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Perturbations of Normally Hyperbolic Manifolds with Applications to the Navier–Stokes Equations[1]

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There are two objectives in this paper. First we develop a theory which is valid in the infinite dimensional setting and which shows that, under reasonable conditions, if M is a normally hyperbolic, compact, invariant manifold for a semiflow $S_0(t)$ generated by a given evolutionary equation on a Banach space W, then for every small perturbation G of the given evolutionary equation, there is a homeomorphism $h_G: M \to W$ such that $M^G = h_G(M)$ is a normally hyperbolic, compact, invariant manifold for the perturbed semiflow $S_G(t)$, and that h_G converges to the identity mapping (on M), as G converges to 0. The second objective is to develop a methodology which is rich enough to show that this theory can be easily applied to a wide range of applications, including the Navier–Stokes equations. It is noteworthy in this regard that, in order to be able to apply this theory in the analysis of numerical schemes used to study discretizations of partial differential equations, one needs to use a new measure or norm of the perturbation term G that arises in these schemes. © 2001 Academic Press

Key Words: approximation dynamics; Bubnov–Galerkin approximations; Couette–Taylor flow; evolutionary equations; exponential dichotomy; exponential trichotomy; ordinary differential equations; Navier–Stokes equations; normal hyperbolicity; numerical schemes; partial differential equations; reaction diffusion equations; robustness.

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1. INTRODUCTION

One of the major challenges arising in the study of nonlinear dynamics is the problem of trying to explain the robustness of chaotic dynamics. At first glance this seems like an oxymoron. How can one hope to explain chaotic (nonstable) dynamics in terms of robustness (stability)? The quest seems to be a quixotic exercise into the impossible. But, is that really the case? We see many examples of chaotic dynamics around us, be it in the complicated phenomena of regional weather patterns, or in the very simple case of two coupled nonlinear oscillators. The chaotic behavior persists even with small changes in the system or in the model. The challenge to dynamicists is to explain this persistence.

During the last several decades, a dynamical theory based on the study of certain hyperbolic structures within a nonlinear dynamical system, has proven to be a useful tool for addressing this problem. The oldest aspect of this theory goes back to the time of Lyapunov and Poincaré, and it involves the persistence of the saddle point property for a hyperbolic fixed point under small perturbations. At a later time, a similar result was shown in the case of a hyperbolic periodic orbit. However, in these classical situations, one does not really encounter chaotic dynamics.

Chaotic dynamics can appear when one tries to extend these earlier perturbation theories to the study of perturbations of certain compact, invariant manifolds. This is where the notion of a normally hyperbolic manifold enters the picture. We postpone for now the precise definition of this very important concept. Suffice it to say here that it forms the center piece of the perturbation theory of invariant manifolds developed, for example, in Krylov and Bogoliubov (1934); Levinson (1950); Bogoliubov and Mitropolsky (1955); Hale (1961); Pliss (1966, 1977); Sacker (1969); Fenichel (1971); Hirsch, Pugh, and Shub (1977); Mañé (1988), Pilyugin (1992); and Wiggins (1994). A related use of a hyperbolic structure arises in the theory of structural stability, see Smale (1967), for example. One of the upshots of this study was the theorem that all "hyperbolic sets" are structurally stable, see Anosov (1967), and Arnold (1983), for example. More recently, it is shown in Pliss and Sell (1991, 1998) that certain foliated invariant sets, which are not hyperbolic sets in the sense described in Arnold (1983), but which have a suitable hyperbolic structure, vary *continuously* under small perturbations in the underlying ordinary differential equation.

While it is not our major concern here, it should be noted that there is another important aspect of normal hyperbolicity which is of interest, and that is the role this concept plays in the onset of bifurcation phenomena, see for example, Marsden and McCracken (1976), Chenciner and Iooss (1979), Sell (1979), Chow and Hale (1982), and Chossat and Iooss (1994).

Our primary goal in this paper is to derive a theory of persistence for perturbations of normally hyperbolic, compact, invariant manifolds in an infinite dimensional setting. Simply stated, we set out to prove that, under reasonable conditions, if M is a normally hyperbolic, compact, invariant manifold for a semiflow $S_0(t)$ generated by a given evolutionary equation on a Banach space W, then for every "small" perturbation G of the given evolutionary equation, there is a homeomorphism $h_G: M \to W$ such that $M^G = h_G(M)$ is a normally hyperbolic, compact, invariant manifold for the perturbed semiflow $S_G(t)$, and that h_G converges to the identity mapping (on M), as G converges to 0.

In the infinite dimensional setting, there are a number of related dynamical theories of invariant manifolds which have been studied in earlier works. The local behavior, for example, the center manifold theory, can be found in many sources. See Hale (1969, 1988); Henry (1981); Chow and Lu (1988); Vanderbauwhede and Iooss (1992); Chow and Yi (1994); Chen, Hale, and Tan (1997); Chow, Liu, and Yi (1999); and Sell and You (2001), for example. The global theories, such as inertial manifolds, can be found in Foias, Sell, and Temam (1988); Mallet-Paret and Sell (1988); Constantin, Foias, Nicolaenko, and Temam (1988); Temam (1988); Foias, Nicolaenko, and Temam (1989); Mallet-Paret, Sell, and Shao (1993); and Rosa and Temam (1996); for example. Related work on inertial sets (i.e., exponential attractors) appears in Eden, Foias, Nicolaenko, and Temam (1994); Eden, Foias, and Nicolaenko (1996); and Dung and Nicolaenko (2001); and related contributions to nonautonomous dynamics can be found in Sell (1967a, 1967b, 1971); Sacker and Sell (1978, 1980); Henry (1981); Meyer and Sell (1989); Raugel and Sell (1993); Chow and Yi (1994); Chepyzhov and Vishik (1995); Shen and Yi (1995, 1998); and Yi (1998), for example. More recently, one finds finds theories in infinite dimensional dynamics on compact, invariant manifolds, see Jones and Titi (1996); and Jones and Shkoller [37].

However, the paper closest to the theory we present here is Bates, Lu, and Zeng (1998), or BLZ, for short. Our proof of the existence of the perturbed manifold M^G differs significantly from that of BLZ. This new proof offers several advantages over the BLZ approach: it is shorter; probably less complex; but most importantly, the methodology developed herein may be more suitable for applications. In particular, we seek to build our theory by using the physical parameters of the problem. This is accomplished via the traditional route of (1) converting a given system of partial differential equations into a nonlinear evolutionary equation and (2) using the resulting theory of linear and nonlinear semiflows for the analysis. Of special interest in this work is the role that the theory of persistence of invariant manifolds plays in the context of the numerical analysis of the longtime dynamics of solutions of partial differential equations. In one of our applications, we study the connection between the Bubnov–Galerkin approximants and the solutions of the Navier–Stokes equations. It is in this connection where we describe the importance of a new seminorm $\|G\|_{\{A;\, C^1(\Omega)\}}$ for measuring the size of the perturbation term G. While the traditional C^1-seminorm $\|G\|_{C^1(\Omega)}$ appears *not* to decrease as the order n of the Bubnov–Galerkin approximant grows, we do show that the alternate seminorm $\|G\|_{\{A;\, C^1(\Omega)\}}$ converges to 0, and $n \to \infty$. This discovery of the importance of the new seminorm illustrates the value of using the approach advocated in this work. We hope that this point of view will be of use to others in the future.

The issue of what is meant by a "small" perturbation of the given evolutionary equation is a technical matter, and it is a major concern in this work. To put this into context, we let the given evolutionary equation have the form

$$\partial_t u + Au = F(u), \tag{1.1}$$

while the perturbed equation is given by

$$\partial_t y + Ay = F(y) + G(y). \tag{1.2}$$

The precise assumptions on the linear operator A and the "nonlinear" terms F and G will be given shortly. For applications to partial differential equations, the operator A is typically a uniformly elliptic operator, such as $A = -\Delta$, the (negative) Laplacian, or $A = \Delta^2$, the biharmonic operator, with suitable boundary conditions. Since we wish to treat problems like the Navier–Stokes equations, we allow for the fact that the nonlinear terms F and G may depend on lower order spatial derivatives. As a result, these terms need not be Fréchet differentiable as a mapping of a Banach space W into itself. However, when A is a positive sectorial operator on W, then the nonlinear mappings are good mappings from $V^{2\beta}$ into W, where $V^{2\beta} = \mathscr{D}(A^\beta)$ is the domain of A^β, and A^β is a suitable fractional power of A with $0 \leqslant \beta < 1$. Thus we require that F and G be Fréchet differentiable mappings of $V^{2\beta}$ into W, where the derivatives DF and DG are now bounded linear operators. By using a suitable notion of Fréchet differentiability for this problem, we are able to show that many of the dynamical techniques developed for ordinary differential equations have suitable extensions to the infinite dimensional setting. For example, in Section 5 we show that if the perturbation G is the result of a small change in the physical parameters of the problem, then both G and DG are small.

For numerical analytical issues, on the other hand, the situation is somewhat different. In this case the perturbation term G is not arbitrary. Instead, it is determined entirely by the numerical scheme used to approximate the original evolutionary equation. Since a numerical scheme, for a partial differential equation, is inevitably a finite dimensional approximation to an infinite dimensional problem, one oftentimes encounters a situation where the sizes of G and DG do not decrease as one moves from a coarse grid to a fine grid, or equivalently, as one adds more modes in a Bubnov–Galerkin approximation. (See Section 5.) A different measure, or norm, of the sizes of G and DG is needed, in order to develop a good theory of longtime dynamics. (See Section 4.)

In order to give a better idea of the Main Result derived in this paper, it is convenient to focus first on the Navier–Stokes equations. Assume for now that Eq. (1.1) represents the nonlinear evolutionary equation one obtains from the Navier–Stokes equations, after applying the Helmholz projection onto the space of divergent-free vector fields. (See Section 5.) In this case, the linear operator A is the Stokes operator, and $F(u)$ contains the (nonlinear) inertial term and the forcing term for these equations. In this setting, both nonlinear terms F and G belong to the space C^1_{Lip}, where

$$C^1_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} C_{\mathrm{Lip}}(V^{2\beta}, W) \cap C^1_{\mathrm{F}}(V^{2\beta}, W). \tag{1.3}$$

(These spaces are defined in Section 2.) In Eq. (1.3), W is a suitable Hilbert space, $V^{2\beta} = \mathscr{D}(A^\beta)$ is the domain of A^β, and A^β is a fractional power of A. We assume that M is a given compact, invariant manifold for Eq. (1.1), and we let Ω denote some prescribed open, bounded neighborhood of M. In the following we will refer to the norm $\|G\|_{\{A;\, C^1(\Omega)\}}$, which is defined in Section 4. Also the concepts of normal hyperbolicity and Lipschitz class are defined in Section 4, as well. The following is a special case of our Main Theorem.

Theorem NSE. *Let M be a compact, connected, invariant C^2-manifold in $V^{2\beta}$ for the Navier–Stokes equations* (1.1). *Assume that M is normally hyperbolic and that the associated exponential trichotomy is of Lipschitz class. Then for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that if the perturbation term $G \in C^1_{\mathrm{Lip}}$ satisfies $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, then there is a Lipschitz homeomorphism $h\colon M \to V^{2\beta}$ with the following properties*:

(1) *The manifold $M^G \stackrel{\mathrm{def}}{=} h(M)$ is an invariant manifold for the perturbed Eq.* (1.2).

(2) *Both manifolds M and M^G lie in $\mathscr{D}(A)$, the domain of A. Furthermore, M^G is of class C^1, and it is normally hyperbolic for Eq.* (1.2).

(3) *One has $\|A^\beta(h(v) - v)\| \leqslant 2\varepsilon$, for all $v \in M$.*

(4) *For each $u_0 \in M$ and $y_0 \in M^G$, the mild solutions $S(t)\, u_0$ and $y(t, y_0)$ are strong solutions.*

While our theory can be applied in many contexts, we will focus on the Navier–Stokes equations here. In particular, we will use this theory to show the persistence of some patterns seen in the Couette–Taylor flow, which is the fluid flow between two right circular cylinders where the angular velocity ω of one of the cylinders (say the inner cylinder) varies. Of special interest here is the scenario which describes a secondary $T^1 \to T^2$ Hopf bifurcation, as the parameter ω varies. A second application is in the role played by our Main Theorem in the Bubnov–Galerkin approximations. It is this application, in particular, which illustrates the full significance of the norm $\|G\|_{\{A;\, C^1(\Omega)\}}$ used in our theory.

This paper is organized as follows. In Section 2 we present a quick overview of the basic theory of linear and nonlinear evolutionary equations. In Section 3 we describe the key hyperbolic structure considered in this paper: an exponential trichotomy. A precise statement of our Basic Theorems is given in Section 4. In Section 5 we present the applications. Finally in Section 6 we present the proofs of our theorems.

2. LINEAR AND NONLINEAR EVOLUTIONARY EQUATIONS

In this section we examine a number of the classical issues arising in the study of the dynamics of linear and nonlinear evolutionary equations. In particular, we consider various features of the longtime dynamics of the evolutionary Eq. (1.1) on a Banach space W, with norm $\|w\| = \|w\|_W$. A brief overview of the theory of solutions of Eq. (1.1) is presented here. Additional information can be found in Henry (1981), Pazy (1983), and Sell and You (2001). We begin with the linear problem,

$$\partial_t u + Au = 0. \tag{2.1}$$

2.1. *Linear Theory.* We assume here that the linear operator A is a positive sectorial operator on W. As a result, A is a closed operator on W, with a domain $\mathcal{D}(A)$ which is dense in W, and $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on W. Thus for each $u_0 \in W$, the function $u(t) = e^{-At}u_0$ is the (unique) solution of Eq. (2.1) with $u(0) = u_0$. Since the operator A is positive, there exist constants $a > 0$ and $M_0 \geqslant 1$, such that

$$\|e^{-At}u_0\| \leqslant M_0 \|u_0\| e^{-at}, \qquad \text{for all} \quad u_0 \in W \quad \text{and} \quad t \geqslant 0. \tag{2.2}$$

For each $\alpha \geqslant 0$, we let A^α denote the fractional power of A, and we set $V^{2\alpha} = \mathcal{D}(A^\alpha)$, the domain of A^α. Each A^α is a closed, densely defined, linear operator on W, and one has the continuous imbedding $\mathcal{D}(A^\alpha) \mapsto \mathcal{D}(A^\beta)$, whenever, $\alpha \geqslant \beta$. This means that there is a constant $c = c_{\alpha,\beta} > 0$ such that $\|u\|_{V^{2\alpha}} \leqslant c\, \|u\|_{V^{2\beta}}$, for all $u \in \mathcal{D}(A^\alpha)$, where the norm $\|u\|_{V^{2\alpha}}$ is the graph norm, $\|u\|_{V^{2\alpha}} = \|A^\alpha u\| = \|A^\alpha u\|_W$. Thus the identity mapping I is in $\mathscr{L} = \mathscr{L}(V^{2\alpha}, V^{2\beta})$, the space of bounded linear transformations from $V^{2\alpha}$ into $V^{2\beta}$. For $M \in \mathscr{L}$, the operator norm $\|M\|_{\mathscr{L}}$ is defined by

$$\|M\|_{\mathscr{L}} \stackrel{\text{def}}{=} \sup\{\|A^\beta Mu\|: \|A^\alpha u\| \leqslant 1\}.$$

To summarize, we make the following Standing Hypothesis:

STANDING HYPOTHESIS A. *Let A be a positive, sectorial operator on a Banach space W with associated analytic semigroup e^{-At}. Let $V^{2\alpha}$ be the family of interpolation spaces generated by the fractional powers of A, where $V^{2\alpha} = \mathscr{D}(A^\alpha)$, for $\alpha \geqslant 0$. Let $\|A^\alpha u\| = \|A^\alpha u\|_W = \|u\|_{V^{2\alpha}}$ denote the norm on $V^{2\alpha}$.*

There is no loss in generality in assuming that the sectorial operator A is positive. Indeed, if A is any sectorial operator, there is a real number $a \in \mathbb{R}$ such that the linear operator $\mathrm{B} = \mathrm{A} + \mathrm{aI}$ is a positive, sectorial operator. In this case, Eq. (1.1) is equivalent to the equation $\partial_t u + Bu = H(u)$, where $H(u) = F(u) + au$. The positivity of the sectorial operator A offers a convenient way for describing the fractional power spaces $V^{2\alpha}$. If A is not positive, then one can achieve the same goals indirectly by using B and its fractional powers, since the semigroups satisfy $e^{-Bt} = e^{-at}e^{-At}$, for $t \geqslant 0$, see Henry (1981).

In some applications, for example the Navier–Stokes equations, the ambient space W is a Hilbert space. In this case, we will use H in place of W, and we let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm on H. In this setting, we are especially interested in those problems where the operator A is a positive, selfadjoint, linear operator, with compact resolvent. In this case, A satisfies the Standing Hypothesis A, but there is further information we get from the spectral theory for A. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ denote the eigenvalues of A, repeated with their multiplicities, and let $\{e_1, e_2, e_3, ...\}$ denote the corresponding orthonormal basis (in H) of eigenvectors. Then one has $\langle u, Au \rangle \geqslant \lambda_1 \|u\|^2$, for all $u \in H$. For $\alpha \geqslant 0$, the domain $\mathscr{D}(A^\alpha)$ of the fractional power A^α is defined by

$$V^{2\alpha} \stackrel{\text{def}}{=} \mathscr{D}(A^\alpha) \stackrel{\text{def}}{=} \left\{ u \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} |\langle u, e_i \rangle|^2 < \infty \right\}, \tag{2.3}$$

and the operator A^α is given by $A^\alpha u = \sum_{i=1}^{\infty} \lambda_i^\alpha \langle u, e_i \rangle e_i$, for all $u \in \mathscr{D}(A^\alpha)$. For $\alpha > 0$, we let $V^{-2\alpha}$ denote the dual space of $V^{2\alpha}$. Thus $V^{2\alpha}$ and $V^{-2\alpha}$ are related by the duality property

$$\langle\langle v, w \rangle\rangle = \langle A^\alpha v, A^{-\alpha} w \rangle_H, \qquad \text{for} \quad v \in V^{2\alpha}, \quad w \in V^{-2\alpha},$$

see Dautry and Lions (1990) and Sell and You (2001).

This duality property extends the definition of the fractional powers $V^{2\beta}$ to $\beta < 0$. Furthermore, V^{α} itself becomes a Hilbert space with the V^{α}-inner product and the V^{α}-norm given by

$$\begin{aligned} \langle u, v \rangle_{\alpha} &\stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \lambda_i^{\alpha} u_i \bar{v}_i, \\ \|v\|_{\alpha}^2 &\stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \lambda_i^{\alpha} \, |v_i|^2, \qquad u, v \in V^{\alpha}, \quad \text{and} \quad \alpha \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Moreover, if $\alpha \geqslant \beta$, then $V^{\alpha} \mapsto V^{\beta}$ and $\|v\|_{V^\beta}^2 \leqslant \lambda_1^{\beta-\alpha} \|v\|_{V^\alpha}^2$, for all $v \in V^{\alpha}$. Since A has compact resolvent, one obtains the compact imbedding $V^{2\alpha} \hookrightarrow V^{2\beta}$, whenever $\alpha > \beta$. For any $\alpha \in \mathbb{R}$, Eq. (2.3) implies that $A \colon V^{2+\alpha} \to V^{\alpha}$, and $\|v\|_{2+\alpha}^2 = \|Av\|_{\alpha}^2$, for all $v \in V^{2+\alpha}$, i.e., A is an isometry from $V^{2+\alpha}$ onto V^{α}. Similarly the analytic semigroup e^{-At} on H extends to an analytic semigroup on each space V^{α}, $\alpha \in \mathbb{R}$, by means of the formula $e^{-At}u = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle u, e_i \rangle \, e_i$. We summarize this by saying that:

Standing Hypothesis B. *The operator A is a positive, selfadjoint, linear operator, with compact resolvent, on a Hilbert space H. Consequently A satisfies the Standing Hypothesis A. Moreover, the fractional power spaces V^{α} are defined for all $\alpha \in \mathbb{R}$, and Eq.* (2.4) *defines the Hilbert space structure on each V^{α}.*

In addition to viewing $A \colon V^{2+\alpha} \to V^{\alpha}$ as an isometry, in the sense described above, one also has $A \colon \mathscr{D}_H(A) \to H$, where $\mathscr{D}_H(A) \stackrel{\text{def}}{=} \mathscr{D}(A) = V^2$ is the domain of A in H. By using the fact that $A^{\alpha} \colon V^{2\alpha} \to H$ is also an isometry, one can lift the domain $\mathscr{D}_H(A)$ to any of the fractional power spaces by the identity

$$\mathscr{D}_{V^{2\alpha}}(A) \stackrel{\text{def}}{=} A^{-\alpha}(\mathscr{D}_H(A)) = V^{2\alpha+2} \subset V^{2\alpha}.$$

We refer to $\mathscr{D}_{V^{2\alpha}}(A)$ as the *domain of A in $V^{2\alpha}$*. Note that for $v_0 \in \mathscr{D}_{V^{2\alpha}}(A)$, one has $v_0 = A^{-\alpha} u_0$, for some $u_0 \in \mathscr{D}_H(A)$. We let $Av_0 = AA^{-\alpha}u_0 = A^{-\alpha}Au_0$, which is in $V^{2\alpha}$, since $Au_0 \in H$.

Assume now that the Standing Hypothesis A is satisfied. The Fundamental Theorem on Sectorial Operators gives some very valuable information about the connection between the analytic semigroup e^{-At} and the fractional powers A^{α}, for $\alpha \geqslant 0$. In particular, one has $A^{\alpha}e^{-At}u = e^{-At}A^{\alpha}u$, for all $u \in \mathscr{D}(A^{\alpha})$ and $t \geqslant 0$, and e^{-At} is an analytic semigroup on V^{α}, for each $\alpha \geqslant 0$. In this setting, for $\beta \geqslant 0$, there is a constant $M_1 \geqslant 1$, such that

$$\|A^{\beta}e^{-At}u_0\| \leqslant M_1 e^{-at} \, \|A^{\beta}u_0\|, \qquad \text{for all} \quad u_0 \in V^{2\beta}. \tag{2.5}$$

In addition, for any $\alpha \geqslant 0$ and $t > 0$, the semigroup e^{-At} maps W into $\mathscr{D}(A^{\alpha})$, and there is a constant $M_{\alpha} > 0$ such that

$$\|e^{-At}\|_{\mathscr{L}(W, \mathscr{D}(A^{\alpha}))} = \|A^{\alpha} e^{-At}\|_{\mathscr{L}(W, W)} \leqslant M_{\alpha} t^{-\alpha} e^{-at}, \qquad \text{for all} \quad t > 0, \tag{2.6}$$

where $a > 0$ is given by (2.2). Moreover, for $0 < \alpha \leqslant 1$, there is a constant $K_{\alpha} > 0$ such that

$$\|e^{-At} w - w\| \leqslant K_{\alpha} t^{\alpha} \, \|A^{\alpha} w\|, \qquad \text{for } t \geqslant 0 \quad \text{and} \quad w \in \mathscr{D}(A^{\alpha}). \tag{2.7}$$

Furthermore, the solutions $e^{-At} w$ are Lipschitz continuous in t, for $t > 0$. More precisely, for every $\beta \geqslant 0$, there is a constant $C_{\beta} > 0$ such that

$$\|A^{\beta}(e^{-A(t+h)} - e^{-At}) \, w\| \leqslant C_{\beta} \, |h| \, t^{-(1+\beta)} \, \|w\|, \tag{2.8}$$

for all $t > 0$ and $w \in W$, see Pazy (1983) or Sell and You (2001).

The coefficients M_{α}, K_{α}, and C_{β} appearing above depend on properties of the spectrum of the linear operator A. However, in the case that A satisfies the stronger Standing Hypothesis B, then one can sometimes use the spectral properties of A in (2.3) and (2.4) to derive good estimates of these quantities. For example, instead of inequality (2.6), one obtains

$$\|A^{\beta} e^{-At}\|_{\mathscr{L}(H, H)} = \begin{cases} \beta^{\beta} e^{-\beta} t^{-\beta}, & \text{for} \quad 0 < t \leqslant \beta \lambda_1^{-1}, \\ \lambda_1^{\beta} \, e^{-\lambda_1 t}, & \text{for} \quad \beta \lambda_1^{-1} \leqslant t < \infty. \end{cases}$$

It follows that if $0 \leqslant \beta < 1$, then one obtains the useful estimate

$$\int_0^t \|A^{\beta} e^{-A\tau}\|_{\mathscr{L}(H, H)} \leqslant (1-\beta)^{-1} \, e^{-\beta} \lambda_1^{\beta - 1}, \qquad \text{for all} \quad t \geqslant 0. \tag{2.9}$$

On the other hand, when A satisfies the Standing Hypothesis A, then it follows from inequality (2.6) and the definition of the Gamma function Γ that

$$\begin{aligned} \int_0^t \|A^{\beta} e^{-A\tau}\|_{\mathscr{L}(H, H)} &\leqslant M_{\beta} \int_0^t \tau^{-\beta} e^{-a\tau} \, d\tau \\ &\leqslant M_{\beta} \, a^{\beta - 1} \Gamma(1-\beta), \qquad \text{for all} \quad t \geqslant 0. \end{aligned}$$

In the last inequality, both M_{β} and $a^{\beta-1}$ depend on the spectrum of A.

2.2. *Nonlinear Theory.* In this section, we define the function spaces C_{Lip}, C^1_F, and C^1_{Lip}, which contain the nonlinear terms F and G in Eqs. (1.1) and (1.2). We also introduce the related topologies on these spaces, and we outline the basic theory concerning the properties of the solutions of these equations. For the nonlinear problem (1.1) we assume that the Standing Hypothesis A holds and that the nonlinear term $F = F(u)$ satisfies either $F \in C_{\mathrm{Lip}}$, where

$$C_{\mathrm{Lip}} \overset{\mathrm{def}}{=} C_{\mathrm{Lip}}(V^{2\beta}, W), \qquad \text{for some } \beta \quad \text{with } 0 \leqslant \beta < 1, \tag{2.10}$$

or $F \in C^1_{\mathrm{Lip}}$, see (1.3). We define $C_{\mathrm{Lip}}(V^{2\beta}, W)$ to be the collection of all functions $F\colon V^{2\beta} \to W$ such that, for every bounded set B in $V^{2\beta}$, there exist constants $k_0 = k_0(B) = k_0(F, B) \geqslant 0$ and $k_1 = k_1(B) = k_1(F, B) \geqslant 0$, such that

$$\|F(u)\| \leqslant k_0, \qquad \text{for all} \quad u \in B, \tag{2.11}$$

and

$$\|F(u_1) - F(u_2)\| \leqslant k_1 \, \|A^\beta(u_1 - u_2)\|, \qquad \text{for all} \quad u_1, u_2 \in B. \tag{2.12}$$

Also we define $C^1_{\mathrm{F}}(V^{2\beta}, W)$ to be the collection of all functions $F\colon V^{2\beta} \to W$ such that F is continuously Fréchet differentiable on $V^{2\beta}$, where the derivative $DF(u_0)$ satisfies $DF(u_0) \in \mathscr{L}(V^{2\beta}, W)$, for each $u_0 \in V^{2\beta}$, and F satisfies

$$F(u_0 + v) = F(u_0) + DF(u_0)\, v + E(u_0, v), \qquad \text{for all } u_0, \quad v \in V^{2\beta}, \tag{2.13}$$

where the error term E satisfies

$$\lim_{\|A^\beta v\| \to 0} \frac{\|A^\beta E(u_0, v)\|}{\|A^\beta v\|} = 0, \qquad \text{for each} \quad u_0 \in V^{2\beta}. \tag{2.14}$$

The continuity of DF means that the mapping $u_0 \to DF(u_0)$ is a strongly continuous mapping of $V^{2\beta}$ into $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$. Note that if $F \in C^1_{\mathrm{Lip}}$, see Eq. (1.3), then the Fréchet derivative $DF(u)$ is uniformly bounded on each bounded set B in $V^{2\beta}$, i.e., one has $\sup\{\|DF(u)\|_{\mathscr{L}} \colon u \in B\} < \infty$.

Let the Standing Hypothesis A be satisfied and let $F = F(u) \in C_{\mathrm{Lip}}$, see (2.10). We say that a function $u = u(t)$ is a *mild solution* of

$$\partial_t u + Au = F(u), \qquad u(0) = u_0 \in V^{2\beta}, \tag{2.15}$$

on some interval $0 \leqslant t < T$, where $0 < T \leqslant \infty$, provided that

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} F(u(s))\, ds, \qquad \text{for} \quad 0 \leqslant t < T. \tag{2.16}$$

The basic theory on the existence, uniqueness, continuity, and regularity of mild solutions of (2.15) can be found in Sell and You (2001). Also see, Henry (1981) and Pazy (1983). We present here a summary of this theory for the convenience of the reader.

