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Approximating the Tutte polynomial of a binary matroid and other related combinatorial polynomials [☆]

Leslie Ann Goldberg ^{a,*}, Mark Jerrum ^b

^a Department of Computer Science, University of Liverpool, Ashton Building, Liverpool L69 3BX, United Kingdom

^b School of Mathematical Sciences Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

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ABSTRACT

We consider the problem of approximating certain combinatorial polynomials. First, we consider the problem of approximating the Tutte polynomial of a binary matroid with parameters $q \geq 2$ and γ . (Relative to the classical (x, y) parameterisation, $q = (x - 1)(y - 1)$ and $\gamma = y - 1$.) A graph is a special case of a binary matroid, so earlier work by the authors shows inapproximability (subject to certain complexity assumptions) for $q > 2$, apart from the trivial case $\gamma = 0$. The situation for $q = 2$ is different. Previous results for graphs imply inapproximability in the region $-2 \leq \gamma < 0$, apart from at two “special points” where the polynomial can be computed exactly in polynomial time. For binary matroids, we extend this result by showing (i) there is no FPRAS in the region $\gamma < -2$ unless $\text{NP} = \text{RP}$, and (ii) in the region $\gamma > 0$, the approximation problem is hard for the complexity class $\#\text{RHP}_1$ under approximation-preserving (AP) reducibility. The latter result indicates a gap in approximation complexity at $q = 2$: whereas an FPRAS is known in the graphical case, there can be none in the binary matroid case, unless there is an FPRAS for all of $\#\text{RHP}_1$. The result also implies that it is computationally difficult to approximate the weight enumerator of a binary linear code, apart from at the special weights at which the problem is exactly solvable in polynomial time. As a consequence, we show that approximating the cycle index polynomial of a permutation group is hard for $\#\text{RHP}_1$ under AP-reducibility, partially resolving a question that we first posed in 1992.

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1. Introduction

The multivariate Tutte polynomial (in q^{-1} and $\boldsymbol{\gamma}$) of a matroid \mathcal{M} with ground set E and rank function $r_{\mathcal{M}}$ is defined as follows (see [19, (1.3)])

$$\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma}) = \sum_{A \subseteq E} q^{-r_{\mathcal{M}}(A)} \prod_{e \in A} \gamma_e, \quad (1)$$

where $\boldsymbol{\gamma} = \{\gamma_e\}_{e \in E}$. The tilde in \tilde{Z}_{Tutte} is for consistency with [19]; in general we follow the notation used there, so we can conveniently access various useful identities.

An important class of matroids are the graphic matroids, i.e., those that arise as the cycle matroid $\mathcal{M}(G)$ of some graph G (see Section 1.2 for details). The Tutte polynomial of graphic matroids in particular has received much attention.

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* Corresponding author.

E-mail address: L.A.Goldberg@liverpool.ac.uk (L.A. Goldberg).

For convenience, we often speak about the Tutte polynomial of the graph G rather than the Tutte polynomial of its cycle matroid $\mathcal{M}(G)$. Indeed, this polynomial was first defined for graphs, and only later generalised to matroids.

The Tutte polynomial encodes a large quantity of combinatorial information about the matroid [14,19,21] and the complexity of computing the polynomial has been much studied [1,8–10,14,20]. One important motivation for this study of the Tutte polynomial is that it includes as a special case (when q is a positive integer) the problem of computing the partition function of the Potts model in statistical physics. To be more precise about the computational task, parameters q and γ are fixed, and the problem is to compute $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$ for an input matroid \mathcal{M} , where $\boldsymbol{\gamma}$ is the constant function with $\gamma_e = \gamma$ for every ground set element e . In order to introduce the topic, we assume in this introduction that the parameters q and γ are rational, though we shall see below that this can be generalised.

Jaeger, Vertigan and Welsh [14] investigated the complexity of exactly computing $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$ given an input matroid \mathcal{M} . They used a different parameterisation of the polynomial, but the problem that they studied is identical to the one that we describe here. For the record, they [14, (2.2)] define

$$T(\mathcal{M}; x, y) = \sum_{A \subseteq E} (x-1)^{r_{\mathcal{M}}(E) - r_{\mathcal{M}}(A)} (y-1)^{|A| - r_{\mathcal{M}}(A)}. \quad (2)$$

Now substitute $\gamma = y - 1$ and $q = (x - 1)(y - 1)$. Let $\boldsymbol{\gamma}$ be the constant function with $\gamma_e = \gamma$ for every $e \in E$. Assuming that $q \neq 0$ (hence $\gamma \neq 0$), Eq. (1) gives

$$T(\mathcal{M}; x, y) = (x-1)^{r_{\mathcal{M}}(E)} \tilde{Z}_{\text{Tutte}}(\mathcal{M}; (x-1)(y-1), \boldsymbol{\gamma}) = (q/\gamma)^{r_{\mathcal{M}}(E)} \tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma}). \quad (3)$$

Unfortunately, any two-parameter version of the Tutte polynomial will omit some points. On the one hand, setting $y = 1$ in (2) forces $q = (x - 1)(y - 1)$ to be 0 but setting $\gamma_e = \gamma = y - 1 = 0$ in (1) does not force q to be 0. On the other hand, the single point $q = 0, \gamma = 0$ in (1) corresponds to an entire line in the (x, y) -coordinate system, where $y = 1$ but x can have any value. For this reason, it is sometimes convenient [19, §2.3] to treat the $q \rightarrow 0$ case as a limit case. We will not need to do this here.

Jaeger et al. showed that, even when the input is restricted to be a *graphic* matroid, exact evaluation is #P-hard, apart from when $q = 1$ and at four “special points”. The first three of these are $(q, \gamma) = (4, -2), (2, -2)$ and $(2, -1)$. The fourth is the point $(x, y) = (1, 1)$ for which Eq. (3) is invalid due to division by 0 – evaluation at this point corresponding to counting spanning trees in the input graph. As already noted, the line $\gamma = 0$ is also easy in our parameterisation. Thus, exactly evaluating the Tutte polynomial of a graph is #P-hard, apart from when $q = 1$ and at these special points. Jaeger et al. also considered the case in which the parameters are complex numbers, where there are additional special points, but we do not consider this case here. As they noted, exact evaluation can be done in polynomial time for $q = 1$ and it can be done in polynomial time at some of the special points for large classes of matroids. For binary matroids, which are a focus of this paper, the first three special points at least are polynomial-time computable. A definition of binary matroid is given in Section 1.2.

Our earlier work [8,10] investigates the complexity of *approximately* computing $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$ when \mathcal{M} is restricted to be graphic. We are interested in determining for which points (q, γ) there is a *fully polynomial randomised approximation scheme* (FPRAS) for the polynomial. An FPRAS is a polynomial-time randomised approximation algorithm achieving arbitrarily small relative error. A precise definition is provided in Section 1.3. We survey the main results now, partly because we build on them in this article, and partly to highlight the differences in computational complexity between the graphic and binary cases.

