.
\]

which, for \( t < t^* \), can be written

\[
\psi_t \leq \Delta \psi + n \left[ \psi_M(t) \right]^{\frac{1}{n}} \psi,
\]

where

\[
\psi_M(t) = \max_{\Omega(t)} \psi(x,t).
\]

By means of an integrating factor, the function

\[
w(x,t) = \psi(x,t) \exp \left\{ -n \int_0^t \left[ \psi_M(\eta) \right]^{\frac{1}{n}} d\eta \right\}
\]

(3.3)
satisfies the initial-boundary value problem
\[ \begin{align*}
    w_t &\leq \Delta w \quad \text{in } \Omega \times (0, t^*), \\
    w & = 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
    w(x, 0) & = g(x) \quad \text{in } \Omega. 
\end{align*} \] (3.4)

We now introduce another function
\[ \chi(x, t) = \left[ \frac{g}{\varphi_1} \right]_M \varphi_1 e^{-\lambda_1 t}, \] (3.5)
where \( \lambda_1 \) is the first positive eigenvalue and \( \varphi_1 \) the associated eigenfunction of the fixed membrane problem
\[ \begin{align*}
    \Delta \varphi + \lambda \varphi & = 0 \quad \text{in } \Omega, \\
    \varphi & = 0 \quad \text{on } \partial \Omega, \\
    \varphi & > 0 \quad \text{in } \Omega, 
\end{align*} \] (3.6)

and
\[ \left[ \frac{g}{\varphi_1} \right]_M = \max_{\Omega} \left\{ \frac{g(x)}{\varphi_1(x)} \right\}. \]

It is easy to see that \( \chi \) satisfies
\[ \begin{align*}
    \chi_t - \Delta \chi & = 0 \quad \text{in } \Omega \times (0, t^*), \\
    \chi & = 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
    \chi(x, 0) & = \max_{\Omega} \left\{ \frac{g(x)}{\varphi_1(x)} \right\} \varphi_1 \quad \text{in } \Omega. 
\end{align*} \] (3.7)

From (3.4) and (3.7) it follows that
\[ \begin{align*}
    (w - \chi)_t - \Delta (w - \chi) & \leq 0 \quad \text{in } \Omega \times (0, t^*), \\
    w - \chi & = 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
    w - \chi & \leq 0 \quad \text{in } \Omega \times \{ t = 0 \}, 
\end{align*} \] (3.8)
and consequently, by the maximum principle, that
\[ \psi \exp \left\{ -n \int_0^t \left[ \psi_M(\eta) \right]_M^\frac{1}{n} \, d\eta \right\} \leq \left[ \frac{g}{\varphi_1} \right]_M \varphi_1 \exp(-\lambda_1 t). \]

Taking the \( n \)th root and normalizing \( \varphi_1 \) by
\[ \max_{\Omega} \varphi_1(x) = 1, \]
we have
\[ \psi \frac{1}{n} \exp \left\{ -n \int_0^t \left[ \psi_M(\eta) \right]_M^\frac{1}{n} \, d\eta \right\} \leq \left[ \frac{g}{\varphi_1} \right]_M^\frac{1}{n} \exp \left\{ -\frac{\lambda_1}{n} t \right\}. \]
We now evaluate at the point where $\psi$ takes its maximum so that

$$
\psi_{\text{M}}^\frac{1}{n} \exp\left\{-\int_0^t \left[\psi_M(\eta)\right]^\frac{1}{n} \, d\eta\right\} \leq \left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M \exp\left\{-\frac{\lambda_1}{n} t\right\}.
$$

Upon integration from 0 to $t$, we obtain

$$
1 - \exp\left\{-\int_0^t \left[\psi_M(\eta)\right]^\frac{1}{n} \, d\eta\right\} \leq \frac{n}{\lambda_1} \left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M \left(1 - \exp\left\{-\frac{\lambda_1}{n} t\right\}\right)
$$

and on taking the limit as $t \to t^*$,

$$
1 \leq \frac{n}{\lambda_1} \left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M \left(1 - e^{-\frac{\lambda_1}{n} t^*}\right). \tag{3.9}
$$

It follows that in order to have blow-up we must have

$$
\left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M > \frac{\lambda_1}{n},
$$

in which case we have a lower bound for blow-up time

$$
t^* \geq -\frac{n}{\lambda_1} \ln\left[1 - \frac{\lambda_1}{n} \left(\frac{g}{\varphi_1}\right)^{-\frac{1}{n}}_M\right]. \tag{3.10}
$$

A further consequence of (3.9) is that we have global existence in time if

$$
\left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M \leq \frac{\lambda_1}{n},
$$

We summarize our result in the following theorem.

**Theorem 3.1.** If $u$ is a nonnegative solution of (2.1) and

$$
\left[\frac{g}{\varphi_1}\right]^\frac{1}{n}_M > \frac{\lambda_1}{n},
$$

where $(\lambda_1, \varphi_1)$ is the first eigenpair in (3.6), then if $u$ blows up at $t = t^*$ a lower bound for $t^*$ is given by (3.10). If (3.10) does not hold, then a global solution exists.

We note that if $n = \lambda_1$, the conditions and bound in Theorem 3.1 can be simplified somewhat. We further remark that if the eigenpair $(\lambda_1, \varphi_1)$ is unknown for $\Omega$, then one may use the pair $(\tilde{\lambda}_1, \tilde{\varphi}_1)$ for any circumscribing region. In that case, both the function $\chi(x,t)$ in (3.7) may be nonnegative and $(w - \chi)(x,t)$ in (3.8) may be nonpositive on $\partial\Omega \times (0, t^*)$, and the $\lambda_1$ and $\varphi_1$ are replaced by $\tilde{\lambda}_1$ and $\tilde{\varphi}_1$, respectively, in the bound for $t^*$.

**References**