First we note that for every $u_0 \in V^{2\beta}$ and every $F \in C_{\mathrm{Lip}}$, there is a unique mild solution $u = u(t) = S(t)\, u_0 = S(F, t)\, u_0$ of (2.15) on some interval $0 \leqslant t < T = T(u_0) = T(F, u_0)$, where $0 < T \leqslant \infty$, and one has $u \in C[0, T; V^{2\beta})$. Without loss of generality, we assume that this solution is maximally defined in the sense that either $T = \infty$, or one has $\lim_{t \to T^-} \|A^\beta u(t)\| = \infty$. Furthermore, the mild solution u is a *strong solution* (in W) in the sense that for every τ with $0 < \tau < T$, the functions $Au(\cdot)$ and $\partial_t u(\cdot)$ are in $L^1(0, \tau; W)$, and $\partial_t u(t) + Au(t) \overset{\text{a.e.}}{=} F(u(t))$, on $(0, T)$.

The mapping $S\colon (F, u_0, t) \to S(F, t)\, u_0$ is a continuous mapping of the space

$$\Xi = \{(F, u_0, t) \in C_{\mathrm{Lip}} \times V^{2\beta} \times [0, \infty) : 0 \leqslant t < T(F, u_0)\}$$

into $V^{2\beta}$. Since this continuity property is very important for our theory, we will outline the basic argument here. For the space $V^{2\beta}$ we use the strong topology. On the space C_{Lip}, we will use a Fréchet space topology which is generated by a countable family of pseudonorms. For each bounded set B in $V^{2\beta}$ and $F \in C_{\mathrm{Lip}}$, we define

$$\|F\|_{\{A;\, C^0(B)\}} \overset{\text{def}}{=} \sup_{t \geqslant 0} \int_0^t \|\sup_{u \in B} A^\beta e^{-A(t-s)} F(u)\|\, ds.$$

Note that (2.6), (2.11), and the the definition of Gamma function, imply that

$$\|F\|_{\{A;\, C^0(B)\}} \leqslant M_\beta\, a^{\beta - 1} \Gamma(1 - \beta)\, \|F\|_{C^0(B)}, \tag{2.17}$$

where $\|F\|_{C^0(B)} = \sup_{u \in B} \|F(u)\|$. We see then that, if F_m is a sequence in C_{Lip} with the property that $F_m \to 0$, uniformly on each bounded set B in $V^{2\beta}$, then one has $\|F\|_{C^0(B)} \to 0$, which in turn implies that $\|F_m\|_{\{A;\, C^0(B)\}} \to 0$, as $m \to \infty$. Since every Banach space is the countable union of bounded sets, say $V^{2\beta} = \bigcup_{n=1}^{\infty} B_n$, where $B_n \subset B_{n+1}$, one can use a countable number of pseudonorms $\|\cdot\|_{\{A;\, C^0(B_n)\}}$ to construct a metric d_A; for example,

$$d_A(F_1 - F_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|F_1 - F_2\|_{\{A;\, C^0(B_n)\}}}{1 + \|F_1 - F_2\|_{\{A;\, C^0(B_n)\}}},$$

or $d_A(F_1 - F_2) = \sum_{n \in Z^+} 2^{-|n|} \min(\|F_1 - F_2\|_{\{A;\, C^0(B_n)\}}, 1)$, which is an equivalent metric, see Kelley and Namioka (1976). We will denote the topology on C_{Lip} generated by the pseudonorms $\|\cdot\|_{\{A;\, C^0(B)\}}$ as $\mathscr{T}^0_A$. The subscript refers to the role played by the sectorial operator A in the definition of these pseudonorms. A second topology on C_{Lip} is generated by the pseudonorms $\|F\|_{C^0(B)}$ and is denoted by $\mathscr{T}^0_{\mathrm{bo}}$, i.e., it is the topology of uniform convergence on bounded sets, the bounded-open (bo)-topology. It follows from inequality (2.17) that the topologies satisfy $\mathscr{T}^0_A \subset \mathscr{T}^0_{\mathrm{bo}}$. We let $(\Xi, \mathscr{T}^0_A)$ and $(\Xi, \mathscr{T}^0_{\mathrm{bo}})$ denote the space Ξ with the respective topologies on C_{Lip}. We now show that the mild solution mapping $S: (\Xi, \mathscr{T}^0_A) \to V^{2\beta}$ is continuous, from which it follows that the mapping $S: (\Xi, \mathscr{T}^0_{\mathrm{bo}}) \to V^{2\beta}$ is continuous, as well.

The proof of the continuity of S uses the Gronwall–Henry inequality, see the Appendix. We present the key idea here. Let $F_i \in C_{\mathrm{Lip}}$ and let $u_i = u_i(t)$ be mild solutions of the problems

$$\partial_t u_i + A u_i = F_i(u_i), \qquad u_i(0) = u_{i0} \in V^{2\beta}, \quad \text{for } i = 1, 2,$$

and assume that there is a bounded set B in $V^{2\beta}$ and a time $\tau > 0$ such that $u_i(t) \in B$, for $0 \leqslant t < \tau$ and $i = 1, 2$. Let $w = u_1 - u_2$ and set $w_0 = w(0) = u_{10} - u_{20}$. Let $F_i(u_i) = F_i(u_i(t))$, for $i = 1, 2$. Then $w = w(t)$ is a solution of

$$\begin{aligned} w(t) &= e^{-At} w_0 + \int_0^t e^{-A(t-s)} [F_1(u_1) - F_2(u_2) \pm F_1(u_2)]\, ds \\ &= e^{-At} w_0 + \int_0^t e^{-A(t-s)} [F_1(u_2) - F_2(u_2)]\, ds \\ &\quad + \int_0^t e^{-A(t-s)} [F_1(u_1) - F_1(u_2)]\, ds. \end{aligned}$$

Let n satisfy $0 \leqslant n < \tau$. Since $w_0 \in V^{2\beta}$, inequality (2.5) implies that, for $0 \leqslant t \leqslant n$, one has

$$\begin{aligned} \|A^\beta w(t)\| \leqslant M_1 e^{-at} \|A^\beta w_0\| &+ \int_0^t \|A^\beta e^{-A(t-s)} [F_1(u_2) - F_2(u_2)]\|\, ds \\ &+ \int_0^t \|A^\beta e^{-A(t-s)}\|\, \|F_1(u_1) - F_1(u_2)\|\, ds. \end{aligned}$$

It then follows from (2.6) and (2.12) that

$$\begin{aligned} \|A^\beta w(t)\| \leqslant M_1 e^{-at} \|A^\beta w_0\| &+ \|F_1 - F_2\|_{\{A;\, C^0(B)\}} \\ &+ M_\beta k_1 \int_0^t (t-s)^{-\beta} e^{-a(t-s)} \|A^\beta w(s)\|\, ds. \end{aligned}$$

With $v(t)=e^{at}\,\|A^{\beta}w(t)\|$, $h(t)=M_1\,\|A^{\beta}w_0\|+e^{at}\,\|F_1-F_2\|_{\{A;\,C^0(B)\}}$, $r=1-\beta$, and $M=M_{\beta}k_1$, it follows from the Gronwall–Henry inequality that there are positive constants C_n and D_n such that

$$\|A^{\beta}w(t)\|\leqslant C_n\,\|A^{\beta}w_0\|+D_n\,\|F_1-F_2\|_{\{A;\,C^0(B)\}},\qquad \text{for}\quad 0\leqslant t\leqslant n. \tag{2.18}$$

There are other issues one needs to verify, such as showing that $T(F,u_0)$ satisfies the relation $T(F,u_0)\leqslant \liminf_{n\to\infty} T(F^n,u_n)$, where $F^n\to F$ (in $\mathscr{T}^o_A$) and $u_n\to u_0$ (in $V^{2\beta}$). However, we note that inequality (2.29) essentially completes the proof of the continuity of S on the space $(\Xi,\mathscr{T}^0_A)$. In fact, this shows that $S(F,u_0,t)$ is locally Lipschitz continuous in F and u_0, uniformly for t in compact sets.

With $F\in C_{\mathrm{Lip}}$ fixed, it then follows that $S(t)\,u_0=S(F,t)\,u_0$ generates a semiflow on the phase space $\mathscr{M}\overset{\mathrm{def}}{=}\{u_0\in V^{2\beta}:T(F,u_0)=\infty\}$. Thus one has that (1) $S(0)\,w=w$, for all $w\in\mathscr{M}$; (2) the *semigroup property* holds, i.e.,

$$S(t)\,S(s)\,w=S(t+s)\,w,\qquad \text{for all}\quad w\in\mathscr{M},\quad\text{and}\quad s,t\in[0,\infty);$$

and (3) the mapping $S:\mathscr{M}\times[0,\infty)\to\mathscr{M}$ is a continuous.

A set $\mathscr{K}$ is an *invariant* set provided that $S(t)\mathscr{K}=\mathscr{K}$, for all $t\geqslant 0$. If $\mathscr{K}$ is a compact, invariant set, then for any T with $0<T<\infty$ and any $\varepsilon>0$, there is a $\delta>0$, such that if $v\in\mathscr{K}$ and if u satisfies $d(u,v)\leqslant\delta$, then one has $d(S(t)\,u,\mathscr{K})\leqslant d(S(t)\,u,S(t)\,v)<\varepsilon$, for $0\leqslant t\leqslant T$, where $d(u,\mathscr{K})=\inf\{d(u,w):w\in\mathscr{K}\}$. Note that if $\mathscr{K}$ be a bounded, invariant set in $V^{2\beta}$, then one has $\mathscr{K}\subset\mathscr{D}(A)=V^2$, and for every $u_0\in\mathscr{K}$, the global mild solution $S(t)\,u_0$ is a strong solution of Eq. (1.1) in W, for all $t\in\mathbb{R}$, and one has

$$S(\cdot)\,u_0\in C^{0,1-r}_{\mathrm{loc}}(\mathbb{R};V^{2r})\cap C(\mathbb{R};\mathscr{D}(A)),\qquad \text{for each}\quad u_0\in\mathscr{K}, \tag{2.19}$$

and $\mathscr{K}$ is a bounded, invariant set in V^{2r}, for each r with $0\leqslant r<1$. The proof of this regularity property can be found in Sell and You (2001, Theorem 47.6).

2.3. *Linear Skew Product Semiflows.* The next objective is to present a typical construction of a linear skew-product semiflow. We begin with the linear time-varying evolutionary equation

$$\partial_t v+Av=B(t)\,v,\qquad v(0)=v_0\in V^{2\beta}, \tag{2.20}$$

where A satisfies the Standing Hypothesis A. We are interested here in the case where the time-varying linear operator $B = B(t)$ is in any one of three spaces:

(1) $B \in L^\infty(0, T; \mathscr{L}) \cap C[0, T; \mathscr{L})$, where $0 < T < \infty$; or

(2) $B \in \mathscr{M}^\infty(0, \infty; \mathscr{L}) \stackrel{\text{def}}{=} L^\infty(0, \infty; \mathscr{L}) \cap C[0, \infty; \mathscr{L})$; or

(3) $B \in \mathscr{M}^\infty(\mathbb{R}; \mathscr{L}) \stackrel{\text{def}}{=} L^\infty(\mathbb{R}; \mathscr{L}) \cap C(\mathbb{R}; \mathscr{L})$,

where $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$. We will use the symbol $\mathscr{M}^\infty$ to refer to either of the two spaces: $\mathscr{M}^\infty(0, \infty; \mathscr{L})$ or $\mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$. For our description of the basic theory of the solutions of Eq. (2.20), we will focus on Cases (2) and (3). The modification needed for Case (1) is a simple exercise.

In this paper we will study a class of linear skew product semiflows on a product space $\mathscr{E} = V \times \mathscr{W}$, where V is a Banach space, and $\mathscr{W}$ is a metric space. Typically we consider $\mathscr{W} = (\mathscr{M}^\infty, \mathscr{T}_A)$ or $\mathscr{W} = (\mathscr{M}^\infty, \mathscr{T}_{\mathrm{bo}})$, where the topologies $\mathscr{T}_A$ and $\mathscr{T}_{\mathrm{bo}}$ are described below. In these cases, $\mathscr{W}$ is a Fréchet space consisting of time-varying functions $B = B(t)$.

We will show that the mapping $(B, \tau) \to B_\tau$, where $B_\tau(t) = B(\tau + t)$, generates a semiflow/flow on $\mathscr{W}$, i.e., a semiflow on $\mathscr{M}^\infty(0, \infty; \mathscr{L})$ and a flow on $\mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$. In the applications, there are many possibilities for the choice of the Banach space V, see Sell and You (2001). However, for this paper, we will use the natural choice, $V = V^{2\beta}$.

A *linear skew-product semiflow* on $\mathscr{E} = V \times \mathscr{W}$ is a mapping $\pi = (\Phi, \sigma)$ of the form $\pi: \mathscr{E} \to \mathscr{E}$, where

$$\pi(v, B, t) = (\Phi(B, t)\, v, B_t), \qquad \text{for} \quad t \geqslant 0, \tag{2.21}$$

with the following properties:

(1) The mapping $\sigma(B, \tau) = B_\tau$ is a semiflow/flow on $\mathscr{W}$;

(2) $\Phi(B, 0) = I$, the identity operator, for all $B \in \mathscr{W}$;

(3) $\Phi(B, t)$ is an element of $\mathscr{L}(V, V)$ that satisfies the cocycle identity:

$$\Phi(B, s+\tau) = \Phi(B_\tau, s)\, \Phi(B, \tau), \qquad \text{for} \quad B \in \mathscr{W}, \quad \text{and} \quad \tau, s \geqslant 0. \tag{2.22}$$

(4) The mapping from $\mathscr{E} \times (0, \infty)$ into V given by $(v, B, t) \to \Phi(B, t)\, v$ is continuous.

(5) For each $(v, B) \in \mathscr{E}$ the mapping $t \to \Phi(B, t)\, v$ is continuous at $t = 0$, and for each $v \in V$ the limit $\lim_{t \to 0^+} \Phi(B, t)\, v = v$ is uniform for B in compact sets, i.e., for every compact set $\mathscr{K}_0 \subset \mathscr{W}$, $v \in V$, and $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\Phi(B, t)\, v - v\| \leqslant \varepsilon$, for all $B \in \mathscr{K}_0$ and $t \in [0, \delta]$.

A typical construction of a linear skew product semiflow begins by finding a mild solution of Eq. (2.20). In particular, for each $B \in \mathcal{W}$ and $v_0 \in V^{2\beta}$, we let $v(t) = \Phi(B, t)\, v_0$ denote the mild solution of (2.20), i.e., $\Phi(B, t)\, v_0$ satisfies

$$\Phi(B, t)\, v_0 = e^{-At} v_0 + \int_0^t e^{-A(t-s)} B(s)\, \Phi(B, s)\, v_0\, ds. \tag{2.23}$$

It is a straightforward application of the contraction mapping theorem and inequality (2.6) to show that Eq. (2.23) has a unique solution, and that this solution is defined for all $t \geqslant 0$. Moreover, one has

$$\Phi(B, \cdot)\, v_0 \in C[0, \infty; V^{2\beta}) \cap C^{0,\theta}_{\mathrm{loc}}(0, \infty; V^{2r}),$$

for all r with $0 \leqslant r < 1$, where $0 < \theta < 1 - r$. As usual, $C^{0,\theta}$ denotes a space of Hölder continuous functions.

It is easily seen that, for $B \in L^\infty$, the mapping $v_0 \to \Phi(B, t)\, v_0$ is linear for each $t \geqslant 0$. For $-\infty \leqslant a \leqslant b \leqslant \infty$, we define

$$\|B\|_{\infty; [a, b]} = \sup\{ \|B(t)\, v\| \colon \|A^\beta v\| \leqslant 1 \text{ and } a \leqslant t \leqslant b\}, \tag{2.24}$$

and $\|B\|_\infty = \|B\|_{\infty; [0, \infty)}$. A direct application of the Gronwall–Henry inequality yields

$$\|A^\beta \Phi(B, t)\, v_0\| \leqslant M_1 e^{-at} E_{\varepsilon, 1}(\mu t)\, \|A^\beta v_0\|, \qquad \text{for} \quad t \geqslant 0, \tag{2.25}$$

where $\varepsilon = 1 - \beta$ and $\mu^\varepsilon = M_\beta \|B\|_\infty \Gamma(\varepsilon)$. Thus $\Phi(B, t)$ is a bounded linear operator on $V^{2\beta}$, for each $t \geqslant 0$, with $\Phi(B, 0)\, v_0 = v_0$, for all $v_0 \in V^{2\beta}$. Furthermore, there is a unique continuous extension of $\Phi(B, t)\, v_0$, to all $v_0 \in W$, and there is a constant $M > 0$ such that one has

$$\|\Phi(B, t)\, v_0\| \leqslant M e^{-at} E_{\varepsilon, 1}(\mu t)\, \|v_0\|, \qquad \text{for all} \quad v_0 \in W.$$

Since the solutions of (2.23) are unique, one can readily verify the cocycle identity

$$\Phi(B, \tau + t) = \Phi(B_\tau, t)\, \Phi(B, \tau), \qquad \text{for} \quad t, \tau \geqslant 0.$$

We will refer to $\Phi(B, t)$ as the *solution operator* for Eq. (2.20) generated by $B \in \mathcal{W}$.

In order to show that the mapping π defined by Eq. (2.21) is a linear skew product semiflow on $V^{2\beta} \times \mathcal{W}$, it remains only to verify the continuity of π. For this purpose, we now introduce the two topologies $\mathscr{T}_A$ and $\mathscr{T}_{\mathrm{bo}}$ on $\mathscr{M}^\infty$. For $B \in \mathscr{M}^\infty(0, \infty; \mathscr{L})$ and $\tau > 0$, we define the pseudonorm

$$\|B\|_{\{A;\,[0,\,\tau]\}} = \sup_{0 \leqslant t \leqslant \tau} \int_0^t \|A^\beta e^{-A(t-s)} B(s)\|_{\mathscr{L}}\, ds,$$

see Pliss and Sell (1999). For $B \in \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$ and $\tau > 0$, we set $\|B\|_{\{A;\,[-\tau,\,0]\}} = \|B_{-\tau}\|_{\{A;\,[0,\,\tau]\}}$. The pseudonorms $\|B\|_{\infty;\,[0,\,\tau]}$ and $\|B\|_{\infty;\,[-\tau,\,0]}$ are given by Eq. (2.24). It follows from inequality (2.6) and the definition of the Gamma function Γ that, for any $B \in \mathscr{M}^\infty(0, \infty; \mathscr{L})$ and $\tau > 0$, one has

$$\|B\|_{\{A;\,[0,\,\tau]\}} \leqslant M_\beta\, a^{\beta-1} \Gamma(1-\beta)\, \|B\|_{\infty;\,[0,\,\tau]}, \tag{2.26}$$

with a similar inequality valid for negative time when $B \in \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$.

We let $\mathscr{T}_A$ denote the topology on $\mathscr{M}^\infty(0, \infty; \mathscr{L})$ (or $\mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$) generated by the countable family of pseudonorms $\|\cdot\|_{\{A;\,[0,\,n]\}}$ (or $\|\cdot\|_{\{A;\,[-n,\,n]\}}$), where n is an integer. Similarly. we let $\mathscr{T}_{\mathrm{bo}}$ denote the topology on $\mathscr{M}^\infty(0, \infty; \mathscr{L})$ (or $\mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$) generated by $\|\cdot\|_{\infty;\,[0,\,n]}$ (or $\|\cdot\|_{\infty;\,[-n,\,n]}$). The argument leading to inequality (2.26) implies that $\mathscr{T}_A \subset \mathscr{T}_{\mathrm{bo}}$.

The proof of the continuity of the translation mapping $\sigma(B, \tau) = B_\tau$ is based on the simple observation: if $B \in \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$, then B is uniformly continuous on compact subsets of $\mathbb{R}$. Consequently, if $\lim B^m = B$ (in $(\mathscr{M}^\infty, \mathscr{T}_A)$) and $\lim \tau_m = \tau$ (in $\mathbb{R}$), then

$$\|B^m_{\tau_m} - B_\tau\|_{\{A;\,[0,\,n]\}} \leqslant \|B^m_{\tau_m} - B_{\tau_m}\|_{\{A;\,[0,\,n]\}} + \|B_{\tau_m} - B_\tau\|_{\{A;\,[0,\,n]\}}$$

implies that $\lim B^m_{\tau_m} = B_\tau$ (in $(\mathscr{M}^\infty, \mathscr{T}_A)$). A similar argument works in $(\mathscr{M}^\infty, \mathscr{T}_{\mathrm{bo}})$.

In order to show the continuity of the solution $\Phi(B, t)\, v_0$, we can repeat the argument leading to inequality (2.18), in the case of the topology $\mathscr{T}_A$. The inclusion $\mathscr{T}_A \subset \mathscr{T}_{\mathrm{bo}}$ then establishes the continuity in terms of the topology $\mathscr{T}_{\mathrm{bo}}$. We invite the reader to verify the validity of Item (5) in the definition of a linear skew product semiflow. We then have the following conclusion:

THEOREM 2.1. *Let the Standing Hypothesis A be satisfied, and let $\mathscr{M}^\infty$ and $\mathscr{W}$ be given as above. Then the following statements are valid*:

(1) *The mapping $(B, \tau) \to B_\tau$ is a semiflow/flow on the space $\mathscr{W}$.*

(2) *The function $\pi(v_0, B, \tau) \stackrel{\text{def}}{=} (\Phi(B, \tau)\, v_0, B_\tau)$ is a linear skew product semiflow on $V^{2\beta} \times \mathscr{W}$.*

These conclusions hold in both topologies $\mathscr{T}_A$ and $\mathscr{T}_{\mathrm{bo}}$ on $\mathscr{W}$.

In the following definitions of several distinguished subsets of $\mathscr{E} = V^{2\beta} \times \mathscr{W}$, we will make use of the concept of a negative continuation, which we now define. A mapping $\phi(t) = (v(t), B_t)\colon \mathbb{R} \to \mathscr{E}$ is said to be a *globally defined solution* through (v_0, B) provided that ϕ is continuous, that $v(0) = v_0$, and that

$$\Phi(B_\tau, t)\, v(\tau) = v(\tau + t), \qquad \text{for all} \quad \tau \in \mathbb{R} \quad \text{and} \quad t \geqslant 0. \tag{2.27}$$

In this case, the restriction of ϕ to $(-\infty, 0]$, which we will denote by $\phi^{v_0, B}$, is said to be a *negative continuation* of (v_0, B). We do not assume the uniqueness of the negative continuations, but as noted later, when one has an exponential dichotomy, then in a qualified sense, *some* negative continuations are unique.

Let us now restrict to the linear skew product semiflows $\pi = (\Phi, \sigma)$ on $\mathscr{E}(\mathscr{K}) = V^{2\beta} \times \mathscr{K}$, where $\mathscr{K}$ is an invariant set in $\mathscr{W}$ and $\mathscr{M}^\infty = \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$. Thus one has $\sigma(\mathscr{K}, t) = \mathscr{K}$, for all $t \geqslant 0$. We define next the following subsets of $\mathscr{E}(\mathscr{K})$:

(1) $\mathscr{U}$: The set of points $(v, B) \in \mathscr{E}(\mathscr{K})$ such that there is a negative continuation $\phi^{v, B}(t) = (v(t), B_t)$ that satisfies $\|v(t)\| \to 0$, as $t \to -\infty$.

(2) $\mathscr{B}^-$: The set of points $(v, B) \in \mathscr{E}(\mathscr{K})$ such that there is a negative continuation $\phi^{v, B}(t) = (v(t), B_t)$ that satisfies $\sup_{t \leqslant 0} \|v(t)\| < \infty$.

(3) $\mathscr{B}_u^-$: The set of points $(v, B) \in \mathscr{E}(\mathscr{K})$ such that there is a *unique* bounded negative continuation $\phi^{v, B}$.

(4) $\mathscr{B}^+$: The set of points $(v, B) \in \mathscr{E}(\mathscr{K})$ such that $\sup_{t \geqslant 0} \|\Phi(B, t)\, v\| < \infty$.

(5) $\mathscr{S}$: The set of points $(v, B) \in \mathscr{E}(\mathscr{K})$ such that $\|\Phi(B, t)\, v\| \to 0$, as $t \to \infty$.

(6) $\mathscr{B} \overset{\text{def}}{=} \mathscr{B}^- \cap \mathscr{B}^+$.

One refers to $\mathscr{U}$ as the *unstable set*, to $\mathscr{S}$ as the *stable set*, and to $\mathscr{B}$ as the *bounded set*. Note that the set $\mathscr{B}$ is an invariant set, and it is the set of all points $(v_0, B) \in \mathscr{E}(\mathscr{K})$ such that there is a globally defined solution ϕ through (v_0, B) with $\sup_{t \in \mathbb{R}} \|A^\beta \phi(t)\| < \infty$.

2.4. *The Linearized Equation.* In this section we begin with a given solution $u = u(t)$ of the nonlinear evolutionary Eq. (2.15), and we seek to linearize this equation along the solution $u(t)$. For this purpose we assume that the Standing Hypothesis A is satisfied and that the nonlinearity F is in C^1_{Lip}, or more generally,

$$F \in C_{\mathrm{Lip}}(U, W) \cap C^1_{\mathrm{F}}(U, W), \qquad \text{where} \quad 0 \leqslant \beta < 1,$$

and U is an bounded, open set in $V^{2\beta}$. We say that the derivative DF satisfies the *Lipschitz property* (on U) if for every compact set $\mathscr{K}$ in U there exist constants $C = C(\mathscr{K}) > 0$ and $r_0 = r_0(\mathscr{K}) > 0$, such that

$$\|DF(u_1) - DF(u_2)\|_{\mathscr{L}(V^{2\beta},\, W)} \leqslant C\, \|A^\beta(u_1 - u_2)\|, \qquad \text{for all} \quad u_1, u_2 \in \mathscr{K},$$

whenever $\|A^\beta(u_1 - u_2)\| \leqslant r_0$.

Next we let $u = u(t)$ be a continuous mapping from $[0, \infty)$ into U. For example, $u(t)$ might be a mild solution $u(t) = S(t)\, u_0$ of Eq. (2.15). Let DF denote the Fréchet derivative of F. Then $B(t) \stackrel{\text{def}}{=} DF(u(t))$ is well defined for all $t \geqslant 0$, and the mapping $t \to B(t)$ is a continuous mapping of $[0, \infty)$ into $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$, i.e., $B(\cdot) \in C[0, \infty; \mathscr{L})$. Let us look first at the case where the initial condition u_0, for the mild solution $S(t)\, u_0$, is a point in a compact, invariant set $\mathscr{K}$ for Eq. (2.15). In this case, there is a global solution, which we will denote by $S(t)\, u_0$, passing through u_0, and the linear operator $B(t)$ is defined for all $t \in \mathbb{R}$. Since the derivative DF is continuous and bounded in U, one has $B(\cdot) \in \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$, where $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$. Hence, the theory described above applies to the linear equation

$$\partial_t v + Av = DF(u(t))\, v.$$

The last equation is referred to as the *linearized equation* associated with (2.15).

However, before turning to the linearized equation, *per se*, we should make note of additional information which can be brought to bear on the problem. Since $F \in C_{\text{Lip}}(V^{2\beta}, W)$, it follows that every bounded, invariant set $\mathscr{K}$ in $V^{2\beta}$ satisfies $\mathscr{K} \subset \mathscr{D}(A)$, and the mild solution $u(t) = S(t)\, u_0$ is a strong solution of Eq. (2.15). Also $u(t)$ is Hölder continuous in t, i.e., $u(\cdot) \in C^{0,\, 1-\beta}_{\text{loc}}(\mathbb{R}, V^{2\beta})$, see (2.19). Furthermore, if the derivative DF satisfies the Lipschitz property, then $B(t) = DF(u(t))$ is also Hölder continuous in t, since one has $\|DF(u(t_1)) - DF(u(t_2))\|_{\mathscr{L}} \leqslant K_1\, \|A^\beta(u(t_1) - u(t_2))\| \leqslant K_2\, |t_1 - t_2|^{(1-\beta)}$.

In addition to the Standing Hypothesis A, we now assume that $F \in C^1_{\text{Lip}}$, see (1.3). For a bounded set U in $V^{2\beta}$ we define two pseudonorms $\|F\|_{\{A;\, C^1(U)\}}$ and $\|F\|_{C^1(U)}$, as

$$\|F\|_{\{A;\, C^1(U)\}} \stackrel{\text{def}}{=} \|F\|_{\{A;\, C^0(U)\}} + \sup_{t \geqslant 0} \int_0^t \|\sup_{u \in B} A^\beta e^{-A(t-s)} DF(u)\|_{\mathscr{L}}\, ds$$

$$\|F\|_{C^1(U)} \stackrel{\text{def}}{=} \|F\|_{C^0(U)} + \sup_{u \in U} \|DF(u)\|_{\mathscr{L}},$$

where $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$. As argued above, one has

$$\|F\|_{\{A;\, C^1(U)\}} \leqslant M_\beta\, a^{\beta-1} \Gamma(1-\beta)\, \|F\|_{C^1(U)}.$$

Let $\mathcal{T}^1_A$ and $\mathcal{T}^1_{\text{bo}}$ denote the topologies generated on C^1_{Lip} by the two pseudonorms $\|\cdot\|_{\{A;\, C^1(U)\}}$ and $\|\cdot\|_{C^1(U)}$, respectively. The last inequality then shows that one has $\mathcal{T}^1_A \subset \mathcal{T}^1_{\text{bo}}$.

