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Physics Letters B

L.R.S. Bianchi type II stiff fluid cosmological model with decaying vacuum energy density Λ in general relativity

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A B S T R A C T

Locally rotationally symmetric (L.R.S.) Bianchi type II stiff fluid cosmological model is investigated. To get the deterministic model of the universe, we have assumed a supplementary condition $A = B^m$ between metric potentials A and B, where m is a constant. It is shown that the vacuum energy density Λ is positive and proportional to $\frac{1}{t^2}$. The values of deceleration parameter q, matter-energy density Ω_m and dark-energy density Ω_Λ are found to be in good agreement with the values obtain from 5-years WMAP observations. The predicted value of the jerk parameter is in agreement with the SNLS SNIa and X-ray galaxy cluster distance data but it is not match with the SNIa gold sample data. In general, the model represent accelerating, shearing and non-rotating universe. The physical and geometrical behavior of this model is also discussed.

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1. Introduction

One of the outstanding problems in cosmology is the so-called "cosmological constant" problem. Recent observations of type Ia supernovae (the Supernova Cosmology Project and the High-Z Supernova Team) [1–4] presented evidence that the expansion of the universe is accelerating. These teams have measured the distances to cosmological supernovae by using the fact that the intrinsic luminosity of type Ia supernovae is closely correlated to their decline rate from maximum brightness, which can be independently measured. These measurements, combined with red-shift data for the supernovae, led to the prediction of an accelerating universe. Both teams obtained $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$ and strongly ruled out the traditional $(\Omega_m, \Omega_\Lambda) = (1, 0)$ universe. This value of the density parameter Ω_Λ corresponds to a cosmological constant that is small, nevertheless, nonzero and positive.

Cosmological (or vacuum energy) constant [5–8] is one of the most theoretical candidates for dark energy. Unfortunately there is a huge difference of order 10^{120} between observational ($\Lambda \sim 10^{-55}$ cm^{-2}) and the particle physics prediction value for Λ. This discrepancy is known as cosmological constant problem. Carmeli and Kuzmenko [9] have recently shown that the cosmological relativistic theory [10] predicts $\Lambda = 1.934 \times 10^{-35}$ s^{-2} which is in agreement with the measurements recently obtained by the High-Z Supernova Team and Supernova Cosmological Project [1–4]. There have been several ansätze suggested in which the Λ term decays with time [11–25]. Chen and Wu [21] have suggested.

The special ansatz $\Lambda \propto R^{-2}$ (where R is the scale factor of the Robertson–Walker-metric) which has been modified by several authors [26–31]. However, an accelerating universe cannot be predicted by all vacuum decaying cosmological models. Several authors have argued in favor of the dependence $\Lambda \propto \frac{1}{t^2}$ in different context [13–15]. The relation $\Lambda \propto \frac{1}{t^2}$ seems to play a major role in cosmology [15]. Recently Pradhan et al. [32] have obtained some L.R.S. Bianchi type II bulk viscous fluid universe with decaying vacuum energy density.

A convenient method to describe models close to ΛCDM is based on the cosmic jerk parameter j, a dimensionless third derivative of the scale factor with respect to the cosmic time [33,34]. A deceleration-to-acceleration transition occurs for models with a positive value of j_0 and negative q_0. Flat ΛCDM models have a constant jerk $j = 1$.

Stiff fluid cosmological models create more interest in the study of the universe because for these models the speed of light is equal to the speed of sound and its governing equations have the same characteristics as those of gravitational field (Zel'dovich [35]). Barrow [36] has discussed the relevance of stiff equation of state $\rho = p$ to the matter content of the universe in the early state of evolution of universe. Wesson [37] has investigated an exact solution of Einstein's field equation with stiff equation of state. Mohanty et al. [38] have investigated cylindrically symmetric Zel'-dovich fluid distribution in General Relativity. Götz [39] obtained a plane symmetric solution of Einstein's field equation for stiff perfect fluid distribution. Bali and Tyagi [40] have investigated Bianchi type I magnetized stiff fluid cosmological model in General Relativity.

