Lacunary Interpolation for Entire Functions

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While the problem of lacunary interpolation for polynomials and trigonometric polynomials has drawn the attention of many mathematicians (see, for example, [1, 2, 5, 6, 8–10]) in the case of entire functions of exponential type, being the natural counterpart for approximants in the case of infinite interval, only the problem of interpolation where the value of the function and its consecutive derivatives are given at prescribed points is so far known. We shall here initiate the study of lacunary interpolation for entire functions. The most elementary case of (0, 2) interpolation is being pursued at the moment.

We first notice that a trigonometric polynomial of order \( n \) is an entire function of exponential type \( n \). Our (0, 2) lacunary interpolation problem can be stated as follows:

Given the distinct points

\[ k\pi/\sigma \quad (k = 0, \pm 1, \pm 2, \ldots), \]

where \( \sigma \) is a given positive number and arbitrary sequence of numbers

\[ \{ a_k \}_{k = -\infty}^{\infty} \quad \text{and} \quad \{ b_k \}_{k = -\infty}^{\infty}, \]

it is to be decided whether or not there exists an entire function \( f_\sigma(x) \) of exponential type \( 2\sigma \) satisfying the conditions

\[ f_\sigma(k\pi/\sigma) = a_k \quad \text{and} \quad f_\sigma''(k\pi/\sigma) = b_k. \]
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As in the case of polynomials, the natural questions that would arise are
(i) Does there exist such an \( f_\sigma \) at all? (ii) If there is such an \( f_\sigma \), is it uniquely determined? (iii) If \( f_\sigma \) exists and is unique, can it be represented in a convenient form, and (iv) assuming that the \( a_k \)'s are the values of a given function \( f(x) \) defined on the real line \( \mathbb{R} \) at the set of points \( x_k = k\pi/\sigma \) what are the conditions under which \( f_\sigma(x) \) converges to \( f(x) \) uniformly as \( \sigma \) tends to infinity?

Since the given points are in arithmetic progression, it is easily seen that if \( U(x) \) is an entire function of exponential type \( 2\sigma \), taking the value 1 at \( x = 0 \) and \( U(k\pi/\sigma) = 0 \) for \( k \) different from zero and \( U''(k\pi/\sigma) = 0 \) for all \( k \), then \( U(x - (k\pi/\sigma)) \) will take the value 1 at \( k\pi/\sigma \) and will be zero at other points. Moreover, the second derivative will vanish at all points of sequence (1). Similarly if \( V(x) \) is an entire function, 0 at all points and \( V''(0) = 1 \) while \( V''(k\pi/\sigma) = 0 \) for \( k \neq 0 \), then \( V(x - (k\pi/\sigma)) \) has the same properties except that the second derivatives take the value 1 at \( k\pi/\sigma \) instead of at 0. If we set

\[
F(x) = \sum_{-\infty}^{\infty} a_k U \left( x - \frac{k\pi}{\sigma} \right) + \sum_{-\infty}^{\infty} b_k V \left( x - \frac{k\pi}{\sigma} \right),
\]

then \( F(x) \) will satisfy the conditions:

\[
F(k\pi/\sigma) = a_k; \quad F''(k\pi/\sigma) = b_k. \tag{3}
\]

The required function therefore will be of the form (2) provided it is an entire function of exponential type \( 2\sigma \). This automatically implies that the series on the r.h.s. of (2) will have to be convergent and suitable conditions on \( a_k \)'s and \( b_k \)'s are necessary. Our problem first reduces to the problem of finding the entire functions \( U \) and \( V \) as mentioned above.