For $F \in C^1_{\text{Lip}}$, we let $S(v, t)$ denote the maximally defined mild solution of

$$S(v, t) = e^{-At}v + \int_0^t e^{-A(t-s)}F(S(v, s))\, ds, \qquad \text{for} \quad v \in V^{2\beta}. \tag{2.28}$$

Recall that $S(v, t)$ is a strong solution in W of Eq. (2.15), for every $v \in V^{2\beta}$, and one has $S(\cdot, t)\colon V^{2\beta} \to V^{2\beta} \mapsto W$, for each $t \geqslant 0$. For such a solution we let $\Phi(v, t)$ be the solution operator generated by the equation,

$$\partial_t w + Aw = DF(S(v, t))\, w, \tag{2.29}$$

which agrees with Eq. (2.20) where $B(t) = DF(S(v, t))$. If in addition, the Fréchet derivative DF has the Lipschitz property, then $B(t)$ is Hölder continuous in t and $\Phi(v, t)\, w$ is a strong solution of Eq. (2.29), for $w \in V^{2\beta}$ and $0 \leqslant t < T$. For each $w \in W$, we define the derivative DS by

$$DS(v_0, t)\, w = \frac{\partial}{\partial v} S(v, t)\bigg|_{v = v_0} w, \qquad \text{for} \quad v, v_0 \in V^{2\beta},$$

where the limit for the derivative exists in W, i.e., $DS(v_0, t)$ is a strong derivative. We note that, for each $t \geqslant 0$ and each $v \in V^{2\beta}$, the mild solution $S(v, t)$ is Fréchet differentiable and the derivative $DS(v, t)$ satisfies

$$DS(v, t)\, w = \Phi(B, t)\, w = \Phi(v, t)\, w, \qquad \text{for all} \quad w \in V^{2\beta},$$

in the sense that

$$DS(v, t)\, w = e^{-At}w + \int_0^t e^{-A(t-s)}DF(S(v, s))\, DS(v, s)\, w\, ds,$$

for $v, w \in V^{2\beta}$. If $v = u_0 \in \mathcal{K}$, where $\mathcal{K}$ is a compact, invariant set for the mild solutions of Eq. (1.1), then (2.19) is valid, and one has

$$B_\tau(t) = DF(S(\tau + t)\, u_0) = DF(S(t)\, S(\tau)\, u_0), \qquad \text{for all} \quad \tau, t \in \mathbb{R}.$$

When $\mathcal{K}$ is a compact, invariant set in $V^{2\beta}$, we have the following generalization of inequality (2.5): There is an $a_0 \geqslant 0$ and $K_0 \geqslant 1$ such that

$$\|A^\beta \Phi(u_0, t)\, w\| \leqslant K_0 e^{a_0 t}\, \|A^\beta w\|, \qquad \text{for all} \quad u_0 \in \mathcal{K} \quad \text{and} \quad t \geqslant 0.$$

Furthermore, the following generalization of inequality (2.6) is valid,

$$\|A^\beta \Phi(u_0, t)\, v_0\| \leqslant M_\beta t^{-\beta} e^{-at} E_{\varepsilon, 1}(\mu t)\, \|v_0\|, \tag{2.30}$$

for all $u_0 \in \mathscr{K}$, $v_0 \in W$ and $t > 0$, where $\varepsilon = 1 - \beta$ and $\mu^\varepsilon = M_\beta \|B\|_\infty \Gamma(\varepsilon)$, see the Appendix, Henry (1981), or Sell and You (2001, Chap. 4).

2.5. *Inhomogeneous Equations.* Next we examine the linear inhomogeneous equation

$$\partial_t v + Av = B(t)\, v + g(t), \tag{2.31}$$

where the Standing Hypothesis A is satisfied, $B = B(t) \in \mathscr{W}$, $g \in L^\infty_{\mathrm{loc}}[0, T; W)$, and $0 < T \leqslant \infty$. We will say that $v = v(t)$ is a *mild solution* of Eq. (2.31) in $V^{2\beta}$ on the interval $[0, T)$, if $v(0) = v_0 \in V^{2\beta}$, $v(\cdot) \in C[0, T; V^{2\beta})$, and one has

$$v(t) = e^{-At} v_0 + \int_0^t e^{-A(t-s)} [B(s)\, v(s) + g(s)]\, ds, \qquad \text{for} \quad 0 \leqslant t < T. \tag{2.32}$$

We note that, for every $v_0 \in V^{2\beta}$ and every $g \in L^\infty_{\mathrm{loc}}[0, T; W)$, there is a unique mild solution $v = v(t)$ of Eq. (2.31) in $V^{2\beta}$ on the interval $[0, T)$.

If $g \equiv 0$, then the mild solution v is given by $v(t) = \Phi(B, t)\, v_0$, for all $t \geqslant 0$. For the general case, we claim that the mild solution v of Eq. (2.31) satisfies a second Variation of Constants Formula,

$$v(t) = \Phi(B, t)\, v_0 + \int_0^t \Phi(B_s, t-s)\, g(s)\, ds, \qquad \text{for} \quad t \geqslant 0. \tag{2.33}$$

where $v_0 \in V^{2\beta}$, and of course B_s is the translate $B_s(t) = B(s+t)$, for $s, t \in \mathbb{R}$. In order to show that v satisfies Eq. (2.33), it suffices to show that the particular solution of (2.32), where $v_0 = 0$, satisfies

$$v(t) = \int_0^t \Phi(B_s, t-s)\, g(s)\, ds, \qquad \text{for} \quad t \geqslant 0. \tag{2.34}$$

For this purpose, we define $w(t) \stackrel{\text{def}}{=} \int_0^t \Phi(B_r, t-r)\, g(r)\, dr$. One then shows that

$$w(t) = \int_0^t e^{-A(t-s)} [B(s)\, w(s) + g(s)]\, ds, \qquad \text{for} \quad 0 \leqslant t < T, \tag{2.35}$$

in which case it follows, from the uniqueness of the mild solutions that $w(t) \equiv v(t)$. The proof that $w(t)$ satisfies Eq. (2.35) involves a straightforward change of variables along with the fact that $\Phi(B_r, t-r)$ is given by

$$\Phi(B_r, t-r)\, w_0 = e^{-A(t-r)} w_0 + \int_0^{t-r} e^{-A(t-r-\sigma)} B_r(\sigma)\, \Phi(B_r, \sigma)\, w_0 \, d\sigma,$$

for $w_0 \in V^{2\beta}$. We will omit the details.

Finally we note that, if $v_0 \in V^{2\beta}$, then the mild solution v of Eq. (2.31) satisfies

$$v(\cdot) \in C[0, T; V^{2\beta}) \cap C^{0,\theta}_{\text{loc}}(0, T; V^{2r}),$$

for every r with $0 \leqslant r < 1$, where $0 < \theta < 1 - r$. If in addition, B and g satisfy

$$B \in C^{0,\theta_1}_{\text{loc}}(0, \infty; \mathscr{L}(V^{2\beta}, W)) \qquad \text{and} \qquad g \in L^1_{\text{loc}}[0, T; W) \cap C^{0,\theta_2}_{\text{loc}}(0, T; W), \tag{2.36}$$

for some θ_i with $0 < \theta_i \leqslant 1$, for $i = 1, 2$, then v is a strong solution of Eq. (2.31) in W on $0 \leqslant t < T$, i.e., one has: (1) v is (strongly) differentiable in W almost everywhere (a.e.) in $(0, T)$; (2) $\partial_t v \in L^1_{\text{loc}}[0, T; W)$; (3) $v(t) \in \mathscr{D}(A)$ a.e. on $(0, T)$; and (4) v satisfies the equation $\partial_t v(t) + Av(t) \overset{\text{a.e.}}{=} B(t)\, v(t) + g(t)$, on $(0, T)$, in the space W.

3. HYPERBOLIC STRUCTURE: EXPONENTIAL TRICHOTOMY

Our objective in this section is to present the basic theory of exponential trichotomies on a Banach space. The definition of this concept, as well as the related concept of an exponential dichotomy, is similar to that used in the theory of ordinary differential equations, see Pliss and Sell (1991, 1998, 1999). Also see, Sacker and Sell (1994); and Chow, Lu, and Mallet-Paret (1995). However, since we are considering semiflows here, it is essential that we exercise special care in dealing with the negative continuations.

Let V be a given Banach space and let $\mathscr{W}$ be a metric space. For any set $\mathscr{V}$ in the product space $\mathscr{E} = V \times \mathscr{W}$, we define the *fiber* $\mathscr{V}(B_0)$ over the point $B_0 \in \mathscr{W}$ by

$$\mathscr{V}(B_0) \overset{\text{def}}{=} \{(w, B) \in \mathscr{E} : (w, B) \in \mathscr{V} \text{ and } B = B_0\}.$$

Similarly for any set $\mathscr{K} \subset \mathscr{W}$ we define the *restriction of* $\mathscr{V}$ *to* $\mathscr{K}$ as

$$\mathscr{V}(\mathscr{K}) \overset{\text{def}}{=} \{(w, B) \in \mathscr{E} : B \in \mathscr{K}\} = \bigcup_{B \in \mathscr{K}} \mathscr{V}(B).$$

Notice that $\mathscr{E}(B) = V \times \{B\}$, and $\mathscr{E}(B)$ is a Banach space, with the same structure as V. A mapping $P: \mathscr{E} \to \mathscr{E}$ is said to be a *projector* if P is continuous and has the form $P(w, B) = (P(B)\, w, B)$, where $P(B)$ is a continuous (linear) projection[2] on the fiber $\mathscr{E}(B)$. This means that $P(B): \mathscr{E}(B) \to \mathscr{E}(B)$ is a bounded linear mapping that satisfies $P(B)\, P(B) = P(B)$. For any projector $P: \mathscr{E} \to \mathscr{E}$ we define the *range* and *null space* by

$$\mathscr{R} = \mathscr{R}(P) = \{(w, B) \in \mathscr{E} : P(B)\, w = w\}$$

and

$$\mathscr{N} = \mathscr{N}(P) = \{(w, B) \in \mathscr{E} : P(B)\, w = 0\}.$$

Note that the fibers $\mathscr{R}(B)$ and $\mathscr{N}(B)$ are closed linear subspaces of $\mathscr{E}(B)$, since $P(B)$ is a continuous linear mapping. Furthermore, these fibers vary continuously in B, which implies that $P(B)$ varies continuously in the operator norm in $\mathscr{L}(V, V)$. The following result is easily proven

LEMMA 3.1. *Let P be a projector on $\mathscr{E}$. Then $\mathscr{R}$ and $\mathscr{N}$ are closed subsets in $\mathscr{E}$, and one has*

$$\mathscr{R}(B) \cap \mathscr{N}(B) = \{0\} \qquad \textit{and} \qquad \mathscr{R}(B) + \mathscr{N}(B) = \mathscr{E}(B), \qquad \textit{for all} \quad B \in \mathscr{W}. \tag{3.1}$$

If P is a projector on $\mathscr{E}$, then the mapping $Q = I - P$, where $Q: \mathscr{E} \to \mathscr{E}$ is defined by $Q(w, B) = ((I - P(B))\, w, B)$, is also a projector on $\mathscr{E}$. The projector Q is called the *complementary projector* to P, and one has $\mathscr{R}(Q) = \mathscr{N}(P)$ and $\mathscr{N}(Q) = \mathscr{R}(P)$.

The range and nullspace of a projector are subbundles of $\mathscr{E}$. Recall that a subset $\mathscr{X}$ in $\mathscr{E}$ is said to be a *subbundle* of $\mathscr{E}$ if there is a projector P on $\mathscr{E}$ with the property that $\mathscr{X} = \mathscr{R}(P)$, see Sacker and Sell (1974, 1976ab), for an equivalent definition. In this case, $\mathscr{Y} = \mathscr{N}(P)$ is a *complementary* subbundle, and one has $\mathscr{E} = \mathscr{X} + \mathscr{Y}$, in the sense that (3.1) is valid. The equation $\mathscr{E} = \mathscr{X} + \mathscr{Y}$ is sometimes referred to as a *Whitney sum* of subbundles. The *trivial* subbundle $\mathscr{E}_0 = \{0\} \times \mathscr{Y}$ plays a role in the theory of exponential dichotomies.

Now let

$$\pi(w, B, \tau) = (\Phi(B, \tau)\, w, \sigma(B, \tau)), \qquad \text{where} \quad \sigma(B, \tau) = B_\tau, \tag{3.2}$$

[2] There is a small ambiguity in this notation since we use both $P(w, B)$ and $P(B)w$ to represent P. These two formulations really offer two ways of viewing the concept of a projector.

be a linear skew product semiflow on $\mathscr{E} = V \times \mathscr{W}$. We assume here that the mapping $(B, \tau) \to \sigma(B, \tau) = B_\tau$ is defined for all $(B, \tau) \in \mathscr{W} \times \mathbb{R}$ and that it is a flow on $\mathscr{W}$. Next let $\mathscr{K}$ be an invariant set in $\mathscr{W}$, i.e., $\sigma(\mathscr{K}, \tau) = \mathscr{K}$, for all $\tau \geqslant 0$. In this setting, a projector P on $\mathscr{E}(\mathscr{K}) = V \times \mathscr{K}$ is said to be *invariant* if one has

$$P(B_t)\, \Phi(B, t) = \Phi(B, t)\, P(B), \qquad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}. \tag{3.3}$$

The invariance of a projector is equivalent to the assertion that both subbundles, $\mathscr{R}$ and $\mathscr{N}$, are positively invariant under the linear skew product semiflow π. Note that P is invariant if and only if the complementary projector Q is invariant.

In the next definition and lemma, we will use the unstable set $\mathscr{U}$, the stable set $\mathscr{S}$, and the bounded set $\mathscr{B}$, which are defined in Subsection 2.3. We say that a linear skew product semiflow π has an *exponential dichotomy* in V and over an invariant set $\mathscr{K} \subset \mathscr{W}$ (or π has an *exponential dichotomy on* $V \times \mathscr{K}$), if there is a projector P on the restriction $\mathscr{E}(\mathscr{K}) = V \times \mathscr{K}$, and constants $K \geqslant 1$ and $\alpha > 0$ such that the following hold:

(1) The projectors P and Q are invariant on $\mathscr{E}(\mathscr{K})$, where $Q = I - P$.

(2) One has $\mathscr{R}(P(B)) \subset \mathscr{U}(B)$, for each $B \in \mathscr{K}$. For each $w \in \mathscr{R}(P(B))$, we let

$$\phi^{w, B}(t) = (\Phi(m, t)\, w, B_t), \qquad \text{for} \quad t \leqslant 0,$$

denote *any* negative continuation with $\|\Phi(B, t)\, w\| \to 0$, as $t \to -\infty$.

(3) The following inequalities are valid for all $w \in W$,

$$\|\Phi(B, t)\, Q(B)\, w\| \leqslant K\, \|w\|\, e^{-\alpha t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.4}$$

and

$$\|\Phi(B, t)\, P(B)\, w\| \leqslant K\, \|w\|\, e^{\alpha t}, \qquad \text{for} \quad t \leqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.5}$$

where (3.5) is valid for any negative continuation through $(P(B)\, w, B)$ that remains in $\mathscr{U}$, for all $t \leqslant 0$.

We will refer to (K, α) as the *characteristics* of the dichotomy, and we will call (P, Q) the *associated projectors*. We now have the following result wherein, among other things, we show that certain negative continuations are unique.

LEMMA 3.2. *Let* $\pi = (\Phi, \sigma)$ *be a linear skew product semiflow on* $\mathscr{E} = V \times \mathscr{W}$, *where* σ *is a flow on* $\mathscr{W}$. *Assume that* π *has an exponential dichotomy over an invariant set* $\mathscr{K} \subset \mathscr{W}$. *Then the following statements are valid*:

(1) *The bounded set satisfies* $\mathscr{B}(\mathscr{K}) = \mathscr{E}_0(\mathscr{K}) = \{0\} \times \mathscr{K}$.

(2) *One has* $\mathscr{S}(B) \cap \mathscr{U}(B) = \{0\}$, *for all* $B \in \mathscr{K}$.

(3) *One has* $\mathscr{S}(B) = \mathscr{N}(P(B)) = \mathscr{R}(Q(B))$ *and* $\mathscr{U}(B) = \mathscr{R}(P(B))$, *for all* $B \in \mathscr{K}$, *and the convergence rates in* $\mathscr{S}$ *and* $\mathscr{U}$ *are exponential over* $\mathscr{K}$.

(4) *The subbundles* $\mathscr{S} = \mathscr{S}(\mathscr{K}) = \mathscr{R}(Q)$ *and* $\mathscr{U} = \mathscr{U}(\mathscr{K}) = \mathscr{R}(P)$ *satisfy*

$$\pi(\mathscr{S}, t) \subset \mathscr{S} \qquad \text{and} \qquad \pi(\mathscr{U}, t) = \mathscr{U}, \qquad \text{for} \quad t \geqslant 0, \tag{3.6}$$

that is, $\mathscr{S}$ *is positively invariant and* $\mathscr{U}$ *is invariant under* π.

(5) *For all* $B \in \mathscr{K}$, *the restriction of* $\Phi(B, t)$ *to* $\mathscr{R}(P(B)) = \mathscr{U}(B)$ *is an isomorphism of* $\mathscr{R}(P(B))$ *onto* $\mathscr{R}(P(B_t))$, *for each* $t \geqslant 0$. *Moreover, for each* $w \in V$, *the function* $\Phi(B, t)\, P(B)\, w$ *has a unique negative continuation satisfying*

$$\Phi(B, t)\, P(B)\, w \in \mathscr{R}(P(B_t)), \qquad \text{for all} \quad t \leqslant 0,$$

and the extended cocycle identity

$$\Phi(B, \tau + t)\, P(B) = \Phi(B_\tau, t)\, \Phi(B, \tau)\, P(B), \qquad \text{for all} \quad \tau, t \in \mathbb{R}, \tag{3.7}$$

is valid, for all $B \in \mathscr{K}$.

(6) *For each* $w \in \mathscr{R}(P(B)) = \mathscr{U}(B)$, *there is a unique negative continuation through* (w, B), *where the* w-*coordinate is uniformly bounded, for* $t \leqslant 0$, *and this negative continuation is* $\Phi(B, t)\, w$.

Since the proof of this lemma can be found in Pliss and Sell (1999), we will not include it here. Also see Sell and You (2001). One should compare Eq. (3.7) with (2.22) and note the important role played by the projector P.

There is more information contained in inequality (3.5). Because of the extended cocycle identity (3.7), one can reverse time in this case and thereby obtain the inequality

$$\|\Phi(B, t)\, P(B)\, v\| \geqslant K^{-1}\, \|P(B)\, v\|\, e^{\alpha t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}. \tag{3.8}$$

Indeed, let $w \in \mathscr{R}(P(B))$, so that $P(B)\, w = w$, and let $\tau < 0$. Set $t = -\tau$ and $v = \Phi(B, \tau)\, P(B)\, w$. Then Eq. (3.7) implies that $\Phi(B_\tau, t)\, v = \Phi(B_\tau, t)\, \Phi(B, \tau)\, w = w$. Thus $v \in \mathscr{R}(P(B_\tau))$ (from Lemma 3.2) and $P(B_\tau)\, v = v$. The invariance property (3.3) and Lemma 3.2 imply that $v = \Phi(B, \tau)\, P(B)\, w$. Hence, one can rewrite inequality (3.5) in terms of v. By replacing B_τ with B, one then obtains (3.8).

A given linear skew product flow $\pi = (\Phi, \sigma)$ on $\mathscr{E} = V \times \mathscr{W}$ can be imbedded into the family $\pi_\lambda = (\Phi_\lambda, \sigma)$, for $\lambda \in \mathbb{R}$, by defining $\Phi_\lambda(B, t) \stackrel{\text{def}}{=} e^{-\lambda t}\Phi(B, t)$. One refers to π_λ as the *shifted semiflow*. The reason for this teminology can be appreciated by assuming that $\Phi(B, t)$ is a fundamental solution operator for the nonautonomous linear evolutionary equation $\partial_t u = B(t) u$. In this case, $\Phi_\lambda(B, t)$ is a fundamental solution operator for the shifted equation $\partial_t v = (B(t) - \lambda I) v$. The set of all $\lambda \in \mathbb{R}$ for which π_λ admits an exponential dichotomy on $\mathscr{E}$ is called the *resolvent set* for π. The *dynamical spectrum* $\Sigma(\pi)$ of π is the complement in $\mathbb{R}$ of the resolvent set. The unstable set, the stable set, the bounded set, etc are all defined for the shifted flow π_λ and will be denoted by $\mathscr{U}_\lambda$, $\mathscr{S}_\lambda$, $\mathscr{B}_\lambda$, etc. The two sets $\mathscr{U}_\lambda$ and $\mathscr{S}_\lambda$ are monotone in λ; $\mathscr{U}_\lambda$ is nonincreasing, and $\mathscr{S}_\lambda$ is nondecreasing, see Sacker and Sell (1978, 1980), Chow, Lu, and Mallet-Paret (1995), and Pliss and Sell (1999), for more details.

We are interested in the situation where the shifted linear skew product semiflow π_λ has an exponential dichotomy for different values of the parameter λ. More precisely, let λ and μ be given, where $\lambda < \mu$, and assume that π_λ and π_μ each has an exponential dichotomy over an invariant set $\mathscr{K}$ in $\mathscr{W}$, with invariant projectors (P_λ, Q_λ) and (P_μ, Q_μ), and characteristics $(K_\lambda, \alpha_\lambda)$ and (K_μ, α_μ). By replacing these characteristics with (K, α), where $K = \max(K_\lambda, K_\mu)$ and $\alpha = \min(\alpha_\lambda, \alpha_\mu)$, we see that (K, α) serve as characteristics for both π_λ and π_μ. As noted above, one has $\mathscr{S}_\lambda \subset \mathscr{S}_\mu$ and $\mathscr{U}_\mu \subset \mathscr{U}_\lambda$.

The existence of the exponential dichotomies for the two linear skew product semiflows π_λ and π_μ, where $\lambda < \mu$, has an equivalent formulation in terms of a trichotomy. In particular, we will say that π has an *exponential trichotomy* over an invariant set $\mathscr{K} \subset \mathscr{W}$, with *characteristics* λ_1, λ_2, λ_3, λ_4, and K, where $\lambda_1 < \lambda_2 \leqslant 0 \leqslant \lambda_3 < \lambda_4$ and $K \geqslant 1$, if there exist three projectors P, Q, and R defined over $\mathscr{K}$ such that the following properties hold:

(1) Each of the projectors P, Q, and R is invariant on $\mathscr{E}(\mathscr{K})$.

(2) For each $B \in \mathscr{K}$, the projections $P(B)$, $Q(B)$, and $R(B)$ commute and one has

$$I = P(B) + Q(B) + R(B) \quad \text{and} \quad P(B)\, Q(B) = P(B)\, R(B) = Q(B)\, R(B) = 0.$$

(3) For each $w \in \mathscr{R}(P(B))$ there is a negative continuation

$$\phi^{w, B}(t) = (\Phi(B, t)\, w, B_t), \qquad \text{for} \quad t \leqslant 0,$$

such that $\|e^{-\lambda_4 t}\Phi(B, t)\, w\| \to 0$, as $t \to -\infty$. (We do not require that this negative continuation be unique, and we let $(\Phi(B, t)\, w, B_t)$, denote any negative continuation that satisfies $\|e^{-\lambda_4 t}\Phi(B, t)\, w\| \to 0$, as $t \to -\infty$.)

(4) For each $w \in \mathscr{R}(R(B))$ there is a negative continuation

$$\phi^{w, B}(t) = (\Phi(B, t)\, w, B_t), \qquad \text{for} \quad t \leqslant 0,$$

such that $\|e^{-\lambda_2 t}\Phi(B, t)\, w\| \to 0$, as $t \to -\infty$. (We do not require that this negative continuation be unique, and we let $(\Phi(B, t)\, w, B_t)$, denote any negative continuation that satisfies $\|e^{-\lambda_2 t}\Phi(B, t)\, w\| \to 0$, as $t \to -\infty$.)

(5) The following four inequalities are valid for all $w \in V$:

$$\|\Phi(B, t)\, Q(B)\, w\| \leqslant K\, \|w\|\, e^{\lambda_1 t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.9}$$

$$\|\Phi(B, t)\, P(B)\, w\| \leqslant K\, \|w\|\, e^{\lambda_4 t}, \qquad \text{for} \quad t \leqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.10}$$

$$\|\Phi(B, t)\, R(B)\, w\| \leqslant K\, \|w\|\, e^{\lambda_3 t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.11}$$

and

$$\|\Phi(B, t)\, R(B)\, w\| \leqslant K\, \|w\|\, e^{\lambda_2 t}, \qquad \text{for} \quad t \leqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.12}$$

where (3.10) and (3.12) are valid for any negative continuation, as described above.

Since inequalities (3.9)–(3.12) hold at $t = 0$, one has

$$\|P(B)\, w\|, \|Q(B)\, w\|, \|R(B)\, w\| \leqslant K\, \|w\|, \qquad \text{for all} \quad B \in \mathscr{K} \quad \text{and} \quad w \in W. \tag{3.13}$$

An illustration of an exponential trichotomy is given by the linear problem

$$\partial_t u = Lu, \qquad \text{where} \quad u \in W$$

on a Banach space W. We assume here that $L = -A + B$, where A satisfies the Standing Hypothesis A, A has compact resolvent, and $B \in \mathscr{L}(V^{2\beta}, W)$. In this case, L has compact resolvent and e^{Lt} is an analytic semigroup. This linear equation has an exponential trichotomy with characteristics $\lambda_1 < \lambda_2 \leqslant 0 \leqslant \lambda_3 < \lambda_4$ and $K \geqslant 1$ if and only if the eigenvalues λ of L split into 3 bands in the complex plane $\mathbb{C}$:

$$\operatorname{Re} \lambda \leqslant \lambda_1, \qquad \text{or} \quad \lambda_2 \leqslant \operatorname{Re} \lambda \leqslant \lambda_3, \qquad \text{or} \quad \lambda_4 \leqslant \operatorname{Re} \lambda.$$

The range $\mathscr{R}(R)$ of the projection R is the algebraic sum of the generalized eigenspaces corresponding to eigenvalues λ with $\lambda_2 \leqslant \operatorname{Re} \lambda \leqslant \lambda_3$, and similar characterizations apply to $\mathscr{R}(Q)$ and $\mathscr{R}(P)$. Thus $\mathscr{R}(R)$ would include any eigenvectors with eigenvalues λ, with $\operatorname{Re} \lambda = 0$. The characteristic K, which is dependent on the norm on W, has the property that $K \to \infty$, as the angle between any two of the spaces $\mathscr{R}(Q)$, $\mathscr{R}(R)$, or $\mathscr{R}(P)$ goes to 0. The following property is easily verified.

