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# Projective equivalence of ideals in Noetherian integral domains

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**Abstract**

Let $I$ be a nonzero proper ideal in a Noetherian integral domain $R$. In this paper we establish the existence of a finite separable integral extension domain $A$ of $R$ and a positive integer $m$ such that all the Rees integers of $IA$ are equal to $m$. Moreover, if $R$ has altitude one, then all the Rees integers of $J = \mathrm{Rad}(IA)$ are equal to one and the ideals $J^m$ and $IA$ have the same integral closure. Thus $\mathrm{Rad}(IA) = J$ is a projectively full radical ideal that is projectively equivalent to $IA$. In particular, if $R$ is Dedekind, then there exists a Dedekind domain $A$ having the following properties: (i) $A$ is a finite separable integral extension of $R$; and (ii) there exists a radical ideal $J$ of $A$ and a positive integer $m$ such that $IA = J^m$. In this case the extension $A$ also has the property that for each maximal ideal $N$ of $A$ with $I \subseteq N$, the canonical inclusion $R/(N \cap R) \hookrightarrow A/N$ is an isomorphism, and the integer $m$ is a multiple of $[A_{(0)} : R_{(0)}]$.


*Keywords:* Projective equivalence of ideals; Rees valuations; Rees integers; Noetherian integral domains

## 1. Introduction

All rings in this paper are commutative with a unit $1 \neq 0$. Let $I$ be a regular proper ideal of the Noetherian ring $R$, that is, $I$ contains a regular element of $R$ and $I \neq R$. An ideal $J$ of $R$ is **projectively equivalent** to $I$ if there exist positive integers $m$ and $n$ such that $I^m$ and $J^n$ have the

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same integral closure, that is, $(I^m)_a = (J^n)_a$, where $K_a$ denotes the integral closure in $R$ of an ideal $K$ of $R$. The concept of projective equivalence of ideals and the study of ideals projectively equivalent to $I$ was introduced by Samuel in [14] and further developed by Nagata in [11].

Making use of interesting work of Rees in [13], McAdam, Ratliff, and Sally in [10, Corollary 2.4] prove that the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to $I$ is discrete and linearly ordered with respect to inclusion. They also prove the existence of a fixed positive integer $d$ such that for every ideal $J$ projectively equivalent to $I$, $(J^d)_a = (I^n)_a$ for some positive integer $n$. If $J$ and $K$ are in $\mathbf{P}(I)$ and $m$ and $n$ positive integers, then $(J^m K^n)_a \in \mathbf{P}(I)$. Thus there is naturally associated to $I$ a unique subsemigroup $S(I)$ of the additive semigroup of nonnegative integers $\mathbb{N}_0$ such that $S(I)$ contains all sufficiently large integers. A semigroup having these properties is called a **numerical semigroup**. The numerical semigroup $S(I)$ is an invariant of the projective equivalence class $\mathbf{P}(I)$ of $I$ in the sense that if $I$ is projectively equivalent to $J$, then $S(I) = S(J)$, cf. [1, Remark 4.3]. It is observed in [2, Remark 3.11] that every numerical semigroup is realizable as $S(M)$ for an appropriate local domain $(R, M)$.

The set $\mathbf{P}(I)$ is said to be **projectively full** if $S(I) = \mathbb{N}_0$, or equivalently, if every element of $\mathbf{P}(I)$ is the integral closure of a power of the largest element $K$ of $\mathbf{P}(I)$, i.e., every element of $\mathbf{P}(I)$ has the form $(K^n)_a$, for some positive integer $n$. If this holds, then each ideal $J$ in $R$ such that $J_a = K$ is said to be **projectively full**. A number of results about, and examples of, projectively full ideals are given in [1–3], and [4]. Several characterizations of such ideals are given in [1, (4.11) and (4.12)], and in [2, Section 3] relations between projectively full ideals in $R$ and in factor rings of $R$, localizations of $R$, and extension rings of $R$ are proved.

The set Rees $I$ of Rees valuation rings of $I$ is a finite set of rank one discrete valuation rings (DVRs) that determine the integral closure $(I^n)_a$ of $I^n$ for every positive integer $n$ and are the unique minimal set of DVRs having this property. Consider the minimal primes $z$ of $R$ such that $IR/z$ is a proper nonzero ideal. The set Rees $I$ is the union of the sets Rees $IR/z$. Thus one is reduced to describing the set Rees $I$ in the case where $I$ is a nonzero proper ideal of a Noetherian integral domain $R$. Consider the Rees ring $\mathbf{R} = R[t^{-1}, It]$. The integral closure $\mathbf{R}'$ of $\mathbf{R}$ is a Krull domain, so $W = \mathbf{R}'_p$ is a DVR for each minimal prime $p$ of $t^{-1}\mathbf{R}'$, and $V = W \cap F$, where $F$ is the field of fractions of $R$, is also a DVR. The set Rees $I$ of Rees valuation rings of $I$ is the set of DVRs $V$ obtained in this way, cf. [15, Section 10.1].

If $(V_1, N_1), \ldots, (V_n, N_n)$ are the Rees valuation rings of $I$, then the integers $(e_1, \ldots, e_n)$, where $IV_i = N_i^{e_i}$, are the **Rees integers** of $I$. Necessary and sufficient conditions for two regular proper ideals $I$ and $J$ to be projectively equivalent are that (i) Rees $I$ = Rees $J$ and (ii) the Rees integers of $I$ and $J$ are proportional [1, Theorem 3.4]. If $I$ is integrally closed and each Rees integer of $I$ is one, then $I$ is a projectively full radical ideal.[1]

A main goal in the papers [1–4], and [6], and also in the present paper, is to answer the following question:

**Question 1.1.** Let $I$ be a nonzero proper ideal in a Noetherian domain $R$. Under what conditions does there exist a finite integral extension domain $A$ of $R$ such that $\mathbf{P}(IA)$ contains an ideal $J$ whose Rees integers are all equal to one?

Progress is made on Question 1.1 in [3]. To describe this progress, let $b_1, \ldots, b_g$ be regular elements in $R$ that generate $I$ and for each positive integer $m > 1$ let $A_m = R[x_1, \ldots, x_g] =$

[1] There exist local domains $(R, M)$ for which $M$ is not projectively full. A sufficient, but not necessary, condition in order that $I$ be projectively full is that the gcd of the Rees integers of $I$ be one.

$R[X_1, \dots, X_g]/({X_1}^m - b_1, \dots, {X_g}^m - b_g)$ and let $J_m = (x_1, \dots, x_g)A_m$. Then the main result in [3] establishes the following:

**Theorem 1.2.** *Let $R$ be a Noetherian ring, let $I$ be a regular proper ideal in $R$, let $b_1, \dots, b_g$ be regular elements in $I$ that generate $I$, and let $(V_1, N_1), \dots, (V_n, N_n)$ be the Rees valuation rings of $I$. Assume that*:

(a) *$b_i V_j = I V_j$ ($= {N_j}^{e_j}$, say) for $i = 1, \dots, g$ and $j = 1, \dots, n$; and,*
(b) *the greatest common divisor $e$ of $e_1, \dots, e_n$ is a unit in $R$.*

*Then $A_e = R[x_1, \dots, x_g]$ is a finite free integral extension ring of $R$ and the ideal $J_e = (x_1, \dots, x_g)A_e$ is projectively full and projectively equivalent to $IA_e$. Thus $\mathbf{P}(IA_e) = \mathbf{P}(J_e)$ is projectively full. Also, if $R$ is an integral domain and if $z$ is a minimal prime ideal in $A_e$, then $((J_e + z)/z)_a$ is a projectively full ideal in $A_e/z$ that is projectively equivalent to $(IA_e + z)/z$, so $\mathbf{P}((IA_e + z)/z)$ is projectively full.*