**Theorem 1.** Let \( \sigma > 0 \) be given. If

\[
x_k = k\pi/\sigma, \quad k = 0, \pm 1, \pm 2, \ldots,
\]

then the even entire functions \( U_\sigma \) and \( V_\sigma \) of exponential type \( 2\sigma \) satisfying the conditions

\[
U_\sigma(0) = 1, \quad U_\sigma \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = \pm 1, \pm 2, \ldots, \tag{4}
\]

\[
U_\sigma'' \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{5}
\]
and

\[ V_\sigma \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = 0, \pm 1, \pm 2, \ldots \quad (6) \]

\[ V''_\sigma (0) = 1, \quad V''_\sigma \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = \pm 1, \pm 2, \ldots \quad (7) \]

are given by

\[ U_\sigma(x) = \frac{\sin \sigma x}{\sigma x} \left\{ 1 - x \int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt \right\} \quad (8) \]

and

\[ V_\sigma(x) = \frac{\sin \sigma x}{2\sigma} \int_0^x \frac{\sin \sigma t}{\sigma t} \, dt. \quad (9) \]

Moreover, \( U_\sigma \) and \( V_\sigma \) are also unique.

**Proof.** Let us first obtain the explicit form of the function \( U_\sigma(x) \). By condition (4), \( U_\sigma \) has the form

\[ U_\sigma(x) = \frac{\sin \sigma x}{\sigma x} \phi(x), \]

where \( \phi(x) \) is an even entire function of exponential type \( \sigma \) with \( \phi(0) = 1 \). Using condition (5), we obtain

\[ \phi''(0) = \sigma^2/3 \]

and

\[ (k\pi/\sigma) \phi'(k\pi/\sigma) - \phi(k\pi/\sigma) = 0, \quad k = \pm 1, \pm 2, \ldots \]

If we set

\[ G(x) = x\phi'(x) - \phi(x) \]

we see that

\[ G(0) = -1, \quad G \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = \pm 1, \pm 2, \ldots \]

Since \( G \) is an even entire function of exponential type \( \sigma \) we may take \( G(x) = -\sin(\sigma x)/\sigma x \). Hence \( \phi \) satisfies the differential equation

\[ x\phi'(x) - \phi(x) = -\sin(\sigma x)/\sigma x. \quad (10) \]
Making the substitution $g(x) = \phi(x) - 1$, (10) reduces to

$$xg'(x) - g(x) = -(\sin(\sigma x) - \sigma x)/\sigma x$$

with

$$g(0) = 0 \quad \text{and} \quad g'(x) = \phi'(x).$$

Solving the above linear equation, we have

$$\frac{g(x)}{x} = -\int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt + C.$$ 

Since $g$ is an even entire function, the constant of integration $C$ is zero from which we obtain

$$\phi(x) = 1 - x \int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt.$$ 

Therefore

$$U_\sigma(x) = \frac{\sin \sigma x}{\sigma x} \left\{ 1 - x \int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt \right\}.$$ 

To find $V_\sigma$, because of condition (6) we may assume

$$V_\sigma(x) = \sin \sigma x \cdot \psi(x), \quad (11)$$

where $\psi(x)$ is an odd entire function of exponential type $\sigma$. Condition (7) then gives us

$$\psi'(0) = 1/2\sigma \quad \text{and} \quad \psi'(k\pi/\sigma) = 0, \quad k = \pm 1, \pm 2, \ldots$$

and so we have

$$\psi'(x) = (1/2\sigma)(\sin \sigma x)/\sigma x \quad (12)$$

which gives

$$\psi(x) = \frac{1}{2\sigma} \int_0^x \frac{\sin \sigma t}{\sigma t} \, dt;$$

the constant of integration being zero because $\psi(x)$ is odd. Hence,

$$V_\sigma(x) = \frac{\sin \sigma x}{2\sigma} \int_0^x \frac{\sin \sigma t}{\sigma t} \, dt.$$
It is clear that the integrands given in (8) and (9) are entire functions and so are the functions $U_{\sigma}$ and $V_{\sigma}$. We shall prove that $U_{\sigma}$ and $V_{\sigma}$ are unique. We prove the uniqueness of $U_{\sigma}$ only, the proof being similar for $V_{\sigma}$.