LEMMA 3.3. *The linear skew product semiflow* π *has an exponential trichotomy over an invariant set* $\mathscr{K} \subset \mathscr{W}$ *with characteristics* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, *and* K *if and only if both of the following two properties hold*:

(1) *For any* λ *with* $\lambda_1 < \lambda < \lambda_2$, *the semiflow* π_λ *has an exponential dichotomy with characteristics* (K, α_λ) *and associated projections* (P_λ, Q_λ), *where*

$$2\alpha_\lambda = \min(\lambda - \lambda_1, \lambda_2 - \lambda), \qquad P_\lambda = P + R, \qquad \textit{and} \qquad Q_\lambda = Q.$$

(2) *For any* μ *with* $\lambda_3 < \mu < \lambda_4$, *the semiflow* π_μ *has an exponential dichotomy with characteristics* (K, α_μ) *and associated projections* (P_μ, Q_μ), *where*

$$2\alpha_\mu = \min(\mu - \lambda_3, \lambda_4 - \mu), \qquad P_\mu = P, \qquad \textit{and} \qquad Q_\mu = R + Q.$$

As a result, we see that Lemma 3.2 is applicable to each of the exponential dichotomies, for π_λ and for π_μ. Consequently the negative continuations $(\Phi(B, t)\, u, B_t)$, for $t \leqslant 0$, which are defined when $u \in \mathscr{R}(P(B)) \cup \mathscr{R}(R(B))$, are uniquely determined. Furthermore, there are unique negative continuations, denoted by $(\Phi(B, t)\, u, B_t)$, similarly defined for $u \in \mathscr{R}(P(B)) + \mathscr{R}(R(B))$, for $t \leqslant 0$. The argument leading up to inequality (3.8) applies in the case of the exponential dichotomies for π_λ and π_μ. In this case, inequalities (3.10) and (3.12) imply that

$$\|\Phi(B, t)\, P(B)\, v\| \geqslant K^{-1}\, \|P(B)\, v\|\, e^{\lambda_4 t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}, \tag{3.14}$$

and

$$\|\Phi(B, t)\, R(B)\, v\| \geqslant K^{-1}\, \|R(B)\, v\|\, e^{\lambda_2 t}, \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad B \in \mathscr{K}. \tag{3.15}$$

Likewise, one has

$$\|\Phi(B, t)\, R(B)\, v\| \geqslant K^{-1}\, \|R(B)\, v\|\, e^{\lambda_3 t}, \qquad \text{for} \quad t \leqslant 0 \quad \text{and} \quad B \in \mathscr{K}. \tag{3.16}$$

In the infinite dimensional setting, one is unable to convert either inequality (3.4), or (3.9), into a statement about the behavior of solutions in the stable bundle $\mathscr{S}$, or $\mathscr{S}_\lambda$, for time $t \leqslant 0$. The reason is that, in general, the solution $\Phi(B, t)\, Q(B)\, v$ need not exist for $t < 0$. However, a partial extension is possible in the following case: Assume that there is a $w \in V$

and a $\tau>0$, such that $\Phi(B_{-\tau},\tau)\,w=v$. Now Eqs. (2.22) and (3.3) and inequalities (3.9) and (3.13) imply that, for $t\geqslant 0$, one has

$$\|Q(B_t)\,\Phi(B,t)\,v\|=\|\Phi(B_{-\tau},\tau+t)\,Q(B_{-\tau})\,w\|\leqslant K\,\|w\|\,e^{\lambda_1(\tau+t)},$$

which goes to 0, as $t\to\infty$, since $\lambda_1<0$. Furthermore, if $w\in\mathscr{R}(Q(B_{-\tau}))$, then one has $v\in\mathscr{R}(Q(B))$, by (3.6). We now adopt the convention of defining $\Phi(B,t)\,v$, for $-\tau\leqslant t\leqslant 0$, by the formula

$$\Phi(B,t)\,v\overset{\text{def}}{=}\Phi(B_{-\tau},\tau+t)\,w,\qquad\text{for}\quad -\tau\leqslant t\leqslant 0.\tag{3.17}$$

In this case, the cocycle identity (2.22) admits the extension

$$\Phi(B,s+t)\,v=\Phi(B_t,s)\,\Phi(B,t)\,v,\tag{3.18}$$

for $-\tau\leqslant s,t<\infty$ with $-\tau\leqslant s+t$. Consequently, the argument leading to inequalities (3.14) and (3.15) now extends to give us

$$\|\Phi(B,t)\,Q(B)\,v\|\geqslant K^{-1}\,\|Q(B)\,v\|\,e^{\lambda_1 t},\qquad\text{for}\quad -\tau\leqslant t\leqslant 0.\tag{3.19}$$

It should be emphasized that Eqs. (3.17) and (3.18) depend on the point w. There may be several such points w which can be used, since the negative continuations are not assumed to be unique. However, in every case, inequality (3.19) is valid.

In our definition of an exponential trichotomy, we allow for the case where $P\equiv 0$. In this case, the inequalities (3.10) and (3.14) are deleted, and only the characteristics K, λ_1, and λ_2 play any significant role in our theory. An exponential trichotomy with the property that $P\equiv 0$ is said to be a *stable exponential trichotomy*. Such a trichotomy typically arises in the theory of inertial manifolds, see Foias, Sell, and Temam (1988), Mallet-Paret and Sell (1988), Foias, Nicolaenko, Sell, and Temam (1988), Foias, Sell, and Titi (1989), and Mallet-Paret, Sell, and Shao (1993), for example. As noted in the last lemma, a stable exponential *trichotomy* is equivalent to a single exponential *dichotomy*, with $\lambda<0$.

4. PRECISE STATEMENTS OF THEOREMS

Let us return now to the nonlinear evolutionary Eq. (1.1) and consider the flow generated by the mild solutions of Eq. (1.1) in the vicinity of a smooth, compact, invariant, finite dimensional manifold M of class C^2 in $V^{2\beta}$. Let $k=\dim M$ denote the dimension of M. As before, we assume that A satisfies the Standing Hypothesis A, and that the nonlinearity $F=F(u)$

is in C^1_{Lip}, see (1.3). Recall that one then has $M \subset \mathscr{D}(A)$, and that (2.19) is valid for each $u_0 \in M$. Also we require that the perturbation term G be in C^1_{Lip}.

4.1. *The Tangent Bundle Flow.* Some important dynamical properties of compact, invariant manifolds for Eq. (1.1) are presented in the following theorem. In particular, we examine here the induced linear skew product semiflow on the tangent bundle TM. The proof of the following result can be found in Sell and You (2001). (Also see Lemma 6.5.)

THEOREM 4.1. *Let the Standing Hypothesis A be satisfied for Eq.* (1.1), *and assume that* $F \in C^1_{\mathrm{Lip}}$. *Let M be a compact, connected, finite dimensional manifold of class* C^1 *in* $V^{2\beta}$, *and assume that M is an invariant set for the semiflow generated by the mild solutions of Eq.* (1.1). *Let* $\pi(w, u; t) = (\Phi(u, t)\, w, S(t)\, u)$ *denote the induced linear skew product semiflow on* $V^{2\beta} \times M$. *Then the following properties are valid*:

(1) *The tangent bundle TM is an invariant set for* π, *i.e.*, $\pi(TM; t) = TM$, *for all* $t \geqslant 0$.

(2) *There is a* $K \geqslant 1$ *and an* $a \geqslant 0$ *such that for any* $(w, u) \in TM$, *the globally defined solution* $\Phi(u, t)\, w$ *satisfies*

$$K^{-1} e^{-a\,|t|}\, \|A^\beta w\| \leqslant \|A^\beta \Phi(u, t)\, w\| \leqslant K e^{a\,|t|}\, \|A^\beta w\|, \qquad \textit{for} \quad t \in \mathbb{R}.$$

(3) *For each* $T > 0$, *the mapping* $u \to S(t)\, u$ *is uniformly, locally Lipschitz continuous in* $u \in M$, *on the intervals* $0 \leqslant t \leqslant T$ *and* $-T \leqslant t \leqslant 0$. *In particular, there exist a* $\rho = \rho(T) > 0$ *and a* $K_0 = K_0(T) > 0$ *such that*

$$\|A^\beta(S(t)\, u_1 - S(t)\, u_2)\| \leqslant K_0\, \|A^\beta(u_1 - u_2)\|,$$

$$\textit{for} \quad t \in [0, T], \quad \textit{or} \;\; t \in [-T, 0],$$

provided that $\|A^\beta(S(t)\, u_1 - S(t)\, u_2)\| \leqslant \rho$, *for all* $t \in [0, T]$, *or* $t \in [-T, 0]$.

4.2. *Basic Perturbation.* In addition to the given Eq. (1.1), we will also study the behavior of solutions of the perturbed Eq. (1.2) in the vicinity of M, where F and G are in C^1_{Lip}, see (1.3). Let $\Omega \overset{\mathrm{def}}{=} N(M, \sigma_2)$ denote the neighborhood (in $V^{2\beta}$) of M of radius $\sigma_2 > 0$. For $0 < \rho \leqslant \operatorname{diam}_{V^{2\beta}}(M)$ and $0 \leqslant \sigma \leqslant \sigma_2$, we define $b^F = b^F(\rho, \sigma)$ by

$$b^F_1(\rho, \sigma) \overset{\mathrm{def}}{=} \sup_{y_1,\, y_2 \in N(M, \sigma)} \{\|DF(y_1) - DF(y_2)\|_{\mathscr{L}} : \|A^\beta(y_1 - y_2)\| \leqslant \rho\}, \tag{4.1}$$

where $N(M, 0) = M$ and $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$. Since DF is bounded on Ω, it follows that $b_1^F(\rho, \sigma)$ is finite-valued, and since DF is continuous and M is compact, one has $b_1^F(\rho, \sigma) \to 0$, as $(\rho, \sigma) \to 0$.

The Class Σ. In the sequel we will use the class Σ consisting of all positive, real-valued functions $\beta_1 = \beta_1(r) = \beta_1(r_1, r_2)$, defined for $r = (r_1, r_2)$ with $0 < r_i < r_{i0}$, where $r_{i0} = r_{i0}(\beta_1) > 0$, for $i = 1, 2$, and satisfying $\beta_1(r) \to 0$, as $r \to 0$. For example, if some real-valued function $\xi(\varepsilon)$ is of order $o(\varepsilon)$, as $\varepsilon \to 0$, then one can write this in the form $|\xi(\varepsilon)| = \varepsilon\beta_1(\varepsilon)$, where $\beta_1 \in \Sigma$. The function $b_1^F(\rho, \sigma)$ is an element of Σ. Other examples which arise below are $\beta_1(\varepsilon, \delta)$, $\beta_2(\varepsilon, \delta)$, and $b_2^F(\rho)$. We will let $\beta_1, \beta_2, \ldots$, and $b_0, b_1^F, \ldots$ denote various elements of Σ.[3] We will use the terms $\beta_1, \beta_2, \ldots$ as local variables, which may be redefined from time-to-time. Global variables, which have a unique definition in this work, will be denoted by b_0, b_1^F, b_2^F, ... The superscript F will be used to denote elements of Σ which depend on F, but which are independent of the perturbation term G.

In the theory presented below we will be assuming that the perturbation term G satisfies

$$\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta, \tag{4.2}$$

where δ will be assumed to be small. In Section 5 we will show that such perturbations arise naturally when studying the approximation dynamics associated with the Bubnov–Galerkin approximations.

In order to study the mild solutions of Eqs. (1.1) and (1.2), we introduce the time-dependent change of variables $y = S(t)\, u_0 + w$ into Eq. (1.2), where $S(t)\, u_0$ is a given mild solution of Eq. (1.1). Observe that $y = y(t)$ is a mild solution of (1.2) if and only if $w = w(t)$ is a mild solution of

$$\begin{aligned} \partial_t w + Aw &= F(S(t)\, u_0 + w) - F(S(t)\, u_0) + G(S(t)\, u_0 + w) \\ &= B(t)\, w + H(S(t)\, u_0, w), \end{aligned} \tag{4.3}$$

where $B(t) = DF(S(t)\, u_0)$, $H(S(t)\, u_0, w) = E(S(t)\, u_0, w) + G(S(t)\, u_0 + w)$, and E satisfies (2.13) and (2.14), with $E(S(t)\, u_0, 0) = 0$, and the Fréchet derivative satisfies $DE(S(t)\, u_0, 0) = \frac{\partial}{\partial w} E|_{(S(t)\, u_0,\, 0)} = 0$. Furthermore, E has the following property: For every $T > 0$ there is a function $\gamma = \gamma(\sigma) \in \Sigma$ such

[3] Note that the subscript in β_i will distinguish an element $\beta_i \in \Sigma$ from the special parameter β used in (2.10) and the A^β-norm on $V^{2\beta} = \mathscr{D}(A^\beta)$.

that the following two inequalities are valid for all $u_0 \in M$ and all t with $0 \leqslant t \leqslant 2T$,

$$\|E(S(t)\,u_0, w_1) - E(S(t)\,u_0, w_2)\| \leqslant \gamma(\sigma)\ \|A^\beta(w_1 - w_2)\|, \tag{4.4}$$

whenever $\|A^\beta w_1\|, \|A^\beta w_2\| \leqslant \sigma$, and

$$\|E(S(t)\,u_0, w)\| \leqslant \gamma(\sigma)\ \|A^\beta w\| \leqslant \sigma\,\gamma(\sigma), \qquad \text{for} \quad \|A^\beta w\| \leqslant \sigma. \tag{4.5}$$

For each $u_0 \in M$, we let $\Phi(u_0, t)$ denote the fundamental solution operator of the linear system

$$\partial_t w + Aw = DF(S(t)\,u_0)\,w \tag{4.6}$$

that satisfies $\Phi(u_0, 0) = I$, where I is the identity operator on $V^{2\beta}$. As a result of this, it follows that

$$\pi(v, u, t) = (\Phi(u, t)\,v, S(t)\,u)$$

is a linear skew product semiflow on $V^{2\beta} \times M$. When π has an exponential trichotomy, there exist three invariant projectors $\{P, Q, R\}$, which we will write as $P^u = P$, $P^o = R$, and $P^s = Q$. The invariance property (3.3) now assumes the form

$$P^i(S(t)\,u)\,\Phi(u, t) = \Phi(u, t)\,P^i(u), \qquad \text{for} \quad i = s, o, u \quad \text{and} \quad t \geqslant 0. \tag{4.7}$$

This linear skew product semiflow is closely related to the version introduced in Subsection 2.3. Indeed, since M is invariant under the semiflow generated by the mild solutions of Eq. (1.1), for each $u \in M$, there is a globally defined solution $S(t)\,u$ with $S(t)\,u \in M$, for all $t \in \mathbb{R}$. We now define a mapping $J\colon M \to \mathscr{W} = \mathscr{M}^\infty(\mathbb{R}; \mathscr{L})$, where $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$, by $J(u)(t) = DF(S(t)\,u)$, for $(u, t) \in M \times \mathbb{R}$. Note that J is a continuous mapping from M into $\mathscr{W}$, and it commutes with the flow in the sense that $J(u)_\tau = J(S(\tau)\,u)$, for all $\tau, t \in \mathbb{R}$, i.e.,

$$J(u)_\tau\,(t) = J(u)(\tau + t) = DF(S(\tau + t)\,u) = DF(S(t)\,S(\tau)\,u) = J(S(\tau)\,u)(t).$$

Thus J maps the compact, invariant manifold M in $V^{2\beta}$ onto a compact, invariant set $J(M)$ in $\mathscr{W}$.

We note that if $w = w(t)$ is a mild solution of Eq. (4.3) with $w(0) = w_0 = y_0 - u_0$ and $u_0 \in M$, then as noted in Subsection 2.5, w satisfies two Variation of Constants formula:

$$w(t) = e^{-At} w_0 + \int_0^t e^{-A(t-s)}[B(s)\,w(s) + H(S(s)\,u_0, w(s))]\,ds \tag{4.8}$$

and

$$w(t) = \Phi(u_0, t)\, w_0 + \int_0^t \Phi(S(s)\, u_0, t-s)\, H(S(s)\, u_0, w(s))\, ds. \tag{4.9}$$

4.3. *Statement of Theorems.* We say that the compact, invariant manifold M for Eq. (1.1) is *normally hyperbolic* if the linear skew product semiflow π has an exponential trichotomy over M with characteristics $K \geqslant 1$ and $\lambda_1 < \lambda_2 \leqslant \lambda_3 < \lambda_4$, where $\lambda_2 \leqslant 0 \leqslant \lambda_3$, and associated projectors $\{P, Q, R\}$ such that the neutral space $\mathscr{R}(R(v))$ satisfies

$$\mathscr{R}(R(v)) = T_v M, \qquad \text{for all} \quad v \in M. \tag{4.10}$$

We allow here for the projection P to satisfy $P \equiv 0$. In this case, the exponential trichotomy is stable, and we say that the manifold M is normally hyperbolic *and* stable. In the case of a manifold that is normally hyperbolic and stable, only the characteristics K, λ_1, and λ_2 play any role in the perturbation theory described below.

The exponential trichotomy over M is said to be of *Lipschitz class* if the projections $P(v)$, $Q(v)$, and $R(v)$ are locally Lipschitz continuous functions on M. Since we assume M to be of class C^2, the projections $R(v)$, $P^o(v)$ and $Q^o(v) = I - P^o(v) = P^s(v) + P^u(v)$ are (Fréchet) differentiable mappings with respect to $v \in M$. We will denote the derivative of P^o at $v \in M$ by $DP^o(v)$. Since one has $P^o(v)\, P^o(v) = P^o(v)$, it follows from a simple calculation that $P^o(v)\, DP^o(v)\, P^o(v) = 0$, for all $v \in M$.

We are now prepared to describe the main results of this section. Thus we assume that M is a compact, connected C^2-manifold for the unperturbed Eq. (1.1), and we require that M be normally hyperbolic, where the associated exponential trichotomy is of Lipschitz class. We then argue that if the perturbation term G is in C^1_{Lip} and inequality (4.2) holds, where $\delta > 0$ is sufficently small, then the perturbed Eq. (1.2) has a normally hyperbolic invariant manifold M^G. Furthermore, M^G is homeomorphic to M, and the homeomorphism $h\colon M \to M^G$ is close to the identity mapping. The second result is a Shadow Theorem which compares the nonlinear dynamics on the two manifolds M and M^G.

MAIN THEOREM. *Let the Standing Hypothesis A be satisfied. Let $F \in C^1_{\mathrm{Lip}}$ and let M be a compact, connected, finite dimensional, invariant C^2-manifold in $V^{2\beta}$ for the unperturbed Eq.* (1.1). *Assume that M is normally hyperbolic and that the associated exponential trichotomy is of Lipschitz class.*

Then for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, then there is a Lipschitz homeomorphism $h\colon M \to V^{2\beta}$ with the following properties:

(1) *The manifold* $M^G \stackrel{\text{def}}{=} h(M)$ *is an invariant manifold for the perturbed Eq.* (1.2).

(2) *Both manifolds* M *and* M^G *lie in* $\mathcal{D}(A) = V^2$. *Furthermore,* M^G *is of class* C^1, *and it is normally hyperbolic for Eq.* (1.2).

(3) *One has* $\|A^\beta(h(v) - v)\| \leqslant 2\varepsilon$, *for all* $v \in M$.

Moreover, for each $u_0 \in M$, $y_0 \in M^G$, *and* $0 \leqslant r < 1$, *the mild solutions* $S(t)\, u_0$ *and* $y(t, y_0)$ *are strong solutions with* $S(\cdot)\, u_0,\ y(\cdot, y_0) \in C^{0, 1-r}_{\text{loc}}(\mathbb{R}; V^{2r}) \cap C(\mathbb{R}; \mathcal{D}(A))$.

For the next result, we introduce the concept of a shadow semiflow $S_1^G(t)$ on the unperturbed manifold M. The terminology arises because this semiflow acts as a "shadow" to the nonlinear dynamics on the perturbed manifold M^G. In particular, we let $S_1(t)$ and $S_2(t) = S_2^G(t)$ denote the semiflows in $V^{2\beta}$ generated by the maximally defined mild solutions of Eqs. (1.1) and (1.2), respectively. Let $h\colon M \to V^{2\beta}$ be a continuous mapping, where $M^G \stackrel{\text{def}}{=} h(M)$ is an invariant set for the perturbed equation (1.2). We say that a continuous mapping $S_1^G(t)\, u_0\colon M \times [0, \infty) \to M$ is a *shadow semifow* for the nonlinear dynamics $S_2^G(t)$ on M^G, if it satisfies

$$S_2^G(t)\, h(u_0) = h(S_1^G(t)\, u_0), \qquad \text{for all} \quad (u_0, t) \in M \times [0, \infty). \tag{4.11}$$

As noted in the Main Theorem, we will show the existence of such a mapping $h = h^G\colon M \to V^{2\beta}$, for each G satisfying inequality (4.2), with δ sufficiently small. In this case, we say that the shadow semiflow $S_1^G(t)$ is *G-continuous* if it is continuous in the $\mathcal{T}^1_A$ topology generated by $\|G\|_{\{A;\, C^1(\Omega)\}}$. This means that if G_n and u_n are convergent sequences in $C_{\text{Lip}}(V^{2\beta}, W)$ and $V^{2\beta}$, respectively, with $\|G_n - G\|_{\{A;\, C^1(\Omega)\}} \to 0$ and $\|A^\beta(u_n - u)\| \to 0$, as $n \to \infty$, then

$$\|A^\beta(S_1^{G_n}(t)\, u_n - S_1^G(t)\, u)\| \to 0, \qquad \text{as} \quad n \to \infty,$$

uniformly for t in compact sets in $[0, \infty)$. We will prove the following result:

Shadow Theorem. *Let the hypotheses of the Main Theorem be satisfied, and let* $h = h^G\colon M \to V^{2\beta}$ *and* $M^G = h(M)$ *satisfy the conclusions. Then there exists a G-continuous shadow semiflow* $S_1^G(t)$ *on* M, *for every* G *satisfying inequality* (4.2), *where* δ *is given by the Main Theorem. Furthermore, when* $G \equiv 0$, *then* $S_1^0(t) = S_1(t)$ *on* M.

5. APPLICATIONS TO THE NAVIER–STOKES EQUATIONS

In this section we will examine two applications of the dynamical theories presented in the preceeding four sections. While both of these applications are taken from the theory of the Navier–Stokes equations, the methodology developed here is widely applicable. We first seek to develop a rigorous foundation for the dynamical features seen in the classical Couette–Taylor flow. The second application arises in the numerical analysis of fluid flows. What we want to show here is that numerical methods based on the Bubnov–Galerkin approximations can lead to good information about the longtime dynamics of the underlying problem.

The Couette–Taylor flow refers to two patterns which arise in one of the classical fluid dynamics experiments, see Taylor (1923); Batchelor (1996); and the references in Chossat and Iooss (1994). With the modern tool of laser optics for making measurements of observables in the fluid flows and the related spectral analysis of the resulting data, one now has a good picture of some of the longtime dynamical features which occur before the onset of turbulence in real systems, see Gollub and Swinney (1975), and Andereck, Liu, and Swinney (1986). While good information is coming in at the experimental level, what can be said about the model itself? The model is described in terms of the Navier–Stokes equations. We begin with a brief overview of these equations.

5.1. *The Navier–Stokes Equations.* The Navier–Stokes equations for an incompressible, viscous fluid motion assume the form

$$\begin{aligned} \partial_t u - \nu\, \Delta u + (u \cdot \nabla)\, u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \tag{5.1}$$

on an open, bounded domain Ω_0 in $\mathbb{R}^d$ of class C^2, where $d=2$ or 3. In (5.1) the term $u=u(x,t)$ is an d-dimensional vector field on Ω_0, and it represents the velocity of the fluid at the point $x \in \Omega_0$ and time t. The pressure term $p=p(x,t)$ is a scalar field on Ω_0. The function $f=f(x,t)$ is a d-dimensional vector field on Ω_0. It represents a force (like gravity or buoyancy) acting on the fluid. The viscosity term ν is a positive constant. The first equation in (5.1) is sometimes called the *momentum equation*, and the second equation $\nabla \cdot u = 0$ is referred to as the *conservation equation*. The nonlinear term $(u \cdot \nabla) u$ in (5.1) is referred to as the *inertial term*. In the sequel, we examine the solutions of (5.1), where u also satisfies the initial value problem

$$u(x,0) = u_0(x). \tag{5.2}$$

The ordered pair (u_0, f) is referred to as the *data* of the Navier–Stokes equations. The term u_0 is the *initial condition* and f is the *forcing function*. See Ladyzhenskaya (1963, 1972); Joseph (1976); Temam (1977, 1983); Constantin and Foias (1988); or Sell and You (2001); for more details.

In order for the problem (5.1) to be well-posed we require that appropriate boundary conditions be satisfied. For many physical problems, the boundary conditions describe a force acting on the boundary, such us fluid entering or leaving a cavity, or a non-slip condition caused by the motion of a segment of the boundary of Ω_0, for example, the Dirichlet boundary conditions

$$u(x, t) = 0, \qquad x \in \partial\Omega_0 \qquad \text{and} \qquad t \geqslant 0, \tag{5.3}$$

may hold. Other than the obvious requirement that the pressure $p = p(x, t)$ lie in a space where the gradient ∇p makes sense, there are no other side conditions imposed upon p. In the case that Ω_0 is a rectangle (in $\mathbb{R}^2$) or a parallelepiped (in $\mathbb{R}^3$), one can also study (5.1) where the periodic boundary conditions hold, see Temam (1983).

Next we turn to the linear problem, where the inertial term $(u \cdot \nabla)u$ is set equal to 0, to obtain the Stokes equations:

$$\begin{aligned} \partial_t u - \nu\, \Delta u + \nabla p &= f \\ \nabla \cdot u &= 0. \end{aligned} \tag{5.4}$$

The Stokes operator A arises when one projects (5.4) into the real Hilbert space

$$H = \mathrm{Cl}_{L^2(\Omega_0)}\{u \in C_0^\infty(\Omega_0) : \nabla \cdot u = 0\},$$

or equivalently, $H = \{u \in L^2(\Omega_0, \mathbb{R}^d) : \nabla \cdot u = 0 \text{ in } \Omega_0 \text{ and } u \cdot n = 0 \text{ on } \partial\Omega_0\}$, for Dirichlet boundary conditions. For periodic boundary conditions, one uses a different choice for H, see Temam (1983).

Let $\mathbb{P}$ denote the Helmholtz (Leray) projection, i.e., the orthogonal projection of $L^2(\Omega_0)$ onto H. Recall that $H^\perp$, the orthogonal complement of H, is

$$H^\perp = \mathrm{Cl}_{L^2(\Omega_0)}\{\nabla p : p \in C^1(\bar{\Omega}_0, \mathbb{R})\}.$$

We assume that the forcing function $f = f(t)$ satisfies $f \in L^\infty(0, \infty; H)$. In this case one has $\mathbb{P}f = f$. By applying $\mathbb{P}$ to (5.4) one obtains

$$\partial_t u + \nu A u = f, \tag{5.5}$$

where $Au = -\mathbb{P}\,\Delta u$ is the Stokes operator, and Δ satisfies the appropriate boundary conditions. It is important to note that the Stokes operator A satisfies the Standing Hypothesis B on the space H. We let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ denote the eigenvalues of A, and $\{e_1, e_2, ...\}$ is an orthonormal basis in H of eigenvectors of A that satisfy $Ae_i = \lambda_i e_i$, for $i \geqslant 1$.

The three spaces $V = V^1$, $H = V^0$, and V^{-1} have special significance in the theory of the Navier–Stokes equations. First the space V satisfies $\mathscr{D}(A^{1/2}) = V$, and the imbeddings $V \hookrightarrow H \hookrightarrow V^{-1}$ are compact. From Eq. (2.4), one has

$$\langle u, v\rangle_V = \langle A^{1/2}u, A^{1/2}v\rangle = \langle A^{1/2}u, A^{1/2}v\rangle_{L^2}, \qquad \text{for} \quad u, v \in V,$$

and

$$\langle u, v\rangle_{V^{-1}} = \langle A^{-1/2}u, A^{-1/2}v\rangle = \langle A^{-1/2}u, A^{-1/2}v\rangle_{L^2}, \qquad \text{for} \quad u, v \in V^{-1}.$$

Furthermore the imbeddings $V \hookrightarrow H \hookrightarrow V^{-1}$ gives rise to a duality and a bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle = \langle\!\langle \cdot, \cdot \rangle\!\rangle_{(V^{-1}, V)}$, where

$$\langle\!\langle u, v\rangle\!\rangle \overset{\text{def}}{=} \langle A^{-1/2}u, A^{1/2}v\rangle = \langle A^{-1/2}u, A^{1/2}v\rangle_{L^2}, \quad \text{for } u \in V^{-1} \text{ and } v \in V.$$

The Helmholtz projection $\mathbb{P}$ can be applied to the Navier–Stokes equations (5.1) as well. Since $\nabla \cdot u = 0$ one has $\mathbb{P}\,\partial_t u = \partial_t u$. As a result, one obtains

$$\partial_t u + vAu + B(u, u) = f, \tag{5.6}$$

which is referred to as the *Navier–Stokes* (*evolutionary*) *equation*, where $B(u, v)$ is the bilinear form $B(u, v) \overset{\text{def}}{=} \mathbb{P}(u \cdot \nabla)\, v$.

By using the Sobolev imbedding theorems to study the trilinear form $b(u, v, w)$, it follows that there are positive constants C_0, C_1, and C_2, such that the following inequalities hold when $d = 2$, or 3,

$$\begin{aligned} \|B(u, v)\|, \|B(v, u)\| &\leqslant C_0\, \|A^{5/8}u\|\, \|A^{5/8}v\|, && \text{for} \quad u, v \in V^{5/4},\\ \|A^{-1/4}B(u, v)\|, \|A^{-1/4}B(v, u)\| &\leqslant C_1\, \|A^{1/2}u\|\, \|A^{1/2}v\|, && \text{for} \quad u, v \in V^1,\\ \|A^{1/2}B(u, v)\|, \|A^{1/2}B(v, u)\| &\leqslant C_2\, \|Au\|\, \|Av\|, && \text{for} \quad u, v \in V^2, \end{aligned} \tag{5.7}$$

see Sell and You (2001). (Also see Constantin and Foias (1988).) As a result, we see that the bilinear term $B = B(u, v)$ satisfies

$$B\colon V^2 \times V^2 \to V^1, \qquad B\colon V^{5/4} \times V^{5/4} \to H, \qquad \text{and} \qquad B\colon V^1 \times V^1 \to V^{-1/2}.$$

Furthermore, $B = B(u, u)$ satisfies

$$B \in C^1_{\mathrm{Lip}}(V^2, V^1) \cap C^1_{\mathrm{Lip}}(V^{5/4}, H) \cap C^1_{\mathrm{Lip}}(V^1, V^{-1/2}). \tag{5.8}$$

If the forcing term f is sufficiently smooth, e.g., $f \in V^1$, then the evolutionary equations generated by the nonlinear, or the linearized, Navier–Stokes equations are local evolutionary equations on the spaces, $V^{-1/2}$, H, and V^1.