We prove in [6, (3.19) and (3.20)] that if either (i) $R$ contains an infinite field, or (ii) $R$ is a local ring with an infinite residue field, then it is possible to choose generators $b_1, \dots, b_g$ of $I$ that satisfy assumption (a) of Theorem 1.2. We prove in [6, (3.7)] that if "greatest common divisor" is replaced with "least common multiple," then the integral closure of the ideal $J_e$ in Theorem 1.2 is a radical ideal with all Rees integers equal to one. Specifically:

**Theorem 1.3.** *With the notation of Theorem* 1.2*, assume that*: *assumption* (a) *of Theorem* 1.2 *holds*; *and,* (b′) *the least common multiple $c$ of $e_1, \dots, e_n$ is a unit in $R$. Then for each positive multiple $m$ of $c$ that is a unit in $R$ the ideal $(J_m)_a$ is projectively full and $(J_m)_a$ is a radical ideal that is projectively equivalent to $IA_m$. Also, the Rees integers of $J_m$ are all equal to one and $x_i U$ is the maximal ideal of $U$ for each Rees valuation ring $U$ of $J_m$ and for $i = 1, \dots, g$. Moreover, if $R$ is an integral domain and if $z$ is a minimal prime ideal in $A_m$, then $((J_m + z)/z)_a$ is a projectively full radical ideal that is projectively equivalent to $(IA_m + z)/z$.*

Examples [6, (3.22) and (3.23)] show that condition (b′) of Theorem 1.3 is needed for the proof of this result given in [6]. We show in [6, (2.6)] that every basis consisting of regular elements of $I$ can be used to find an integral extension ring $A_m$ of $R$ having a radical ideal $J_m$ that is projectively equivalent to $IA_m$. Specifically:

**Theorem 1.4.** *With notation as in Theorem* 1.2*, if $b_1, \dots, b_g$ are arbitrary regular elements in $I$ that generate $I$ and if $m$ is an integer greater than or equal to $\max(\{e_i \mid i = 1, \dots, n\})$, then $J_m$ is projectively equivalent to $IA_m$, $(J_m)_a = \mathrm{Rad}(J_m)$, and $A_m/(J_m)_a \cong R/\mathrm{Rad}(I)$. Further, if $R$ is an integral domain and if $z$ is a minimal prime ideal in $A_m$, then $((J_m + z)/z)_a$ is a radical ideal that is projectively equivalent to $(IA_m + z)/z$.*

The following result is the main result in the present paper:

**Theorem 1.5.** *Let $I$ be a nonzero proper ideal in a Noetherian integral domain $R$.*

1. *There exists a finite separable integral extension domain $A$ of $R$ and a positive integer $m$ such that all the Rees integers of $IA$ are equal to $m$.*

2. *If $R$ has altitude one, then there exists a finite integral extension domain $A$ of $R$ such that $\mathbf{P}(IA)$ contains an ideal $H$ whose Rees integers are all equal to one. Therefore $H = \mathrm{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to $IA$.*

Observe that Theorem 1.5.2, answers Question 1.1 in the affirmative for each nonzero proper ideal $I$ in an arbitrary Noetherian integral domain $R$ of altitude one with no additional conditions; therefore the conclusions of Theorems 1.2 and 1.3 are valid without the assumption of conditions (a), (b), and (b′) if $R$ is a Noetherian integral domain of altitude one. In particular, Theorem 1.5.2 shows that these conclusions hold for the above mentioned examples [6, (3.22) and (3.23)].

A classical theorem of Krull is an important tool in our work. By successively applying this theorem of Krull, we construct a finite integral extension domain $A$ of $R$ such that $H = \mathrm{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to $IA$. Moreover, the Rees integers of $H$ are all equal to one. If, in addition, $R$ is integrally closed, then $A$ is the integral closure of $R$ in a finite separable algebraic field extension and $H^m = IA$, where $m$ is a multiple of $[A_{(0)} : R_{(0)}]$; and for each maximal ideal $N$ of $A$ with $I \subset N$, the canonical inclusion map $R/(N \cap R) \hookrightarrow A/N$ is an isomorphism.

In Section 3 we consider the question of extending Theorem 1.5.2 to the case of regular principal ideals $bR$ of a Noetherian domain $R$ of altitude greater than one. A complicating factor here is the possibility that $\mathrm{Rad}(bA)$ may have embedded asymptotic prime divisors. In Section 4 we present an application that partially extends Theorem 1.5.1 to certain finite sets of ideals.

Our notation is mainly as in Nagata [12], so, for example, the term **altitude** refers to what is often called dimension or Krull dimension, and a **basis** for an ideal is a set of elements that generate the ideal.

## 2. Finite integral extensions of a Noetherian domain

To prove our main result, we use a theorem of Krull; before stating Krull's Theorem, we recall the following terminology from [5].

**Definition 2.1.** Let $(V_1, N_1), \ldots, (V_n, N_n)$ be distinct rank one discrete valuation domains of a field $F$ and for $i = 1, \ldots, n$ let $K_i = V_i/N_i$ denote the residue field of $V_i$. Let $m$ be a positive integer. By an **m-consistent system for** $\{V_1, \ldots, V_n\}$, we mean a collection of sets $S = \{S_1, \ldots, S_n\}$ satisfying the following conditions:

(1) $S_i = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \ldots, s_i\}$, where $K_{i,j}$ is a simple algebraic field extension of $K_i$ with $f_{i,j} = [K_{i,j} : K_i]$, and $s_i, e_{i,j} \in \mathbb{N}_+$ (the set of positive integers).
(2) For each $i$, the sum $\sum_{j=1}^{s_i} e_{i,j} f_{i,j} = m$.

**Definition 2.2.** The $m$-consistent system $S$ as in Definition 2.1 is said to be **realizable** if there exists a separable algebraic extension field $L$ of $F$ such that:

(a) $[L : F] = m$.
(b) For $1 \leqslant i \leqslant n$, $V_i$ has exactly $s_i$ extensions $V_{i,1}, \ldots, V_{i,s_i}$ to $L$.
(c) The residue field of $V_{i,j}$ is $K_i$-isomorphic to $K_{i,j}$, and the ramification index of $V_{i,j}$ over $V_i$ is $e_{i,j}$ (so $N_i V_{i,j} = {N_{i,j}}^{e_{i,j}}$).

If $S$ and $L$ are as above, we say the field $L$ **realizes** $S$ or that $L$ is a **realization** of $S$.

**Theorem 2.3.** *(See Krull* [8].*) Let* $(V_1, N_1), \dots, (V_n, N_n)$ *be distinct rank one discrete valuation domains of a field* $F$ *with* $K_i = V_i/N_i$ *for* $i = 1, \dots, n$*, let* $m$ *be a positive integer, and let* $S = \{S_1, \dots, S_n\}$ *be an* $m$*-consistent system for* $\{V_1, \dots, V_n\}$ *with* $S_i = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ *for* $i = 1, \dots, n$*. Then* $S$ *is realizable if one of the following conditions is satisfied.*

(i) $s_i = 1$ *for at least one* $i$.
(ii) $F$ *has at least one rank one discrete valuation domain* $V$ *distinct from* $V_1, \dots, V_n$.
(iii) *For each monic polynomial* $X^t + a_1X^{t-1} + \cdots + a_t$ *with* $a_i \in \bigcap_{i=1}^n V_i = D$*, and for each* $h \in \mathbb{N}$*, there exists an irreducible separable polynomial* $X^t + b_1X^{t-1} + \cdots + b_t \in D[X]$ *with* $b_l - a_l \in {N_i}^h$ *for each* $l = 1, \dots, t$ *and* $i = 1, \dots, n$.

Observe that condition (i) of Theorem 2.3 is a property of the $m$-consistent system $S = \{S_1, \dots, S_n\}$, whereas condition (ii) is a property of the family of rank one discrete valuation domains of the field $F$, and condition (iii) is a property of the family $(V_1, N_1), \dots, (V_n, N_n)$.