Suppose $U_1$ is another even entire function of exponential type $2\sigma$ satisfying the conditions

$$U_1(0) = 1, \quad U_1 \left( \frac{k\pi}{\sigma} \right) = 0, \quad k = \pm 1, \pm 2, \ldots,$$

$$U''_1(k\pi/\sigma) = 0, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Let

$$H(z) = U_{\sigma}(z) - U_1(z).$$

Then

$$H(k\pi/\sigma) = 0, \quad H''(k\pi/\sigma) = 0 \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots.$$

Also $H$ is an even entire function of exponential type $2\sigma$. Since an entire function of exponential type $\sigma$ vanishing at the points

$$k\pi/\sigma, \quad k = 0, \pm 1, \pm 2, \ldots,$$

is given by

$$C \sin \sigma z$$

(see [1, p. 180]), by the condition on $H(z)$ we have

$$H(z) = C \cdot \sin \sigma z \cdot h(z), \quad (A)$$

where $h(z)$ is an odd entire function of exponential type $\sigma$. The condition $H''(k\pi/\sigma) = 0$ for all $k$ implies

$$h' \left( \frac{k\pi}{\sigma} \right) = 0 \quad \text{for all} \quad k.$$

and hence

$$h''(z) = C_1 \sin \sigma z$$

which is an odd function. This contradicts the fact that $h$ is odd. Hence the constant $C$ of Eq. (A) must be zero which proves that

$$U_{\sigma}(z) = U_1(z).$$

This completes the proof.
Let us now proceed to estimate the bounds of $|U_{\sigma}(x)|$ and $|V_{\sigma}(x)|$.

**Theorem 2.** Functions $U_{\sigma}(x)$ and $V_{\sigma}(x)$ have the following bounds:

1. $|U_{\sigma}(x)| \leq \frac{(e^2 + 4e - 1)}{2e}$ \hspace{1cm} (13)
2. $|V_{\sigma}(x)| \leq \frac{(8 + \pi^2)}{4\pi \sigma^2}$ \hspace{1cm} (14)

**Proof.** From (8), we have

$$U_{\sigma}(x) = \frac{\sin \sigma x}{\sigma x} - \int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt.$$  

Now

$$\left| \int_0^1 \frac{\sin(t) - t}{t^3} \, dt \right| \leq \int_0^1 \left| \frac{\sin(t) - t}{t^3} \right| \, dt \leq \int_0^1 (\sinh(1) - 1) \, dt = \frac{e^2 - 2e - 1}{2e}.$$  

If $x \geq 1$

$$\left| \int_1^x \frac{\sin(t) - t}{t^3} \, dt \right| \leq \frac{e^2 - 2e - 1}{2e} + 2 \left( \frac{1 - \frac{1}{x}}{x^2} \right) \leq 2.$$  

Hence

$$\left| \int_0^x \frac{\sin(t) - t}{t^3} \, dt \right| \leq \frac{e^2 - 2e - 1}{2e} + 2 = \frac{e^2 + 2e - 1}{2e}$$

so that

$$\left| \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3} \, dt \right| \leq \sigma \cdot \frac{e^2 + 2e - 1}{2e}$$

and

$$|U_{\sigma}(x)| \leq \left| \frac{\sin \sigma x}{\sigma x} \right| + \left| \frac{\sin \sigma x}{\sigma} \right| \cdot \left| \int_0^x \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} \, dt \right| \leq 1 + \frac{e^2 + 2e - 1}{2e} = \frac{e^2 + 4e - 1}{2e}.$$  

This gives (13).
To estimate $|V_{\sigma}(x)|$ we see from (12) that

$$\psi'(x) = \frac{1}{2\sigma} \sin(\sigma x)$$

so that, because $\psi(x)$ is odd, we have

$$\psi(x) = \frac{1}{2\sigma} \int_0^x \frac{\sin t}{t} dt = \frac{1}{2\sigma} \int_0^x \frac{\sin t}{t} dt.$$ 

If $0 \leq x \leq \pi/2$, then

$$\int_0^x \frac{\sin t}{t} dt \leq \int_0^\pi \frac{\sin t}{t} dt \leq \frac{\pi}{2}$$

so that

$$\left| \int_0^x \frac{\sin t}{t} dt \right| \leq \frac{\pi}{2}.$$ 

If $x > \pi/2$, then

$$\int_0^x \frac{\sin t}{t} dt = \int_0^{\pi/2} \frac{\sin t}{t} dt + \int_{\pi/2}^x \frac{\sin t}{t} dt.$$ 