If the forcing term f is sufficiently smooth, e.g., $f \in V^1$, then the evolutionary Eq. (5.6) for the Navier–Stokes equations generates a semiflow on $V^{2\beta}$, where $V^{2\beta}$ is any space: $V^{-1/2}$, H, or V^1, see Subsection 2.2 and Eq. (2.15).

5.2. *The Couette–Taylor Flow.* We now consider a fluid flow between two concentric right circular cylinders, and the two cylinders are allowed to rotate with independent, but fixed, angular velocities, see Andereck, Liu, and Swinney (1986) and Chossat and Iooss (1994). While the two angular velocities lead to a two-parameter bifurcation problem, our purpose here is adequately served by considering the outer cylinder to be held fixed while the inner cylinder is made to rotate with a constant angular velocity ω, see Taylor (1923), Gollub and Swinney (1975), and Golubitsky and Stewart (1986). We will study the longtime dynamics of this problem for various choices of the parameter ω.

The fluid motion in the region Ω_0 between the two concentric cylinders is given by the Navier–Stokes equations

$$\begin{aligned} \partial_t u - \nu\, \Delta u + (u \cdot \nabla)\, u + \nabla p &= f_g \\ \nabla \cdot u &= 0. \end{aligned} \tag{5.9}$$

Both cartesian coordinates (u_1, u_2, u_3) and cylindrical coordinates (u_r, u_θ, u_z) will be used to describe a vector field u. We fix the region $\Omega_0 = An \times [0, L]$, so that An is the annular region $0 \leqslant \theta \leqslant 2\pi$, $\eta \leqslant r \leqslant 1$, where $0 < \eta < 1$, and $0 \leqslant z \leqslant L$.

For the Couette–Taylor flow, the external force f_g is a gravitational force, which we assume to satisfy $f_g = -g\vec{k}$, where g is the gravitational constant. It is convenient to replace the pressure p in (5.9) with $p + p_0$, where $p_0(x, y, z) = -gz$ is the hydrostatic pressure. In this case, one has $\nabla p_0 = f_g$, and these two terms can be cancelled. As a result, (5.9) reduces to

$$\begin{aligned} \partial_t u - \nu\, \Delta u + (u \cdot \nabla)\, u + \nabla p &= 0 \\ \nabla \cdot u &= 0. \end{aligned} \tag{5.10}$$

The boundary conditions for the problem (5.10) are as follows. On $\partial An \times [0, L]$, one has

$$\begin{aligned} \text{angular velocity} = u_\theta = \omega, &\qquad \text{at} \quad r = \eta \\ \text{radial velocity} = u_r = 0, &\qquad \text{at} \quad r = \eta \\ u_\theta = u_r = 0, &\qquad \text{at} \quad r = 1, \end{aligned} \tag{5.11}$$

and we assume that all terms are spatially periodic in z. We refer to Eq. (5.10), with the boundary conditions (5.11) as the *spatially periodic CT model.*

The first step in reducing this problem to the theory considered above is to “homogenize” the boundary conditions. This is accomplished by making a change of variables $u = v + w^c$ and $p = q + p^c$ in Eq. (5.10), where w^c and p^c will be chosen so that the following conditions hold: (1) $\nabla \cdot w^c = 0$ and $\partial_t w^c = 0$ in Ω_0, (2) w^c satisfies the boundary conditions (5.11), and (3) (w^c, p^c) is periodic in z. In addition, we require that $\partial_t p^c = 0$ and that (w^c, p^c) be a stationary solution of Eq. (5.10), i.e., one has

$$-\nu \, \Delta w^c + (w^c \cdot \nabla) \, w^c + \nabla p^c = 0. \tag{5.12}$$

While it can happen that Eq. (5.12) has many solutions satisfying the boundary conditions (5.11), there is one solution of special interest: the Couette solution, or the Couette flow. In particular, we define $w^c = w_\theta(r) \vec{\theta}$, where $w_\theta = C(r^{-1} - r)$, with $C = \omega\eta(1 - \eta^2)^{-1}$ and

$$p^c = \frac{1}{2} C^2 \left(-\frac{1}{r^2} - 4 \log r + r^2 \right) = C^2 \int \frac{1}{r} w_\theta^2 \, dr.$$

Note that (w^c, p^c) is a solution of (5.12), for every $\nu > 0$.

In terms of the new variable $v = u - w^c$, Eq. (5.10) can be rewritten as

$$\begin{aligned} \partial_t v - \nu \, \Delta v + (v \cdot \nabla) \, v + (v \cdot \nabla) \, w^c + (w^c \cdot \nabla) \, v + \nabla q = 0 \\ \nabla \cdot v = 0, \end{aligned} \tag{5.13}$$

where the vector field v satisfies the homogeneous Dirichlet boundary conditions $v(x, y, z, t) = 0$, for $(x, y) \in \partial An \times [0, L]$, and v is periodic in z. We now use the Helmholtz projection $\mathbb{P}$ on (5.13) to obtain

$$\partial_t v + \nu A v + B(v, v) + \omega [B(v, \hat{w}) + B(\hat{w}, v)] = 0, \tag{5.14}$$

where $\hat{w}$ is given by $\omega \hat{w} = w_\theta(r) \vec{\theta}$. Due to the special boundary conditions on (5.13), the range H of the Helmholtz projection differs slightly from the construction given in Subsection 5.1. In particular, H is now the closure in

$L^2(\Omega_0, \mathbb{R}^3)$ of the collection of all functions $u = u(x, y, z)$ in $C^\infty(\bar{\Omega}_0, \mathbb{R}^3)$ with the property that u is periodic in z and the set $\{(x, y, z) \in \bar{\Omega}_0 : u(x, y, z) = 0\}$ contains a neighborhood of $\partial An \times [0, L]$, see Chossat and Iooss (1994).

Since the Stokes operator A is a positive, self adjoint operator, with compact resolvent, we see that at $\omega = 0$, the stationary solution $v \equiv 0$ of Eq. (5.14) is stable and hyperbolic. It then follows from the Stable Manifold Theorem, that there is an $\omega_1 > 0$ such that the zero solution $v \equiv 0$ of Eq. (5.14) is stable and hyperbolic for $0 \leqslant \omega < \omega_1$. This stability and hyperbolicity will persist for $\omega > 0$ until an eigenvalue of the linear operator

$$L^c v = \nu A v + \omega[B(v, \hat{w}) + B(\hat{w}, v)]$$

crosses the imaginary axis in the complex plane. We will denote this value of ω as $\omega = \omega_1$. For $0 \leqslant \omega < \omega_1$, the experimental results and the rigorous analytical results are essentially in full agreement. The laminar flow pattern associated with the Couette flow is denoted by T_c^0 in Fig. 1.

Next we review some of the experimental results which appear in the literature, see Gollub and Swinney (1975); Lvov, Predtechensky, and Chernykh (1983); and Andereck, Liu, and Swinney (1986). However, before doing this, it is important to emphasize a basic difference between the CT model described above, and the real world situation one encounters in the experimental setup. In particular, the assumption of spatial periodicity in the z-variable is not a realistic assumption for the experimental setup. In the experiments, one encounters different boundary conditions on the top and bottom, and consequently, the CT model does not directly apply to this situation. The conventional wisdom, which is quoted in this case, is that if the aspect ratio $\frac{L}{1-\eta}$ is large enough, then the spatially periodic CT model should be a "reasonable" approximation to the real world phenomena seen in the experiments. It would be nice to have a theorem and proof which might substantiate a part of this conventional wisdom, but that remains an open problem. Nevertheless, we do accept the conventional wisdom at this point.

In the experimental results one sees the following scenarios: As ω crosses ω_1, a bifurcation of the Couette flow occurs, and for $\omega > \omega_1$, one observes a new pattern T_t^0, the Taylor vortex flow, see Kirchgässner and Sorger

$$\begin{pmatrix}\text{Couette}\\ \text{flow}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Taylor}\\ \text{vortex}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Wavy}\\ \text{vortex}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Modulated}\\ \text{vortex}\end{pmatrix} \longrightarrow ???$$

$$T_c^0 \longrightarrow T_t^0 \longrightarrow T^1 \longrightarrow T^2 \longrightarrow ???$$

FIG. 1. Bifurcations in the Couette–Taylor flow.

(1969) and Chossat and Iooss (1994). The latter flow is a new stationary solution $u = w^t$ for the original problem (5.10) and (5.11). Next for ω in the range $\omega_1 < \omega < \omega_2$, the Taylor vortex flow is stable and hyperbolic. In this case, we consider the linearization of the Navier–Stokes Eq. (5.13) along the Taylor flow w^t, where the linear operator L^c is now replaced by

$$L^t v = \nu A v + [B(v, w^t) + B(w^t, v)], \qquad \text{for} \quad \omega > \omega_1.$$

Since the Taylor solution w^t depends on the parameter ω, the linear operator L^t depends on ω, as well. As ω crosses the value ω_2, i.e., when an eigenvalue of L^t crosses into the unstable zone, another bifurcation occurs and a new pattern, the "wavy vortex flow" appears, see Fig. 1. In this case, the bifurcation appears to be a Hopf bifurcation, and for $\omega > \omega_2$, one obtains a time-periodic solution of the Navier–Stokes equations. The orbit of this time-periodic solution is a 1D manifold T^1, i.e., a circle. The T^1-pattern resulting from this periodic solution persists for $\omega_2 < \omega < \omega_3$. As ω crosses ω_3 a secondary bifurcation occurs resulting in another pattern, called the "modulated waves". What is generally believed is that the time-periodic orbit T^1 bifurcates into a 2D torus T^2, as ω crosses ω_3. For this reason, we will denote the modulated wave pattern by T^2. The pattern T^2 persists over a parameter range $\omega_3 < \omega < \omega_4$.

As ω crosses ω_4, it is not clear what occurs. It may be a Hopf–Landau bifurcation where the 2D torus T^2 bifurcates into a 3D torus T^3, see Chenciner and Iooss (1979), Sell (1979), and Haken (1981, 1983). Or it may be a more complicated behavior, including strange attractors, see Ruelle and Takens (1971) and Sell (1981). Or it may be something quite different. No one really knows. Each opinion is, at best, an educated guess, and each comes with its pros and cons.

What is "really known" about these scenarios for $0 < \omega < \omega_4$? On the assumption that the three bifurcations, (where ω crosses ω_1, ω_2, and ω_3) are as described, then the Main Theorem guarantees the persistence of the patterns depicted in Fig. 1, for ω in the ranges $\omega_1 < \omega < \omega_2$, $\omega_2 < \omega < \omega_3$, and $\omega_3 < \omega < \omega_4$. In order to apply the Main Theorem, one begins with the linearized equation along the manifold T^1 or T^2, and then one studies the perturbed problem which arises after making a small change in the parameter ω. The reason that the Main Theorem is applicable is that the terms in Eq. (5.14) depend continuously in ω in both the $\mathscr{T}^1_A$ and the $\mathscr{T}^1_{\text{bo}}$ topologies. (We assume here that the manifolds have the required smoothness, and that the exponential trichotomy is of Lipschitz class.) Consequently inequality (4.2) is valid in each of these ranges, for small values of δ, provided that one makes a very small change in the parameter ω. It should be noted that the basic Hopf bifurcation theory assures the normal hyperbolicity of the periodic orbit T^1 and the 2D torus T^2 immediately after the bifurcation, see Sacker (1964, 1969), Ruelle and Takens (1971), Marsden and McCracken (1976), and Chow and Hale (1982).

What then is "really known" about the bifurcations themselves? This has been a subject of serious study over the last 30 years, see Kirchgässner (1975) and Chossat and Iooss (1994), for example. Since one has an explicit formula for the Couette solution (w^c, p^c), it should be not surprising that much of the analytical work has focused on bifurcations in the vicinity of this solution. For example, by using (1) a smooth center manifold theorem and the associated normal forms and (2) the spatial symmetries enjoyed by the solutions of the Navier–Stokes equations in the region Ω_0, Chossat and Iooss (1994) have made an indepth analysis of such bifurcations. Some of this analysis also applies to the Taylor solution w^t. While the existence of the bifurcation values ω_1 and ω_2 can be argued successfully based on such analyses, the numerical computation of these values, in any specific case, is a challenge.

The analysis of the CT model for ω in the region $\omega > \omega_2$ is much more complicated. Even at the computational level, this offers great challenges. In principle, one can develop a center manifold for the study of bifurcations near a periodic orbit, and to some extent, for the study of the dynamics near a compact, invariant manifold, see Sell (1978) and Chow, Liu, and Yi (1999). For example, in the case of a periodic orbit, one can mimic the finite dimensional setting and construct a Poincaré mapping for a suitable normal cross section to the periodic orbit. However, whatever method one chooses, for applications to the CT model, one will need to use sophisticated numerical methods to locate the bifurcation values ω_3 and ω_4. The related numerical issues form our next concern.

5.3. *The Bubnov–Galerkin Approximations.* A common concern arising in the study of fluid flows is the connection between the model problem, which we will assume here as being governed by the Navier–Stokes equations, and the numerical study of these equations. While a complete study of this connection lies in the domain of approximation dynamics, see Sell (2001), we can give some insight into the basic theory by focusing on the Bubnov–Galerkin approximations.

For this purpose, let us return to Eq. (5.14) and assume that, for some $\omega > 0$, $M = M_\omega$ is a compact, invariant manifold of class C^2 in V^1. We note that Eq. (5.14) is of the form (1.1), where

$$F = F(u) = -[B(u, u) + \omega B(u, \hat{w}) + \omega B(\hat{w}, u)]. \tag{5.15}$$

Also F satisfies (5.8). Then $F(u)$ is Fréchet differentiable, and the derivative DF has the Lipschitz property. Consequently, if M is normally hyperbolic in V^1, and the associated trichotomy is of Lipschitz class, then the Main Theorem is applicable in the study of the strong solutions of (5.14). We will now examine the significance of this observation in the context of the Bubnov–Galerkin approximations of (5.14).

For each integer $n \geqslant 1$ we let $P = P_n$ denote the orthogonal projection of H onto Span$\{e_1, ..., e_n\}$, and set $Q = Q_n = I - P$. For each $u \in H$ we define p and q by $p = Pu$ and $q = Qu$. Notice that $p = p_n$ and $q = q_n$ depend on the n modes $\{e_1, ..., e_n\}$ used to define the spectral projections P and Q. When we apply P and Q to the evolutionary equation (5.14), we obtain the equivalent system

$$\begin{cases} \partial_t p + \nu Ap = PF(p+q), \\ \partial_t q + \nu Aq = QF(p+q), \end{cases} \tag{5.16}$$

where F is given by Eq. (5.15). Notice that the p-equation in (5.16) is an n-dimensional equation, while the q-equation is infinite dimensional. The nth order Bubnov–Galerkin approximation of (5.14) is given by the solutions of the nth order ordinary differential equation

$$\partial_t p + \nu Ap = PF(p), \tag{5.17}$$

which is obtained from (5.16) by setting $q = 0$ in the p-equation and ignoring the q-equation.

Since Eq. (5.17) is an ordinary differential equation on a finite dimensional space PH, and since Eq. (5.16) is an infinite dimensional system on H, it may appear initially that there is no hope to compare favorably the longtime dynamics of the solutions of these two equations. Nevertheless, a good comparison can be made by imbedding Eq. (5.17) into a suitable system on the full space H. This imbedding is accomplished by appending a q-equation to the p-equation in (5.17) in such a way that the longtime dynamics of the new (p, q)-system is *identical* to the longtime dynamics of the ordinary differential equation (5.16). While such an imbedding is not unique, the system

$$\begin{cases} \partial_t p + \nu Ap = PF(p+q), \\ \partial_t q + \nu Aq = 0, \end{cases} \tag{5.18}$$

suits our needs. Let $q(t) = e^{-\nu Aqt} q_0$ be any solution of the q-equation in (5.18), where $q_0 \in QH \cap V^1$. Since $\|A^{1/2}q(t)\|^2 \leqslant e^{-\lambda_{n+1}t} \|A^{1/2}q_0\|^2$, for $t \geqslant 0$, Eqs. (5.17) and (5.18) have the same *longtime* dynamics in the space V^1.

One can rewrite Eqs. (5.16) and (5.18) in the form (1.1) and (1.2), where $F = F(u)$ is given by Eq. (5.15), and $G(u) = -QF(u)$. Thus one has $PF(u) = F(u) + G(u)$. In this way we see that Eq. (5.18) is a perturbation of Eq. (5.16), or equivalently, Eq. (5.13). This leads us to the following question:

Is the Main Theorem applicable to the perturbation problem (1.1)–(1.2)? As we now show, the answer is yes, *provided that* the eigenvalue λ_{n+1} is sufficiently large. In order to verify this, let us begin with a compact, invariant manifold M of class C^2 in V^1 for the original Navier–Stokes Eqs. (5.14) and assume that M is normally hyperbolic and the associated exponential trichotomy is of Lipschitz class. For example, M may be the torus T^2 seen in the Couette–Taylor flow. Next let $U = N_R(0)$ be an open, neighborhood of the origin in V^1, of radius $R > 0$, where $M \subset U$ and $w^c \in U$.

As noted above, the nonlinear term $F = F(u)$ satisfies (5.8), i.e., $F \in C^1_{\mathrm{Lip}}$, with $V^{2\beta} = V^1$, $W = V^{-1/2}$, and $2\beta = \frac{3}{2}$. Consequently, inequality (5.7) implies that there exist $K_0 = K_0(\omega) > 0$ and $K_1 = K_1(\omega) > 0$ such that

$$\|A^{-1/4}F(u)\| \leqslant K_0 \quad \text{and} \quad \|A^{-1/4}DF(u)\|_{\mathscr{L}(V^1, H)} \leqslant K_1, \qquad \text{for all} \quad u \in U. \tag{5.19}$$

Since Q is an orthogonal projection on V^α, for every $\alpha \in \mathbb{R}$, it follows that the perturbation term $G(u) = -QF(u)$ satisfies (5.19), as well. This implies that $\|G\|_{C^1(U)} \leqslant K_0 + K_1$, which need not be small! However, since

$$A^\beta e^{-At} QF = (AQ)^\beta \, e^{(AQ)\,t} \, QF \quad \text{and} \quad A^\beta e^{-At} QDF = (AQ)^\beta \, e^{-(AQ)\,t} \, QDF,$$

inequality (2.9) implies that

$$\|G\|_{\{A;\, C^1(U)\}} \leqslant (1-\beta)^{-1} \, e^{-\beta} \, \lambda_{n+1}^{\beta-1} \, \|G\|_{C^1(U)}, \qquad \text{where} \quad \beta = 3/4.$$

We see that for λ_{n+1} sufficiently large, or equivalently, for n sufficiently large (say $n \geqslant N_\omega$), one has $\|G\|_{\{A;\, C^1(U)\}} \leqslant \delta$, where δ is given by the Main Theorem. As a result, we see that for any $n \geqslant N_\omega$, where $\lambda_n < \lambda_{n+1}$, the ordinary differential equation (5.17) has a compact, invariant, normally hyperbolic manifold M_n with $\dim M_n = \dim M$. Furthermore, the manifold M_n, as imbedded in the problem (5.18), is “close to” M.

Remarks. (1) Since the nonlinearity F satisfies (5.8), one can use this to generate a “boot-strap” argument to conclude that the solutions on the manifold M have greater regularity than stated in (2.19). In order to prove this, one needs to note that the exponential trichotomy on the space $V^{2\beta} = V^1$ induces an exponential trichotomy on V^2 with good characteristics, see Sell and You (2001), and Henry (1981). As a result one can conclude that $M \subset V^3 \subset H^3(\Omega_0)$, with a corresponding change in (2.19).

(2) The fact, that the pseudonorm $\|G\|_{\{A;\, C^1(U)\}}$ may be small, even when the pseudonorm $\|G\|_{C^1(U)}$ is large, plays an important role even in the study of some ordinary differential equations. For example, this issue

arises when comparing the longtime dynamics generated by two multigrid methods used for the calculation of approximate solutions of a partial differential equation. Under the appropriate setup, a normally hyperbolic, invariant manifold for a fine grid calculation can be well approximated by a nearby normally hyperbolic, invariant manifold for a coarse grid calculation. This theory is developed further in the forthcoming monograph Sell (2001).

6. PROOFS OF THEOREMS

The initial stage in the proofs of the theorems consist of (1) constructing a good local coordinate system in the vicinity of the manifold M; and deriving a few basic lemmas for (2) the unperturbed dynamics on M and (3) the perturbed dynamics near M. We will use the notation developed above. In particular, we let M denote the given compact, connected, finite dimensional, invariant manifold of class C^2 for Eq. (1.1). We assume that the manifold M is normally hyperbolic and the associated exponential trichotomy has the Lipschitz property. Let K, λ_1, λ_2, λ_3, and λ_4 denote the characteristics of the exponential trichotomy, and let $\{P^s, P^o, P^u\}$ denote the associated projectors on M. Let $\Omega = N(M, \sigma_2) \subset V^{2\beta}$ be given as in Subsection 4.2.

6.1. *Local Coordinates near M.* First we note that by using a larger value of the characteristic K, if necessary, we can assume that, for some $a_0 \geqslant \lambda_3$, one has

$$\|A^\beta \Phi(v_0, t)\, w\| \leqslant K e^{a_0 t}\, \|A^\beta w\|, \qquad \text{for all} \quad t \geqslant 0, \tag{6.1}$$

as well as

$$\|A^\beta \Phi(v_0, t)\, w\| \leqslant K t^{-\beta} e^{a_0 t}\, \|w\|, \qquad \text{for all} \quad t \geqslant 0. \tag{6.2}$$

For $v_0 \in M$, we let $D(v_0, \rho)$ denote the closed, k-dimensional disk in $\mathscr{R}(P^o(v_0))$, centered at the origin, of radius ρ, i.e.,

$$D(v_0, \rho) = \{p \in \mathscr{R}(P^o(v_0)) : \|A^\beta p\| \leqslant \rho\}.$$

We will use $D(v_0, \rho)$ to define local coordinates in the vicinity of a point $v_0 \in M$. Since $v_0 + D(v_0, \rho)$ is tangent to M at v_0, there is a ρ small enough, say $0 < \rho \leqslant \rho_2$, where ρ_2 does not depend on $v_0 \in M$, and a function f: $D(v_0, \rho) \to \mathscr{R}(Q^o(v_0)) \subset V^{2\beta}$ such that the image

$$\mathscr{D}_\rho(v_0) \stackrel{\text{def}}{=} \{v_0 + p + f(p) : p \in D(v_0, \rho)\} = \text{Graph}\, f \tag{6.3}$$

is an open neighborhood of v_0 in M. The Lipschitz property for the exponential trichotomy is equivalent to saying that there is an $L_0 > 0$ such that one has

$$\|P^i(u_1) - P^i(u_2)\|_{\mathscr{L}} \leqslant L_0 \|A^\beta(u_1 - u_2)\|, \qquad \text{for} \quad i = s, o, u, \tag{6.4}$$

for all $u_1, u_2 \in \mathscr{D}_\rho(v_0)$ and $v_0 \in M$, where $\mathscr{L} = \mathscr{L}(V^{2\beta}, V^{2\beta})$. The complementary projector $Q^o \stackrel{\text{def}}{=} I - P^o$ is invariant and $Q^o(u) = P^s(u) + P^u(u)$, for all $u \in M$. It follows from inequality (3.13) that

$$\|P^i(u)\|_{\mathscr{L}} \leqslant K, \qquad \text{for} \quad u \in M \quad \text{and} \quad i = s, o, u. \tag{6.5}$$

Because of the Lipschitz property for M, the tangent space to the curve $r = f(p)$ is Lipschitz continuous, which implies that the function f, itself, is of class $C^{1,1}$. Since M is of class C^2, the function f is of class C^2, as well. Since the space $r = 0$ coincides with the neutral space $U^o(v_0) = \mathscr{R}(R(v_0))$, see (4.10), it follows that $f(0) = 0$ and the derivative $D_p f = \frac{\partial f}{\partial p}$ satisfies $D_p f(0) = 0$. Furthermore, the second derivative $D_p^2 f = \frac{\partial^2 f}{\partial p^2}$ satisfies

$$\|D^2 f(p)\|_{\mathscr{BL}} \leqslant \hat{L}, \qquad \text{for} \quad \|A^\beta p\| \leqslant \rho \quad \text{and all } v_0 \in M,$$

where $\|D^2 f(p)\|_{\mathscr{BL}}$ denotes the bilinear operator norm, i.e.,

$$\|D^2 f(p)\|_{\mathscr{BL}} = \max\{\|A^\beta D^2 f(p)(u, v)\| : \|A^\beta u\|, \|A^\beta v\| \leqslant 1\}.$$

(The constant $\hat{L}$ depends on the Lipschitz coefficient for the mapping $v \to P^o(v)$, for $v \in M$, and is independent of the base point v.) By using a larger value for L_0, if necessary, one then has the validity of

$$\|A^\beta(f(p_1) - f(p_2))\| \leqslant L_0 \, \rho \, \|A^\beta(p_1 - p_2)\|, \qquad \text{for} \quad \|A^\beta p_1\|, \|A^\beta p_2\| \leqslant \rho, \tag{6.6}$$

and

$$\|A^\beta f(p)\| \leqslant L_0 \|A^\beta p\|^2 \leqslant L_0 \rho \, \|A^\beta p\| \leqslant L_0 \rho^2, \qquad \text{for} \quad \|A^\beta p\| \leqslant \rho, \tag{6.7}$$

as well as, inequality (6.4). Consequently, one has the following result, which treats the radii ρ of the disks $\mathscr{D}_\rho(v_0)$ as a parameter.

LEMMA 6.1. *Let the hypotheses of the Main Theorem be satisfied. Then there exists a* $\rho_2 > 0$ *with* $4K^2 L_0 \rho_2 \leqslant 1$, *such that for all* $v_0 \in M$ *and all*

$u_1, u_2 \in \mathscr{D}_\rho(v_0)$, *where* $0 < \rho \leqslant \rho_2$, *inequalities* (6.4), (6.6), *and* (6.7) *are valid, and one has*

$$\begin{aligned}\tfrac{3}{4}\,\|A^\beta P^o(v_0)(u_1 - v_0)\| &\leqslant (1 - L_0\rho)\,\|A^\beta P^o(v_0)(u_1 - v_0)\|\\ &\leqslant \|A^\beta(u_1 - v_0)\|\\ &\leqslant (1 + L_0\rho)\,\|A^\beta P^o(v_0)(u_1 - v_0)\|\\ &\leqslant \tfrac{5}{4}\,\|A^\beta P^o(v_0)(u_1 - v_0)\|.\end{aligned}$$

In addition, one obtains

$$\left.\begin{cases}\|A^\beta P^s(v_0)(u_1 - u_2)\|\\ \|A^\beta P^u(v_0)(u_1 - u_2)\|\\ \|A^\beta Q^o(v_0)(u_1 - u_2)\|\end{cases}\right. \leqslant K^2 L_0\,\|A^\beta(u_1 - u_2)\|^2 \leqslant K^2 L_0\,\rho\,\|A^\beta(u_1 - u_2)\|. \tag{6.8}$$

We will denote a typical point $v \in \mathscr{D}_\rho(v_0)$ in the form $v = v_0 + p + f(p)$, where $\|A^\beta p\| < \rho \leqslant \rho_2$. One then has

$$\begin{aligned}\|A^\beta(v - v_0)\| &\leqslant (1 + L_0\rho)\,\rho,\\ \|A^\beta(v_1 - v_2)\| &\leqslant (1 + L_0\rho)\,\|A^\beta(p_1 - p_2)\| \leqslant \tfrac{5}{4}\,\|A^\beta(p_1 - p_2)\|,\end{aligned} \tag{6.9}$$

where $v_i = v_0 + p_i + f(p_i)$ and $\|A^\beta p_i\| \leqslant \rho$, for $i = 1, 2$. By combining Lemma 6.1 with the definition of normal hyperbolicity and inequalities (3.9), (3.11), (3.15), (6.1), (6.5), and (6.8), we obtain the following result:

LEMMA 6.2. *Let the hypotheses of the Main Theorem be satisfied. Then for all* $v_0 \in M$, *all* $v_1 \in \mathscr{D}_\rho(v_0)$, *where* $0 < \rho \leqslant \rho_2$, *the following are valid,*

$$\|A^\beta \Phi(v_0, t)\,P^o(v_0)(v_1 - v_0)\| \leqslant K^2\,\|A^\beta(v_1 - v_0)\|\,e^{\lambda_3 t},$$

$$\|A^\beta \Phi(v_0, t)\,P^s(v_0)(v_1 - v_0)\| \leqslant K^3 L_0\,\rho\,\|A^\beta(v_1 - v_0)\|\,e^{\lambda_1 t},$$

$$\|A^\beta \Phi(v_0, t)\,P^u(v_0)(v_1 - v_0)\| \leqslant K^3 L_0\,\rho\,\|A^\beta(v_1 - v_0)\|\,e^{a_0 t},$$

$$\begin{aligned}\|A^\beta \Phi(v_0, t)(v_1 - v_0)\| \leqslant{}& K^2\,\|A^\beta(v_1 - v_0)\|\,e^{\lambda_3 t}\\ &+ K^3 L_0\,\rho\,\|A^\beta(v_1 - v_0)\|\,e^{a_0 t},\end{aligned}$$

$$\|A^\beta \Phi(v_0, t)\,P^o(v_0)(v_1 - v_0)\| \geqslant K^{-1}\,\|A^\beta P^o(v_0)(v_1 - v_0)\|\,e^{\lambda_2 t},$$

$$\|A^\beta \Phi(v_0, t)(v_1 - v_0)\| \geqslant (4(5K)^{-1}\,e^{\lambda_2 t} - K^2 L_0 \rho e^{a_0 t})\,\|A^\beta(v_1 - v_0)\|,$$

for all $t \geqslant 0$, *where* ρ_2 *is given by Lemma* 6.1.