**Remark 2.4.** Let $D$ be a Dedekind domain with quotient field $F \neq D$, let $M_1, \dots, M_n$ be distinct maximal ideals of $D$, let $I = {M_1}^{e_1} \cdots {M_n}^{e_n}$ be an ideal in $D$, where $e_1, \dots, e_n$ are positive integers, and let $S = \{S_1, \dots, S_n\}$ be a realizable $m$-consistent system for $\{D_{M_1}, \dots, D_{M_n}\}$, where $S_i = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ for $i = 1, \dots, n$. Let $L$ be a field that realizes $S$ and let $E$ be the integral closure of $D$ in $L$. Then:

(2.4.1) $L$ has rank one discrete valuation domains $(V_{i,1}, N_{i,1}), \dots, (V_{i,s_i}, N_{i,s_i})$ such that for each $i, j$: $V_{i,j} \cap F = D_{M_i}$; $V_{i,j}/N_{i,j}$ is $D/M_i$-isomorphic to $K_{i,j}$; $[K_{i,j} : (D/M_i)] = f_{i,j}$; and, $M_iV_{i,j} = {N_{i,j}}^{e_{i,j}}$. Also, for $i = 1, \dots, n$, $V_{i,1}, \dots, V_{i,s_i}$ are all of the extensions of $D_{M_i}$ to $L$.
(2.4.2) $E$ is a Dedekind domain that is a finite separable integral extension domain of $D$, and $IE = {M_1}^{e_1} \cdots {M_n}^{e_n}E = {P_{1,1}}^{e_1e_{1,1}} \cdots {P_{n,s_n}}^{e_ne_{n,s_n}}$, where $P_{i,j} = N_{i,j} \cap E$ for $i = 1, \dots, n$ and $j = 1, \dots, s_i$.

**Proof.** (2.4.1) follows immediately from (a)–(c) of Definition 2.2.

For (2.4.2), $E$ is a Dedekind domain, by [16, Theorem 19, p. 281], and $E$ is a finite separable integral extension domain of $D$, by [16, Corollary 1, p. 265], since $L$ is a finite separable algebraic extension field of $F$. Also, $V_{i,j} = E_{P_{i,j}}$, so $IV_{i,j} = (IE)V_{i,j} = (ID_{M_i})V_{i,j} = ({M_i}^{e_i}D_{M_i})V_{i,j} = (M_iV_{i,j})^{e_i} = {N_{i,j}}^{e_ie_{i,j}}$. Since the ideals $P_{i,j}$ are the only prime ideals in $E$ that lie over $M_i$ (for $i = 1, \dots, n$ and $j = 1, \dots, s_i$) and since the $P_{i,j}$ are comaximal, it follows that $IE = {P_{1,1}}^{e_1e_{1,1}} \cdots {P_{n,s_n}}^{e_ne_{n,s_n}}$. □

We use the following two lemmas in the proof of Theorem 2.8.

**Lemma 2.5.** *Let* $D$ *be a Dedekind domain and let* $I = {M_1}^{e_1} \cdots {M_n}^{e_n}$ $(n > 1)$ *be an irredundant primary decomposition of a nonzero proper ideal* $I$ *in* $D$*. Assume that the integers* $e_i$ *have no common factor* $d > 1$*. Let* $p$ *be a prime integer dividing at least one of the* $e_i$*. Then there exists a Dedekind domain* $E_1$ *that is a finite separable integral extension domain of* $D$ *with an ideal* $J_1$ *such that*: ${J_1}^{p^h} = IE_1$ *for some positive integer* $h$; *and, if* $J_1 = {N_1}^{c_1} \cdots {N_g}^{c_g}$ *is an irredundant primary decomposition of* $J_1$*, then* $\prod_{j=1}^g c_j$ *has fewer distinct prime integer factors than does* $\prod_{j=1}^n e_j$*. Moreover, the canonical inclusion map* $D/(N_i \cap D) \hookrightarrow E_1/N_i$ *is an isomorphism for each* $i \in \{1, \dots, g\}$.

**Proof.** Let $e_i = p^{h_i} d_i$, where $p \nmid d_i$, and $h_i \geqslant 0$. We may assume that the $e_i$ are ordered so that $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n$. Our hypotheses imply that $h_1 > 0$ and $h_n = 0$. Let $S = \{S_1, \ldots, S_n\}$ with $S_i = \{(K_{i,j}, 1, p^{h_1 - h_i}) \mid j = 1, \ldots, p^{h_i}\}$. We show that $S$ is a realizable $p^{h_1}$-consistent system for $\{D_{M_1}, \ldots, D_{M_n}\}$. Observe that $\sum_{j=1}^{s_i} e_{i,j} f_{i,j} = \sum_{j=1}^{p^{h_i}} p^{h_1 - h_i} \cdot 1 = p^{h_1}$. Therefore $S$ is a $p^{h_1}$-consistent system. Since $s_n = p^{h_n} = p^0 = 1$, $S$ is realizable, by Theorem 2.3(i). Therefore, by Remark 2.4 (especially (2.4.2)), the integral closure $E_1$ of $D$ in a realization $L$ of $S$ for $\{D_{M_1}, \ldots, D_{M_n}\}$ is a Dedekind domain such that

$$\begin{aligned} IE_1 &= \prod_{i=1}^{n} \bigl(M_i{}^{e_i} E_1\bigr) \\ &= \prod_{i=1}^{n} \bigl(N_{i,1}{}^{e_i e_{i,1}} \cdots N_{i,s_i}{}^{e_i e_{i,s_i}}\bigr) = \prod_{i=1}^{n} \bigl(N_{i,1}{}^{(p^{h_i} d_i)(p^{h_1 - h_i})} \cdots N_{i,s_i}{}^{(p^{h_i} d_i)(p^{h_1 - h_i})}\bigr) = J_1{}^{p^{h_1}}, \end{aligned}$$

where $J_1 = \prod_{i=1}^{n} (N_{i,1}{}^{d_i} \cdots N_{i,s_i}{}^{d_i})$. Also, $\prod_{i=1}^{n} d_i{}^{s_i} = \prod_{i=1}^{n} d_i{}^{p^{h_i}}$ has fewer distinct prime integer factors than does $\prod_{i=1}^{n} e_i$. Finally, since all $f_{i,j}$ are equal to one, it follows that $E_1/N \cong D/(N \cap D)$ for all maximal ideals $N$ of $E_1$ that contain $I$. □

**Lemma 2.6.** *Let $R$ be a Noetherian domain of altitude one with quotient field $F$ and let $I$ be a nonzero proper ideal in $R$. Let $L$ be a finite algebraic extension field of $F$, and let $E$ denote the integral closure of $R$ in $L$. If there exist distinct maximal ideals $N_1, \ldots, N_n$ of $E$ and positive integers $k_1, \ldots, k_n, h$ such that $IE = (N_1{}^{k_1} \cdots N_n{}^{k_n})^h$, then there exists a finite integral extension domain $A$ of $R$ with quotient field $L$ and an ideal $H$ of $A$ such that*

(i) *$H$ has Rees valuation rings $E_{N_1}, \ldots, E_{N_n}$ with corresponding Rees integers $k_1, \ldots, k_n$; and*
(ii) *$(H^h)_a = (IA)_a$.*

**Proof.** If $E$ is a finite $R$-module, then let $A = E$ and $H = N_1{}^{k_1} \cdots N_n{}^{k_n}$. Otherwise, let $A_0$ be a subring of $E$ that is a finite integral extension domain of $R$ and that has quotient field $L$. For $i = 1, \ldots, n$ let $G_i \subseteq E$ be a finite set such that $G_i E = N_i$, let $A = A_0[G_1 \cup \cdots \cup G_n]$, and for $i = 1, \ldots, n$ let $P_i = N_i \cap A$. Then $A$ is a finite integral extension domain of $R$, $E$ is the integral closure of $A$ in its quotient field $L$, and $P_i$ is a maximal ideal in $A$ such that $P_i E = N_i$ for $i = 1, \ldots, n$. Let $H = P_1{}^{k_1} \cdots P_n{}^{k_n}$. Since

$$\begin{aligned} HE \cap A &= \bigl[\bigl(P_1{}^{k_1} \cdots P_n{}^{k_n}\bigr)E\bigr] \cap A = \bigl[\bigl(P_1{}^{k_1} E\bigr) \cap \cdots \cap \bigl(P_n{}^{k_n} E\bigr)\bigr] \cap A \\ &= \bigl[N_1{}^{k_1} \cap \cdots \cap N_n{}^{k_n}\bigr] \cap A, \end{aligned}$$

our hypotheses imply that $H$ has Rees valuation rings $E_{N_1}, \ldots, E_{N_n}$ with corresponding Rees integers $k_1, \ldots, k_n$. Also $(H^h)_a = (H^h E)_a \cap A = (H^h E) \cap A = (N_1{}^{k_1 h} \cdots N_n{}^{k_n h}) \cap A = IE \cap A = (IE)_a \cap A = (IA)_a$. □

We also use the following well-known fact concerning the Rees valuation rings of an ideal (cf. the proof of [3, Theorem 2.5]).