For $q > 2$ we gave inapproximability results both for $\gamma < 0$ and for $\gamma > 0$. As already noted, the case $\gamma = 0$ is trivial. In the “antiferromagnetic” case $\gamma < 0$, we showed [8] that, apart from at the special point $(q = 4, \gamma = -2)$, there is no FPRAS for approximately evaluating the Tutte polynomial of a graph unless $\text{NP} = \text{RP}$. In the “ferromagnetic” case $\gamma > 0$, we showed [10] that the approximation problem is hard for the logically-defined complexity class $\#\text{RH}\Pi_1$ under approximation-preserving “AP-reductions”.

The complexity class $\#\text{RH}\Pi_1$ of counting problems was introduced by Dyer, Goldberg, Greenhill and Jerrum [5] as a means to classify a wide class of approximate counting problems that were previously of indeterminate computational complexity. The problems in $\#\text{RH}\Pi_1$ are those that can be expressed in terms of counting the number of models of a logical formula from a certain syntactically restricted class which is also known as “restricted Krom SNP” [4]. $\#\text{RH}\Pi_1$ has a completeness class with respect to AP-reductions which includes a wide range of natural counting problems – see Section 1.3 for some examples. Either all of these problems admit an FPRAS, or none do. No FPRAS is known for any of them at the time of writing, despite much effort having been expended on finding one. We conjecture that none exists. Proving counting problems to be hard for $\#\text{RH}\Pi_1$ with respect to AP-reductions is similar to working with the Unique Games Conjecture in the area of approximation algorithms for optimisation problems, or employing the class PPAD in analysing the complexity of Nash equilibria. Since a graphical matroid is a binary matroid, both of the hardness results for $q > 2$ mentioned earlier (for $\gamma < 0$ and for $\gamma > 0$) extend to the class of binary matroids.

The paper [8] also includes hardness results for $q < 2$ which extend to the binary matroid case. For example, there is no FPRAS unless $\text{NP} = \text{RP}$ if either γ or q/γ is less than -2 . The interested reader is referred to [8].

The situation is different for $q = 2$. In this case, we showed [8] that in the region $-2 < \gamma < 0$ (apart from at special points) there is no FPRAS for approximately evaluating the Tutte polynomial of a graph unless $\text{NP} = \text{RP}$. However, the most

that is known for $\gamma < -2$ (see [8]) is that the problem is as difficult as approximately counting perfect matchings in a graph, a well-known open problem. For $\gamma > 0$, the Tutte polynomial of a graph *can* be approximated efficiently – Jerrum and Sinclair have given an FPRAS [16]. In this paper, we show that the problem of approximating the Tutte polynomial of a binary matroid is apparently harder. In particular, we show in Theorem 1 that there is no FPRAS in the region $\gamma < -2$ unless $\text{NP} = \text{RP}$ and that the problem is hard for $\#\text{RHP}_1$ with respect to AP-reductions for $\gamma > 0$.

The results in Theorem 1 have interesting consequences for the problem of approximating related polynomials. It is well known that the Tutte polynomial of a binary matroid contains as a special case the weight enumerator of a binary linear code, which will be defined in Section 3. Hence, we immediately get a complexity classification (Corollary 7) for this problem. This, in turn, allows us to make progress on a long-standing open problem about the complexity of approximating the cycle index polynomial of a permutation group (see Section 4). We had previously shown [6] that there is no FPRAS for this problem, unless $\text{NP} = \text{RP}$, if the parameter, x , is a non-integer. Using our result about the weight enumerator of a binary linear code, we show that the cycle index polynomial is as difficult to approximate as $\#\text{RHP}_1$ for every parameter value $x > 1$ (Corollary 9). As we will explain in Section 4, it is at the integer points that the cycle index polynomial has combinatorial meaning.

1.1. Matroid preliminaries

A matroid \mathcal{M} is a combinatorial structure defined by a set E (the “ground set”) together with a “rank function” $r_{\mathcal{M}} : E \rightarrow \mathbb{N}$. The rank function satisfies the following conditions for all subsets $A, B \subseteq E$: (i) $0 \leq r_{\mathcal{M}}(A) \leq |A|$, (ii) $A \subseteq B$ implies $r_{\mathcal{M}}(A) \leq r_{\mathcal{M}}(B)$ (monotonicity), and (iii) $r_{\mathcal{M}}(A \cup B) + r_{\mathcal{M}}(A \cap B) \leq r_{\mathcal{M}}(A) + r_{\mathcal{M}}(B)$ (submodularity).

A subset $A \subseteq E$ satisfying $r_{\mathcal{M}}(A) = |A|$ is said to be *independent*. Every other subset $A \subseteq E$ is said to be *dependent*. A maximal (with respect to inclusion) independent set is a *basis*, and a minimal dependent set is a *circuit*. A circuit with one element is a *loop*.

Suppose that \mathcal{M} is a matroid with ground set E . Then \mathcal{M} is associated with a *dual matroid* \mathcal{M}^* with the same ground set E but rank function $r_{\mathcal{M}^*}$ given by $r_{\mathcal{M}^*}(A) = |A| + r_{\mathcal{M}}(E - A) - r_{\mathcal{M}}(E)$. A *cocircuit* in \mathcal{M} is a set that is a circuit in \mathcal{M}^* ; equivalently, a cocircuit is a minimal set that intersects every basis. A cocircuit with one element is a *coloop*.

We will use the matroid operations *contraction* and *deletion*. Suppose $e \in E$ is a member of the ground set of matroid \mathcal{M} . The *contraction* \mathcal{M}/e of e from \mathcal{M} is the matroid on ground set $E - \{e\}$ with rank function given by $r_{\mathcal{M}/e}(A) = r_{\mathcal{M}}(A \cup \{e\}) - r_{\mathcal{M}}(\{e\})$, for all $A \subseteq E - \{e\}$. The *deletion* $\mathcal{M} \setminus e$ of $\{e\}$ from \mathcal{M} is the matroid on ground set $E - \{e\}$ with rank function given by $r_{\mathcal{M} \setminus e}(A) = r_{\mathcal{M}}(A)$, for all $A \subseteq E - \{e\}$. We refer the reader to Oxley’s book [18] for a thorough exposition of matroid theory.

1.2. The Tutte polynomial of a binary matroid

Let M be a matrix over a field F with row set V and column set E . M “represents” a matroid \mathcal{M} with ground set E . The rank $r_{\mathcal{M}}(A)$ of a set of columns A in the matroid is defined to be the rank of the submatrix consisting of those columns. It is easy to see (see [18]) that a rank function defined in this way satisfies the three conditions (i)–(iii) for a matroid rank function presented in the previous subsection. Therefore, a set $A \subseteq E$ is dependent in the matroid if and only if the columns in A are linearly dependent as vectors. A matroid is said to be *representable* over the field F if it can be represented in this way. It is said to be *binary* if it is representable over $\text{GF}(2)$.