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In this Letter, a new anisotropic L.R.S. (Locally Rotationally Symmetric) Bianchi type II stiff fluid cosmological model with variable Λ has been investigated by assuming a supplementary condition $A = B^m$ between metric potentials A and B, where m is a constant. The out line of this Letter is as follows: In Section 2, the metric and the field equations are described. Section 3 deals with the solutions of the field equations. In Section 3.1 some physical and geometric properties of the model are described. Section 4 the jerk parameter of this model is driven. Finally, conclusions are summarized in the last Section 5.

2. The metric and field equations

The metric for L.R.S. Bianchi type II model in an orthogonal frame is given by

$$ds^2 = g_{ij}\theta^i\theta^j, \quad g_{ij} = \mathrm{diag}(-1, 1, 1, 1), \tag{1}$$

where the Cartan bases θ^i are given by

$$\theta^0 = dt, \qquad \theta^1 = B\omega^1, \qquad \theta^2 = A\omega^2, \qquad \theta^3 = A\omega^3. \tag{2}$$

Here, A and B are the time-dependent metric functions. Assuming (x, y, z) as local coordinates, the differential one forms ω^i are given by

$$\omega^1 = dy + x\,dz, \qquad \omega^2 = dz, \qquad \omega^3 = dx. \tag{3}$$

The Einstein's cosmological field equations are given by (with $8\pi G = 1$ and $c = 1$)

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -T_{ij}. \tag{4}$$

We consider the energy–momentum tensor in the form

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij}. \tag{5}$$

Hence, for energy–momentum tensor and L.R.S. Bianchi type II model the Einstein's field equations (4) lead to the following system of equations:

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} - \frac{3}{4}\frac{B^2}{A^4} = -p + \Lambda, \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{1}{4}\frac{B^2}{A^4} = -p + \Lambda, \tag{7}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} - \frac{1}{4}\frac{B^2}{A^4} = \rho + \Lambda, \tag{8}$$

where an overdot stands for the first and double overdot for second derivative with respect to t.

The spatial volume for L.R.S. Bianchi type II model is given by

$$V = A^2 B. \tag{9}$$

We define $a = (A^2 B)^{\frac{1}{3}}$ as the average scale factor of L.R.S. Bianchi type II model (1) so that the Hubble's parameter is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{2\dot{A}}{A} + \frac{\dot{B}}{B}\right). \tag{10}$$

We define the generalized mean Hubble's parameter H as

$$H = \frac{1}{3}(H_x + H_y + H_z), \tag{11}$$

where $H_x = \frac{\dot{B}}{B}$, $H_y = H_z = \frac{\dot{A}}{A}$ are the directional Hubble's parameters in the directions of x, y and z respectively.

The deceleration parameter q is conventionally defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{12}$$

The scalar expansion θ, shear scalar σ^2 and the average anisotropy parameter Am are defined by

$$\theta = \frac{2\dot{A}}{A} + \frac{\dot{B}}{B}, \tag{13}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - \frac{1}{3}\theta^2\right), \tag{14}$$

$$Am = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\triangle H_i}{H}\right)^2, \tag{15}$$

where $\triangle H_i = H_i - H$ $(i = 1, 2, 3)$.

3. Solution of the field equations

The field equations (6)–(8) are a system of three equations with five unknown parameters A, B, p, ρ, Λ. Two additional constraints relating these parameters are required to obtain explicit solutions of the system. Following Bali and Jain [41] and Pradhan et al. [42], I assume that the expansion (θ) in the model is proportional to the eigen value σ_1^1 of the shear tensor σ_j^i. This condition leads to

$$A = B^m, \tag{16}$$

where m is a constant.

In order to overcome the under-determinacy we have here because of the five unknown involved in three independent field equations, I assume that the fluid obeys the stiff fluid equation of state, i.e.

$$p = \rho. \tag{17}$$

From (6)–(8), (16) and (17) we obtain

$$2\ddot{B} + 4m\frac{\dot{B}^2}{B} = \frac{B^{-4m+3}}{m} + \frac{2\Lambda}{m}B. \tag{18}$$

Let $\dot{B} = f(B)$ which implies that $\ddot{B} = ff'$, where $f' = \frac{df}{dB}$. Hence (18) can be written as

$$\frac{d}{dB}f^2 + 4m\frac{f^2}{B} = \frac{B^{-4m+3}}{m} + \frac{2\Lambda}{m}B. \tag{19}$$

After integrating, Eq. (19) leads to

$$f^2 = \frac{B^{-4(m-1)}}{4m} + \frac{\Lambda}{m(1+2m)}B^2 + kB^{-4m}, \tag{20}$$

where k is an integrating constant.