**Remark 2.7.** Let $I$ be a nonzero proper ideal in a Noetherian integral domain $R$, let $\operatorname{Rees} I = \{V_1, \ldots, V_n\}$, and let $A$ be a finite integral extension domain of $R$. Then $\operatorname{Rees} IA = \{V_{1,1}, \ldots, V_{n,c_n}\}$, where for each $i \in \{1, \ldots, n\}$, $V_{i,1}, \ldots, V_{i,c_i}$ are all the extensions of $V_i$ to the quotient field of $A$.

Theorem 2.8.2, answers Question 1.1 in the affirmative for each nonzero proper ideal in an arbitrary Noetherian integral domain of altitude one with no additional conditions.

**Theorem 2.8.** *Let $I$ be a nonzero proper ideal in a Noetherian integral domain $R$.*

1. *There exists a finite separable integral extension domain $A$ of $R$ and a positive integer $m$ such that all the Rees integers of $IA$ are equal to $m$.*
2. *If $R$ has altitude one, then there exists a finite integral extension domain $A$ of $R$ such that $\mathbf{P}(IA)$ contains an ideal $H$ whose Rees integers are all equal to one. Therefore $H = \operatorname{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to $IA$.*

**Proof.** For part 2, if $R$ has altitude one, then the integral closure of $R$ is a Dedekind domain $D$ and there exist distinct maximal ideals $M_1, \ldots, M_n$ ($n \geqslant 1$) in $D$ and positive integers $e_1, \ldots, e_n$ such that $ID = {M_1}^{e_1} \cdots {M_n}^{e_n}$. The $D_{M_i}$ are the Rees valuation rings of $I$ and $e_1, \ldots, e_n$ are the Rees integers of $I$. If either $n = 1$ or $e_1 = \cdots = e_n$, then the conclusions of part 2 follow from Lemma 2.6 with $L = F$, so we may assume that $n > 1$ and that not all the $e_i$ are equal. Let $d$ be the greatest common divisor of $e_1, \ldots, e_n$. Then the ideal $I_0 = {M_1}^{\frac{e_1}{d}} \cdots {M_n}^{\frac{e_n}{d}}$ is such that $ID = {I_0}^d$, so the ideal $I_0$ may be used in place of $ID$. Thus we may assume that the $e_i$ have no common factor $d > 1$. Let $k$ be the number of distinct prime integers dividing $\prod_{j=1}^{n} e_j$. By induction on $k$, it suffices to show that there exists a finite integral extension domain $A$ of $R$, an ideal $H$ of $A$, and a positive integer $h$ such that: $(H^h)_a = (IA)_a$; $H$ has Rees integers $c_1, \ldots, c_g$; and, there are at most $k - 1$ distinct prime integers dividing $\prod_{j=1}^{g} c_j$. Therefore Theorem 2.8.2 follows from Lemmas 2.5 and 2.6.

For the proof of part 1, let $\operatorname{Rees} I = \{(V_1, N_1), \ldots, (V_n, N_n)\}$, let $e_1, \ldots, e_n$ be the Rees integers of $I$, and let $D = V_1 \cap \cdots \cap V_n$. Then $D$ is a Dedekind domain with maximal ideals $M_1, \ldots, M_n$ and $D_{M_i} = V_i$ for each $i$ with $1 \leqslant i \leqslant n$. Also $ID = M_1^{e_1} \cdots M_n^{e_n}$. If either $n = 1$ or $e_1 = \cdots = e_n$, then the assertion of part 1 is obvious. Thus we may assume that $n > 1$ and that not all the $e_i$ are equal. The argument in the paragraph above for part 2 implies that there exists a finite separable algebraic field extension $L$ of the quotient field $F$ of $D$ such that if $E$ is the integral closure of $D$ in $L$, then $IE = J^m$, where $J$ is a radical ideal of $E$. There exists $\theta \in L$ such that $L = F[\theta]$ and there exists a nonzero $r \in R$ such that $r\theta$ is integral over $R$. Let $A = R[r\theta]$. Remark 2.7 implies that each of the Rees integers of $IA$ is $m$. □

**Corollary 2.9.** *Let $I$ be a nonzero proper ideal in a Dedekind domain $D$. There exists a Dedekind domain $E$ having the following properties*:

(i) *$E$ is a finite separable integral extension of $D$*; *and*
(ii) *there exists a radical ideal $J$ of $E$ and a positive integer $m$ such that $IE = J^m$.*

*Therefore $J$ is a projectively full radical ideal that is projectively equivalent to $IE$, and the Rees integers of $J$ are all equal to one. The extension $E$ also has the property that for each maximal* *ideal $N$ of $E$ with $I \subseteq N$, the canonical inclusion $D/(N \cap D) \hookrightarrow E/N$ is an isomorphism, and $m$ is a multiple of $[E_{(0)} : D_{(0)}]$.*

**Proof.** Everything but the last sentence of Corollary 2.9 is immediate from Theorem 2.8.2. The application of Lemma 2.5 to the integral closure $D$ of the Noetherian domain $R$ of Theorem 2.8.2 implies that $D/(N \cap D) \hookrightarrow E/N$ is an isomorphism. That $m$ is a multiple of $[E_{(0)} : D_{(0)}]$ follows from Remark 2.11.3. □

In Lemma 2.10, we give a different consistent system for $D_{M_1}, \ldots, D_{M_n}$ that may be used in place of Lemma 2.5 to inductively complete an alternative proof of Theorem 2.8. The proof of Lemma 2.10 is described more fully in Remark 2.11.1. (Concerning the hypothesis "$k > 1$" in Lemma 2.10, if $k = 0$, then $I = M_1 \cdots M_n$ is a radical ideal and the lemma holds with $E_1 = D$, $J_1 = I$, and $h = 1$.)

**Lemma 2.10.** *Let $D$ be a Dedekind domain, let $I = M_1{}^{e_1} \cdots M_n{}^{e_n}$ $(n > 1)$ be an irredundant primary decomposition of a nonzero proper ideal $I$ in $D$, and assume that $e_i > 1$ for at most $k$ $(1 \leqslant k \leqslant n)$ of the integers $e_i$. Then there exists a Dedekind domain $E_1$ that is a finite separable integral extension domain of $D$ with an ideal $J_1$ such that: $J_1{}^h = IE_1$ for some positive integer $h$; and, if $J_1 = N_1{}^{c_1} \cdots N_g{}^{c_g}$ is an irredundant primary decomposition of $J_1$, then $c_j > 1$ for at most $k - 1$ of the integers $c_1, \ldots, c_g$. Moreover, the canonical inclusion map $D/(N_i \cap D) \hookrightarrow E_1/N_i$ is an isomorphism for each $i \in \{1, \ldots, g\}$.*