The *cycle matroid* of an undirected graph $G = (V, E)$ is the matroid $\mathcal{M}(G)$ represented by the vertex-edge incidence matrix M of G . In this case, $r_{\mathcal{M}(G)}(A) = |V| - \kappa(V, A)$ where $\kappa(V, A)$ is the number of connected components of the graph (V, A) . We simplify notation by writing $\tilde{Z}_{\text{Tutte}}(G; q, \boldsymbol{\gamma})$ in place of $\tilde{Z}_{\text{Tutte}}(\mathcal{M}(G); q, \boldsymbol{\gamma})$. Since the Tutte polynomial of a binary matroid generalises the Tutte polynomial of a graph, any hardness result for the latter immediately translates to the former. In this context, it should be noted that there is a slight mismatch between the definition of the Tutte polynomial given here, and the one used in the papers we cite, e.g., [10,19]. There, the Tutte polynomial of a graph is defined using the “random cluster” formulation:

$$Z_{\text{Tutte}}(G; q, \boldsymbol{\gamma}) = \sum_{A \subseteq E} q^{\kappa(V, A)} \prod_{e \in A} \gamma_e = q^{|V|} \tilde{Z}_{\text{Tutte}}(G; q, \boldsymbol{\gamma}). \quad (4)$$

(Note the absence of a tilde!) But since the two formulations differ only by an easily-computable factor $q^{|V|}$, all complexity results, whether about approximate or exact computation, translate directly.

For fixed real numbers q and γ we define the following computational problem, which is parameterised by q and γ .

Problem BINARYMATROIDTUTTE(q, γ).

Instance A matrix M over $\text{GF}(2)$ with rows V and columns E representing a binary matroid \mathcal{M} .

Output $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is the constant function with $\gamma_e = \gamma$ for every $e \in E$.

1.3. Standard definitions: approximation schemes and approximation-preserving reductions

We are interested in the complexity of *approximately* solving the problem BINARYMATROIDTUTTE(q, γ). We start with the relevant definitions. The reader who is already familiar with the complexity of approximate counting can skip this section. We use the presentation from [10].

A *randomised approximation scheme* is an algorithm for approximately computing the value of a function $f : \Sigma^* \rightarrow \mathbb{R}$. The approximation scheme has a parameter $\varepsilon > 0$ which specifies the error tolerance. A *randomised approximation scheme* for f is a randomised algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., for the problem $\text{BINARYMATROIDTUTTE}(q, \gamma)$, the input would be a matrix M over $\text{GF}(2)$ representing a binary matroid \mathcal{M}) and a rational error tolerance $\varepsilon > 0$, and outputs a rational number z (a random variable depending on the “coin tosses” made by the algorithm) such that, for every instance x , $\Pr[e^{-\varepsilon} f(x) \leq z \leq e^{\varepsilon} f(x)] \geq \frac{3}{4}$. The randomised approximation scheme is said to be a *fully polynomial randomised approximation scheme*, or *FPRAS*, if it runs in time bounded by a polynomial in $|x|$ and ε^{-1} . As in [10], we say that a real number z is *efficiently approximable* if there is an FPRAS for the constant function $f(x) = z$.

Our main tool for understanding the relative difficulty of approximation counting problems is *approximation-preserving reductions*. We use Dyer, Goldberg, Greenhill and Jerrum’s notion of approximation-preserving reduction [5]. Suppose that f and g are functions from Σ^* to \mathbb{R} . An “approximation-preserving reduction” from f to g gives a way to turn an FPRAS for g into an FPRAS for f . Here is the definition. An *approximation-preserving reduction* from f to g is a randomised algorithm \mathcal{A} for computing f using an oracle for g . The algorithm \mathcal{A} takes as input a pair $(x, \varepsilon) \in \Sigma^* \times (0, 1)$, and satisfies the following three conditions: (i) every oracle call made by \mathcal{A} is of the form (w, δ) , where $w \in \Sigma^*$ is an instance of g , and $0 < \delta < 1$ is an error bound satisfying $\delta^{-1} \leq \text{poly}(|x|, \varepsilon^{-1})$; (ii) the algorithm \mathcal{A} meets the specification for being a randomised approximation scheme for f (as described above) whenever the oracle meets the specification for being a randomised approximation scheme for g ; and (iii) the run-time of \mathcal{A} is polynomial in $|x|$ and ε^{-1} .

If an approximation-preserving reduction from f to g exists we write $f \leq_{\text{AP}} g$, and say that f is *AP-reducible to g* . Note that if $f \leq_{\text{AP}} g$ and g has an FPRAS then f has an FPRAS. (The definition of AP-reduction was chosen to make this true). If $f \leq_{\text{AP}} g$ and $g \leq_{\text{AP}} f$ then we say that f and g are *AP-interreducible*, and write $f \equiv_{\text{AP}} g$. A word of warning about terminology: Subsequent to [5], the notation \leq_{AP} has been used to denote a different type of approximation-preserving reduction which applies to optimisation problems. We will not study optimisation problems in this paper, so hopefully this will not cause confusion.

Dyer et al. [5] studied counting problems in $\#P$ and identified three classes of counting problems that are interreducible under approximation-preserving reductions. The first class, containing the problems that admit an FPRAS, are trivially AP-interreducible since all the work can be embedded into the reduction (which declines to use the oracle). The second class is the set of problems that are AP-interreducible with $\#\text{SAT}$, the problem of counting satisfying assignments to a Boolean formula in CNF. Zuckerman [22] has shown that $\#\text{SAT}$ cannot have an FPRAS unless $\text{RP} = \text{NP}$. The same is obviously true of any problem to which $\#\text{SAT}$ is AP-reducible.

The third class appears to be of intermediate complexity. It contains all of the counting problems expressible in a certain logically-defined complexity class, $\#\text{RH}\Pi_1$. Typical complete problems include counting the downsets in a partially ordered set [5], computing the partition function of the ferromagnetic Ising model with local external magnetic fields [7], and counting the independent sets in a bipartite graph, which is defined as follows.

Problem $\#\text{BIS}$.

Instance A bipartite graph B .

Output The number of independent sets in B .

In [5] it was shown that $\#\text{BIS}$ is complete for the logically-defined complexity class $\#\text{RH}\Pi_1$ with respect to approximation-preserving reductions. We conjecture that there is no FPRAS for $\#\text{BIS}$.

2. Approximating the Tutte polynomial of a binary matroid

This section provides the proof of the following theorem.

Theorem 1. *Suppose that $q \geq 2$ and γ are efficiently approximable.*

- (1) *If $\gamma = 0$, or if (q, γ) is one of the special points $(4, -2)$, $(2, -2)$ or $(2, -1)$, then $\text{BINARYMATROIDTUTTE}(q, \gamma)$ can be solved exactly in polynomial time.*
- (2) *Otherwise, if $\gamma < 0$ then there is no FPRAS for $\text{BINARYMATROIDTUTTE}(q, \gamma)$ unless $\text{NP} = \text{RP}$.*
- (3) *If $\gamma > 0$ then $\#\text{BIS} \leq_{\text{AP}} \text{BINARYMATROIDTUTTE}(q, \gamma)$.*

Some of the parts of the theorem follow from our earlier work in [8] and [10]. The main new result is item (3). Its proof follows from

- (1) the AP-reduction from $\#\text{BIS}$ to $\text{UNIFORMHYPERTUTTE}(2, 1)$ from our paper [10],
- (2) an AP-reduction from $\text{UNIFORMHYPERTUTTE}(2, 1)$ to the problem of computing the Tutte polynomial of a binary matroid, where the values γ_e depend on the input (Lemma 3), and
- (3) implementation of these values γ_e using series-parallel extensions on binary matroids (Lemma 6).