Now, integrating by parts, we have

$$\int_\pi^{x/2} \frac{\sin t}{t^2} dt = -\frac{\cos x}{x^2} - \int_{\pi/2}^x \frac{\cos t}{t^2} dt.$$ (i)

It is easy to observe that

$$|\cos x/x| \leq 2/\pi.$$ (ii)

But

$$\left| \int_{\pi/2}^x \frac{\cos t}{t^2} dt \right| \leq \frac{2}{\pi}.$$ (iii)

Combining (i)–(iii), we obtain

$$\left| \int_0^x \frac{\sin t}{t} dt \right| \leq \frac{\pi}{2} + \frac{2}{\pi} + \frac{2}{\pi} = \frac{\pi^2 + 8}{2\pi}$$

and so

$$|\psi(x)| \leq \frac{1}{2\sigma^2} \left( 8 + \pi^2 \right)/2\pi = \frac{8 + \pi^2}{4\pi\sigma^2}.$$
Now using (11), we have
\[ |V_\sigma(x)| = |\sin \sigma x \cdot \psi(x)| \leq (8 + \pi^2)/4\pi \sigma^3. \]

**Theorem 3.** If \( \{a_k\} \) and \( \{b_k\} \) are two sequences of complex numbers such that
\[
\sum_{k=-\infty}^{\infty} \left| \frac{a_k}{k} \right| < \infty \tag{15}
\]
and
\[
\sum_{k=-\infty}^{\infty} |b_k| < \infty, \tag{16}
\]
then
\[
R_\sigma(x) = \sum_{k=-\infty}^{\infty} a_k U_\sigma(x - (k\pi/\sigma)) + \sum_{k=-\infty}^{\infty} b_k V_\sigma(x - (k\pi/\sigma)) \tag{17}
\]
represents an entire function of exponential type \( 2\sigma \) satisfying the conditions
\[
R_\sigma(k\pi/\sigma) = a_k; \quad R_\sigma''(k\pi/\sigma) = b_k. \tag{18}
\]

**Proof.** That conditions (18) are satisfied is obvious from representation (17) and it is enough to prove that \( R_\sigma(x) \) does represent an entire function of exponential type \( 2\sigma \). Setting
\[
F_1(z) = \sum_{k=-\infty}^{\infty} a_k U_\sigma \left( z - \frac{k\pi}{\sigma} \right) \quad \text{and} \quad F_2(z) = \sum_{k=-\infty}^{\infty} b_k V_\sigma \left( z - \frac{k\pi}{\sigma} \right),
\]
we shall show that \( F_1 \) and \( F_2 \) are entire functions of exponential type \( 2\sigma \). Now
\[
\phi(z) = 1 - z \int_0^z \frac{\sin(\sigma t) - \sigma t}{\sigma t^3} dt.
\]
Since
\[
\frac{\sin(\sigma t) - \sigma t}{\sigma t^3} = O \left( \frac{1}{t^4} \right)
\]
for large \( t \), it follows that \( \phi(z) \) is bounded for real \( z \). Since \( \phi(z) \) has the same order and type as \( (\sin(\sigma z) - \sigma z)/\sigma z^3 \) we conclude that \( \phi(z) \) is an entire
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function of exponential-type $\sigma$ bounded on the real axis. Hence there exists a constant $M > 0$ such that

$$|\phi(x + iy)| \leq Me^{\sigma|y|}. \quad (19)$$

Now

$$F_1(z) = \sum_{k=-\infty}^{\infty} a_k U_\sigma \left( z - \frac{k\pi}{\sigma} \right)$$

$$= \sum_{k=-\infty}^{\infty} a_k \cdot \frac{\sin \sigma(z - (k\pi/\sigma))}{\sigma(z - (k\pi/\sigma))} \phi \left( z - \frac{k\pi}{\sigma} \right)$$

$$= \frac{\sin \sigma z}{\sigma z} \phi(z) + \sum_{k=-\infty}^{\infty} \frac{(-1)^k a_k \sin \sigma z}{\sigma z - k\pi} \phi \left( z - \frac{k\pi}{\sigma} \right).$$