**Proof.** Let $S = \{S_1, \ldots, S_n\}$, where: $S_1 = \{(K_{1,j}, 1, 1) \mid j = 1, \ldots, e_1\}$; and, for $i = 2, \ldots, n$, $S_i = \{(K_{i,1}, 1, e_1)\}$. Then a proof similar to the proof of Lemma 2.5 shows that $S$ is a realizable $e_1$-consistent system for $\{D_{M_1}, \ldots, D_{M_n}\}$ and that $IE_1 = J_1{}^{e_1}$, where $J_1 = (N_{1,1} \cdots N_{1,e_1})N_{2,1}{}^{e_2}N_{3,1}{}^{e_3} \cdots N_{n,1}{}^{e_n}$. Finally, since all $f_{i,j}$ are equal to one, it follows that $E_1/N \cong D/(N \cap D)$ for all maximal ideals $N$ of $E_1$ that contain $I$. □

**Remark 2.11.** (2.11.1) In Theorem 2.8.2, assume that the exponents $e_1, \ldots, e_n$ are arranged so that $e_i > 1$ if and only if $i \in \{1, \ldots, k\}$, where $k \leqslant n$. If we successively carry out the separate steps of the induction in the proof of Theorem 2.8.2 using Lemma 2.10, then we get a chain of rings $D = E_0 \subset E_1 \subset \cdots \subset E_k = E$, where each $E_i$ $(i = 1, \ldots, k)$ is a Dedekind domain that is a finite separable integral extension of $E_{i-1}$ and for which $(E_i)_U$ has exactly $e_i - 1$ more maximal ideals than $(E_{i-1})_U$, where $U = D \setminus (M_1 \cup \cdots \cup M_n)$. In fact, for $i = 1, \ldots, k$, $E_i$ is obtained as the integral closure of $E_{i-1}$ in a realization $L_i$ of a realizable $e_i$-consistent system $S^{(i)}$ for $\{(E_{i-1})_N \mid N \in \mathbf{N}(E_{i-1})\}$, where $\mathbf{N}(E_{i-1}) = \{N \mid N$ is a maximal ideal in $E_{i-1}$ and $N \cap D \in \{M_1, \ldots, M_n\}$. Here, the $e_i$-consistent system $S^{(i)}$ completely splits (into $e_i$ components) the unique maximal ideal in $E_{i-1}$ that contracts in $D$ to $M_i$, and it completely ramifies (of index $e_i$) all the remaining maximal ideals in $\mathbf{N}(E_{i-1})$, so $E_i/N \cong E_{i-1}/(N \cap E_{i-1}) \cong D/(N \cap D)$ for all $N$ in $\mathbf{N}(E_i)$. ($S^{(i)}$ is realizable, by Theorem 2.3(i), since for all but one $N \in \mathbf{N}(E_{i-1})$, the corresponding component $S^{(i)}{}_j$ of $S^{(i)}$ contains a single ordered triple $(E_{i-1}/N, 1, e_i)$.) Therefore: (a) exactly $e_i$ of the maximal ideals in $E_i$ contract in $E_{i-1}$ to the unique maximal ideal in $E_{i-1}$ that contracts in $D$ to $M_i$; and (b) the remaining maximal ideals in $\mathbf{N}(E_i)$ are in one-to-one correspondence with the remaining $e_1 + \cdots + e_{i-1} + (n - i)$ maximal ideals in $\mathbf{N}(E_{i-1})$. Further, for each maximal ideal $N$ of (b) it holds that $E_i/N \cong D/(N \cap D)$ and $(N \cap E_{i-1})(E_i)_N = N^{e_i}(E_i)_N$, while for the $e_i$ maximal ideals $N$ of (a) it holds that $E_i/N \cong D/(N \cap D)$ and $(N \cap E_{i-1})(E_i)_N = N(E_i)_N$.

It follows that, in $E = E_k$, $\mathbf{N}(E)$ has exactly $e_1 + \cdots + e_k + (n-k)$ maximal ideals, and of these, exactly $e_i$ of them contract in $D$ to $M_i$ for $i = 1, \ldots, n$. Also, if $N$ is a maximal ideal in $E$ and $N \cap D = M_i$ (with $i \in \{1, \ldots, n\}$), and if $e_i{}^* = \frac{e_1 \cdots e_n}{e_i}$, then $N \in \mathbf{N}(E_k)$ $(= \mathbf{N}(E))$, $ME_n = N^{e_i{}^*} E_N$, and $E/N \cong D/M_i$. It therefore follows that:

(*1) *the quotient field* $L = L_k$ *of* $E$ *is a realization of the realizable* $e_1 \cdots e_n$*-consistent system* $S = \{S_1, \ldots, S_n\}$ *for* $\{D_{M_1}, \ldots, D_{M_n}\}$, *where*: *for* $i = 1, \ldots, k$, $S_i = \{(K_{i,j}, 1, e_i{}^*) \mid j = 1, \ldots, e_i\}$; *and, for* $i = k+1, \ldots, n$, $S_i = \{(K_{i,1}, 1, e_1 \cdots e_n)\}$,[2] *so* $IE = J^{e_1 \cdots e_n}$, *where* $J = \bigcap\{N \mid N \in \mathbf{N}(E)\}$ (since $IE_N = N^{e_1 \cdots e_n} E_N$ for each maximal ideal $N$ in $\mathbf{N}(E)$), hence $JE_U$ is the Jacobson radical of $E_U$.

(2.11.2) Assume that $I = M_1{}^{e_1} \cdots M_n{}^{e_n}$, that no prime integer divides each $e_i$, and that the least common multiple of $e_1, \ldots, e_n$ is $d = p_1{}^{m_1} \cdots p_k{}^{m_k}$, where $p_1, \ldots, p_k$ are distinct prime integers and $m_1, \ldots, m_k$ are positive integers. Then it follows as in (2.11.1) that if we successively carry out the separate steps of the induction in the proof of Theorem 2.8.2 using Lemma 2.5, then we get a chain of rings $D = E_0 \subset E_1 \subset \cdots \subset E_k = E$, where each $E_i$ $(i = 1, \ldots, k)$ is a Dedekind domain that is obtained as the integral closure of $E_{i-1}$ in a realization $L_i$ of a realizable $p_i{}^{m_i}$-consistent system $S^{(i)}$ for $\{(E_{i-1})_N \mid N \in \mathbf{N}(E_{i-1})\}$, where $\mathbf{N}(E_{i-1}) = \{N \mid N$ is a maximal ideal in $E_{i-1}$ and $N \cap D \in \{M_1, \ldots, M_n\}\}$. It therefore follows that:

(*2) *the quotient field* $L = L_k$ *of* $E$ *is a realization of the realizable* $d$*-consistent system* $S = \{S_1, \ldots, S_n\}$ *for* $\{D_{M_1}, \ldots, D_{M_n}\}$, *where*: *for* $i = 1, \ldots, n$, $S_i = \{(K_{i,j}, 1, \frac{d}{e_i}) \mid j = 1, \ldots, e_i\}$, *so* $IE = J^d$, *where* $J = \prod_{i=1}^{n}(N_{i,1} \cdots N_{i,e_i})$ *with* $\mathbf{N}(E) = \{N_{1,1}, \ldots, N_{n,e_n}\}$.

(2.11.3) It follows from the last part of (2.11.1) that, in Corollary 2.9, the extension domain $E$ of $D$ and the integer $h$ such that $IE = J^h$ can be chosen such that: $h = e_1 \cdots e_n$; and, the quotient field $L$ of $E$ is a realization of an $h$-consistent system for the Rees valuation rings of $I$. And it follows from the last part of (2.11.2) that, in Corollary 2.9, if $d$ is the greatest common divisor of $e_1, \ldots, e_n$, if $I_0 = M_1{}^{\frac{e_1}{d}} \cdots M_n{}^{\frac{e_n}{d}}$ (so $I_0$ is projectively equivalent to $I$), and if $c$ is the least common multiple of $\frac{e_1}{d}, \ldots, \frac{e_n}{d}$, then the extension domain $E$ of $D$ and the integer $h$ such that $I_0 E = J^h$ can be chosen such that: $h = c$; and, the quotient field $L$ of $E$ is a realization of an $h$-consistent system for the Rees valuation rings of $I$ (and of $I_0$).