The details are in the following sections.

2.1. The Tutte polynomial of a uniform hypergraph

We have seen one possible generalisation of the Tutte polynomial of a graph, namely to binary matroids. Another natural generalisation is to hypergraphs. The two generalisations are different, but the relationship between them is interesting, and will be exploited in one of our reductions.

It is typical to define a hypergraph as a pair $(\mathcal{V}, \mathcal{E})$ in which \mathcal{V} is a set of vertices, and \mathcal{E} is a set of non-empty subsets of \mathcal{V} , called hyperedges. For our work on the Tutte polynomial, it will be more convenient to extend this definition. Thus, we will use the term “hypergraph” to refer to a pair $(\mathcal{V}, \mathcal{E})$ in which \mathcal{V} is a set of vertices, and \mathcal{E} is a *multiset* of non-empty subsets of \mathcal{V} , called hyperedges. The reason that the collection \mathcal{E} of hyperedges is a multiset, rather than a set, is that it is useful for the Tutte polynomial to allow “parallel” edges so that certain operations, such as parallel extensions, which we shall define below, can be freely applied [19]. A hypergraph is *uniform* if all hyperedges have the same cardinality.

Let $H = (\mathcal{V}, \mathcal{E})$ be a hypergraph. The multivariate Tutte polynomial of H was studied (under a different name) by Grimmett [12]. A definition can be found, for example, in [10]. In this paper we will use the Potts model version. Suppose that q is a positive integer and that $\boldsymbol{\gamma} = \{\gamma_f\}_{f \in \mathcal{E}}$. Let

$$Z_{\text{Potts}}(H; q, \boldsymbol{\gamma}) = \sum_{\sigma: \mathcal{V} \rightarrow [q]} \prod_{f \in \mathcal{E}} (1 + \gamma_f \delta(\{\sigma(v) \mid v \in f\})), \quad (5)$$

where $[q] = \{0, \dots, q-1\}$ is a set of q spins or colours, and $\delta(S)$ is 1 if its argument is a singleton and 0 otherwise. Identity (5) extends the Tutte polynomial from graphs to hypergraphs, but only for positive integer q . It is possible to provide a formulation for general q along the lines of (4), but this is not needed in what follows.

We consider the following computational problem.

Problem UNIFORMHYPERTUTTE($q, \boldsymbol{\gamma}$).

Instance A uniform hypergraph $H = (\mathcal{V}, \mathcal{E})$.

Output $Z_{\text{Potts}}(H; q, \boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is the constant function with $\gamma_f = \gamma$ for every $f \in \mathcal{E}$.

The will use the following lemma, which is an easy consequence of the results of [10].

Lemma 2. #BIS \leq_{AP} UNIFORMHYPERTUTTE(2, 1).

Proof. This follows from Observation 2, Lemma 14 and Lemma 15 of [10]. We note that [10] stated a more general definition of “hypergraph” in which hyperedges were taken to be multisets, rather than sets. Nevertheless, the construction in Lemma 15 actually produces a hypergraph that conforms to the definition that we use here. \square

2.2. A Potts model characterisation

Just as we used the Potts model version of the multivariate Tutte polynomial of a hypergraph, it will be helpful to have a representation of $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$ in terms of the (multivariate) partition function of the Potts model. See [19, Theorem 3.1]. Let \mathcal{M} be a matroid represented by a matrix M over $\text{GF}(q)$ with rows V and columns E . For every column $e \in E$, let Λ_e be the linear equation $\sum_{i \in V} M_{i,e} \sigma(i) = 0$ where the arithmetic is in $\text{GF}(q)$. The Potts partition function of \mathcal{M} is defined as follows:

$$Z_{\text{Potts}}(\mathcal{M}; q, \boldsymbol{\gamma}) = \sum_{\sigma: V \rightarrow [q]} \prod_{e \in E} (1 + \gamma_e \delta_e(\sigma)), \quad (6)$$

where

$$\delta_e(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ satisfies } \Lambda_e, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $Z_{\text{Ising}}(\mathcal{M}; \boldsymbol{\gamma})$ be a synonym for $Z_{\text{Potts}}(\mathcal{M}; 2, \boldsymbol{\gamma})$. The Ising model is the special case $q = 2$ of the Potts model. Note that

$$\begin{aligned} Z_{\text{Ising}}(\mathcal{M}; \boldsymbol{\gamma}) &= \sum_{\sigma: V \rightarrow \{0,1\}} \sum_{A \subseteq E} \prod_{e \in A} \gamma_e \delta_e(\sigma) \\ &= \sum_{A \subseteq E} \gamma_A \sum_{\sigma: V \rightarrow \{0,1\}} \prod_{e \in A} \delta_e(\sigma), \end{aligned}$$

where $\gamma_A = \prod_{e \in A} \gamma_e$. The number of configurations $\sigma: V \rightarrow \{0, 1\}$ for which $\prod_{e \in A} \delta_e(\sigma) = 1$ is the number of solutions to the system of linear equations $\Lambda_A = \{\Lambda_e \mid e \in A\}$, which we denote $\#\Lambda_A$. Thus,

$$\begin{aligned}
 Z_{\text{Ising}}(\mathcal{M}; \boldsymbol{\gamma}) &= \sum_{A \subseteq E} \gamma_A \# \Lambda_A \\
 &= \sum_{A \subseteq E} \gamma_A 2^{|V| - r_{\mathcal{M}}(A)} \\
 &= 2^{|V|} \tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}).
 \end{aligned} \tag{7}$$

It is interesting to compare definitions (5) and (6) in the case $q = 2$ to see how they both arise as natural generalisations of the Ising partition function of a graph. In the classical Ising model on a graph, each edge (u, v) contributes a factor depending on whether $\sigma(u)$ and $\sigma(v)$ are equal. If we think of this condition as asserting that the edge (u, v) is monochromatic, then the extension (5) to hypergraphs is immediate. On the other hand, we can equally think of the same condition as asserting $\sigma(u) + \sigma(v) = 0 \pmod{2}$, which leads us naturally to definition (6) for binary matroids.

2.3. Reduction from UNIFORMHYPERTUTTE(2, γ)

Consider the following computational problem, which is similar to BINARYMATROIDTUTTE(q, γ) except that the weight γ is part of the input.

Problem VARBINARYMATROIDTUTTE(q).

Instance A matrix M over GF(2) with rows V and columns E representing a binary matroid \mathcal{M} . A positive integer N , given in unary.

