Numerical range and product of matrices
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ABSTRACT
In this paper, we prove the converse of a well known result in the field of the numerical range. In fact, we show that for a matrix $A \in M_n$, if the inclusion $\sigma(AB) \subseteq W(A)W(B)$ holds for all matrices $B \in M_n$, then $A$ is a scalar multiple of a positive semidefinite matrix.

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1. Introduction

The study of the numerical range of bounded operators on finite [3] and infinite [2] dimensional Hilbert spaces has a long history. Properties, geometry, location and generalizations of the numerical range are most interesting research areas in this field [3]. Meanwhile the study of relationship between the numerical range and matrix products [1,5] and also submultiplicativity of the numerical range have an important position [3, Section 1.7]. Unfortunately even very weak multiplicative properties do not hold. For example the property $\sigma(AB) \subseteq W(A)W(B)$ dose not hold in general, where $\sigma$ and $W$ denote the spectrum and the numerical range, respectively. A well known result shows that if $A$ is a scalar multiple of a positive semidefinite matrix, then the inclusion $\sigma(AB) \subseteq W(A)W(B)$ holds for all matrices $B \in M_n$ [3, Corollary 1.7.7]. In this paper we prove the converse of this theorem. Before this, we need to introduce some basic notations which will be used in this paper.

Let $M_n$ be the algebra of all $n \times n$ matrices with complex entries, $E_{ij}$, $1 \leq i, j \leq n$ the matrix units in $M_n$ and $\text{diag}(x_1, \ldots, x_n)$ the diagonal matrix with the entries $x_1, \ldots, x_n$ on the main diagonal. The permutation matrix $P_{ij}$ is the matrix obtained by swapping rows $i$ and $j$ of the identity matrix. For $A \in M_{n_1}$ and $B \in M_{n_2}$ the direct sum of $A$ and $B$ is the matrix

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{n_1+n_2}.$$
If $z_1$ and $z_2$ are two complex numbers, the segment with two end points $z_1$ and $z_2$ is displayed by $[z_1, z_2]$. For a set $S$ in the plane, $Co(S)$ denotes the convex hull of $S$.

The numerical range of $A$ is a compact and convex region [3] of the complex plane which is defined by

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathbb{C}^n, x^*x = 1 \}.$$  

The numerical radius of $A$ is displayed by $r(A)$ and is defined as follow:

$$r(A) = \sup \{ |\lambda| : \lambda \in W(A) \}.$$  

Also we define the set $K_A$ by

$$K_A = \{ \lambda : |\lambda| \leq r(A) \}.$$  

It is clear that $K_A$ contains $W(A)$.

2. **Main result**

At first we remember that a matrix $A \in M_n$ is radial if $r(A) = \|A\|$, where

$$\|A\| = \sup \{ |Ax| : x \in \mathbb{C}^n, x^*x = 1 \}.$$  

**Lemma 2.1.** Let $A$ be an element of $M_n$ such that for every $B \in M_n$ the inclusion $\sigma(AB) \subseteq K_AK_B$ holds. Then $A$ is a radial matrix.

**Proof.** By the polar decomposition, we can find a unitary matrix $U$ with $(AA^*)^{\frac{1}{2}} = AU$. Now we have

$$\sigma(|A|) = \sigma((AA^*)^{\frac{1}{2}}) = \sigma(AU) \subseteq K_AK_U.$$  

But $\|A\| \in \sigma(|A|)$ and $K_U$ is equal to the unit disk. Therefore $K_A$ has an element $\lambda$ such that $|\lambda| \geq \|A\|$. Since $|\lambda| \leq r(A) \leq \|A\|$, we have $r(A) = \|A\|$ and hence $A$ is radial. $\square$

**Lemma 2.2.** Let $A$ and $\tilde{A}$ be two unitarily similar matrices in $M_n$. Then $\sigma(AB) \subseteq K_AK_B$, for every $B \in M_n$ if and only if $\sigma(\tilde{A}B) \subseteq K_{\tilde{A}}K_B$, for every $B \in M_n$.