While the inequalities in this lemma are valid for all $t \geqslant 0$, we will be using them when t is restricted to a finite interval $0 \leqslant t \leqslant 2T$, where $T > 0$ is fixed as follows: With the characteristics K, λ_1, λ_2, λ_3, and λ_4 of the compact, invariant set M given by the exponential trichotomy on M, we seek a real numbers $\tau > 0$ and $T > 0$ such that

$$\begin{cases} 4K^2 e^{\lambda_1 \tau} < 1, \\ 96K^2 e^{(\lambda_1 - \lambda_2)\tau} < 1, \\ 16K^2 e^{-\lambda_4 \tau} < 1, \\ 48K^3 e^{(\lambda_3 - \lambda_4)\tau} < 1, \end{cases} \qquad \text{for} \quad T \leqslant \tau \leqslant 2T. \tag{6.10}$$

Note that each of the exponents in the inequalities (6.10) is negative. Consequently, there does exist a time $T > 0$ such that, for all $\tau \geqslant T$, these inequalities are satisfied. We fix one such T for the sequel. We will use the fact that (6.10) is valid for all τ with $T \leqslant \tau \leqslant 2T$.

The next step is to construct a local coordinate system near M by restricting this coordinate system to a suitable neighborhood of each disk $\mathscr{D}_\rho(v_0)$. We begin by choosing ρ_1 and σ_1 with $0 < \rho_1 \leqslant \rho_2$, $0 < \sigma_1 \leqslant \sigma_2$, and for each $v_0 \in M$ one has

$$\text{Convex Hull}(N(\mathscr{D}_{\rho_1}(v_0), \sigma_1)) \subset \Omega = N(M, \sigma_2), \tag{6.11}$$

and $M \cap N(\mathscr{D}_{\rho_1}(v_0), \sigma_1) \subset \mathscr{D}_{\rho_2}(v_0)$. The relationship (6.11) is important because it enables us to get a good estimate for the *effective* Lipschitz coefficient for the nonlinear perturbation term G in terms of $\|G\|_{\{A;\, C^1(\Omega)\}}$. In particular, let $w_i = w_i(t)$ denote two continuous functions with $w_i(t) \in N(\mathscr{D}_{\rho_1}(v_0), \sigma_1)$, for $0 \leqslant t < t_0$ and $i = 1, 2$, where $0 < t_0 \leqslant \infty$. Assume that $G \in C^1_{\text{Lip}}$ satisfies inequality (4.2). Since $\lambda w_1(s) + (1 - \lambda)\, w_2(s) \in \Omega$, for $0 \leqslant s < t_0$ and $0 \leqslant \lambda \leqslant 1$ (by (6.11)), and since

$$G(w_1) - G(w_2) = \int_0^1 DG(w_2 + \theta(w_1 - w_2))\, d\theta (w_1 - w_2), \qquad \text{for} \quad 0 \leqslant s < t_0,$$

which implies that

$$e^{-A(t-s)}[G(w_1) - G(w_2)] = \int_0^1 e^{-A(t-s)} DG(w_2 + \theta(w_1 - w_2))\, d\theta (w_1 - w_2), \tag{6.12}$$

for $0 \leqslant s < t_0$, we claim that

$$\int_0^t \|A^\beta e^{-A(t-s)}[G(w_1) - G(w_2)]\|\, ds \leqslant \delta \sup_{0 \leqslant s \leqslant t} \|A^\beta (w_1(s) - w_2(s))\|. \tag{6.13}$$

Indeed from Eq. (6.12) one has

$$\int_0^t \|A^\beta e^{-A(t-s)}[G(w_1)-G(w_2)]\|\,ds$$

$$\leqslant \int_0^t \int_0^1 \|A^\beta e^{-A(t-s)} DG\|_{\mathscr{L}}\,d\theta\,\|A^\beta(w_1-w_2)\|\,ds$$

$$\leqslant \|G\|_{\{A;\,C^1(\Omega)\}} \sup_{0\leqslant s\leqslant t} \|A^\beta(w_1(s)-w_2(s))\|,$$

thus inequality (6.13) now follows from (4.2).

There is a variation of inequality (6.13) which arises when one uses the alternate Variation of Constants Formula (2.34), or (4.9), where $v_0 \in M$. In this case, we let

$$w(t) \stackrel{\text{def}}{=} \int_0^t \Phi(S(r)\,v_0, t-r)\,g(r)\,dr,$$

where $g(r)=G(w_1(r))-G(w_2(r))$, and w_1 and w_2 are given as in the last paragraph. As noted in Subsection 2.5, $w=w(t)$ satisfies

$$w(t)=\int_0^t e^{-A(t-s)}[DF(S(s)\,v_0)\,w(s)+g(s)]\,ds.$$

It then follows from inequalities (2.6) and (6.13) that

$$\|A^\beta w(t)\| \leqslant \delta \sup_{0\leqslant s\leqslant t} \|A^\beta(w_1(s)-w_2(s))\| + M_\beta K_1 \int_0^t (t-s)^{-\beta}\,\|A^\beta w(s)\|\,ds,$$

for $t \geqslant 0$, where $\|DF(u)\|_{\mathscr{L}} \leqslant K_1$, for $u \in M$. It then follows from the Gronwall–Henry inequality that there is a constant $C(2T)>0$ such that

$$\|A^\beta w(t)\| \leqslant C(2T)\,\delta \sup_{0\leqslant s\leqslant 2T} \|A^\beta(w_1(s)-w_2(s))\|, \qquad \text{for} \quad 0\leqslant t\leqslant 2T. \tag{6.14}$$

We will require that σ_1 and (especially) ρ_1 satisfy a few auxiliary properties. In particular, by using the Lipschitz property, one can show that if the radius ρ of the disks $\mathscr{D}_\rho(v_0)$ and the number σ_1 are replaced by smaller values, if necessary, then one can construct a new (local) coordinate system in the vicinity of each disk $\mathscr{D}_\rho(v_0)$.

LEMMA 6.3. *Let M be a compact manifold of class C^2 in $V^{2\beta}$. There exist $\rho_1 > 0$ and $\sigma_1 > 0$, such that $0 < \rho_1 \leqslant \frac{3}{10} \rho_2$, $0 < \sigma_1 \leqslant \sigma_2$, relation (6.11) is valid and, for every $v_0 \in M$ and every $y \in N(\mathcal{D}_{\rho_1}(v_0), \sigma_1)$, the following properties are valid:*

(1) *There is one and only one point $v \in \mathcal{D}_{\rho_2}(v_0)$ such that $y - v \in U^s(v) \oplus U^u(v)$. Furthermore, the mapping ψ: $y \to v \stackrel{\text{def}}{=} \psi(y) = \psi(v_0, y)$ is of class C^2 on $N(\mathcal{D}_{\rho_1}(v_0), \sigma_1)$ with $\psi(v_0, v_1) = \psi(v_1) = v_1$, for all $v_1 \in \mathcal{D}_{\rho_1}(v_0)$.*

(2) *If in addition, one has $\|A^\beta(y - v_0)\| < 2\sigma_1$, then $v = \psi(y)$ satisfies $v \in \mathcal{D}_{\rho_1}(v_0)$.*

(3) *Moreover, the Fréchet derivative $D\psi(y)$, of $\psi(y)$ with respect to y, where $y \in N(\mathcal{D}_{\rho_1}(v_0), \sigma_1)$, satisfies*

$$D\psi(y) = R(v) = P^o(v) = P^o(\psi(y)). \tag{6.15}$$

(4) *The mapping ψ satisfies $\psi(y) = \psi(v_0, y) = y - \phi(v_0, y) = y - \phi(y)$, where $D\phi(y) = Q^o(v) = Q^o(\psi(y))$. The mapping ϕ has the property that for all $v \in M$, one has*

$$\phi(v + n) = n, \qquad \textit{for} \quad n \in \mathcal{R}(Q^o(v)). \tag{6.16}$$

(5) *Let $y_i \in N(\mathcal{D}_{\rho_1}(v_0), \sigma_1)$, for some $v_0 \in M$ and set $v_i = \psi(y_i)$, for $i = 1, 2$. Then one has*

$$v_1 - v_2 = \psi(y_1) - \psi(y_2) = P^o(v_2)(y_1 - y_2) + e_3, \tag{6.17}$$

and there is a $b_2^F \in \Sigma$ such that $e_3 = e_3(y_2, y_1 - y_2)$ satisfies

$$\|A^\beta e_3\| \leqslant b_2^F(\rho) \, \|A^\beta(y_1 - y_2)\|, \tag{6.18}$$

whenever $\|A^\beta(y_1 - y_2)\| \leqslant \rho \leqslant \rho_1$.

(6) *In the sequel, we will require that*

$$C_2 \, b_2^F(\rho_1) \leqslant \frac{1}{144K^2} \, e^{2\lambda_2 T} \qquad (\leqslant K), \tag{6.19}$$

in which case, one has $\|v_1 - v_2\| \leqslant 2K \, \|y_1 - y_2\|$. The constant C_2 is defined in Lemma 6.6, and it satisfies $C_2 \geqslant 1$.

Proof. The proofs of Items (3) and (4) follow directly from Items (1) and (2), and the proof of Items (1) and (2) is a direct application of a Collared or Tubular Neighborhood Theorem for M. For the convenience of the reader, we will present here the essence of this argument.

For $0 < \rho \leqslant \rho_1 \leqslant \rho_2$, we let $y \in N(\mathcal{D}_\rho(v_0), \sigma_1)$. The defining relationships for the point $v \in \mathcal{D}_{\rho_2}(v_0)$ is that (1) $v = v_0 + p + f(p)$, for some point $p \in \mathcal{R}(P^o(v_0))$ with $\|A^\beta p\| \leqslant \rho_2$, and (2) $y - v \in \mathcal{R}(Q^o(v))$. We now define $P_0 = P^o(v_0)$, $Q_0 = Q^o(v_0)$, $P = P^o(v)$, and $Q = Q^o(v)$. Given v_0 and y, our first goal is to find a point $p \in \mathcal{R}(P_0)$ so that

$$y - v_0 - p - f(p) = y - v = Q(y - v) = Q(y - v_0) - Qp - Qf(p).$$

In other words, p must satisfy

$$p = J(p, y) \stackrel{\text{def}}{=} P(y - v_0) + QP_0 p + Pf(P_0 p). \tag{6.20}$$

We will now show that Eq. (6.20) has a unique fixed point p in a suitable space. Since $p = P_0 p$, $Q_0 P_0 = 0$ and $P_0 f(P_0 p) = 0$, one has

$$J(p, y) = P(y - v_0) + (Q - Q_0)\, P_0 p - (P - P_0)\, f(P_0 p).$$

Assume for the moment that $\|A^\beta(y - v_0)\| < 2\sigma_1$. In this case, we will show that there is such a p, where $\|A^\beta p\| \leqslant \rho_1$ and $v = v_0 + p + f(p) \in \mathcal{D}_{\rho_1}(v_0)$. A direct calculation, using inequalities (6.4) and (6.7), leads to

$$\|A^\beta J(p, y)\| \leqslant 2K\sigma_1 + L_0(\|A^\beta p\| + \|A^\beta f(p)\|)^2 \leqslant 2K\sigma_1 + L_0\, \rho_1^2 (1 + L_0\, \rho_1)^2.$$

Since $p = P_0 p$, the fixed point $p = J(p, y)$ will satisfy $\|A^\beta p\| \leqslant \rho_1$, provided that

$$L_0\, \rho_1 (1 + L_0\, \rho_1)^2 \leqslant \frac{1}{2} \qquad \text{and} \qquad 2K\sigma_1 \leqslant \frac{\rho_1}{2}. \tag{6.21}$$

After a lengthy calculation which uses $v_i = u_0 + p_i + f(p_i)$, for $i = 1, 2$, and inequalities (6.4) and (6.6), one finds that

$$\|A^\beta(J(p_1, y) - J(p_2, y))\| \leqslant K_1\, \|A^\beta(p_1 - p_2)\|,$$

where $K_1 = 2(1 + L_0\, \rho_1)\, \sigma_1 + 2(1 + L_0\, \rho_1)^2\, \rho_1$. When ρ_1 and σ_1 are chosen so that (6.11) and (6.21) hold, as well as,

$$2(1 + L_0 \rho_1)\, \sigma_1 \leqslant \tfrac{1}{3} \qquad \text{and} \qquad 2(1 + L_0 \rho_1)^2\, \rho_1 \leqslant \tfrac{1}{3}, \tag{6.22}$$

then $J(p, y)$ is a strict contraction on $\mathcal{D}_{\rho_1}$ and there is a unique fixed point for $p = J(p, y)$. Since $P_0 p$ is also a fixed point of Eq. (6.20), where $\|A^\beta p\| \leqslant \rho_1$ and $\|A^\beta P_0 p\| \leqslant \rho_1$, it follows that $p = P_0 p$, i.e., $p \in \mathcal{R}(P_0)$. Thus p also satisfies the equation $p = P_0 J(p, y)$, and the point $v = v_0 + p + f(p)$ satisfies $v \in \mathcal{D}_{\rho_1}(v_0)$. This in turn implies that $y - v = Q(y - v)$.

It remains to verify the conclusion when y does not satisfy the added requirement that $\|A^{\beta}(y-v_0)\|<2\sigma_1$. Since $y\in N(\mathscr{D}_\rho(v_0),\sigma_1)$, there is a point $v_1\in\mathscr{D}_\rho(v_0)$ such that $\|A^{\beta}(y-v_1)\|<2\sigma_1$ and $y\in N(\mathscr{D}_\rho(v_1),\sigma_1)$. From the argument of the last paragraph, there is then a point $v_2\in\mathscr{D}_{\rho_1}(v_1)$ with $y-v_2\in\mathscr{R}(Q^o(v_2))$. Now inequality (6.9) implies that

$$\|A^{\beta}(v_2-v_0)\|\leqslant\|A^{\beta}(v_2-v_1)\|+\|A^{\beta}(v_1-v_0)\|\leqslant 2(1+L_0\,\rho_1)\,\rho_1.$$

With $p=v-u_0-f(p)$, it follows from (6.7) that

$$\|A^{\beta}p\|\leqslant 2(1+L_0\,\rho_1)\,\rho_1+L_0\,\rho_1\,\|A^{\beta}p\|.$$

One finds that $(1+L_0\,\rho_1)\leqslant\frac{5}{4}$ and $(1-L_0\rho_1)^{-1}\leqslant\frac{4}{3}$, since Lemma 6.1 implies $4L_0\,\rho_1\leqslant 4L_0\,\rho_2\leqslant 4K^2L_0\,\rho_2\leqslant 1$. As a result, one obtains $\|A^{\beta}p\|\leqslant\frac{10}{3}\,\rho_1$. Since $\rho_1\leqslant\frac{3}{10}\,\rho_2$, we see that $v\in\mathscr{D}_{\rho_2}(u_0)$, as desired.

If $v_0\in M$ and $y_0\in N(\mathscr{D}_\rho(v_0),\sigma_1)$ are chosen so that $Q_0(y_0-v_0)=y_0-v_0$, then one has $\psi(y_0)=v_0$. In other words, one has $J(p,y_0)=0$ whenever $P_0(y_0-v_0)=0$. Next let w be given where $y=y_0+w\in N(\mathscr{D}_\rho(v_0),\sigma_1)$ and $\|A^{\beta}w\|$ is small and set $v=\psi(y)$. Let $e_3=e_3(y_0,w)$ be defined by

$$e_3(y_0,w)\stackrel{\text{def}}{=}\psi(y_0+w)-\psi(y_0)-P_0w=v-v_0-P_0w.$$

By means of a straightforward calculation, which uses the identity $P_0(DP_0)\,P_0=0$, where $DP_0=DP^o(v_0)$, one shows that $\lim_{\|A^{\beta}w\|\to 0}\|A^{\beta}w\|^{-1}e_3(y_0,w)=0$. This proves that $\psi(y)$ is Fréchet differentiable, and Eqs. (6.15) and (6.17) hold, that inequality (6.18) is valid, and that $b_2^F\in\Sigma$. The second inequality in (6.19) follows from the facts that $K\geqslant 1$, $\lambda_2\leqslant 0$, and $T>0$. ∎

Notation. We will denote the new (nonlinear) coordinates of the point y by

$$y=v+s+u=v+n, \tag{6.23}$$

where $\|y-v\|<\sigma_1$, $v\in M$, $s\in U^s(v)$, $u\in U^u(v)$, and $n=s+u$. By (6.16) one then has $\phi(y)=n=s+u$. In the sequel we fix ρ_1 and σ_1 so that inequalities (6.11), (6.19), (6.21), (6.22), and $\rho_1\leqslant\frac{3}{10}\,\rho_2$ hold. This new coordinate system for the point y depends on the base point $v_0\in M$. If two disks $\mathscr{D}_{\rho_1}(v_0)$ and $\mathscr{D}_{\rho_1}(v_1)$ have a nontrivial intersection and $y\in N(\mathscr{D}_{\rho_1}(v_0),\sigma_1)\cap N(\mathscr{D}_{\rho_1}(v_1),\sigma_1)$, then $\psi(v_0,y)=\psi(v_1,y)$, i.e., the coordinate representations agree. More generally, let $y=y(t)=y(t,y_0)$ be a solution of the perturbed Eq. (1.2), where $y_0=v_0+n_0$, $v_0\in M$, and $n_0\in\mathscr{R}(Q^o(v_0))$. Assume that one has $y(t)\in N(\mathscr{D}_{\rho_1}(S(t)\,v_0),\sigma_1)$ for t in some interval I. Then the local coordinate representation

$$y(t)=v(t)+n(t),\qquad\text{for all}\quad t\in I, \tag{6.24}$$

where $v(t) \in \mathscr{D}_{\rho_2}(S(t)\, v_0)$, $P^o(v(t))\, n(t) = 0$, and $Q^o(v(t))\, n(t) = n(t)$ are well-defined.

In addition to (6.11), (6.19), (6.21), (6.22), and $\rho_1 \leqslant \frac{3}{10}\, \rho_2$, we require that ρ_1 satisfy

$$3K^3 L_0\, \rho_1 e^{a_0 t} \leqslant K^2 e^{\lambda_3 t} \qquad \text{and} \qquad 10K^3 L_0\, \rho_1 e^{a_0 t} \leqslant e^{\lambda_2 t}, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T, \tag{6.25}$$

and that ρ_1 and σ_1 satisfy

$$C_3\, b_1^F(\rho_1, \sigma_1) \leqslant \frac{1}{144K^2}\, e^{2\lambda_2 T}, \tag{6.26}$$

where $C_3 = C_3(2T) > 0$ is defined in Lemma 6.6 and b_1^F is given by Eq. (4.1). Since $\lambda_2 \leqslant \lambda_3 \leqslant a_0$, it then follows from inequality (6.25) and Lemmas 6.1 and 6.2 that for $v_1 \in \mathscr{D}_{\rho_1}(v_0)$, for $0 \leqslant t \leqslant 2T$, one has

$$\frac{1}{2K}\, \|A^\beta(v_1 - v_0)\|\, e^{\lambda_2 t} \leqslant \|A^\beta \Phi(v_0, t)(v_1 - v_0)\| \leqslant 2K^2\, \|A^\beta(v_1 - v_0)\|\, e^{\lambda_3 t}. \tag{6.27}$$

6.2. *The Dynamics on M.* The results of the last three lemmas give valuable information about the geometry of compact, connected, finite dimensional, invariant manifolds M of class C^2. We will also be interested in such manifolds which may be lacking in smoothness, but which have the property that the induced linear skew product semiflow π has an exponential trichotomy. This is developed in the following lemma wherein we assume M to be a Lipschitz manifold, and not necessarily of class C^2. Such manifolds arise when the representation $f(p)$ given by Eq. (6.3) is a Lipschitz continuous function. While we do assume here that π has an exponential trichotomy on M, we do not require that the tangency condition (4.10) be satisfied.

LEMMA 6.4. *Let the Standing Hypothesis A be satisfied and let $F \in C^1_{\mathrm{Lip}}$. Let M be a compact, connected, finite dimensional, invariant manifold of Lipschitz class in $V^{2\beta}$ for the unperturbed Eq.* (1.1). *Assume that the induced linear skew product semiflow π has an exponential trichotomy on M. Then there is a $b_3^F \in \Sigma$ such that for any two points $u_1, u_2 \in M$ with $\|A^\beta(u_1 - u_2)\| \leqslant \rho$, where $0 < \rho \leqslant \rho_1$, there is a function $H_2(t)$ with the property that*

$$S(t)\, u_1 - S(t)\, u_2 = \Phi(u_2, t)(u_1 - u_2) + H_2(t), \qquad \textit{for all} \quad t \in [0, 2T], \tag{6.28}$$

and

$$\|A^\beta H_2(t)\| \leqslant b_3^F(\rho)\, \|A^\beta(u_1-u_2)\|, \qquad \textit{for all} \quad t\in[0,2T]. \tag{6.29}$$

Proof. Let $u_1, u_2 \in M$ satisfy $\|A^\beta(u_1-u_2)\| \leqslant \rho$, where $0<\rho\leqslant\rho_1$, and set $w=w(t)=S(t)\,u_1 - S(t)\,u_2$, for $0\leqslant t\leqslant 2T$. Then w is a mild solution of Eq. (4.3) with $G\equiv 0$ and $H\equiv E$. It follows from Eq. (4.9) that

$$w(t)=\Phi(u_2,t)(u_1-u_2)+\int_0^t \Phi(S(s)\,u_2, t-s)\, E(S(s)\,u_2, w(s))\, ds,$$

for $0\leqslant t\leqslant 2T$. Hence Eq. (6.28) is valid, for $0\leqslant t\leqslant 2T$, with

$$H_2(t)=\int_0^t \Phi(S(s)\,u_2, t-s)\, E(S(s)\,u_2, w(s))\, ds.$$

In order to verify that inequality (6.29) is valid, we note that inequalities (4.5) and (6.2) imply that there is a $\gamma\in\Sigma$ such that

$$\|A^\beta H_2(t)\| \leqslant \gamma(\sigma)\int_0^t (t-s)^{-\beta}\, e^{a_0(t-s)}\, \|A^\beta w(s)\|\, ds,$$

provided that $\|A^\beta w(s)\|\leqslant\sigma$, for $0\leqslant s\leqslant t$. Now inequality (2.18), with $F=F_1=F_2$, implies that there is a constant $C_2=C_2(2T)\geqslant 1$ such that, for $0\leqslant t\leqslant 2T$,

$$\|A^\beta w(t)\| = \|A^\beta(S(t)\,u_1 - S(t)\,u_2))\| \leqslant C_2\, \|A^\beta(u_1-u_2)\|. \tag{6.30}$$

If $\|A^\beta(u_1-u_2)\|\leqslant\rho$ and $C_2\,\rho\leqslant\sigma$, then $\|A^\beta w(t)\|\leqslant\sigma$, for $0\leqslant t\leqslant 2T$. With $b_3^F(\rho)=C(T)\,\gamma(C_2\rho)$, one obtains inequality (6.29), for a constant $C(T)>0$. ∎

LEMMA 6.5. *Let the Standing Hypothesis A be satisfied and let $F\in C^1_{\mathrm{Lip}}$. Let M be a compact, connected, finite dimensional, invariant C^2-manifold in $V^{2\beta}$ for the unperturbed Eq. (1.1). Assume that M is normally hyperbolic and that the associated exponential trichotomy is of Lipschitz class. Then there is a ρ_0, with $0<\rho_0\leqslant\rho_1$, such that for any two points $v_1, v_2\in M$ with $\|A^\beta(v_1-v_2)\|\leqslant\rho_0$, one has $\|A^\beta(S(t)\,v_1 - S(t)\,v_2)\|\leqslant\rho_1$, for $-2T\leqslant t\leqslant 2T$; and*

$$\frac{1}{4K}\|A^\beta(v_1-v_2)\|\, e^{\lambda_2 t} \leqslant \|A^\beta(S(t)\,v_1-S(t)\,v_2)\| \leqslant 4K^2\, \|A^\beta(v_1-v_2)\|\, e^{\lambda_3 t}, \tag{6.31}$$

for $0 \leqslant t \leqslant 2T$; *and for* $-2T \leqslant t \leqslant 0$, *one has*

$$\frac{1}{4K^2} \|A^\beta(v_1 - v_2)\| \, e^{\lambda_3 t} \leqslant \|A^\beta(S(t)\, v_1 - S(t)\, v_2)\| \leqslant 4K \, \|A^\beta(v_1 - v_2)\| \, e^{\lambda_2 t}. \tag{6.32}$$

Proof. Let $b_3^F \in \Sigma$ be given by Lemma 6.4 and fix $\rho_0 > 0$ so that $C_2 \rho_0 \leqslant \rho_1$, where $C_2 \geqslant 1$ is given by (6.30), and ρ_1 satisfies the conditions stated above, as well as

$$b_3^F(\rho_1) \leqslant \frac{1}{48K} e^{2\lambda_2 T}. \tag{6.33}$$

Since $\lambda_2 \leqslant 0 \leqslant \lambda_3$ and $K \geqslant 1$, one has

$$b_3^F(\rho) \leqslant \frac{1}{48K} e^{\lambda_2 t} \leqslant 2K^2 e^{\lambda_3 t}, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T \quad \text{and} \quad 0 < \rho \leqslant \rho_1.$$

Now Eq. (6.28) implies that

$$\begin{aligned}\|A^\beta \Phi(v_2, t)(u_1 - v_2)\| - \|A^\beta H_2(t)\| &\leqslant \|A^\beta(S(t)\, u_1 - S(t)\, v_2)\| \\ &\leqslant \|A^\beta \Phi(v_2, t)(u_1 - v_2)\| + \|A^\beta H_2(t)\|,\end{aligned}$$

for $0 \leqslant t \leqslant 2T$. This fact, together with (6.27), imply inequality (6.31). Inequality (6.32) follows directly from (6.31). We will omit these details. ∎

In addition to the functions b_1^F, b_2^F, and b_3^F introduced above, we define b_4^F and b_5^F by

$$\begin{aligned} b_4^F &= b_4^F(\rho, 2\varepsilon) = K(C_3\, b_1^F(\rho, 2\varepsilon) + C_2\, b_2^F(\rho)), \\ b_5^F &= b_5^F(\rho, 2\varepsilon) = 3b_4^F(\rho, 2\varepsilon) + b_3^F(\rho), \end{aligned} \tag{6.34}$$

where the coefficients C_2 and C_3 are defined below in Lemma 6.6.

6.3. *Perturbed Dynamics near M.* Let $C(M, V^{2\beta})$ denote the Banach space of continuous functions $f\colon M \to V^{2\beta}$ with the sup-norm

$$\|f\|_\infty = \sup\{\|A^\beta f(v)\| \colon v \in M\}.$$

Next we define two function classes which are subsets of $C(M, V^{2\beta})$: $\mathscr{F} = \mathscr{F}(\varepsilon, \ell)$ and $\mathscr{G} = \mathscr{G}(\varepsilon, \ell)$, where the parameters $\varepsilon > 0$ and $\ell > 0$ will be chosen later. A vector-valued function f is said to belong to $\mathscr{F}(\varepsilon, \ell)$, if $f \in C(M, V^{2\beta})$ and, for each $v \in M$, one has $f(v) \in U^s(v)$ with $\|A^\beta f(v)\| \leqslant \varepsilon$, and the restriction of f to each disk $\mathscr{D}_{\rho_0}(v_0)$ in M is Lipschitz continuous with Lipschitz coefficient ℓ. Similarly, a vector-valued function g is said to belong to $\mathscr{G}(\varepsilon, \ell)$, if $g \in C(M, V^{2\beta})$ and, for each $v \in M$, one has $g(v) \in U^u(v)$ with $\|A^\beta g(v)\| \leqslant \varepsilon$, and the restriction of g to each disk $\mathscr{D}_{\rho_0}(v)$ in M is Lipschitz continuous with Lipschitz coefficient ℓ. Since $\mathscr{U}^s = \{(v, n)\colon v \in M, n \in U^s(v)\}$ and $\mathscr{U}^u = \{(v, n)\colon v \in M, n \in U^u(v)\}$ are closed subsets of $M \times V^{2\beta}$, see Sacker and Sell (1974, 1976ab), it follows that, for every $\varepsilon > 0$ and $\ell > 0$, the spaces $\mathscr{F}(\varepsilon, \ell)$ and $\mathscr{G}(\varepsilon, \ell)$ are closed sets in $C(M, V^{2\beta})$. Consequently, the product space $\mathscr{F}(\varepsilon, \ell) \times \mathscr{G}(\varepsilon, \ell)$ is a complete metric space with the metric

$$\|(f_1, g_1) - (f_2, g_2)\|_\infty \stackrel{\text{def}}{=} \|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty,$$

where $(f_i, g_i) \in \mathscr{F} \times \mathscr{G}$, for $i = 1, 2$.