Output $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is the constant function with $\gamma_e = 2^{2^N} - 1$ for every $e \in E$.

Lemma 3. UNIFORMHYPERTUTTE(2, 1) \leq_{AP} VARBINARYMATROIDTUTTE(2).

Proof. Let $H = (\mathcal{V}, \mathcal{E})$ be a t -uniform hypergraph, an instance of UNIFORMHYPERTUTTE(2, 1). Without loss of generality, assume $t > 2$ since the result is immediate for $t = 2$ (since a 2-uniform hypergraph is a graph, and a graphic matroid is binary). Let $n = |\mathcal{V}|$ and $m = |\mathcal{E}|$ and assume that these are sufficiently large. Let $\mathbf{1}$ be the constant function which maps every $f \in \mathcal{E}$ to 1. By definition,

$$Z_{\text{Potts}}(H; 2, \mathbf{1}) = \sum_{\sigma: \mathcal{V} \rightarrow \{0,1\}} 2^{\text{mono}(\sigma)},$$

where $\text{mono}(\sigma)$ denotes the number of hyperedges $f \in \mathcal{E}$ that are monochromatic in configuration σ .

Let ε be the desired accuracy of the AP-reduction and let $\delta = \varepsilon / (m \ln 2)$. Let N be any positive integer satisfying

$$N \geq \frac{6m^2(n + \ln(16m))}{\varepsilon^2}.$$

We will construct an $n \times Nm$ matrix M so that N and M constitute an input to VARBINARYMATROIDTUTTE(2). The rows of M correspond to the elements of \mathcal{V} . The matrix contains N columns, f_1, \dots, f_N , for each hyperedge $f \in \mathcal{E}$. When we construct the matrix, we choose each of these columns to be the indicator vector for an even-sized subset of f , chosen independently and uniformly at random.

Given our construction, it is easy to see that a configuration σ which is monochromatic on f will satisfy the equations corresponding to all N columns f_1, \dots, f_N . For this, it is important that the random subsets of f corresponding to these columns have even size since the relevant equation Λ_{f_j} is $\sum_{i \in \mathcal{V}} M_{i, f_j} \sigma(i) = 0 \pmod{2}$.

Suppose that a configuration σ is not monochromatic on a hyperedge f and that it assigns ℓ elements of f to spin 1 and k elements of f to spin 0 for positive integers ℓ and k . Note that $\ell + k = t$ since the hypergraph is t -uniform. The number of even-sized subsets of f is 2^{t-1} and the number of even-sized subsets for which σ restricted to that subset has an even number of 1s is $2^{\ell-1} 2^{k-1} = 2^{t-2}$. Thus, the probability that the equation associated with a column f_i is satisfied is $1/2$. So, by a Chernoff bound, the number of columns f_i with satisfied equations is with high probability in the range $[(N/2)(1 - \delta), (N/2)(1 + \delta)]$. Specifically, the failure probability for this event is at most

$$2 \exp(-\delta^2 N/6) \leq 2 \exp(-n - \ln 16m) \leq 2 \times 2^{-n} \times \frac{1}{16m} = \frac{1}{8m2^n}$$

[17, Cor. 4.6]. From the union bound – ranging over events indexed by the 2^n choices for σ and m choices of column f – we conclude that, with probability at least $7/8$, the following is true. For every configuration σ , for every hyperedge $f \in \mathcal{E}$ on which σ is not monochromatic, the number of columns in $\{f_1, \dots, f_N\}$ with equations satisfied by σ is in the range $[(N/2)(1 - \delta), (N/2)(1 + \delta)]$.

Let \mathcal{M} be the binary matroid represented by M . Let $y = 2^{2^N}$ and $\gamma = y - 1$. Let $\boldsymbol{\gamma}$ be the constant function which maps every element of the ground set of \mathcal{M} to γ as in the definition of VARBINARYMATROIDTUTTE. To complete the verification of our reduction, we need to show that $Z_{\text{Potts}}(H; 2, \mathbf{1})$ may be easily computed given $\tilde{Z}_{\text{Tutte}}(\mathcal{M}, 2, \boldsymbol{\gamma})$; note that the latter

quantity is equal to $2^{-|V|} Z_{\text{Ising}}(\mathcal{M}; \gamma)$ by (7). Now the contribution of a configuration σ to the quantity $2^m Z_{\text{Potts}}(H; 2, \mathbf{1})$ is $2^m 2^{\text{mono}(\sigma)}$. Let Ψ_σ be the contribution of σ to $Z_{\text{Ising}}(\mathcal{M}; \gamma)$. Then

$$\begin{aligned} \Psi_\sigma &\leq y^{N \text{mono}(\sigma)} y^{(m - \text{mono}(\sigma))(N/2)(1+\delta)} \\ &\leq y^{mN(1+\delta)/2} y^{(N/2) \text{mono}(\sigma)} = 2^{m\delta} 2^m 2^{\text{mono}(\sigma)}. \end{aligned}$$

Also, we get a similar lower bound.

$$\begin{aligned} \Psi_\sigma &\geq y^{N \text{mono}(\sigma)} y^{(m - \text{mono}(\sigma))(N/2)(1-\delta)} \\ &\geq y^{mN(1-\delta)/2} y^{(N/2) \text{mono}(\sigma)} = 2^{-m\delta} 2^m 2^{\text{mono}(\sigma)}. \end{aligned}$$

The reduction has the desired accuracy for an AP-reduction, since $2^{m\delta} = e^\varepsilon$. Assume that the oracle call to `VARBINARY MATROIDTUTTE(2)` is powered to have failure probability $\frac{1}{8}$. Then the overall failure probability is bounded by $\frac{1}{4}$, being the sum of $\frac{1}{8}$ from the randomised nature of the reduction itself, and $\frac{1}{8}$ from the single oracle call. \square

2.4. Series-parallel extensions of binary matroids

The standard method for reducing the problem of evaluating the Tutte polynomial with some weight γ' to a Tutte-polynomial evaluation problem with a different weight γ is to “implement” the edge weight γ' using series and parallel extensions of weight- γ edges. See [19, Section 2.3] and [10, Section 10]. Series and parallel extensions of matroids are generalisations of the stretchings and thickenings of graphs used by Jaeger et al. [14]. Sokal [19, Section 2.3] has given the details, both for graphs and for general matroids. It is fairly easy to show that these extensions can be done within the class of binary matroids. We do this here.

Lemma 4 (Parallel extension for binary matroids). *Let \mathcal{M} be a binary matroid represented by the matrix M with rows V and columns E . Let $\gamma = \{\gamma_e\}_{e \in E}$. Let c be any column in E . Suppose $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfy*

$$1 + \gamma_c = (1 + \gamma_1)(1 + \gamma_2). \quad (8)$$

Let e' be a new column. Define $\gamma' = \{\gamma'_e\}_{e \in E \cup \{e'\}}$ as follows. Let $\gamma'_{e'} = \gamma_2$ and $\gamma'_c = \gamma_1$. For every other column $e \in E$, let $\gamma'_e = \gamma_e$. There is a binary matroid \mathcal{M}' represented by a matrix M' with $|V|$ rows and columns $E \cup \{e'\}$ for which $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \gamma) = \tilde{Z}_{\text{Tutte}}(\mathcal{M}'; q, \gamma')$.