To get deterministic solution in terms of cosmic time t, we suppose $k = 0$. In this case Eq. (20) takes the form

$$\frac{dB}{\sqrt{\frac{B^{-4(m-1)}}{4m} + \frac{\Lambda}{m(1+2m)}B^2}} = dt. \tag{21}$$

To get deterministic solution, we assume $m = \frac{3}{2}$. In this case Eq. (21) reduces to

$$\frac{dB}{\sqrt{\frac{1}{6}B^{-2} + \frac{\Lambda}{6}B^2}} = dt. \tag{22}$$

Integrating Eq. (22) we obtain

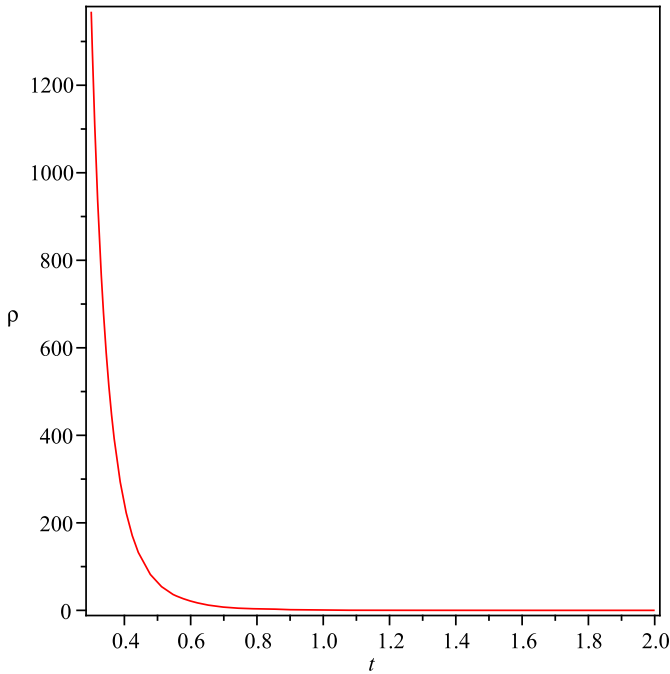$$B^2 = \sqrt{\frac{1}{\Lambda}}\sinh\left(\sqrt{\frac{2\Lambda}{3}}t\right). \tag{23}$$
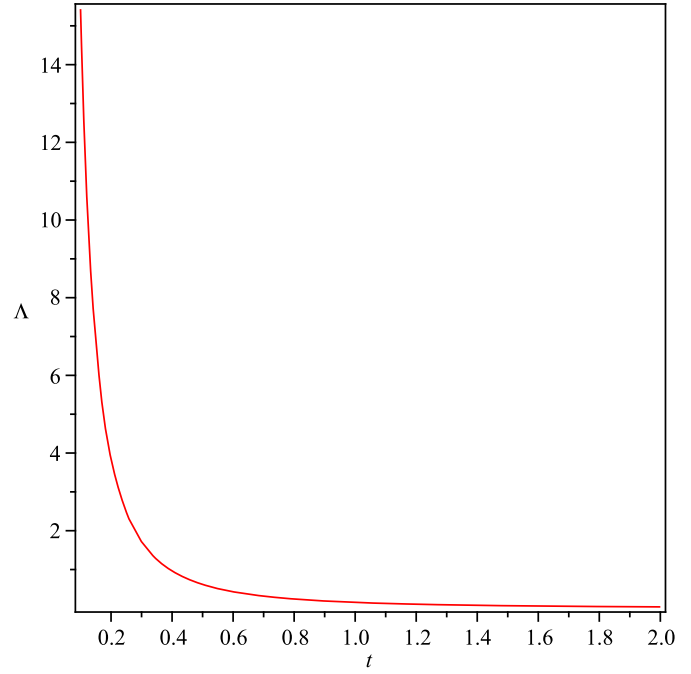
Fig. 1. The plot of energy density ρ vs. t.