Now given $z$, we can choose $k_0$ such that $|\sigma z| \leq k_0 \pi/2$. Then for $|k| \geq k_0$ we have $|\sigma z| \leq k_0 \pi/2 \leq |k| \cdot \pi/2$ so that

$$|\sigma z - k\pi|^{-1} = (\pi |k|)^{-1} \left| 1 - \frac{\sigma z}{k\pi} \right|^{-1} \leq (\pi |k|)^{-1} \left( 1 - \left| \frac{\sigma z}{k\pi} \right| \right)^{-1} = 2/|k| \cdot \pi. \quad (20)$$

Since $|\sin \sigma z| \leq e^{\sigma|y|}$ using (20), (19), and (15) we conclude that $F_1(z)$ does represent an entire function of exponential type $2\sigma$ because of condition (15).

Since $V(z)$ is bounded on the real axis and is an entire function of exponential type $2\sigma$, condition (16) implies that $F_2(z)$ is also an entire function of exponential type $2\sigma$. This completes the proof of the theorem.

We shall now consider the problem of convergence of $R_u(x)$ as $u \to \infty$. The result given here is not claimed to be the best possible, but nevertheless interesting being the first result in this direction.

Let us first consider the class $W$ introduced by Weiner [12] of continuous functions $f$ on $(-\infty, \infty)$ satisfying the condition

$$\sum_{k=-\infty}^{\infty} \max_{0 \leq x \leq 1} |f(k + x)| < \infty.$$ 

With the norm given by

$$\|f\|_W = \sum_{k=-\infty}^{\infty} \max_{0 \leq x \leq 1} |f(k + x)|.$$

$W$ is a Banach space and it has been studied in detail by Goldberg [7].

Define the translation $T_y$ by

$$T_y f(x) = f(x + y)$$
and the one-sided second difference $A^2_h$ by

$$A^2_hf(x) = f(x + 2h) - 2f(x + h) + f(x) = (T_{2h}f - 2T_hf + f)(x).$$

If $h, y \in \mathbb{R}$ and $F \in W$ when $T_yf$ and $A^2_hf$ both belong to $W$. It is also known that

$$\| T_yf \|_W \leq 2 \| f \|_W \tag{21}$$

and

$$\| T_y f - f \|_W \to 0 \quad \text{as} \quad y \to 0, \tag{22}$$

that is, the mapping $y \to T_yf$ is continuous from $\mathbb{R}$ into $W$.

**Definition 4.** If a function $f(x)$ is bounded on $\mathbb{R}$ the modulus of smoothness of order $k \geq 1$ is defined by

$$\omega_k(f; t) = \omega_k(t) = \sup_{|h| \leq t} \| A^k_hf \|_W,$$

where

$$A^k_hf(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(x + rh).$$

The following properties of the modulus of smoothness are needed in the sequel:

(a) If the function $f(x)$ has a bounded derivative of the $r$th order on $\mathbb{R}$, then for any integer $k \geq 0$

$$\omega_{k+r}(f; t) = 2^r \omega_k(f^{(r)}; t).$$

(b) If $n \geq 0$ is an integer, then

$$\omega_k(nt) < 2^n \omega_k(t).$$

The proof is exactly similar to the ones given in Timan [11]. The constant 2 appears in the r.h.s. of (a) and (b) because the translation operator on $W$ has a norm bounded by 2 by (21).