Proof. Let M' be the matrix constructed from M by making column e' a copy of column c . Let \mathcal{M}' be the matroid represented by M' . Note that, for any $A \subseteq E$, $r_{\mathcal{M}}(A) = r_{\mathcal{M}'}(A)$. Thus, from the definitions in Section 1.1, $\mathcal{M} = \mathcal{M}' \setminus e'$. The result now follows from [19, (4.22)], provided that we can show that either (i) c and e' form a two-element circuit of \mathcal{M}' , or (ii) c and e' are both loops of \mathcal{M}' . (These are the two side-conditions for the application of [19, (4.22)].) Now, if c is not the all-zero vector, then c and e' do form a two-element circuit (minimal dependent set) of \mathcal{M}' since $r_{\mathcal{M}'}(\{c, e'\}) = r_{\mathcal{M}'}(\{c\}) = r_{\mathcal{M}'}(\{e'\}) = 1$. Otherwise, $r_{\mathcal{M}'}(\{c\}) = r_{\mathcal{M}'}(\{e'\}) = 0$, so both c and e' are loops of \mathcal{M}' . \square

Lemma 5 (Series extension for binary matroids). *Let \mathcal{M} be a binary matroid represented by the matrix M with rows V and columns E . Let $\gamma = \{\gamma_e\}_{e \in E}$. Let c be any column in E . Assume $q \neq 0$, and suppose $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfy*

$$\left(1 + \frac{q}{\gamma_c}\right) = \left(1 + \frac{q}{\gamma_1}\right) \left(1 + \frac{q}{\gamma_2}\right). \quad (9)$$

Let e' be a new column. Define $\gamma' = \{\gamma'_e\}_{e \in E \cup \{e'\}}$ as follows. Let $\gamma'_{e'} = \gamma_2$ and $\gamma'_c = \gamma_1$. For every other column $e \in E$, let $\gamma'_e = \gamma_e$. There is a binary matroid \mathcal{M}' represented by a matrix M' with $|V| + 1$ rows and columns $E \cup \{e'\}$ for which $(1 + \gamma_1/q + \gamma_2/q) \tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \gamma) = \tilde{Z}_{\text{Tutte}}(\mathcal{M}'; q, \gamma')$.

Proof. Let M' be the matrix constructed from M by adding a new column e' and a new row r' . The new row has ones in column c and column e' only. There are no other ones in column e' . Let \mathcal{M}' be the matroid represented by M' . We first show that, for any $A \subseteq E$,

$$r_{\mathcal{M}}(A) = r_{\mathcal{M}'}(A \cup e') - 1. \quad (10)$$

Eq. (10) can be verified by checking three cases

- $A \subseteq E - \{c\}$,
- $c \in A$ and $r_{\mathcal{M}}(A) = r_{\mathcal{M}'}(A - \{c\})$, and
- $c \in A$ and $r_{\mathcal{M}}(A) = r_{\mathcal{M}'}(A - \{c\}) + 1$.

(The conditions on the rank function of a matroid guarantee that these are the only three cases.) Now note that $r_{\mathcal{M}}(\{e'\}) = 1$, so (10) implies that $r_{\mathcal{M}}(A) = r_{\mathcal{M}}(A \cup \{e'\}) - r_{\mathcal{M}}(\{e'\})$. We conclude (from the definitions in Section 1.1) that $\mathcal{M} = \mathcal{M}'/e'$. The result now follows from [19, (4.28)], provided that we can show either (i) c and e' form a cocircuit of \mathcal{M}' , or (ii) c and e' are both coloops of \mathcal{M}' . (These are the two side-conditions for the application of [19, (4.28)].)

Suppose first that c is a coloop of \mathcal{M} (i.e., an element that is present in every basis of \mathcal{M}). Consider any independent set $A \subseteq E - \{c\}$ of \mathcal{M}' . Since c is linearly independent of the columns in A in M , both c and e' are linearly independent of the columns in A in M' . Thus, both c and e' are coloops of \mathcal{M}' and we have (ii).

Finally, suppose that c is not a coloop of \mathcal{M} . Our goal is to prove (i). First, any independent set in \mathcal{M}' including neither c nor e' can be extended to a larger independent set by adding either one of c or e' , so every basis of \mathcal{M}' intersects $\{c, e'\}$. However, since c is not a coloop of \mathcal{M} , there is a basis B of \mathcal{M} that does not include c . Then $B \cup \{c\}$ and $B \cup \{e'\}$ are both bases of \mathcal{M}' and so $\{c, e'\}$ is a minimal set that intersects every basis of \mathcal{M}' . Thus, we have (i). \square

2.5. Implementing variable weights

Lemma 6. *Suppose that $\gamma > 0$ is efficiently approximable. Then $\text{VARBINARYMATROIDTUTTE}(2) \leq_{\text{AP}} \text{BINARYMATROIDTUTTE}(2, \gamma)$.*

Proof. Let C_γ be a sufficiently large function of the parameter γ . The exact computation of C_γ is from [10]. This will be explained below.

Let M and N be an instance of $\text{VARBINARYMATROIDTUTTE}(2)$. Let \mathcal{M} be the matroid represented by M . Suppose that M has n rows and m columns and that the product Nm is sufficiently large with respect to the constant C_γ . Let $\gamma' = 2^{2/N} - 1$. Let γ^* be the constant function which maps every ground set element of \mathcal{M} to γ^* .

The proof is based on the proof of [10, Lemma 17]. Let ε be the desired accuracy in the approximation-preserving reduction. Let

$$\chi = \frac{\varepsilon^2}{4C_\gamma m^2 N}.$$

Let $\hat{\gamma}$ be a rational in the range $e^{-\chi}\gamma \leq \hat{\gamma} \leq e^{\chi}\gamma$. Since γ is efficiently approximable, the amount of time that it takes to compute $\hat{\gamma}$ is at most a polynomial in m , N and ε^{-1} .