Fig. 2. The plot of vacuum energy density Λ vs. t.

Therefore,

$$A^2 = \left(\frac{1}{\Lambda}\right)^{\frac{3}{4}} \sinh^{\frac{3}{2}}\left(\sqrt{\frac{2\Lambda}{3}}t\right). \tag{24}$$

Eqs. (23) and (24) show that $\Lambda > 0$. In this case the L.R.S. Bianchi type II spacetime can be written as

$$ds^2 = -dt^2 + \sqrt{\frac{1}{\Lambda}} \sinh\left(\sqrt{\frac{2\Lambda}{3}}t\right)(dy + x\,dz)^2$$
$$+ \left(\frac{1}{\Lambda}\right)^{\frac{3}{4}} \sinh^{\frac{3}{2}}\left(\sqrt{\frac{2\Lambda}{3}}t\right)(dx + dz)^2. \tag{25}$$

3.1. The geometric and physical significance of model

The energy density ρ, the pressure p and the vacuum energy density Λ for the model (25) are given by

$$p = \rho = \frac{15}{24}\Lambda \coth^2\left(\sqrt{\frac{2\Lambda}{3}}t\right) - \frac{3\Lambda}{4}, \tag{26}$$

$$\Lambda = \frac{3}{2}\left(\coth^{-1}\sqrt{\frac{20}{13}}\right)^2 \frac{1}{t^2}. \tag{27}$$

From (26), we see that energy conditions, $\rho \geqslant 0$ is satisfied under condition

$$\coth^2\left(\sqrt{\frac{2\Lambda}{3}}t\right) \geqslant \frac{6}{5}. \tag{28}$$

From Eq. (26), it is noted that the proper energy density $\rho(t)$ is a decreasing function of time and it approaches a small positive value at present epoch. This behavior is clearly depicted in Fig. 1. From Eq. (27), we observe that the cosmological term Λ is a decreasing function of time and it approaches a small positive value at late time. From Fig. 2, we note this behavior of cosmological term Λ in the model. Recent cosmological observations suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations on magnitude and

red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus, our model is consistent with the results of recent observations.

The expressions for Hubble parameter H, the scalar of expansion θ, magnitude of shear σ^2, the average anisotropy parameter A_m and the proper volume V for the model (25) are given by

$$H = \left(\frac{2m+1}{3}\right)H_x = \frac{2}{3}\sqrt{\frac{2\Lambda}{3}}\left(\sqrt{\frac{2\Lambda}{3}}t\right), \tag{29}$$

$$\theta = \sqrt{\frac{8}{3}\Lambda}\coth\left(\sqrt{\frac{2\Lambda}{3}}t\right), \tag{30}$$

$$\sigma^2 = \frac{2\Lambda}{9}\coth^2\left(\sqrt{\frac{2\Lambda}{3}}t\right), \tag{31}$$

$$A_m = 2\left(\frac{m-1}{2m+1}\right)^2 = \frac{1}{32}, \tag{32}$$

$$V = \frac{1}{\Lambda}\sinh^2\left(\sqrt{\frac{2\Lambda}{3}}t\right). \tag{33}$$

From (30) and (31) we get

$$\frac{\sigma^2}{\theta^2} = \text{constant.} \tag{34}$$

The deceleration parameter is given by

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\left[\frac{\frac{4\Lambda}{9} - \frac{4\Lambda}{27}\coth^2(\sqrt{\frac{2\Lambda}{3}}t)}{\frac{8\Lambda}{27}\coth^2(\sqrt{\frac{2\Lambda}{3}}t)}\right]. \tag{35}$$

If we put the value of Λ from Eq. (27) we observe that

$$q \simeq -0.96. \tag{36}$$

From Eq. (36) we observe that our model is in accelerating phase and it's behavior is almost the same as de Sitter universe.

Using Eqs. (26)–(29) we can obtain the matter-energy density Ω_m and dark-energy density Ω_Λ as

Table 1
The values of Ω_m and Ω_Λ obtain from our model and WMAP.