## 3. Principal ideals and projective equivalence in finite integral extensions

In this section we consider the question of an extension of Theorem 2.8.2 to regular principal ideals of a Noetherian integral domain of altitude greater than one.

**Discussion 3.1.** Let $b$ be a nonzero nonunit in a Noetherian integral domain $R$, let $R'$ be the integral closure of $R$ in its quotient field $F$, let $p_1, \ldots, p_n$ be the height-one prime ideals in

[2] If at least one of the integers $e_1, \ldots, e_n$ is one, then it follows from Theorem 2.3(i) that $S$ is a realizable $e_1 \cdots e_n$-consistent system, so (*1) readily follows. However, if $e_i \neq 1$ for $i = 1, \ldots, n$, then it is only by this "composition" of realizable consistent systems that we are able to show that $S$ is realizable, and thereby find a finite integral extension domain $E$ of $D$ for which $IE$ is the power of a radical ideal of $E$. This idea of composing realizable consistent systems is further developed in [7].

$R'$ that contain $bR'$, and let $p_1{}^{(e_1)} \cap \cdots \cap p_n{}^{(e_n)}$ (symbolic powers) be an irredundant primary decomposition of $bR'$. It follows (see, for example, [2, (2.3)]) that the rings $V_i = R'_{p_i}$ ($i = 1, \ldots, n$) are the Rees valuation rings of $bR$. Let $D = R'_U$, where $U = R' \setminus (p_1 \cup \cdots \cup p_n)$. Theorem 2.8.2 implies that there exists a finite separable algebraic extension field $L = F[\theta]$ of $F$ such that the integral closure $E$ of $D$ in $L$ is a Dedekind domain having a radical ideal $J$ such that $bE = J^m$ for some positive integer $m$. If altitude$(R) = 1$, then $J \cap R'[\theta]$ is a radical ideal that is projectively full and projectively equivalent to $bR'[\theta]$, by Theorem 2.8.2 and its proof. Thus it seems at least plausible that this may also hold when altitude$(R) > 1$. However, a complication in higher altitude is that powers of $J \cap R'[\theta]$ may have embedded asymptotic prime divisors, as the following example shows.

**Example 3.2.** Let $k$ be a field, let $x, y$ be independent indeterminates, let $R = k[[x^2, xy, y^2]]$, and let $P = (x^2, xy)R$. Then $R$ is an integrally closed local domain of altitude two, the regular local ring $A = k[[x, y]]$ is a finite integral extension domain of $R$, and $P = xA \cap R$ is the radical of the principal ideal $bR = x^2R$, so $V = R_P$ is the only Rees valuation ring of $bR$. Also, $N = PV = (x^2, xy)V$ and $\frac{x}{y} = (xy) \cdot \frac{1}{y^2} \in N$, so $x^2 = (xy) \cdot \frac{x}{y} \in N^2$, so it follows that $N = xyV = \frac{x}{y}V$, hence $N^2 = bV$, so $N$ is a radical ideal that is projectively equivalent to $bV$ and the only Rees integer of $bR$ is two. (In the notation of Discussion 3.1, $D = E = V$, $J = N$, $m = 2$, and $R'[\theta] = R$.) However, $R[P/b]' = R[\frac{y}{x}]' \subseteq k[[x, y]][\frac{y}{x}]$, and the powers of the maximal ideal $M$ of $R$ define a valuation on the quotient field of $R$ that is readily seen to be a Rees valuation ring of $P$, but not a Rees valuation ring of $bR$ (since $V$ is the only Rees valuation ring of $bR$). Therefore $P = J \cap R$ is not projectively equivalent to $bR = bV \cap R$, by [1, (3.4)].

With notation as in Example 3.2, the finite integral extension $A = R[x, y]$ contains an ideal $xA$ that is projectively equivalent to $bA$ and the unique Rees integer of $xA$ is one. Thus, in relation to Question 1.1, it seems natural to ask:

**Question 3.3.** Let $b$ be a nonzero nonunit in a Noetherian integral domain $R$. Does there exist a finite integral extension domain $A$ of $R$ having an ideal $J$ whose Rees integers are all equal to one such that $J$ is projectively equivalent to $bA$?

With notation as in Discussion 3.1, we give in Proposition 3.5 several necessary and sufficient conditions for the radical ideal $J \cap R'[\theta]$ to be projectively equivalent to $bR'[\theta]$. The following definition is used in this result.

**Definition 3.4.** If $I$ is a regular proper ideal in $R$, then $\hat{A}^*(I)$ denotes the set of **asymptotic prime divisors** of $I$; that is, $\hat{A}^*(I) = \{P \in \operatorname{Spec}(R) \mid P \in \operatorname{Ass}(R/(I^k)_a)$ for some positive integer $k\}$.

Concerning the hypothesis "$bR = p_1{}^{(m)} \cap \cdots \cap p_n{}^{(m)}$" in Proposition 3.5, it follows from either Theorem 2.8.1 or Proposition 4.2 below that, for each nonzero nonunit $b$ in each Noetherian integral domain $R$ there exists a positive multiple $m$ of the Rees integers of $bR$ and a finite integral extension domain $A_m$ of $R$ such that $bA_m{}' = p_1{}^{(m)} \cap \cdots \cap p_n{}^{(m)}$, where $A_m{}'$ denotes the integral closure of $A_m$ and $p_1, \ldots, p_n$ are the prime divisors of $bA_m{}'$.

**Proposition 3.5.** *Let $R$ be an integrally closed Noetherian domain, let $m$ be a positive integer, let $b$ be a nonzero nonunit in $R$, let $p_1, \ldots, p_n$ be the (height one) prime divisors of $bR$, let $J =$*

*$p_1 \cap \cdots \cap p_n$, so $J = \mathrm{Rad}(bR)$, and assume that $bR = p_1{}^{(m)} \cap \cdots \cap p_n{}^{(m)}$. Then the following statements are equivalent*:

(3.5.1) *$J$ is projectively equivalent to $bR$.*
(3.5.2) *$J$ is projectively equivalent to some principal ideal in $R$.*
(3.5.3) *$bR = ((p_1 \cap \cdots \cap p_n)^k)_a$ for some positive integer $k$.*
(3.5.4) *$(J^k)_a$ is principal for some positive integer $k$.*
(3.5.5) *$J$ is invertible.*
(3.5.6) *$((p_1 \cap \cdots \cap p_n)^k)_a = p_1{}^{(k)} \cap \cdots \cap p_n{}^{(k)}$ for all positive integers $k$.*
(3.5.7) *$\hat{A}^*(J) = \{p_1, \ldots, p_n\}$ (see (3.4)).*
(3.5.8) *$J^k R_U \cap R = (J^k)_a$ for all positive integers $k$, where $U = R - (p_1 \cup \cdots \cup p_n)$.*

**Proof.** Since $J = p_1 \cap \cdots \cap p_n$ and $bR = p_1{}^{(m)} \cap \cdots \cap p_n{}^{(m)}$, by hypothesis, it follows that if (3.5.6) holds, then $(J^m)_a = bR = (bR)_a$ (since $R$ is integrally closed), so the case $k = m$ of (3.5.6) implies that (3.5.1) holds.

It is clear that (3.5.1) ⇒ (3.5.2)

If (3.5.2) holds, then let $c \in R$ and let $h, k$ be positive integers such that $(J^k)_a = (c^h R)_a$, so $(J^k)_a = c^h R$, since $R$ is integrally closed, so it follows that (3.5.2) ⇒ (3.5.4).