**Theorem 5.** Let $f$ be a function of the class $W$, which has a continuous derivative on $\mathbb{R}$. Suppose

$$g_\sigma(x) = (\sin(\sigma x/2)/\sigma x)^4 \tag{23}$$

$$m_\sigma = \int_{-\infty}^{\infty} \left( \frac{\sin(\sigma t/2)}{\sigma t} \right)^4 dt. \tag{24}$$
If

\[ F_\sigma(x) = \frac{1}{\sigma} \int_{-\infty}^{\infty} g_\sigma(t) [2f(x + t) - f(x + 2t)] \, dt \]

\[ = \frac{1}{\sigma} \int_{-\infty}^{\infty} G_\sigma(x - t) f(t) \, dt, \tag{25} \]

where

\[ G_\sigma(h) = 2g_\sigma(h) - \frac{1}{2} g_\sigma(h/2), \tag{28} \]

then

\[ \| f - F_\sigma \|_W \leq C_1 \cdot \omega_2 \left( f; \frac{1}{\sigma} \right) \]  

(i)

and

\[ \| F_\sigma'' \|_W \leq C_2 \sigma \omega_1 \left( f'; \frac{1}{\sigma} \right), \tag{29} \]

(ii)

where \( C_1, C_2 \) are constants.

**Proof.** Since the convolution product of a function in \( L^1 \) and a function in \( W \) belongs to \( W \), it follows that \( F_\sigma \in W \). Then

\[ f(x) - F_\sigma(x) = \frac{1}{\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \Delta_t^2 f(x) \, dt \]

which gives

\[ \| f - F_\sigma \|_W \leq \frac{1}{\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \| \Delta_t^2 f \|_W \, dt \]

\[ \leq \frac{1}{\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \omega_2(f; t) \, dt. \tag{27} \]

Now using the property that

\[ \omega_2(f; \lambda t) \leq 2(\lambda + 1)^2 \omega_2(f; t), \tag{28} \]

where \( \lambda > 0 \), we see that

\[ \omega_2(f; t) \leq 2(\sigma t + 1)^2 \omega_2(f; 1/\sigma). \]
By virtue of (28), (27) becomes

\[
\|f - F_\sigma\|_w \leq \frac{2}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \left( \frac{1}{\sigma} \right) (\sigma t + 1)^2 \, dt
\]

\[
= \frac{2\sigma^2}{m_\sigma} \omega_2 \left( \frac{1}{\sigma} \right) \left( \int_{-1/\sigma}^{1/\sigma} g_\sigma(t) \left( t + \frac{1}{\sigma} \right)^2 \, dt \right)
\]

\[
\quad + \int_{|t| > 1/\sigma} g_\sigma(t) \left( t + \frac{1}{\sigma} \right)^2 \, dt \] \]

Now

\[
\int_{-1/\sigma}^{1/\sigma} g_\sigma(t) \left( t + \frac{1}{\sigma} \right)^2 \, dt \leq \left( \frac{4}{\sigma^2} \right) \int_{-1/\sigma}^{1/\sigma} g_\sigma(t) \, dt
\]

\[
\leq \left( \frac{4}{\sigma^2} \right) \int_{-\infty}^{\infty} g_\sigma(t) \, dt = 4m_\sigma/\sigma^2
\]

and

\[
\int_{|t| > 1/\sigma} g_\sigma(t)(t + (1/\sigma))^2 \, dt \leq 4 \int_{|t| > 1/\sigma} g_\sigma(t) \cdot t^2 \, dt
\]

\[
= 4 \int_{|t| > 1/\sigma} t^2 \cdot \left( \frac{\sin \sigma t/2}{\sigma t} \right)^4 \, dt
\]

\[
\leq \frac{4}{\sigma^2} \int_{-\infty}^{\infty} \left( \frac{\sin \sigma x/2}{\sigma x} \right)^2 \, dx
\]

\[
= \frac{4m_\sigma}{\sigma^2} \int_{-\infty}^{\infty} \left( \frac{\sin x/2}{x} \right)^4 \, dx
\]

Hence

\[
\|f - F_\sigma\|_w \leq 8 \left[ 1 + \frac{\int_{-\infty}^{\infty} \left( \frac{\sin x/2}{x} \right)^2 \, dx}{\int_{-\infty}^{\infty} \left( \frac{\sin x/2}{x} \right)^4 \, dx} \right] \omega_2(f; 1/\sigma)
\]

\[
= C_1 \omega_2(f; 1/\sigma).
\]