The idea of the proof is to show how to use series and parallel extensions of weight- $\hat{\gamma}$ elements to implement weight γ^* satisfying

$$e^{-\chi}\gamma' \leq \gamma^* \leq e^{\chi}\gamma'. \quad (11)$$

Let $\boldsymbol{\gamma}^*$ be the constant function which maps every ground set element of \mathcal{M} to γ^* . The definition of \tilde{Z}_{Tutte} and the fact that $\chi \leq \varepsilon/(4m)$ imply that

$$e^{-\varepsilon/4}\tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}') \leq \tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}^*) \leq e^{\varepsilon/4}\tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}').$$

Let $\hat{\boldsymbol{\gamma}}$ be the constant function which maps every ground set element to $\hat{\gamma}$. We can think of our implementations as constructing a binary matroid $\widehat{\mathcal{M}}$ such that $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}^*)$ is equal to the product of $\tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \hat{\boldsymbol{\gamma}})$ and an easily-computed function of $\hat{\boldsymbol{\gamma}}$. This easily-computed function arises from the extra factor $(1 + \gamma_1/q + \gamma_2/q)$ in Lemma 5. We will ensure that the matroid $\widehat{\mathcal{M}}$ has at most $C_\gamma m^2 N/\varepsilon$ ground set elements. To finish, we note, using the definition of \tilde{Z}_{Tutte} and the definition of χ , that

$$e^{-\varepsilon/4}\tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \boldsymbol{\gamma}) \leq \tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \hat{\boldsymbol{\gamma}}) \leq e^{\varepsilon/4}\tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma}$ is the constant weight function which assigns every element weight γ . We finish the approximation of $\tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \hat{\boldsymbol{\gamma}})$ by using the oracle to approximate $\tilde{Z}_{\text{Tutte}}(\widehat{\mathcal{M}}; 2, \boldsymbol{\gamma})$ using accuracy parameter $\delta = \varepsilon/2$.

It remains to show how to do the implementation. Take

$$\pi = \frac{\chi}{2}(2^{2/N} - 1) = \frac{\chi}{2}\gamma' \leq \gamma'(1 - e^{-\chi}).$$

The proof of [10, Lemma 17] shows how to use series and parallel extensions of weight $\hat{\boldsymbol{\gamma}}$ elements (from Lemmas 4 and 5) to implement a weight γ^* which satisfies $\gamma' - \pi \leq \gamma^* \leq \gamma'$. This ensures that Eq. (11) holds. The series and parallel extensions in the implementation of γ^* introduce at most $C_\gamma \log(\pi^{-1})$ ground set elements, where C_γ is some quantity depending on γ but not on n , m , or N . Note that $2^{2/N} - 1 \geq 2 \ln(2)/N$ so $\pi^{-1} \leq 2N/\chi$. Thus, $mC_\gamma \log(\pi^{-1}) \leq C_\gamma m^2 N/\varepsilon$ and the matroid $\widehat{\mathcal{M}}$ has at most $C_\gamma m^2 N/\varepsilon$ ground set elements, as required above. \square

2.6. The proof of Theorem 1

Proof of Theorem 1. Item (1) comes from [14]. Item (2) follows quickly from [8] together with an application of matroid duality; here are the details. When $q > 2$, item (2) follows from the corresponding hardness result for graphs, since binary

matroids generalise graphic matroids. The same is true when $q = 2$ and $\gamma > -2$. The complementary case $q = 2$ and $\gamma < -2$ then follows by matroid duality, as can be seen by combining the following observations: Binary matroids are closed under duality [18, 2.2.9] and the representation of the dual of a binary matroid can be constructed efficiently [18, 2.2.8]. Also, if \mathcal{M}^* is the dual of \mathcal{M} then $\tilde{Z}_{\text{Tutte}}(\mathcal{M}; q, \gamma)$ is an easily-computed multiple of $\tilde{Z}_{\text{Tutte}}(\mathcal{M}^*, q, q/\gamma)$ – see [19, (4.14)]. Finally for $q = 2$, if $\gamma < -2$ then $-2 < q/\gamma < 0$.

Item (3) follows from [10, Theorem 1] for $q > 2$ since the cycle matroid of a graph is a binary matroid. For $q = 2$ it follows from Lemmas 2, 3 and 6. \square

3. The weight enumerator of a binary linear code

Given a *generating matrix* M over $\text{GF}(2)$ with r linearly independent rows and c columns, a *code word* w is any vector in the linear subspace \mathcal{Y} generated by the rows of M . For any real number λ , the *weight enumerator* of the code is given by $W_M(\lambda) = \sum_{w \in \mathcal{Y}} \lambda^{\|w\|}$, where $\|w\|$ is the number of non-zero entries in w . We consider the following computational problem, parameterised by λ .

Problem $\text{WE}(\lambda)$.

Instance A generating matrix M over $\text{GF}(2)$.

Output $W_M(\lambda)$.

It is well known (see below) that the weight enumerator of a binary linear code is a special case of the Tutte polynomial of a binary matroid. Thus, Theorem 1 has the following corollary.

Corollary 7. *Suppose that λ is efficiently approximable.*

- (1) *If $\lambda \in \{-1, 0, 1\}$ then $\text{WE}(\lambda)$ is solvable in polynomial time.*
- (2) *If $|\lambda| > 1$ then there is no FPRAS for $\text{WE}(\lambda)$ unless $\text{NP} = \text{RP}$.*
- (3) *If $\lambda \in (-1, 0)$ then there is no FPRAS for $\text{WE}(\lambda)$ unless $\text{NP} = \text{RP}$.*
- (4) *If $\lambda \in (0, 1)$ then $\#\text{BIS} \leq_{\text{AP}} \text{WE}(\lambda)$.*

Corollary 7 follows immediately from Theorem 1 and from Lemma 8 below, which is the $q = 2$ case of a result of Greene [11, Corollary 4.5]. See also Cameron [3, Theorem 4.1], but note that both authors employ a different parameterisation of the Tutte polynomial. We provide a short proof here for completeness, since we have already done almost all of the necessary work.

Lemma 8 (Greene). *Let M be a generating matrix over $\text{GF}(2)$ with rows V and columns E . Let \mathcal{M} be the binary matroid represented by M . Let λ be any non-zero real number and let $\gamma = 1/\lambda - 1$. Let $\boldsymbol{\gamma}$ be the constant function with $\gamma_e = \gamma$ for every column e of M . Then*

$$W_M(\lambda) = \lambda^{|E|} 2^{|V|} \tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}).$$

Proof. As in the proof of Lemma 3, we have, from Eqs. (6) and (7),

$$2^{|V|} \tilde{Z}_{\text{Tutte}}(\mathcal{M}; 2, \boldsymbol{\gamma}) = Z_{\text{Ising}}(\mathcal{M}; \boldsymbol{\gamma}) = \sum_{\sigma: V \rightarrow \{0,1\}} (1 + \gamma)^{\text{sat}(\sigma)} = \sum_{\sigma: V \rightarrow \{0,1\}} \lambda^{-\text{sat}(\sigma)}, \quad (12)$$

where $\text{sat}(\sigma)$ denotes the number of columns $e \in E$ such that

$$\sum_{i \in V} M_{i,e} \sigma(i) = 0 \pmod{2},$$

or, using the notation introduced earlier, $\delta_e(\sigma) = 1$. Similarly,

$$W_M(\lambda) = \sum_{\sigma: V \rightarrow \{0,1\}} \prod_{e \in E} \lambda^{1 - \delta_e(\sigma)} = \sum_{\sigma: V \rightarrow \{0,1\}} \lambda^{|E| - \text{sat}(\sigma)}, \quad (13)$$

where, corresponding to σ , the code word w is the sum of the rows i with $\sigma(i) = 1$, so $1 - \delta_e(\sigma)$ is the bit in position e of the code word. The result follows by comparing (12) and (13). \square