Assume that (3.5.4) holds and let $c \in R$ such that $(J^k)_a = cR$, so $(J^{gk})_a = (c^g R)_a = c^g R$ for all positive integers $g$, hence $\mathrm{Ass}(R/(J^{gk})_a) = \mathrm{Ass}(R/c^g R)$ for all positive integers $g$. Since $R$ is integrally closed, it follows that $\mathrm{Ass}(R/c^g R)$ is the set of height one prime ideals in $R$ that contain $cR$, so it follows that $cR \in p_i$ for $i = 1, \ldots, n$, and $p_1, \ldots, p_n$ are the only height one prime ideals in $R$ that contain $cR$, since $\mathrm{Ass}(R/cR) = \mathrm{Ass}(R/(J^k)_a)$. Therefore, since $\mathrm{Ass}(R/c^g R) = \mathrm{Ass}(R/(J^{gk})_a)$ for all positive integers $g$, it follows from Definition 3.4 that (3.5.7) holds, hence (3.5.4) ⇒ (3.5.7).

Assume that (3.5.7) holds, so $\mathrm{Ass}(R/(J^k)_a) = \{p_1, \ldots, p_n\}$ for all positive integers $k$, since each $p_i$ is a minimal prime divisor of $J$ and of $(J^k)_a$. Therefore for all positive integers $k$, $(J^k)_a = \bigcap\{(J^k)_a R_{p_i} \cap R \mid i = 1, \ldots, n\} = \bigcap\{J^k R_{p_i} \cap R \mid i = 1, \ldots, n\}$ (since $I_a = I$ for all ideals in $R_{p_i}$) $= \bigcap\{(J^k R_{p_i} \cap R_U) \cap R \mid i = 1, \ldots, n\}$ (where $U = R - (p_1 \cup \cdots \cup p_n) = J^k R_U \cap R$, hence (3.5.7) ⇒ (3.5.8).

Let $U$ be as in (3.5.8). Then it is readily checked that $J^k R_U \cap R = p_1{}^{(k)} \cap \cdots \cap p_n{}^{(k)}$ for all positive integers $k$, so (3.5.8) ⇒ (3.5.6) (since $J = p_1 \cap \cdots \cap p_n$).

The case $k = m$ of (3.5.6) implies that (3.5.3) holds (with $k = m$), since $bR = p_1{}^{(m)} \cap \cdots \cap p_n{}^{(m)}$, by hypothesis, and (3.5.3) ⇒ (3.5.4), since $J = p_1 \cap \cdots \cap p_n$.

Finally, if (3.5.2) holds, then $J$ is projectively equivalent to an invertible ideal, so $J$ is invertible, by [3, (2.10)(1)], so (3.5.2) ⇒ (3.5.5). And if (3.5.5) holds, then all ideals that are projectively equivalent to $J$ are invertible, by [3, (2.10)(1)], so $\mathrm{Ass}(R/(J^k)_a) = \{p_1, \ldots, p_n\}$ (the set of minimal prime divisors of $(J^k)_a$), by [3, (3.9)], hence it follows from Definition 3.4 that (3.5.5) ⇒ (3.5.7). □

## 4. An application to asymptotic sequences

The main result in this section, Proposition 4.2, partially extends Theorem 2.8.1 to certain finite sets of ideals, and its Corollary 4.6 applies this to asymptotic sequences. In the proofs we use the following definition.

**Definition 4.1.** Let $I$ be a regular proper ideal in a Noetherian ring $R$ and let $k$ be a positive integer. Then the **multiplicity** of $k$ as a Rees integer of $I$ is the number of DVRs $(V, N) \in \operatorname{Rees} I$ such that $IV = N^k$.

**Proposition 4.2.** *Let $I_1, \dots, I_h$ be nonzero proper ideals in a Noetherian domain $R$ and, for $i = 1, \dots, h$, let $e_{i,1}, \dots, e_{i,n_i}$ be the Rees integers of $I_i$ and $m_i = e_{i,1} \cdots e_{i,n_i}$. Assume that*: (a) $\operatorname{Rees} I_i \cap \operatorname{Rees} I_j = \emptyset$ *for $i \neq j$ in $\{1, \dots, h\}$. Then there exists a simple free separable integral extension domain $A$ of $R$ such that, for $i = 1, \dots, h$, the Rees integers of $I_i A$ are all equal to $m_i$.*

**Proof.** Let $\operatorname{Rees} I_i = \{(V_{i,1}, N_{i,1}), \dots, (V_{i,n_i}, N_{i,n_i})\}$ (for $i = 1, \dots, h$), and let $D = V_{1,1} \cap \dots \cap V_{h,n_h}$, so $D$ is a semi-local Principal Ideal Domain. Also, it follows from assumption (a) that $D$ has exactly $n^* = n_1 + \dots + n_h$ maximal ideals $M_{i,j} = N_{i,j} \cap D$, and $D_{M_{i,j}} = V_{i,j}$ for $i = 1, \dots, h$ and $j = 1, \dots, n_i$.

By hypothesis, for $i = 1, \dots, h$ and $j = 1, \dots, n_i$, $I_i V_{i,j} = {N_{i,j}}^{e_{i,j}}$, so ${M_{i,1}}^{e_{i,1}} \cap \dots \cap {M_{i,n_i}}^{e_{i,n_i}}$ is an irredundant primary decomposition of $I_i D$ (and $e_{i,1}, \dots, e_{i,n_i}$ are the Rees integers of $I_i$). Let

$$ {e_{i,j}}^* = \frac{m_i}{e_{i,j}} \quad \text{for } i = 1, \dots, h \text{ and } j = 1, \dots, n_i \tag{4.2.1} $$

and let

$$ m = {e_{1,1}}^* \cdots {e_{h,n_h}}^*, \tag{4.2.2} $$

so $m = {m_1}^{n_1 - 1} \cdots {m_h}^{n_h - 1}$.

Now resubscript the $M_{i,j}$ as follows: for $i = 1, \dots, h$ and $j = 1, \dots, n_i$ let $M_{n_1 + \dots + n_{i-1} + j} = M_{i,j}$ ($M_{n_1 + \dots + n_{i-1} + j} = M_{1,j}$, if $i = 1$), $e_{n_1 + \dots + n_{h-1} + j} = e_{i,j}$, and ${e_{n_1 + \dots + n_{h-1} + j}}^* = {e_{i,j}}^*$. Then the remainder of the proof of this proposition is similar to the proof of Theorem 2.8.2. We construct a chain of semi-local Principal Ideal Domains

$$ D = E_0 \subseteq E_1 \subseteq \dots \subseteq E_{n^*}, \tag{4.2.3} $$

where, for $k = 1, \dots, n^*$, $E_k$ is the integral closure of $E_{k-1}$ in a realization $L_k$ of a realizable ${e_k}^*$-consistent system $S^{(k)}$ for $\{(E_{k-1})_N \mid N \text{ is a maximal ideal in } E_{k-1}\}$. The systems $S^{(k)}$ are all similar. Specifically: $S^{(k)}$ ramifies to the index ${e_k}^*$ each of the ${e_1}^* \cdots {e_{k-1}}^*$ ($= 1$, if $k = 1$) maximal ideals $N$ in $E_{k-1}$ that contract in $D$ to $M_k$; $S^{(k)}$ splits into ${e_k}^*$ maximal ideals all of the other maximal ideals $N$ in $E_{k-1}$; and, $S^{(k)}$ gives no proper residue field extensions (that is, all of the residue field extensions $K_{i,j}$ (see (1) in Definition 2.1) of each maximal ideal $N$ in $E_{k-1}$ are chosen to be $E_{k-1}/N \cong D/(N \cap D)$).