Parameter	Our model	WMAP
Ω_m	0.155	0.279
Ω_Λ	0.731	0.726

$$\Omega_m = \frac{9}{8}\left[\frac{5}{8} - \frac{3}{4}\tanh^2\left(\coth^{-1}\left(\sqrt{\frac{20}{13}}\right)\right)\right] \simeq 0.155, \qquad (37)$$

and

$$\Omega_\Lambda = \frac{9}{8}\tanh^2\left(\coth^{-1}\left(\sqrt{\frac{20}{13}}\right)\right) \simeq 0.731. \qquad (38)$$

From Eqs. (37) and (38) we observe that the values of matter-energy density Ω_m and dark-energy density Ω_Λ are in good agreement with the values obtain from 5-years WMAP observations for ΛCMD model [43]. The compression of these parameters is shown in Table 1.

4. The jerk parameter of the model

The jerk parameter in cosmology is defined as the dimensionless third derivative of the scale factor with respect to cosmic time

$$j(t) = \frac{1}{H^3}\frac{\dddot{a}}{a} \qquad (39)$$

and in terms of the scale factor to cosmic time

$$j(t) = \frac{(a^2 H^2)''}{2H^2} \qquad (40)$$

where the 'dots' and 'primes' denote derivatives with respect to cosmic time and scale factor, respectively. The jerk parameter appears in the fourth term of a Taylor expansion of the scale factor around a_0

$$\frac{a(t)}{a_0} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \frac{1}{6}j_0 H_0^3(t - t_0)^3$$
$$+ O\left[(t - t_0)^4\right], \qquad (41)$$

where the subscript 0 shows the present value. One can rewrite Eq. (39) as

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H}. \qquad (42)$$

Using Eqs. (29) and (35) in (42) we find

$$j(t) = \frac{3}{2}\left[\sinh\left(\sqrt{\frac{2\Lambda}{3}}t\right)\left[-2 + 2\sinh\left(\sqrt{\frac{2\Lambda}{3}}t\right)\cosh^3\left(\sqrt{\frac{2\Lambda}{3}}t\right)\right.\right.$$
$$\left.\left. - 3\sinh\left(\sqrt{\frac{2\Lambda}{3}}t\right)\right]\right] \times \left[\cosh^4\left(\sqrt{\frac{2\Lambda}{3}}t\right)\right]^{-1}. \qquad (43)$$

Now, putting the value of Λ from Eq. (27) in Eq. (43) we obtain

$$j_0 = 0.88^{+0.21}_{-0.08}. \qquad (44)$$

This value does not overlap with the value $j = 2.16^{+0.81}_{-0.75}$ obtained from the combination of three kinematical data sets: the gold sample of type Ia supernovae [44], the SNIa data from the SNLS project [45], and the X-ray galaxy cluster distance measurements [46]. However, it is in consistent with two of the three data sets separately: the SNLS SNIa set gives $j = 1.32^{+1.37}_{-1.21}$ and the cluster set gives $j = 0.51^{+2.55}_{-2.00}$, and it is the gold sample data that yields larger $j = 2.75^{+1.22}_{-1.10}$ [46].

5. Concluding remarks

A new cosmological model based on L.R.S. Bianchi type II cosmological models with decaying vacuum energy density is obtained. The model (25) starts with a big bang at $t = 0$. The expansion in the model decreases as time increases. The proper volume of the model increases as time increases. Since $\frac{\sigma}{\theta}$ is constant the model does not approach isotropy. There is a point type singularity in the model at $t = 0$ [47]. It is shown that $\Lambda \propto \frac{1}{t^2}$. Therefore, as $t \to 0$, $\Lambda \to \infty$ and when $t \to \infty$ then $\Lambda \to 0$. In Brans–Dicke theories the relation like Eq. (27) can be found when one assumes variable gravitational and cosmological "constant" [13,15] and [17]. Berman [48] also has derived this relation in general relativity. A positive cosmological constant or equivalently the negative deceleration parameter is required to solve the age parameter and density parameter.

The values of deceleration parameter q, matter-energy density Ω_m, dark-energy density Ω_Λ and the jerk parameter for this model are found to be in good agreement with the present values of these parameters obtained from observations. It is reasonable to say that a cosmological model is required to explain acceleration in the present universe. Therefore, the theoretical model found in this Letter is in agreement with the recent observations.

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