In the argument given below, our objective will be to find $(f, g) \in \mathscr{F} \times \mathscr{G}$ so that the mapping h, which is defined by $h(u) = u + f(u) + g(u)$, for $u \in M$, satisfies the conclusions of the Main Theorem. The pair (f, g) will be found as a fixed point of a suitable mapping A_T.

Let $v_0 \in M$ and $y_0 \in \Omega$ be given, and set $w(t) = y(t, y_0) - S(t)\, v_0$, with $w_0 = y_0 - v_0$. It follows that $w(t)$ satisfies Eq. (4.8). Assume that $\|A^\beta w_0\| \leqslant \varepsilon$ and G satisfies inequality (4.2). Then with $F_1 = F$ and $F_2 = F + G$, inequality (2.18) implies that, for $t_0 > 0$, there is a constant $K_1 = K_1(t_0) > 0$ such that

$$\|A^\beta w(t)\| \leqslant K_1(t_0)(\varepsilon + \delta), \qquad \text{for} \quad 0 \leqslant t \leqslant t_0.$$

If ε and δ satisfy $K_2(\varepsilon + \delta) \leqslant \sigma_1 \leqslant \sigma_2$, where $K_2 = K_1(2T)$, then one can choose t_0 so that $t_0 \geqslant 2T$ and

$$\|A^\beta w(t)\| = \|A^\beta(y(t, y_0) - S(t)\, v_0)\| \leqslant K_2(\varepsilon + \delta), \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.35}$$

Next we define

$$e = e(t) = e(t, y_0) = w(t) - \Phi(v_0, t)\, w_0, \qquad \text{where} \quad w(0) = w_0 = y_0 - v_0. \tag{6.36}$$

It follows from Eq. (4.3) that $e(0) = 0$ and $e(t)$ is the mild solution of

$$\partial_t e(t) + Ae(t) = B(t)\, e(t) + E(S(t)\, v_0, w(t)) + G(S(t)\, v_0 + w(t)),$$

where $B(t) = DF(S(t)\, v_0)$. As a result of Eqs. (2.23) and (4.8), the solution e satisfies

$$e(t) = \int_0^t e^{-A(t-s)} B(s)\, e(s)\, ds + \int_0^t e^{-A(t-s)} E(S(s)\, v_0, w(s))\, ds$$
$$+ \int_0^t e^{-A(t-s)} G(S(s)\, v_0 + w(s))\, ds. \tag{6.37}$$

Now inequalities (4.2), (4.5), and (6.2) imply that there is a $\gamma \in \Sigma$, such that, for $0 \leqslant t \leqslant 2T$, one has

$$\|A^\beta e(t)\| \leqslant M_\beta \, \|B\|_\infty \int_0^t (t-s)^{-\beta}\, e^{-a(t-s)}\, \|A^\beta e(s)\|\, ds$$
$$+ \int_0^t (t-s)^{-\beta}\, e^{-a(t-s)}\, \|A^\beta w(s)\|\, \gamma(\|A^\beta w(s)\|)\, ds + \delta.$$

From inequality (6.35), one obtains a $\beta_1 \in \Sigma$, where

$$\|A^\beta e(t)\| \leqslant (\varepsilon + \delta)\, \beta_1(\varepsilon, \delta) + \delta + M_\beta\, \|B\|_\infty \int_0^t (t-s)^{-\beta}\, e^{-a(t-s)}\, \|A^\beta e(s)\|\, ds,$$

for $0 \leqslant t \leqslant 2T$. It then follows from the Gronwall–Henry inequality that there is a constant $C_1 > 0$ and a $b_0 \in \Sigma$ such that

$$\|A^\beta e(t)\| \leqslant (\varepsilon + \delta)\, b_0(\varepsilon, \delta) + C_1 \delta, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.38}$$

Special Notation. The following notation will be used from time-to-time throughout this paper: For $i = 1, 2$, we define $y_i = y_i(t) = y(t, y_{i0})$, $v_i = v_i(t) = v(t, y_{i0})$, $n_i = n_i(t) = n(t, y_{i0})$, $s_i = s_i(t) = s(t, y_{i0})$, $u_i = u_i(t) = u(t, y_{i0})$, and $S_i = S_i(t) = S(t)\, v_{i0}$, where $\psi(y_{i0}) = v_{i0}$ with $v_{i0} \in \mathscr{D}_{\rho_1}(v_0)$, for some $v_0 \in M$, and $n_{i0} = y_{i0} - v_{i0} = s_{i0} + u_{i0}$. Also we define

$$\Delta y = \Delta y(t) = y_1 - y_2, \qquad \Delta v = \Delta v(t) = v_1 - v_2,$$
$$\Delta S = \Delta S(t) = S_1 - S_2, \qquad \Delta n = \Delta n(t) = \Delta y - \Delta v,$$
$$\Delta w = \Delta w(t) = \Delta y - \Delta S, \qquad \Delta z = \Delta z(t) = \Delta v - \Delta S,$$
$$\Delta s = \Delta s(t) = s_1 - s_2, \qquad \text{and} \qquad \Delta u = \Delta u(t) = u_1 - u_2.$$

We let the functions $E_S = E_S(t)$, $E_y = E_y(t)$, $E_v = E_v(t)$, and $E_s = E_s(t)$ be defined, for $0 \leqslant t \leqslant 2T$, by

$$\begin{aligned} \Delta S(t) &= \Phi(v_{20}, t)\, \Delta S(0) + E_S(t), \\ \Delta y(t) &= \Phi(v_{20}, t)\, \Delta y(0) + E_y(t), \\ \Delta v(t) &= \Phi(v_{20}, t)\, \Delta v(0) + E_v(t), \\ \Delta s(t) &= \Phi(v_{20}, t)\, P^s(v_{20})\, \Delta n(0) + E_s(t). \end{aligned} \tag{6.39}$$

This Special Notation is used in the following result and in the sequel.

LEMMA 6.6. *Let the hypotheses of Lemma 6.5 be satisfied, and let $G \in C^1_{\mathrm{Lip}}$. Then there exist σ_0, with $0 < \sigma_0 \leqslant \sigma_1$, $\varepsilon_0 > 0$, $\delta_0 = \delta_0(\varepsilon) > 0$, and nonnegative constants C_0, C_1, C_2, and C_3, which depend on the characteristics of of the exponential trichotomy on M, the time T, such that $C_2 \geqslant 1$, $2\varepsilon_0 \leqslant \sigma_0$, $C_0(\varepsilon_0 + \delta_0) \leqslant \min(\rho_1, \sigma_1)$, and whenever $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \delta \leqslant \delta_0$, and $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, then the conclusions of Lemma 6.5 hold and the following are valid:*

(1) *For any $v_0 \in M$ and $y_0 = v_0 + n_0$, where $n_0 \in U^s(v_0) + U^u(v_0)$ and $\|A^\beta n_0\| \leqslant 2\varepsilon \leqslant \sigma_0$, one has*

$$\left.\begin{cases} \|A^\beta(y(t, y_0) - S(t)\, v_0)\| \\ \|A^\beta(v(t, y_0) - S(t)\, v_0)\| \\ \frac{1}{2}\, \|A^\beta n(t, y_0)\| \end{cases}\right\} \leqslant C_0(\varepsilon + \delta), \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.40}$$

Furthermore, $w(t) = y(t, y_0) - S(t)\, v_0$ satisfies Eq. (4.3), with $w(0) = n_0 = y_0 - v_0$. Also $y(t) = y(t, y_0) = v(t) + n(t)$ satisfies (6.24), and inequality (6.38) holds.

(2) *Assume that $v_{i0} \in M$ with $\|A^\beta\, \Delta v(0)\| \leqslant \rho_0$ and $\|A^\beta(y_{i0} - v_{i0})\| \leqslant 2\varepsilon \leqslant \sigma_0$, for $i = 1, 2$. Then one has $y_i(t) = y(t, y_{i0}) \in N(\mathscr{D}_{\rho_1}(S(t)\, v_{10}), \sigma_1)$, and*

$$\left.\begin{cases} \|A^\beta\, \Delta y(t)\| \\ \|A^\beta\, \Delta v(t)\| \\ \|A^\beta\, \Delta n(t)\| \end{cases}\right\} \leqslant C_2\, \|A^\beta\, \Delta y(0)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.41}$$

(3) *Whenever $\|A^\beta(y_{i0} - v_{i0})\| \leqslant 2\varepsilon \leqslant \sigma_0$, with $v_{i0} \in M$, for $i = 1, 2$, and $\|A^\beta\, \Delta y(0)\| \leqslant C_2^{-1}\rho$, with $0 < \rho \leqslant \rho_0$, then*

$$\left\| A^\beta \int_0^t \Phi(S(s)\, v_{20}, t - s) \int_0^1 [DF(y_2 + \theta(y_1 - y_2)) - DF(y_2)]\, d\theta\, \Delta y(s)\, ds \right\|$$
$$\leqslant C_3\, b_1^F(\rho, 2\varepsilon)\, \|A^\beta \Delta y(0)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T, \tag{6.42}$$

where b_1^F is given by (4.1).

(4) *There exist $\beta_1, \beta_2 \in \Sigma$ such that, for $0 \leqslant t \leqslant 2T$, one has*

$$\begin{aligned} \|A^\beta E_S(t)\| &\leqslant b_3^F(\rho)\, \|A^\beta\, \Delta v(0)\|, \\ \|A^\beta E_y(t)\| &\leqslant (C_3 b_1^F(\rho, 2\varepsilon) + \beta_1(\varepsilon, \delta))\, \|A^\beta\, \Delta y(0)\|, \\ \|A^\beta E_v(t)\| &\leqslant (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta\, \Delta y(0)\|. \end{aligned} \tag{6.43}$$

(5) *Let $y_i = y_{i0}$ and $v_i = v_{i0}$, $i = 1, 2$, be given as in Item* (2), *and that $\|A^\beta\, \Delta y(0)\| \leqslant 3\, \|A^\beta\, \Delta v(0)\|$. Then for $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < \delta \leqslant \delta_0$, one has*

$$\tfrac{1}{2}\, \|A^\beta\, \Delta S(t)\| \leqslant \|A^\beta\, \Delta v(t)\| \leqslant \tfrac{3}{2}\, \|A^\beta\, \Delta S(t)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.44}$$

Define $H_5(t)$ by $\Delta v(t) = \Delta S(t) + H_5(t)$, for $t \in [0, 2T]$. Then there is a $\beta_3 \in \Sigma$ such that, for $0 \leqslant t \leqslant 2T$, one has

$$\|A^\beta H_5(t)\| = \|A^\beta\, \Delta z(t)\| \leqslant (b_5^F(\rho, 2\varepsilon) + \beta_3(\varepsilon, \delta))\, \|A^\beta\, \Delta v(0)\|. \tag{6.45}$$

(6) *Under the conditions stated in Item* (5), *for $0 \leqslant t \leqslant 2T$, one has*

$$\frac{1}{8K} e^{\lambda_2 t}\, \|A^\beta\, \Delta v(0)\| \leqslant \|A^\beta\, \Delta v(t)\| \leqslant 6K^2 e^{\lambda_3 t}\, \|A^\beta\, \Delta v(0)\|. \tag{6.46}$$

Proof. The specifications of the parameters ε_0, δ_0, etc, will be made in the course of the proof. First note that with the exception of the estimate on $z(t) \overset{\text{def}}{=} v(t, y_0) - S(t)\, v_0$, the proof of Item (1) is given in the argument preceeding the statement of the lemma. Indeed, inequality (6.35) implies that $w(t) = y(t, y_0) - S(t)\, v_0$ satisfies (6.40), with $C_0 \geqslant K_2$, $2\varepsilon_0 \leqslant \sigma_0$, and $C_0(\varepsilon_0 + \delta_0) \leqslant \min(\rho_1, \sigma_1)$. In order to prove the second inequality in (6.40), we note that $\psi(y(t, y_0)) = v(t, y_0)$ and $\psi(S(t)\, v_0) = S(t)\, v_0$, by Lemma 6.3. Hence one has

$$\begin{aligned} \|A^\beta z(t)\| &= \|A^\beta (v(t, y_0) - S(t)\, v_0)\| = \|A^\beta (\psi(y(t, y_0)) - \psi(S(t)\, v_0))\| \\ &\leqslant L\, \|A^\beta (y(t, y_0) - S(t)\, v_0)\|, \end{aligned}$$

for $0 \leqslant t \leqslant 2T$, since ψ is Lipschitz continuous on $N(\mathscr{D}_{\rho_1}(v_0), \sigma_1)$. By replacing C_0 with $\max(1, L)\, C_0$, where L is the Lipschitz coefficient of ψ, we see that $z(t)$ satisfies (6.40), as well. This completes the proof of Item (1).

For Item (2), we first note that, since $C_0(\varepsilon_0 + \delta_0) \leqslant \sigma_1$, it follows from Item (1) that $\|A^\beta (y(t, y_{i0}) - S(t)\, v_{i0})\| \leqslant \sigma_1$, for $0 \leqslant t \leqslant 2T$. Since $\|A^\beta\, \Delta S(0)\| \leqslant \rho_0$, one has $\|A^\beta\, \Delta S(t)\| \leqslant \rho_1$, by Lemma 6.5. Hence one has $y(t, y_{i0}) \in N(\mathscr{D}_{\rho_1}(S(t)\, v_{10}), \sigma_1)$, for $0 \leqslant t \leqslant 2T$. The fact that $\Delta y(t)$ satisfies (6.41), for some $C_2 \geqslant 1$ follows directly from the Lipschitz continuity of mild solutions of Eq. (1.2), see inequality (2.18), with $F_1 = F + G$ and $F_2 = F$. Since $v(t, y_{i0}) = \psi(y(t, y_{i0}))$, for $i = 1, 2$, the Lipschitz continuity of ψ implies that $\Delta v(t)$ and

$\Delta n(t)$ satisfy (6.41), as well, with C_2 replaced by $\max(1, L)\, C_2$. Since $0 \leqslant \beta < 1$, the proof of Item (3) then follows from inequalities (4.1), (6.2), (6.26), and (6.41). (Note that $\|A^\beta\, \Delta y(t)\| \leqslant C_2\, \|A^\beta\, \Delta y(0)\| \leqslant \rho$, for $0 \leqslant t \leqslant 2T$.)

For Item (4) we note that, since $E_S(t) = H_2(t)$, see Eq. (6.28), the inequality for $\|A^\beta E_S(t)\|$ in (6.43) follows from inequalities (6.29) and (6.33). For the term E_y, we note that $\Delta y = \Delta y(t)$ is a mild solution of the equation

$$\partial_t\, \Delta y + A\, \Delta y = F(y_1) - F(y_2) + G(y_1) - G(y_2).$$

We next define $\hat{R} = \hat{R}(t)$ so that the last equation becomes

$$\partial_t\, \Delta y + A\, \Delta y = DF(u_2)\, \Delta y + \hat{R},$$

which is a variation of Eq. (2.31). Thus one has

$$\hat{R} = [DF(y_2) - DF(S_2)]\, \Delta y + \int_0^1 [DF(y_2 + \theta\, \Delta y) - DF(y_2)]\, d\theta\, \Delta y$$
$$+ G(y_1) - G(y_2)$$

and

$$E_y(t) = \int_0^t \Phi(S_2(s), t-s)\, \hat{R}(s)\, ds.$$

From inequalities (6.2), (6.40), and (6.41) one obtains a $\beta_4 \in \Sigma$ such that

$$\left\| A^\beta \int_0^t \Phi(S_2(s), t-s)[DF(y_2(s)) - DF(S_2(s))]\, \Delta y(s)\, ds \right\|$$
$$\leqslant \beta_4(\varepsilon, \delta)\, \|A^\beta\, \Delta y(0)\|,$$

for $0 \leqslant t \leqslant 2T$. Next we claim that there is a constant $c_6 = c_6(2T) > 0$ such that

$$\left\| A^\beta \int_0^t \Phi(S_2(s), t-s)[G(y_1(s)) - G(y_2(s))] \right\| ds \leqslant c_6\, \delta\, \|A^\beta\, \Delta y(0)\|, \quad (6.46a)$$

for $0 \leqslant t \leqslant 2T$. Indeed, let $r = r(t)$ be defined by

$$r(t) = \int_0^t \Phi(S_2(s), t-s)[G(y_1(s)) - G(y_2(s))]\, ds.$$

From the alternate Variation of Constants Formula (4.8), one finds that

$$r(t) = \int_0^t e^{-A(t-s)} [B(s)\, r(s) + G(y_1(s)) - G(y_2(s))]\, ds,$$

where $B(s) = DF(S_2(s))$. Now inequalities (6.13) and (6.41) imply that

$$\int_0^t \|A^\beta e^{-A(t-s)} [G(y_1(s)) - G(y_2(s))]\|\, ds \leqslant C_2 \delta\, \|A^\beta\, \Delta y(0)\|,$$

for $0 \leqslant t \leqslant 2T$. By using inequality (2.6), one has

$$\|A^\beta r(t)\| \leqslant C_2 \delta\, \|A^\beta\, \Delta y(0)\| + M_\beta K_1 \int_0^t (t-s)^{-\beta}\, \|A^\beta r(s)\|\, ds,$$

where $\|DF(u)\|_{\mathscr{L}(V^{2\beta}, W)} \leqslant K_1$, for all $u \in M$. Inequality (6.46a) now follows from the Gronwall–Henry inequality, see the Appendix. By combining (6.46a) with the preceeding inequalities given in this proof and with inequality (6.42), we see that the inequality for $\|A^\beta E_y(t)\|$ in (6.43) is valid.

For the term Δv, we will use Eqs. (6.17) and (6.39) and inequality (6.4). It then follows from Eq. (6.39) that E_v satisfies

$$E_v = P^o(S_2)\, \Phi(v_{20}, t)\, \Delta n(0) + P^o(S_2)\, E_y + [P^o(v_2) - P^o(S_2)]\, \Delta y + e_3. \tag{6.47}$$

Since P^o is invariant, one has

$$P^o(S_2)\, \Phi(v_{20}, t)\, \Delta n(0) = \Phi(v_{20}, t)\, P^o(v_{20})\, \Delta n(0).$$

Since $n = Q^o(v)\, n$, one has

$$\begin{aligned} \Delta n(0) &= Q^o(v_{10})\, n_{10} - Q^o(v_{20})\, n_{20} \\ &= Q^o(v_{20})\, \Delta n(0) + [Q^o(v_{10}) - Q^o(v_{20})]\, n_{10}. \end{aligned}$$

Hence, $P^o(v_{20})\, \Delta n(0) = P^o(v_{20}) [Q^o(v_{10}) - Q^o(v_{20})]\, n_{10}$, and it follows from the Lipschitz continuity of Q^o and inequalities (6.2), (6.5), and Item (1) that there is a constant $c_7 = c_7(2T) > 0$ such that

$$\|A^\beta P^o(S_2)\, \Phi(v_{20}, t)\, \Delta n(0)\| \leqslant c_7(\varepsilon + \delta)\, \|A^\beta\, \Delta y(0)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T.$$

Recall that $\|A^\beta P^o(S_2) E_y(t)\| \leqslant K \|A^\beta E_y(t)\|$, for $t \geqslant 0$. The continuity of P^o and inequalities (6.40) and (6.41) imply that there is a $\beta_5 \in \Sigma$ such that

$$\|A^\beta [P^o(v_2) - P^o(S_2)] \,\Delta y(t)\| \leqslant \beta_5(\varepsilon, \delta) \,\|A^\beta \,\Delta y(0)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.48}$$

Using these last three inequalities with inequalities (6.5) and (6.18), and the fact that $\lambda_2 \leqslant 0$, we conclude that the inequality for $\|A^\beta E_v(t)\|$ in (6.43) is valid, as well.

In order to prove Item (5), we note that, since the solutions of (1.2) depend continuously on G in the topology $\mathscr{T}^1_A$, one has

$$\lim_{(\varepsilon, \delta) \to (0, 0)} y(t, y_0) = S(t)\, v_0, \qquad \text{in } V^{2\beta},$$

where the limit is uniform for $(v_0, t) \in M \times [0, 2T]$. In addition, one has

$$\begin{aligned} \lim_{(\varepsilon, \delta) \to (0, 0)} (y(t, y_{10}) - y(t, y_{20})) &= \lim_{(\varepsilon, \delta) \to (0, 0)} (v(t, y_{10}) - v(t, y_{20})) \\ &= S(t)\, v_{10} - S(t)\, v_{20}, \end{aligned} \tag{6.49}$$

in the space $V^{2\beta}$, uniformly for $0 \leqslant t \leqslant 2T$. We claim that, for every $r > 0$, there exist $\varepsilon_1 = \varepsilon_1(r) > 0$ and $\delta_1 = \delta_1(r) > 0$ such that

$$\frac{1}{2} \leqslant \frac{\|A^\beta \,\Delta v(t)\|}{\|A^\beta \,\Delta S(t)\|} \leqslant \frac{3}{2}, \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_1 \quad \text{and} \quad 0 < \delta \leqslant \delta_1, \tag{6.50}$$

whenever, $\|A^\beta \,\Delta v(0)\| \geqslant r$. Indeed, inequalities (6.31) and (6.41) imply that the middle term in (6.50) is bounded and uniformly continuous, for $0 \leqslant t \leqslant 2T$, on the set of $v_{10}, v_{20} \in M$ with $\|A^\beta \,\Delta v(0)\| \geqslant r$. It then follows from Eq. (6.49) that there exist $\varepsilon_1 = \varepsilon_1(r) > 0$ and $\delta_1 = \delta_1(r) > 0$ such that inequality (6.50) is valid, which in turn implies that inequality (6.44) holds under the same conditions.

For the sequel we will fix $r = C_2^{-1} \rho_0$, where ρ_0 is given by Lemma 6.5. It remains to verify inequality (6.44), for $\|A^\beta \,\Delta v(0)\| \leqslant C_2^{-1} \rho$, where $0 < \rho \leqslant \rho_0$. Let us begin with an outline our strategy, where v_{i0} and y_{i0}, for $i = 1, 2$, satisfy

$$\|A^\beta \,\Delta y(0)\| \leqslant 3 \,\|A^\beta \,\Delta v(0)\|. \tag{6.51}$$

Next note that if $\Delta z = \Delta v - \Delta S$ satisfies

$$\|A^\beta \Delta z(t)\| \leqslant \tfrac{1}{2} \,\|A^\beta \,\Delta S(t)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T, \tag{6.52}$$

then inequality (6.44) is valid.

In order to prove inequality (6.52), we note that $\Delta S(0) = \Delta v(0)$ and $z(t) = E_v(t) - E_S(t)$. It then follows from (6.34) and (6.43) that inequality (6.45) holds. Next one chooses $\varepsilon_2 = \delta_2 > 0$ so that $\beta_6(\varepsilon_2, \varepsilon_2) \leqslant \frac{1}{16K} e^{2\lambda_2 T}$. Since $\lambda_2 \leqslant 0$, it follows from inequalities (6.19), (6.26), and (6.33) that one has

$$b_4^F(\rho, 2\varepsilon) \leqslant b_5^F(\rho, 2\varepsilon) \leqslant \frac{1}{16K} e^{2\lambda_2 T} \leqslant \frac{1}{16}, \tag{6.53}$$

for $0 < \rho \leqslant \rho_0$ and $0 \leqslant 2\varepsilon \leqslant \min(\sigma_0, 2\varepsilon_2) \leqslant \sigma_1$. Now set ε_0 and $\delta_0(\varepsilon)$ so that

$$2\varepsilon_0 = \min(\sigma_0, C_0^{-1} \min(\rho_1, \sigma_1), 2\varepsilon_1(r), 2\varepsilon_2) \quad \text{and} \quad \delta_0(\varepsilon) = \min(\varepsilon, \delta_1(r), \delta_2),$$

for $0 < \varepsilon \leqslant \varepsilon_0$. One then has $2\varepsilon_0 \leqslant \sigma_0$ and $C_0(\varepsilon_0 + \delta_0) \leqslant \min(\rho_1, \sigma_1)$. It then follows from inequality (6.31) that inequality (6.52) is valid, for $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < \delta \leqslant \delta_0(\varepsilon)$. Finally, inequality (6.46) follows directly from inequalities (6.31) and (6.44). ▮

For any pair $(f, g) \in \mathscr{F} \times \mathscr{G} = \mathscr{F}(\varepsilon, \ell) \times \mathscr{G}(\varepsilon, \ell)$, we define $h: M \to V^{2\beta}$ by

$$y_0 = h(v_0) = v_0 + f(v_0) + g(v_0), \qquad \text{for} \quad v_0 \in M.$$

We assume that $0 < \varepsilon \leqslant \min(\rho_0, \frac{1}{3}\sigma_0)$. Since $\|A^\beta(y_0 - v_0)\| \leqslant 2\varepsilon \leqslant \sigma_0$, one has $y(t, y_0) \in N(\mathscr{D}_{\rho_1}(S(t) v_0), \sigma_1)$, for $0 \leqslant t \leqslant 2T$. The mapping $v(t, y_0) = \psi(v(t, y_0)) = \psi(v_0, v(t, y_0))$, which is defined by the local coordinate representation and which is valid for $0 \leqslant t \leqslant 2T$, admits a well-defined extension

$$v(t, y_0) = \psi^e(y(t, y_0)) = \psi(S(t) v_0, y(t, y_0)),$$

on a larger time interval, as long as $\|A^\beta(y(t, y_0) - S(t) v_0)\| \leqslant \rho_1$. Furthermore, if t_i satisfies $0 \leqslant t_i \leqslant 2T$, for $i = 1, 2$, and $0 \leqslant t_1 + t_2 \leqslant 2T$, then one has

$$v(t_1, y(t_2, y_0)) = v(t_1 + t_2, y_0) = v(t_2, y(t_1, y_0)).$$

Indeed, we define $S_2(t) y_0 = y(t, y_0)$ to be the unique mild solution of the perturbed equation (1.2) with $S_2(0) y_0 = y_0$. Since one has $S_2(t_1) S_2(t_2) = S_2(t_1 + t_2) = S_2(t_2) S_2(t_1)$, one obtains

$$\begin{aligned} v(t_1, y(t_2, y_0)) &= \psi^e(S_2(t_1) y(t_2, y_0)) = \psi^e(S_2(t_1) S_2(t_2) y_0) \\ &= \psi^e(S_2(t_1 + t_2) y_0) = v(t_1 + t_2, y_0) \\ &= \psi^e(S_2(t_2) S_2(t_1) y_0) = v(t_2, y(t_1, y_0)). \end{aligned} \tag{6.54}$$

Let $(f, g) \in \mathscr{F} \times \mathscr{G}$ be given, where $\ell \leqslant 1$, and consider the collection of all solutions $y(t, y_0)$ of the perturbed equation (1.2) with $y_0 = h(v_0) \stackrel{\text{def}}{=} v_0 + f(v_0) + g(v_0)$, where $v_0 \in M$. For $0 < \varepsilon \leqslant \varepsilon_0$ and $0 \leqslant \delta \leqslant \delta_0$, one has $\|A^\beta(y(t, y_0) - S(t)\, v_0)\| \leqslant \sigma_1$, for $0 \leqslant t \leqslant 2T$. We claim that

$$M_t \stackrel{\text{def}}{=} \{v(t, h(v_0)) : v_0 \in M\} = M, \qquad \text{for each} \quad t \in [0, 2T]. \tag{6.55}$$

Since $v(0, h(v_0)) = v_0$, it follows that $M_0 = M$. For $t > 0$ we note that the mapping $(v_0, t) \to v(t, h(v_0))$ is a continuous mapping of $M \times [0, 2T]$ into M. Since M is compact, it follows that M_t is compact. Furthermore, inequality (6.46) implies that the mapping $v_0 \to v(t, h(v_0))$ is an open mapping. Thus M_t is open. Since M is connected, this implies that $M_t = M$, for all $t \in [0, 2T]$. Furthermore, due to the choice of ρ_1, see (6.11), we see that the mapping $v_0 \to v(t, h(v_0))$ is a homeomorphism of M onto M_t, for each $t \in [0, 2T]$.

6.4. *Proofs of Theorems.* The basic idea in the proofs of the Main Theorem and the Shadow Theorem is to construct a mapping $(f, g; \tau) \to (\bar{f}_\tau, \bar{g}_\tau)$, which is defined on $\mathscr{F} \times \mathscr{G}$, for $T \leqslant \tau \leqslant 2T$. We will show that for ε and δ small, and for $T \leqslant \tau \leqslant 2T$, the following hold:

(1) One has $\|A^\beta \bar{f}_\tau(v)\| \leqslant \frac{3}{4}\varepsilon$ (Lemma 6.7).

(2) The function $\bar{f}_\tau$ is Lipschitz continuous and belongs to $\mathscr{F}$ (Lemma 6.8).

(3) The mapping $(f, g) \to \bar{f}_\tau$ is contracting on $\mathscr{F}$ (Lemma 6.9).