It is readily checked that each $S^{(k)}$ is an ${e_k}^*$-consistent system for $\{(E_{k-1})_N \mid N$ is a maximal ideal in $E_{k-1}\}$, and it is realizable, by Theorem 2.3(i). Therefore their "composition" yields the chain (4.2.3) of separable extensions of degrees ${e_1}^*, \dots, {e_{n^*}}^*$, respectively, so the quotient field $L_{n^*}$ of $E_{n^*}$ is separable over the quotient field $L_0$ of $R$ and $D$, and $[L_{n^*} : L_0] = m$ (with $m$ as in (4.2.2)). It follows that each $M_{i,j}$ is ramified to the index ${e_{i,j}}^*$ in each of the $\frac{m}{{e_{i,j}}^*}$ maximal ideals in $E_{n^*}$ that contain $M_{i,j}$, so ${M_{i,j}}^{e_{i,j}}(E_{n^*})_N = (N^{{e_{i,j}}^*})^{e_{i,j}}(E_{n^*})_N$ for each of these $\frac{m}{{e_{i,j}}^*}$ maximal ideals $N$, and ${e_{i,j}}^* e_{i,j} = m_i$, by (4.2.1). Thus, for $i = 1, \dots, h$ and $j = 1, \dots, n_h$,

${M_{i,j}}^{e_{i,j}} E_{n^*}$ has only the Rees integer $m_i$ with multiplicity $\frac{m}{{e_{i,j}}^*}$ (see Definition (4.1)). Since $I_i = {M_{i,1}}^{e_{i,1}} \cdots {M_{i,n_i}}^{e_{i,n_i}}$, it follows that, for $i = 1, \ldots, h$, the Rees integers of $I_i E_{n^*}$ are all equal to $m_i$.

Since $L_{n^*}$ is a finite separable extension field of $L_0$, there exists an element $\theta$ in $L_{n^*}$ such that $L_{n^*} = L_0[\theta]$. It is readily checked that this implies there exists $r \in R$ such that $r\theta$ is integral over $R$. Therefore $A = R[r\theta]$ has quotient field $L_{n^*}$ and is a simple free separable integral extension domain of $R$. Since the rings $(E_{n^*})_N$ (with $N$ a maximal ideal in $E_{n^*}$) are the Rees valuation rings of the ideals $I_i A$, by Remark 2.7, it follows that, for $i = 1, \ldots, h$, the Rees integers of $I_i A$ are all equal to $m_i$. □

**Remark 4.3.** (4.3.1) A similar proof shows that the conclusion of Proposition 4.2 continues to hold, if the assumption (a) is replaced with "for all $i \neq j$ in $\{1, \ldots, h\}$, if $(V, N) \in \operatorname{Rees} I_i \cap \operatorname{Rees} I_j$, and if $I_i V = N^{e_i}$ and $I_j V = N^{e_j}$, then $m_i$ and $m_j$ are chosen so that $e_j m_i = e_i m_j$."

(4.3.2) For $i = 1, \ldots, h$ and $j = 1, \ldots, n_i$ let ${e_{i,j}}^* = \frac{m_i}{e_{i,j}}$ (as in (4.2.1)), and let $m = {e_{1,1}}^* \cdots {e_{h,n_h}}^*$ (as in (4.2.2)). Then it follows from the proof of Proposition 4.2 that:

(*3) *the field $L_{n^*}$ is a realization of the realizable $m$-consistent system $S^* = \{S_{1,1}, \ldots, S_{h,n_h}\}$,* where, for $i = 1, \ldots, h$ and $j = 1, \ldots, n_i$, $S_{i,j} = \{(K_{i,j,k}, 1, {e_{i,j}}^*) \mid k = 1, \ldots, \frac{m}{{e_{i,j}}^*}\}$.

**Remark 4.4.** There is a simpler proof of Proposition 4.2, if at least one $p_{i,j} = N_{i,j} \cap R$ is not a maximal ideal. Namely, in this case there exists an algebraic extension field of $D/M_{i,j}$ of degree $q$ for all integers $q$. (To see this, there exists $b \in I_{i,j}$ such that $V_{i,j} = {C'}_p$, where $C'$ is the integral closure of $C = R[I/b]$ and $p$ is a (height one) prime divisor of $bC'$, so $V_{i,j}/N_{i,j} \cong D/M_{i,j}$ is a finite extension field of the quotient field $F$ of $R/p_{i,j}$, by [12, (33.10)]. Since $R/p_{i,j}$ is a Noetherian domain and not a field, there exists a DVR with quotient field $F$, so there exists a DVR with quotient field $D/M_{i,j}$, and it is readily seen that there exists an algebraic extension field of $D/M_{i,j}$ of degree $q$ for all integers $q$.) Therefore let $K_{i,j}$ be an algebraic extension field of $D/M_{i,j}$ of degree $\frac{m}{{e_{i,j}}^*}$ and let $S_{i,j} = \{(K_{i,j}, \frac{m}{{e_{i,j}}^*}, {e_{i,j}}^*)\}$. Also, for $(i', j') \neq (i, j)$ let $S_{i',j'} = \{(K_{i',j',k}, 1, {e_{i',j'}}^*) \mid k = 1, \ldots, \frac{m}{{e_{i',j'}}^*}\}$. Then it readily checked that: (a) $S = \{S_{1,1}, \ldots, S_{h,n_h}\}$ is an $m$-consistent system for $\{(V_{1,1}, \ldots, V_{h,n_h}\}$; (b) it is realizable, by Theorem 2.3(i); (c) $I_i E = (\operatorname{Rad}(I_i E))^{m_i}$ for $i = 1, \ldots, h$, where $E$ is the integral closure of $D$ in a realization $L$ of $S$; and, (d) if $A = R[r\theta]$ as in the last paragraph of the proof of Proposition 4.2, then, for $i = 1, \ldots, h$, the Rees valuation rings of $I_i A$ are the rings in $\{E_N \mid N$ is a maximal ideal in $E$ and $I_i \subseteq N\}$, by Remark 2.7, so the Rees integers of $I_i A$ are all equal to $m_i$, by (c).

To prove a corollary of Proposition 4.2, we recall the following definition.

**Definition 4.5.** Let $R$ be a Noetherian ring and let $b_1, \ldots, b_g$ be regular nonunits in $R$. Then $b_1, \ldots, b_g$ are an **asymptotic sequence in** $R$ in case $(b_1, \ldots, b_g)R \neq R$, $b_1$ is not in any minimal prime ideal in $R$, and $b_i \notin \bigcup\{P \mid P \in \hat{A}^*(b_1, \ldots, b_{i-1})R\}$ for $i = 2, \ldots, g$. They are a **permutable asymptotic sequence in** $R$ in case each permutation of them is an asymptotic sequence in $R$.

Concerning Definition 4.5, it is shown in [9, (5.13)] that every $R$-sequence is an asymptotic sequence, and it is shown in [9, (5.3)] that if $R$ is locally quasi-unmixed, then an ideal is generated by an asymptotic sequence if and only if it is an ideal of the principal class.

**Corollary 4.6.** *Let $b_1, \dots, b_g$ be an asymptotic sequence in a Noetherian domain $R$. Then*:

(4.6.1) *For $i = 1, \dots, g$ let $I_i = (b_1, \dots, b_i)R$ and let $m_i$ be a positive common multiple of the Rees integers of $I_i$. Then there exists a simple free separable integral extension domain $A$ of $R$ such that, for $i = 1, \dots, g$, the Rees integers of $I_i A$ are all equal to $m_i$.*

(4.6.2) *Assume that $b_1, \dots, b_g$ is a permutable asymptotic sequence and let $\mathbf{I}$ be the set of all ideals of the form $(b_{\pi(1)}, \dots, b_{\pi(k)})R$, where $k$ varies over $\{1, \dots, g\}$ and where $\pi$ is an arbitrary permutation of $\{1, \dots, g\}$, so there are $h = 2^g - 1$ ideals in $\mathbf{I}$. Let $I_1, \dots, I_h$ be the ideals in $\mathbf{I}$ and for $i = 1, \dots, h$ let $m_i$ be a positive common multiple of the Rees integers of $I_i$. Then there exists a simple free separable integral extension domain $A$ of $R$ such that, for $i = 1, \dots, h$, the Rees integers of $I_i A$ are all equal to $m_i$.*

**Proof.** For (4.6.1), $\text{Rees}\, I_i \cap \text{Rees}\, I_j = \emptyset$ for $i \neq j$ in $\{1, \dots, g\}$, since $b_1, \dots, b_g$ is an asymptotic sequence in $R$, so the conclusion follows from Proposition 4.2.

The proof of (4.6.2) is similar, since $b_1, \dots, b_g$ is a permutable asymptotic sequence in $R$. □

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