**Proof.** Let $U$ be a unitary matrix in $M_n$ with $\tilde{A} = UAU^*$. If for every $B \in M_n$, $\sigma(AB) \subseteq K_AK_B$, since $K_A = K_{\tilde{A}}$, we have

$$\sigma(\tilde{A}B) = \sigma(UAU^*B) = \sigma(U^{*}BU) \subseteq K_{\tilde{A}}K_{U^{*}BU} = K_{\tilde{A}}K_B.$$  

Note that for every matrices $A, B \in M_n$ the equality $\sigma(AB) = \sigma(BA)$ holds [4, Theorem 1.3.20]. The another part can be proved similarly. $\square$

**Lemma 2.3.** Let $A$ be an element of $M_n$ such that for every $B \in M_n$, the inclusion $\sigma(AB) \subseteq K_AK_B$ holds. Then $A$ is a normal matrix.

**Proof.** We argue by induction on $n$. Since every radial matrix in $M_2$ is normal [3, p. 45], by Lemma 2.1, the case $n = 2$ is proved. Now suppose that for $k < n$, the lemma holds and consider $A \in M_n$ which for every $B \in M_n$, $\sigma(AB) \subseteq K_AK_B$. Again by Lemma 2.1, $A$ is radial and hence by [3, p. 45] and Schur triangularization theorem, there exist $1 \leq r \leq n$, complex numbers $t_1, \ldots, t_r$ with $|t_1| = \cdots = |t_r| = 1$ and an upper triangular matrix $B = (b_{ij}) \in M_{n-r}$ with $\|B\| \leq 1$ such that $A$ is unitarily similar to the matrix

$$\tilde{A} = \|A\| \begin{pmatrix} \text{diag}(t_1, \ldots, t_r) & 0 \\ 0 & B \end{pmatrix}.$$
Now let $\tilde{A}$ be the principle submatrix of $\tilde{A}$ obtained by deleting row and column $n$ of the matrix $\tilde{A}$. Using Lemma 2.2, for every matrix $\tilde{B}$ in $M_{n-1}$ we have

$$\sigma(\tilde{A}\tilde{B}) \subseteq \sigma(\tilde{A}(\tilde{B} \oplus 0)) \subseteq K_{\tilde{A}}K_{\tilde{B}}.$$ 

But

$$K_{\tilde{A}} = K_{\tilde{A}} = \{\lambda : |\lambda| \leq \|A\|\}.$$ 

Hence $\sigma(\tilde{A}\tilde{B}) \subseteq K_{\tilde{A}}K_{\tilde{B}}$ and by the assumption of the induction, $\tilde{A}$ is a normal matrix. Since $\tilde{A}$ is an upper triangular matrix, we conclude that it should be a diagonal matrix.

Now we should prove that $b_{1,n-r} = b_{2,n-r} = \cdots = b_{n-r-1,n-r} = 0$. Let

$$\hat{A} = P_{n-1,n}P_{1,n-1}\tilde{A}P_{1,n-1}P_{n-1,n}.$$ 

Then $\hat{A}$ is an upper triangular matrix which is unitarily similar to $\tilde{A}$. Consider $\hat{A}$ as the principle submatrix of $\tilde{A}$ obtained by deleting first row and column of the matrix $\tilde{A}$. For every matrix $\tilde{B}$ in $M_{n-1}$ we have

$$\sigma(\hat{A}\tilde{B}) = \sigma(\tilde{A}\hat{B}) \subseteq \sigma((0 \oplus \hat{B})\hat{A}) = \sigma(\hat{A}(0 \oplus \hat{B})) \subseteq K_{\hat{A}}K_{0 \oplus \hat{B}} = K_{\hat{A}}K_{\hat{B}}.$$ 

and

$$K_{\hat{A}} = K_{\tilde{A}} = \{\lambda : |\lambda| \leq \|A\|\}.$$ 

Hence $\sigma(\hat{A}\tilde{B}) \subseteq K_{\hat{A}}K_{\hat{B}}$ and by the assumption of the induction, $\hat{A}$ is a normal matrix. This implies that $b_{1,n-r} = b_{2,n-r} = \cdots = b_{n-r-2,n-r} = 0$.