Also from (25) we have

\[
F_\sigma(x) = \frac{1}{m_\sigma} \int_{-\infty}^{\infty} G_\sigma(x - t)f(t) \, dt.
\]

Then
\[ F''_\sigma(x) = \frac{1}{m_\sigma} \int_{-\infty}^{\infty} G'_\sigma(x-t) f'(t) \, dt \]
\[ = \frac{2}{m_\sigma} \int_{0}^{\infty} g'_\sigma(t) [f'(x-t) - f'(x+t)] \, dt \]
\[ - \frac{1}{2m_\sigma} \int_{0}^{\infty} g'_\sigma(t) [f'(x-2t) - f'(x+2t)] \, dt \]

so that

\[ \| F''_\sigma \|_{w} \leq \frac{2}{m_\sigma} \int_{0}^{\infty} |g'_\sigma(t)| \| A_2 f' \|_{w} \, dt + \frac{1}{2m_\sigma} \int_{0}^{\infty} |g'_\sigma(t)| \| A_4 f' \|_{w} \, dt \]
\[ \leq \frac{A}{m_\sigma} \int_{0}^{\infty} |g'_\sigma(t)| \omega_1(f', t) \, dt \]
\[ \leq \frac{A}{m_\sigma} \omega_1 \left( f', \frac{1}{\sigma} \right) \int_{0}^{1/\sigma} |g'_\sigma(t)| \left( t + \frac{1}{\sigma} \right) \, dt \]
\[ + \int_{1/\sigma}^{\infty} |g'_\sigma(t)| \left( t + \frac{1}{\sigma} \right) \, dt \]
\[ \leq \frac{2A}{m_\sigma} \omega_1 \left( f', \frac{1}{\sigma} \right) \left( \frac{1}{\sigma} \right) \int_{0}^{1/\sigma} |g'_\sigma(t)| \, dt + \int_{1/\sigma}^{\infty} |g'_\sigma(t)| t \, dt \]
Using these inequalities we obtain

\[ \|F'_{\sigma}\|_{W} \leq C_{2}\sigma\omega_{1}(f'_{\sigma} ; 1/\sigma). \]  

(30)

With these preliminaries we are now in a position to state and prove a theorem on the convergence of sequence of entire functions \( \{R_{\sigma}(x,f)\} \).

**Theorem 6.** Let \( f(x) \) be bounded, uniformly continuous, and differentiable on the whole real axis. Then the sequence of entire functions \( \{R_{\sigma_{n}}(x,f)\} \) defined by

\[
R_{\sigma_{n}}(x,f) = \sum_{k = -\infty}^{\infty} f(x_{k}) \frac{U_{\sigma_{n}}(x-x_{k})}{\omega_{1}(x_{k})} + \sum_{k = -\infty}^{\infty} b_{k} V_{\sigma_{n}}(x-x_{k}),
\]

where \( x_{k} = k\pi/\sigma_{n} \) converges to \( f(x) \) if:

(i) \( \omega_{1}(f'_{\sigma};1/\sigma_{n}) \to 0 \) as \( \sigma_{n} \to \infty \).

(ii) \( \sum_{k = -\infty}^{\infty} |b_{k}| = o(\sigma_{n}^{2}) \).

**Proof.** Introducing \( F_{\sigma_{n}}(x) \) from (25), which has been shown to be an entire function of exponential type \( 2\sigma_{n} \), we write

\[
R_{\sigma_{n}}(x,f) - f(x) = R_{\sigma_{n}}(x) - F_{\sigma_{n}}(x) + F_{\sigma_{n}}(x) - f(x)
\]

\[
= [F_{\sigma_{n}}(x) - f(x)] + \sum_{k = -\infty}^{\infty} [f(x_{k}) - F_{\sigma_{n}}(x_{k})] U_{\sigma_{n}}(x-x_{k})
\]

\[
+ \sum_{k = -\infty}^{\infty} b_{k} V_{\sigma_{n}}(x-x_{k}) - \sum_{k = -\infty}^{\infty} F''_{\sigma_{n}}(x_{k}) V_{\sigma_{n}}(x-x_{k})
\]