4. The cycle index polynomial

Let Γ be a group of permutations of $\{1, \dots, v\}$. Each permutation $g \in \Gamma$ decomposes the set $\{1, \dots, v\}$ into a collection of cycles. $\text{cyc}(g)$ denotes the number of cycles in this decomposition. The single-variable *cycle index polynomial* of Γ is defined as follows.

$$Z_{\text{Cl}}(\Gamma; x) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} x^{\text{cyc}(g)}.$$

When x is a positive integer, $Z_{\text{Cl}}(\Gamma; x)$ counts the number of *orbits* (or equivalence classes) when strings from a size- x alphabet are operated on by permutations in Γ (which permute the character positions in the strings). Two strings are in the same equivalence class if there is a permutation in Γ which maps one into the other. For example, when $x = 2$, $v = 3$, and Γ is the symmetric group (on 3 elements), the elements of Γ are the identity permutation (which has 3 cycles), the three transpositions (1 2), (1 3) and (2 3) (each of which has 2 cycles), and the singleton cycles (1 2 3) and (1 3 2). Thus

$$Z_{\text{Cl}}(\Gamma; 2) = \frac{1}{6}(2^3 + 3 \times 2^2 + 2 \times 2^1) = 4.$$

Thus, there are four orbits (namely the orbits of the strings 000, 001, 011, and 111). For more details, see [2,13,15]. We consider the following problem, in which the parameter x is a positive real number.

Problem CYCLEINDEX(x).

Instance A set of generators for a permutation group Γ .

Output $Z_{\text{Cl}}(\Gamma; x)$.

We showed [6, Theorem 4] that if x is not an integer then there is no FPRAS for CYCLEINDEX(x) unless $\text{RP} = \text{NP}$. In fact, it is NP-hard to approximate $Z_{\text{Cl}}(\Gamma; x)$ within any polynomial factor. However, our technique from [6, Theorem 4] does not say anything about the difficulty of the problem in the more interesting case when x is an integer. We raised this question in [6] but were unable to resolve it (until the present paper). Note that it is easy to compute $Z_{\text{Cl}}(\Gamma; 1)$ exactly in polynomial time. Corollary 7 has the following consequence.

Corollary 9. *Suppose that $x > 0$ is efficiently approximable. Then:*

- (1) *If $x = 1$ then CYCLEINDEX(x) can be solved exactly in polynomial time.*
- (2) *If x is not an integer then there is no FPRAS for CYCLEINDEX(x) unless $\text{NP} = \text{RP}$.*
- (3) *If $x > 1$ is a positive integer then $\#\text{BIS} \leq_{\text{AP}} \text{CYCLEINDEX}(x)$.*

Proof. Items (1) and (2) are from [6, Theorem 4]. We now prove item (3). Let $\lambda = x^{-1}$. Note that $\lambda \in (0, 1)$. We know from Corollary 7 that $\#\text{BIS} \leq_{\text{AP}} \text{WE}(\lambda)$. To finish, we show that $\text{WE}(\lambda) \leq_{\text{AP}} \text{CYCLEINDEX}(x)$.

The reduction is straightforward. Let M be a generating matrix (an instance of $\text{WE}(\lambda)$ with r linearly independent rows and c columns). Let \mathcal{Y} be the subspace generated by the rows of M (this is the set of code words). Let $M_{i,*}$ denote row i of M . For each binary string $m \in \{0, 1\}^r$, let $w(m)$ be the vector $\sum_{i=1}^r m_i M_{i,*}$ (where arithmetic is over $\text{GF}(2)$) and let $k(m)$ be the number of 1's in this vector. Since the rows of M are linearly independent, each element of \mathcal{Y} is uniquely expressible as $w(m)$ with $m \in \{0, 1\}^r$, so

$$W_M(\lambda) = \sum_{m \in \{0, 1\}^r} \lambda^{k(m)}. \tag{14}$$

Let $v = 2c$. Our objective will be to construct a group Γ of permutations of $\{1, \dots, v\}$ so that $W_M(\lambda)$ can be computed from $Z_{\text{Cl}}(\Gamma; x)$. For $i \in [r]$, let g_i be the permutation of $\{1, \dots, v\}$ defined as follows: For all $j \in [c]$, elements $2j - 1$ and $2j$ are mapped to each other by g_i if $M_{i,j} = 1$ and each of these is mapped to itself by g_i if $M_{i,j} = 0$. Let Γ be the group of permutations of $\{1, \dots, v\}$ generated by g_1, \dots, g_r .

For each binary string $m \in \{0, 1\}^r$, let $g(m)$ be the permutation $g_1^{m_1} \dots g_r^{m_r}$, where g_i^ℓ denotes the composition of ℓ copies of the generator g_i so g_i^0 is the identity permutation. Note that, for each $j \in [c]$, elements $2j - 1$ and $2j$ are swapped by $g(m)$ iff $\sum_{i=1}^r m_i M_{i,j} = 1 \pmod 2$. Thus, the number of j for which there is a swap is $k(m)$ so $\text{cyc}(g(m)) = v - k(m)$.

We will show that each permutation $g \in \Gamma$ can be written as $g(m)$ for exactly one $m \in \{0, 1\}^r$. First, suppose that $g = g_{i_1} \dots g_{i_\ell}$ for some $i_1, \dots, i_\ell \in \{1, \dots, r\}^\ell$. Since the generators g_1, \dots, g_r commute, we can re-order so that i_1, \dots, i_ℓ are monotonically non-decreasing. Then, since the generators all have order 2, we can cancel factors that are the identity permutation, making i_1, \dots, i_ℓ distinct. Thus, g can be written as $g(m)$ for at least one $m \in \{0, 1\}^r$. To see that m is unique, suppose that $g(m) = g(m')$ for $m \neq m'$. Then $g_1^{m_1} \dots g_r^{m_r} = g_1^{m'_1} \dots g_r^{m'_r}$. Thus, for each $j \in [c]$, the number of permutations in

$\{g_i \mid m_i = 1\}$ which swap element $2j - 1$ with element $2j$ has the same parity as the number of permutations in $\{g_i \mid m'_i = 1\}$ which swap element $2j - 1$ with element $2j$. Thus,

$$\sum_{i=1}^r m_i M_{i,*} = \sum_{i=1}^r m'_i M_{i,*} \pmod{2}.$$

Since the rows of M are linearly independent, we conclude that $m = m'$. Thus, we have proved that $\Gamma = \{g(m) \mid m \in \{0, 1\}^r\}$ so, from the definition of $Z_{\text{Cl}}(\Gamma; x)$,

$$|\Gamma| Z_{\text{Cl}}(\Gamma; x) = \sum_{m \in \{0,1\}^r} x^{\nu - k(m)} = x^{\nu} \sum_{m \in \{0,1\}^r} \lambda^{k(m)} = x^{\nu} W_M(\lambda),$$

where the last equality uses Eq. (14). \square

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