Also if we set

$$\hat{A} = P_{1,2}P_{n-1,n}P_{1,n-1}\tilde{A}P_{1,n-1}P_{n-1,n}P_{1,2}$$

in (2.1), a same argument shows that $b_{n-r-1,n-r} = 0$. \hfill \Box

**Theorem 2.4.** For a matrix $A \in M_n$, if the inclusion $\sigma(AB) \subseteq W(A)W(B)$ holds for all matrices $B \in M_n$, then $A$ is a scalar multiple of a positive semi definite matrix.

**Proof.** By Lemmas 2.2, 2.3 and Schur triangularization theorem, we can assume that $A = \text{diag}(r_{1}e^{i\theta_{1}}, \ldots, r_{k}e^{i\theta_{k}}, r_{k+1}e^{i\theta_{k+1}}, \ldots, r_{n}e^{i\theta_{n}})$, where $0 \leq \theta_1 \leq \cdots \leq \theta_k < 2\pi$ and $r_{1}e^{i\theta_{1}}, \ldots, r_{k}e^{i\theta_{k}}$ are nonzero extreme points of $W(A)$. To prove the theorem, we should show that $\theta_1 = \cdots = \theta_k$. For $k = 1$, there is nothing to prove. Assuming $k \geq 2$, setting $B = E_{1,2} + E_{2,1}$ and

$$A_s = P_{s,s+1}P_{1,s}AP_{1,s}P_{2,s+1}, \quad s = 1, \ldots, k-1,$$

we have $W(B \oplus 0) = [-1, 1]$ and

$$\sqrt{r_{s}s+1}e^{i(\theta_{s}+\theta_{s+1})/2} \in \sigma(A_s(B \oplus 0)), \quad s = 1, \ldots, k-1.$$

Note that $A_s$ is unitarily similar to $A$, hence $\sigma(A) = \sigma(A_s)$ and $W(A) = W(A_s)$. Also if $0 \in \sigma(A_s)$, then

$$W(A_s) = \text{Co}([r_{1}e^{i\theta_{1}}, r_{2}e^{i\theta_{2}}, \ldots, r_{k}e^{i\theta_{k}}, 0]).$$

Otherwise

$$W(A_s) = \text{Co}([r_{1}e^{i\theta_{1}}, r_{2}e^{i\theta_{2}}, \ldots, r_{k}e^{i\theta_{k}}]).$$
From the inclusion $\sigma(A_s(B \oplus 0)) \subseteq W(A_s)W(B \oplus 0)$, we conclude that there exist $0 \leq h_s$ with

$$h_s e^{i(\theta_s + \theta_{s+1})/2} \in W(A_s)$$

and $0 \leq t_s \leq 1$, such that

$$\sqrt{r_s r_{s+1}} e^{i(\theta_s + \theta_{s+1})/2} = h_s e^{i(\theta_s + \theta_{s+1})/2} t_s, \quad s = 1, \ldots, k - 1.$$ \(2.2\)

But because of (2.2), a simple calculation shows that (Fig. 1)

$$0 \leq h_s \leq \frac{2r_s r_{s+1}}{r_s + r_{s+1}} \left| \cos \left( \frac{\theta_{s+1} - \theta_s}{2} \right) \right|.$$ \(2.3\)

Now using (2.3), we have

$$\sqrt{r_s r_{s+1}} \leq \frac{2r_s r_{s+1}}{r_s + r_{s+1}} \left| \cos \left( \frac{\theta_{s+1} - \theta_s}{2} \right) \right|, \quad s = 1, \ldots, k - 1.$$ \(2.3\)

Therefore

$$1 \leq \frac{r_s + r_{s+1}}{2\sqrt{r_s r_{s+1}}} \leq \left| \cos \left( \frac{\theta_{s+1} - \theta_s}{2} \right) \right|, \quad s = 1, \ldots, k - 1.$$ \(2.3\)

Hence $\theta_s = \theta_{s+1}, \quad s = 1, \ldots, k - 1. \quad \square$

References