so that

\[
|R_{\sigma_{n}}(x) - f(x)| \leq |F_{\sigma_{n}}(x) - f(x)|
\]

\[
+ \sum_{k = -\infty}^{\infty} |f(x_{k}) - F_{\sigma_{n}}(x_{k})| U_{\sigma_{n}}(x-x_{k})
\]

\[
+ \sum_{k = -\infty}^{\infty} |F''_{\sigma_{n}}(x_{k})| V_{\sigma_{n}}(x-x_{k})
\]

\[
+ \sum_{k = -\infty}^{\infty} |b_{k}| V_{\sigma_{n}}(x-x_{k}).
\]

We known from Timan [11] that for any function \( f(x) \) bounded and uniformly continuous on the whole real axis, there exists a sequence of entire function \( g_{n}(x) \) of degree \( \sigma_{n} \) such that

\[
\sup_{-\infty < x < \infty} |f(x) - g_{n}(x)| \to 0 \quad \text{as} \quad \sigma_{n} \to \infty.
\]
Then we get

\[ |F_{\sigma_n}(x) - f(x)| \to 0 \quad \text{as} \quad \sigma_n \to \infty. \quad (A) \]

Since \( x_k = k\pi/\sigma_n \), we have

\[
\sum_{k = -\infty}^{\infty} |f(x_k) - F_{\sigma_n}(x_k)| \leq \sum_{j = -\infty}^{\infty} \sum_{x_k \in [j, j+1]} |f(x_k) - F_{\sigma_n}(x_k)|
\]

\[
\leq \left[ \frac{\sigma_n}{\pi} \right] \sum_{j = -\infty}^{\infty} \max_{x \in [j, j+1]} |f(x) - F_{\sigma_n}(x)|
\]

\[
\leq (\sigma_n/\pi) \|f - F_{\sigma_n}\|_w \leq (\sigma_n/\pi) C \omega_2 \left( f; \frac{1}{\sigma_n} \right).
\]

From the property (a) of the modulus of smoothness, we find

\[ \omega_2(f; 1/\sigma) \leq (1/\sigma) \omega_1(f' ; 1/\sigma) \]

so that

\[
\sum_{k = -\infty}^{\infty} |f(x_k) - F_{\sigma_n}(x_k)| \leq \text{const.} \omega_1 \left( f' ; \frac{1}{\sigma_n} \right).
\]

Hence

\[
\sum_{k = -\infty}^{\infty} |f(x_k) - F_{\sigma_n}(x_k)| |U_{\sigma_n}(x - x_k)|
\]

\[
\leq \text{const.} \omega_1 \left( f' ; \frac{1}{\sigma_n} \right) \to 0 \quad (B)
\]

as \( \sigma_n \to \infty \) from (i). From (30) we find that

\[ \|F_{\sigma_n}''\|_w \leq \text{const.} \sigma_n \omega_1(f' ; 1/\sigma_n) \]

which gives

\[
\sum_{k = -\infty}^{\infty} |F_{\sigma_n}''(x_k)| \leq \left[ \frac{\sigma_n}{\pi} \right] \|F_{\sigma_n}''\|_w \leq \text{const.} \sigma_n^2 \omega_1 \left( f' ; \frac{1}{\sigma_n} \right).
\]

Further, using (14), we get

\[
\sum_{k = -\infty}^{\infty} |F_{\sigma_n}''(x_k)| |V_{\sigma_n}(x - x_k)|
\]

\[
\leq \text{const.} \omega_1 \left( f' ; \frac{1}{\sigma_n} \right) \to 0 \quad \text{as} \quad \sigma_n \to \infty. \quad (C)
\]
From (14) again,

\[ \sum_{k = -\infty}^{\infty} |b_k| |V_{\sigma_n}(x - x_k)| \leq \frac{8 + \pi^2}{4\pi \sigma_n^2} \sum_{k = -\infty}^{\infty} |b_k| \to 0 \quad \text{as} \quad \sigma_n \to \infty. \quad \text{(D)} \]

Combining (A)–(D) the result follows.

REFERENCES