(4) The function $\bar{g}_\tau$ satisfies the same three properties. (Lemmas 6.10, 6.11, and 6.12).

(5) Let (f, g) denote the fixed point of the mapping A_τ: $(f, g) \to (\bar{f}_\tau, \bar{g}_\tau)$, and set $h(v) = v + f(v) + g(v)$, for $v \in M$. Then $M^G = h(M)$ is an invariant set for the perturbed equation (1.2), and the other properties of the Main Theorem and the Shadow Theorem are valid (Theorems 6.13 and 6.14).

For each pair $(f, g) \in \mathscr{F}(\varepsilon, \ell) \times \mathscr{G}(\varepsilon, \ell)$, where $\ell \leqslant 1$, we define a new function $\bar{f}_\tau$ by

$$\bar{f}_\tau(v(\tau, y_0)) = P^s(v(\tau, y_0))(y(\tau, y_0) - v(\tau, y_0)), \qquad \text{for} \quad T \leqslant \tau \leqslant 2T, \tag{6.56}$$

where $y_0 = v_0 + f(v_0) + g(v_0)$. Note that for each $v_0 \in M$, it follows from (6.23) that for $T \leqslant \tau \leqslant 2T$ one has

$$\bar{f}_\tau(v(\tau, y_0)) = P^s(v(\tau, y_0))\, n(\tau, y_0) = s(\tau, y_0) \in U^s(v(\tau, y_0)),$$

since $U^s(u)$ is the range of the projection $P^s(u)$. Equation (6.56) gives the value of $\bar{f}_\tau$ at the point $v(\tau, y_0) \in M$. Due to Lemma 6.3 and (6.55), we see that $\bar{f}_\tau$ is well-defined everywhere on M, and the mapping $(v_0, \tau) \to \bar{f}_\tau(v_0)$ is a continuous mapping of $M \times [T, 2T]$ into $V^{2\beta}$.

LEMMA 6.7. *Let the hypotheses of Lemma* 6.6 *be satisfied, and let* ε_0 *and* δ_0 *be given by Lemma* 6.6. *Then there is an* ε_4, *with* $0 < \varepsilon_4 \leqslant \varepsilon_0$, *such that for all* ε *with* $0 < \varepsilon \leqslant \varepsilon_4$, *there is a* $\delta_4 = \delta_4(\varepsilon)$, *with* $0 < \delta_4 \leqslant \delta_0$, *such that if* $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta = \delta_4$, *and if* $f \in \mathscr{F}(\varepsilon, \ell)$ *and* $g \in \mathscr{G}(\varepsilon, \ell)$, *where* $0 < \ell \leqslant 1$, *then one has*

$$\|A^\beta \bar{f}_\tau(v)\| \leqslant \tfrac{3}{4}\varepsilon, \qquad \text{for all} \quad v \in M \quad \text{and} \quad T \leqslant \tau \leqslant 2T. \tag{6.57}$$

Proof. Due to (6.55), it suffices to verify (6.57) when $v = v(\tau, y_0)$ and $y_0 = v_0 + f(v_0) + g(v_0)$, for some $v_0 \in M$ and $T \leqslant \tau \leqslant 2T$. Let $w(t) = y(t, y_0) - S(t)\, u_0$. From (6.36) one has $w(t) = \Phi(v_0, t)\, w(0) + e(t)$, for $0 \leqslant t \leqslant 2T$, and e satisfies (6.38). Also one has $w(0) = y_0 - u_0 = f(v_0) + g(v_0)$. From the definition of $\bar{f}_\tau$ in (6.56) we have

$$\|A^\beta \bar{f}_\tau(v(\tau, y_0))\| = \|A^\beta P^s(v(\tau, y_0))(w(\tau) + S(\tau)\, v_0 - v(\tau, y_0))\|,$$

and from (6.36) we obtain

$$\|A^\beta \bar{f}_\tau(v(\tau, y_0))\| \leqslant I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \|A^\beta P^s(v(\tau, v_0))\, \Phi(u_0, \tau)\, f(v_0)\|$$

$$I_2 = \|A^\beta P^s(v(\tau, v_0))\, \Phi(u_0, \tau)\, g(v_0)\|$$

$$I_3 = \|A^\beta P^s(v(\tau, v_0))\, e(\tau)\|$$

$$I_4 = \|A^\beta P^s(v(\tau, y_0))(S(\tau)\, v_0 - v(\tau, y_0))\|.$$

From the invariance property (4.7) we obtain

$$P^s(S(\tau)\, v_0)\, \Phi(v_0, \tau)\, f(v_0) = \Phi(v_0, \tau)\, f(v_0),$$

and inequalities (3.9) and (6.10) imply that

$$\|A^\beta \Phi(v_0, t)\, f(v_0)\| \leqslant K e^{\lambda_1 \tau} \varepsilon < \tfrac{1}{4}\varepsilon, \qquad \text{for} \quad T \leqslant \tau \leqslant 2T.$$

By using this fact, with the continuity of P^s and inequality (6.40), we find that there is a $\beta_3 \in \Sigma$ such that

$$I_1 \leqslant \|A^\beta[P^s(v(\tau, y_0)) - P^s(S(\tau) v_0)] \Phi(v_0, \tau) f(v_0)\| + \|A^\beta \Phi(v_0, \tau) f(v_0)\|$$
$$\leqslant (4\beta_3(\varepsilon, \delta) + 1) \|A^\beta \Phi(v_0, \tau) f(v_0)\| \leqslant \beta_3(\varepsilon, \delta)\varepsilon + \tfrac{1}{4}\varepsilon.$$

Similarly (4.7) implies that $P^s(S(\tau) v_0) \Phi(v_0, \tau) g(v_0) = 0$, since $g(v_0) \in U^u(v_0)$. Therefore from (6.40) and the continuity of P^s, we find that there is a $\beta_4 \in \Sigma$ such that

$$I_2 = \|A^\beta(P^s(v(\tau, y_0)) - P^s(S(\tau) v_0)) \Phi(v_0, \tau) g(v_0)\|$$
$$\leqslant \beta_4(\varepsilon, \delta) \|A^\beta g(v_0)\| \leqslant \beta_4(\varepsilon, \delta)\varepsilon.$$

From (6.5) and (6.38) one has $I_3 \leqslant K((\varepsilon + \delta) b_0(\varepsilon, \delta) + C_1\delta)$. Lastly, since $v(\tau, y_0) \in \mathscr{D}_{\rho_1}(S(\tau) v_0)$, it follows from (6.8) and (6.40) that

$$I_4 \leqslant K^2 L_0 \|A^\beta(S(\tau) v_0 - v(\tau, y_0))\|^2 \leqslant C_0^2 K^2 L_0 (\varepsilon + \delta)^2.$$

From these estimates on I_1, I_2, I_3, and I_4, we find that

$$\|A^\beta \bar{f}_\tau(v(\tau, y_0))\| \leqslant (\varepsilon + \delta) \beta_5(\varepsilon, \delta) + \tfrac{1}{4}\varepsilon + KC_1\delta,$$

for some $\beta_5 \in \Sigma$. Next choose $\varepsilon_4 > 0$ so that $\beta_5(\varepsilon_4, \varepsilon_4) \leqslant \frac{1}{4}$ and $0 < \varepsilon_4 \leqslant \varepsilon_0$. Then with

$$\delta_4 = \delta_4(\varepsilon) = \min(\delta_0, \varepsilon_4, \tfrac{1}{4}(KC_1 + \tfrac{1}{4})^{-1} \varepsilon), \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_4,$$

the inequality (6.57) holds whenever $v \in M$ is of the form $v = v(\tau, y_0)$. ∎

In the following result, we argue that $\bar{f}_\tau(v)$ is (locally) Lipschitz continuous in v, on each disk $\mathscr{D}_{\rho_0}(v_0) \subset M$, provided that $T \leqslant \tau \leqslant 2T$.

LEMMA 6.8. *Let the hypotheses of Lemma* 6.6 *be satisfied. Then there is an* ε_5 *with* $0 < \varepsilon_5 \leqslant \varepsilon_4$ *such that for all* ε *with* $0 < \varepsilon \leqslant \varepsilon_5$, *there exist* $\delta_5 = \delta_5(\varepsilon)$ *with* $0 < \delta_5 \leqslant \delta_4$, *where* ε_4 *and* δ_4 *are given in Lemma* 6.7, *and for all* ρ *with* $0 < \rho \leqslant \rho_0$, *there is an* $\ell_5 = \ell_5(\rho, \varepsilon)$ *with* $8KC_2\ell_5(\rho_0, \sigma_0) \leqslant 1$ *and* $0 \leqslant \ell_5(\rho, \varepsilon) < 1$, *such that if* $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta = \delta_5$ *and if* $f \in \mathscr{F} = \mathscr{F}(\varepsilon, \ell)$ *and* $g \in \mathscr{G} = \mathscr{G}(\varepsilon, \ell)$ *with* $\ell \leqslant 1$ *and* $\varepsilon \leqslant \varepsilon_5$, *then the restriction of* $\bar{f}_\tau$ *to the disk* $\mathscr{D}_\rho(v_0)$, *for* $0 < \rho \leqslant \rho_0$, *is Lipschitz continuous with Lipschitz coefficient* ℓ_5, *for every* $v_0 \in M$ *and for* $T \leqslant \tau \leqslant 2T$. *Moreover, one has* $\bar{f}_\tau \in \mathscr{F}$, *and* $\ell_5(\rho, \varepsilon) \in \Sigma$.

Proof. Let $(f, g) \in \mathcal{F} \times \mathcal{G}$ be given. We assume that ε and δ satisfy $0 < \varepsilon \leqslant \varepsilon_4$ and $0 < \delta \leqslant \delta_4$ so that Lemma 6.7 holds. We will use the Special Notation, where $v_{10} \in \mathcal{D}_\rho(v_{20})$, for $0 < \rho \leqslant \rho_0$, and $y_{i0} = h(v_{i0}) = v_{i0} + f(v_{i0}) + g(v_{i0})$, for $i = 1, 2$. Since $\|A^\beta \Delta y(0)\| \leqslant (1 + 2\ell) \|A^\beta \Delta v(0)\|$, and $\ell \leqslant 1$, we see that inequality (6.51) is valid.

Due to (6.55), it will suffice to show that, under the hypotheses stated in this lemma, there is a suitable $\ell_5 = \ell_5(\rho, \varepsilon)$ such that

$$\frac{\|A^\beta(\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_2(\tau)))\|}{\|A^\beta(v_1(\tau) - v_2(\tau))\|} \leqslant \ell_5, \qquad \text{for} \quad T \leqslant \tau \leqslant 2T.$$

Let $\Delta s = \Delta s(t) = \Phi(v_{20}, t)\, P^s(v_{20})\, \Delta n(0) + E_s(t)$ be given as in (6.39), where

$$\Delta s = \bar{f}_\tau(v_1) - \bar{f}_\tau(v_2) = P^s(v_1)\, n_1 - P^s(v_2)\, n_2, \qquad \text{at} \quad t = \tau.$$

By a straightforward calculation, which makes use of the Special Notation in (6.39) and Eq. (6.17), one finds that $E_s = E_s(t)$ satisfies

$$\begin{aligned} E_s = {} & [P^s(v_1) - P^s(v_2)]\, n_1 + [P^s(v_2) - P^s(S_2)]\, \Delta n \\ & + P^s(S_2)[Q^o(v_2) - Q^o(S_2)]\, \Delta y \\ & + P^s(S_2)\, E_y - P^s(S_2)\, e_3(y_2, \Delta y) + P^s(S_2)\, \Phi(v_{20}, t)\, \Delta v(0), \end{aligned}$$

for $0 \leqslant t \leqslant 2T$. By using Lemma 6.6, along with (6.4), (6.5), (6.40), (6.41), (6.51), and $K \geqslant 1$, one can show that each of the terms

$$\|A^\beta(P^s(v_1) - P^s(v_2))\, n_1\|, \qquad \|A^\beta(P^s(v_2) - P^s(S_2))\, \Delta n\|,$$

$$\text{and} \qquad \|A^\beta P^s(S_2)[Q^o(v_2) - Q^o(S_2)]\, \Delta y\|$$

is bounded by $6C_0 C_2 K L_0(\varepsilon + \delta) \|A^\beta \Delta v(0)\|$. Also from the invariance of the projectors and inequality (6.8), one finds that there is a constant $C_4 = C_4(2T) > 0$ such that

$$\|A^\beta P^s(S_2)\, \Phi(v_{20}, t)\, \Delta v(0)\| \leqslant C_4 \rho\, \|A^\beta \Delta v(0)\|.$$

Moreover, Lemma 6.6, (6.5), and (6.18) yield

$$\|A^\beta P^s(S_2)\, e_3(y_2, \Delta y)\| \leqslant K C_2\, b_2^F(\rho)\, \|A^\beta \Delta y(0)\|.$$

From inequalities (6.5) and (6.43), we obtain

$$\|A^\beta P^s(S_2)\, E_y(t)\| \leqslant K(C_3 b_1^F(\rho, 2\varepsilon) + \beta_1(\varepsilon, \delta))\, \|A^\beta\, \Delta y(0)\|, \quad \text{for } 0 \leqslant t \leqslant 2T.$$

Consequently, these estimates and (6.41) imply that there are $\beta_4, \beta_5, \beta_6 \in \Sigma$ such that $\beta_5 \leqslant \beta_6$ and

$$\begin{aligned} \|A^\beta E_s\| &\leqslant (C_4\rho + \beta_4(\varepsilon, \delta))\, \|A^\beta\, \Delta v(0)\| \\ &\quad + (b_4^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \delta))\, \|A^\beta\, \Delta y(0)\| \\ &\leqslant (C_4\rho + 3b_4^F(\rho, 2\varepsilon) + \beta_6(\varepsilon, \delta))\, \|A^\beta\, \Delta v(0)\|, \end{aligned} \tag{6.58}$$

for $0 \leqslant t \leqslant 2T$. It follows from inequality (3.9) that, for $t \geqslant 0$, one has

$$\|A^\beta \Phi(v_{20}, t)\, P^s(v_{20})\, \Delta n(0)\| \leqslant \begin{cases} Ke^{\lambda_1 t}\, \|A^\beta P^s(v_{20})\, \Delta n(0)\|, \\ Ke^{\lambda_1 t}\, \|A^\beta\, \Delta n(0)\|. \end{cases} \tag{6.59}$$

Since $\|A^\beta\, \Delta n(0)\| \leqslant 2\ell\, \|A^\beta\, \Delta v(0)\|$, inequalities (6.58) and (6.59) imply that

$$\|A^\beta\, \Delta s(t)\| \leqslant (2Ke^{\lambda_1 t}\ell + C_4\rho + 3b_4^F(\rho, 2\varepsilon) + \beta_6(\varepsilon, \delta))\, \|A^\beta\, \Delta v(0)\|,$$

for $0 \leqslant t \leqslant 2T$ and $0 < \rho \leqslant \rho_0$. By replacing ρ_0 and σ_0 with a smaller values, if necessary, we may assume that

$$\left.\begin{cases} 16K^3 C_2 L_0\, \rho_0 e^{-\lambda_4 T} \\ 48KC_4 e^{-\lambda_2 T} \rho_0 \\ 144K^2 e^{-\lambda_2 T} b_4^F(\rho_0, \sigma_0) \end{cases}\right\} \leqslant 1. \tag{6.60}$$

Next we choose ε_5 so that $0 < \varepsilon_5 \leqslant \varepsilon_4$ and

$$48Ke^{-2\lambda_2 T}\beta_6(\varepsilon_5, \varepsilon_5) \leqslant 1. \tag{6.61}$$

Set $\delta_5(\varepsilon) = \min(\varepsilon, \delta_4(\varepsilon))$, for $0 < \varepsilon \leqslant \varepsilon_5$. Since $\ell \leqslant 1$, it then follows from the last three inequalities and inequalities (6.10) and (6.46) that

$$\frac{\|A^\beta\, \Delta s(\tau)\|}{\|A^\beta\, \Delta v(\tau)\|} \leqslant \ell_5(\rho, \varepsilon) \leqslant \frac{2}{3}, \qquad \text{for} \quad T \leqslant \tau \leqslant 2T,$$

$0 < \varepsilon \leqslant \varepsilon_5$ and $0 < \rho \leqslant \rho_0$, where

$$\ell_5 = \ell_5(\rho, \varepsilon) = \tfrac{1}{6}\ell + (C_4\rho + 3b_4^F(\rho, 2\varepsilon) + \beta_6(\varepsilon, \varepsilon))\, 8Ke^{-2\lambda_2 T}. \tag{6.62}$$

Equation (6.62) defines a mapping $\ell \to \ell_5$, where $0 \leqslant \ell = \ell(\rho, \varepsilon) \leqslant 1$. Since this mapping is a strict contraction in the L^∞-norm, there is a unique fixed point $\ell_5 = \ell$, where $\ell_5 = \frac{5}{6}(C_4 \rho + 3b_4^F(\rho, 2\varepsilon) + \beta_6(\varepsilon, \varepsilon))\, 8Ke^{-\lambda_2 T}$, i.e., $\ell_5 = \ell_5(\rho, \varepsilon) \in \Sigma$. Since $\ell_5 \in \Sigma$, one can choose smaller values of ρ_0 and σ_0, if necessary, to insure that $8KC_2\, \ell_5(\rho_0, \sigma_0) \leqslant 1$. ▮

LEMMA 6.9. *Let the hypotheses of Lemma 6.8 be satisfied. Then for every ε with $0 < \varepsilon \leqslant \varepsilon_5$ and $\delta \leqslant \delta_5(\varepsilon)$, if $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, then for all $(f_i, g_i) \in \mathscr{F}(\varepsilon, \ell) \times \mathscr{G}(\varepsilon, \ell)$, where $\ell = 1$, one has*

$$\|A^\beta(\bar{f}_{\tau,1}(u) - \bar{f}_{\tau,2}(u))\| \leqslant \tfrac{1}{2} \|(f_1, g_1) - (f_2, g_2)\|_\infty, \qquad \text{for all} \quad u \in M,$$

where $\bar{f}_{\tau,i}$ are given by (6.56), for $i = 1, 2$.

Proof. Let τ be fixed where $T \leqslant \tau \leqslant 2T$. Once again we will use the Special Notation, where now one has

$$v_{10} = v_{20} = v_0 = v_0 \in M \qquad \text{and} \qquad y_{i0} = v_0 + f_i(v_0) + g_i(v_0), \qquad \text{for} \quad i = 1, 2.$$

Recall that $\bar{f}_i(v_i(\tau)) = s_i(\tau) = P^s(v_i(\tau))\, n_i(\tau)$, where $\bar{f}_i = \bar{f}_{\tau,i}$, for $i = 1, 2$.

By Lemma 6.6, the functions $y_i(t)$, $v_i(t)$ and $n_i(t)$, $s_i(t)$, and $u_i(t)$ are Lipschitz continuous functions of the initial data y_{i0}. From the construction of these solutions one has $\Delta v(0) = 0$, $\Delta y(0) = \Delta n(0)$, and

$$\|A^\beta\, \Delta y(0)\| \leqslant \|A^\beta\, \Delta f(v_0)\| + \|A^\beta\, \Delta g(v_0)\| \leqslant \|(f_1, g_1) - (f_2, g_2)\|_\infty, \tag{6.63}$$

where $\Delta f(v_0) = f_1(v_0) - f_2(v_0)$ and $\Delta g(v_0) = g_1(v_0) - g_2(v_0)$. Also note that

$$P^s(v_2)\, \Delta s = \Delta \bar{f}(v_2) + P^s(v_2)[\bar{f}_1(v_1) - \bar{f}_1(v_2)], \qquad \text{at} \quad t = \tau, \tag{6.64}$$

where $\Delta \bar{f}(v_0) = \bar{f}_1(v_0) - \bar{f}_2(v_0)$. From the first inequaliity in (6.58), one finds that

$$\|A^\beta E_s\| \leqslant (b_4^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \varepsilon))\, \|A^\beta\, \Delta y(0)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{6.65}$$

Since $P^s(v_0)\, \Delta n(0) = \Delta f(v_0)$, inequality (6.59) becomes

$$\|A^\beta \Phi(v_{20}, t)\, P^s(v_{20})\, \Delta n(0)\| \leqslant Ke^{\lambda_1 t}\, \|A^\beta\, \Delta f(v_0)\|, \qquad \text{for} \quad t \geqslant 0.$$

Therefore, from (6.65), one obtains

$$\|A^\beta\, \Delta s(t)\| \leqslant Ke^{\lambda_1 t}\, \|A^\beta\, \Delta f(v_0)\| + (b_4^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \delta))\, \|A^\beta\, \Delta y(0)\|,$$

for $0 \leqslant t \leqslant 2T$. Since $K \geqslant 1$, $\lambda_2 \leqslant 0$, and $\beta_5 \leqslant \beta_6$, it then follows from inequalities (6.60), (6.61), (6.63), and (6.64) that, for $t = \tau$, one has

$$\begin{aligned}\|A^\beta \,\Delta \bar{f}(v_2)\| &\leqslant \|A^\beta P^s(v_2)\,\Delta s\| + \|A^\beta P^s(v_2)[\bar{f}_1(v_1) - \bar{f}_1(v_2)]\| \\ &\leqslant K\,\|A^\beta\,\Delta s\| + KC_2\,\ell_5(\rho, \varepsilon)\,\|A^\beta\,\Delta y(0)\| \\ &\leqslant K^2 e^{\lambda_1\tau}\,\|A^\beta\,\Delta f(v_0)\| + (KC_2\ell_5(\rho, \varepsilon) + Kb_4^F(\rho, 2\varepsilon) \\ &\quad + K\beta_5(\varepsilon, \varepsilon))\,\|A^\beta\,\Delta y(0)\| \leqslant (K^2 e^{\lambda_1\tau} + KC_2\,\ell_5(\rho, \varepsilon) + Kb_4^F(\rho, 2\varepsilon) \\ &\quad + K\beta_5(\varepsilon, \delta))\,\|(f_1, g_1) - (f_2, g_2)\|_\infty.\end{aligned}$$

From Lemma 6.8 and inequalities (6.10), (6.60), and (6.61), we see that each of the terms $K^2 e^{\lambda_1\tau}$, $Kb_4^F(\rho, 2\varepsilon)$, $KC_2\,\ell_5(\rho, \varepsilon)$, and $K\beta_5(\varepsilon, \delta)$ is $\leqslant \frac{1}{8}$. ∎

Let $(f, g) \in \mathscr{F} \times \mathscr{G}$. We now seek to define a new function $\bar{g}_\tau$, which is a companion to the function $\bar{f}_\tau$ given by (6.56). Among other things, we want $\bar{g}_\tau(v_0)$ to be in $U^u(v_0)$, for every $v_0 \in M$. Let $v_0 \in M$ be given, and define $y_0 = y_0(V) = v_0 + f(v_0) + V$, where $V \in U^u(v_0)$ will be treated as a parameter. Consider the equation

$$g(v(\tau, y_0(V))) = P^u(v(\tau, y_0(V)))(y(\tau, y_0(V)) - v(\tau, y_0(V))). \tag{6.66}$$

Our objective is to show that if ε and δ are sufficiently small, then Eq. (6.66) has a unique solution $V \in U^u(v_0)$. In this case we will denote this solution by $V = \bar{g}_\tau(v_0)$, thereby defining $\bar{g}_\tau$. Before proving this property, it is convenient to write Eq. (6.66) in the abbreviated form $g(v) = P^u(v)(y - v) = P^u(v)\,n$, where $y = y(\tau, y_0(V))$, $v = v(\tau, y_0(V))$ and $n = y - v$. Note that Eq. (6.66) holds in the subspace $\mathscr{R}(P^u(v))$. By adding and subtracting the three terms $P^u(S)\,y$, $P^u(S)\,S$, and $P^u(S)\,v$ in Eq. (6.66), where $S = S(\tau)\,v_0$, we see that Eq. (6.66) takes on the equivalent form

$$g(v) = P^u(S)(y - S) - P^u(S)(v - S) + [P^u(v) - P^u(S)](y - v). \tag{6.67}$$

Notice that each of the terms y, v, g, and $P^u(v)$ are Lipschitz continuous functions of the parameter V, while the term S does not depend on V.

Lemma 6.10. *Let the hypotheses of Lemma* 6.6 *be satisfied. Then there is an $\varepsilon_7 > 0$ such that $\varepsilon_7 \leqslant \varepsilon_5$ and for all ε with $0 < \varepsilon \leqslant \varepsilon_7$, there is a $\delta_7 = \delta_7(\varepsilon) \in \Sigma$ with $0 < \delta_7 \leqslant \delta_5$, where ε_5 and δ_5 are given in Lemma* 6.8, *such that if $\|G\|_{\{A;\,C^1(\Omega)\}} \leqslant \delta_7$, and if $f \in \mathscr{F}(\varepsilon, \ell)$ and $g \in \mathscr{G}(\varepsilon, \ell)$ for any ℓ with $0 < \ell \leqslant 1$, then for each $v_0 \in M$ and $\tau \in [T, 2T]$, there is a unique solution $V = \bar{g}_\tau(v_0) \stackrel{\text{def}}{=} V(v_0)$ of* (6.66), *where $V \in U^u(v_0)$, with $\|A^\beta V\| \leqslant \frac{3}{4}\varepsilon$. Moreover, $\bar{g}_\tau(v_0)$ is continuous for $v_0 \in M$ and $T \leqslant \tau \leqslant 2T$.*

Proof. Let $w = w(t) = y(t, y_0) - S(t)\, v_0$, where $y_0 = v_0 + f(v_0) + V$, $V \in U^u(v_0)$, and $\|A^\beta V\| \leqslant \varepsilon$. From (6.36) one has $w(t) = \Phi(v_0, t)\, w(0) + e(t, y_0)$, for $0 \leqslant t \leqslant 2T$, where $w(0) = w_0 = f(v_0) + V$. Since the dependence of $e(t, y_0)$ on V is especially important, we will write this as $e(t, y_0) = \hat{e}(V, t)$. Since $f(v_0) \in U^s(v_0)$ and $V \in U^u(v_0)$, one has $P^u(v_0)\, f(v_0) = 0$ and $P^u(v_0)\, V = V$. It then follows from the invariance property (4.7) that

$$\begin{cases} P^u(S)\, \Phi(v_0, \tau)\, f(v_0) = \Phi(v_0, \tau)\, P^u(v_0)\, f(v_0) = 0, \\ P^u(S)\, \Phi(v_0, \tau)\, w(0) = \Phi(v_0, \tau)\, P^u(v_0)\, w(0) = \Phi(v_0, \tau)\, V, \end{cases}$$

$$\text{where} \quad S = S(\tau)\, v_0.$$

Thus by applying $P^u(S)$ to (6.36), one obtains

$$P^u(S)(y - S) = \Phi(v_0, \tau)\, V + P^u(S)\, \hat{e}(V, \tau), \qquad \text{for} \quad T \leqslant \tau \leqslant 2T, \tag{6.68}$$

where $\Phi(v_0, \tau)\, V = \Phi(v_0, \tau)\, P^u(v_0)\, V = P^u(S)\, \Phi(v_0, \tau)\, V$, by the invariance property (4.7). Now Lemma 3.2 implies that $\Phi(v_0, \tau)\, P^u(v_0)$ has a unique extension, for all $\tau \in R$, and the stronger cocycle condition

$$\Phi(S(\tau)\, v_0, s)\, \Phi(v_0, \tau)\, P^u(v_0) = \Phi(v_0, s + \tau)\, P^u(v_0), \qquad \text{for all} \quad s, \tau \in R,$$

is valid. In particular, with $s = -\tau$ and $S = S(\tau)\, v_0$, one has

$$\Phi(S, -\tau)\, P^u(S)\, \Phi(v_0, \tau) = \Phi(S, -\tau)\, \Phi(v_0, \tau)\, P^u(v_0) = P^u(v_0).$$

Consequently one has, $\Phi(S, -\tau)\, P^u(S)\, \Phi(v_0, \tau)\, V = P^u(v_0)\, V = V$. Next we multiply Eq. (6.67) on the left with $\Phi(S, -\tau)\, P^u(S)$, use the last equality and Eq. (6.68), and thereby rewrite Eq. (6.67) in the form

$$V = \Gamma(V) = \Gamma(V, v_0), \tag{6.69}$$

where $v_0 \in M$, $V \in U^u(v_0)$, $(f, g) \in \mathscr{F} \times \mathscr{G}$, and

$$\begin{aligned} \Gamma(V, v_0) &= \Gamma(V, v_0; f, g) \\ &\stackrel{\text{def}}{=} \Phi(S, -\tau)\, P^u(S)[-\hat{e}(V, \tau) + (v - S) + [P^u(S) - P^u(v)]\, n + g(v)]. \end{aligned} \tag{6.70}$$

Notice that the function $\Gamma(V, v_0)$ defined by Eq. (6.70) is well-defined on

$$\mathscr{D}(\Gamma) \stackrel{\text{def}}{=} \{(V, v_0) \in V^{2\beta} \times M : V \in \mathscr{R}(P^u(v_0)) \text{ and } \|A^\beta V\| \leqslant \varepsilon\}.$$

Also note that the range of Γ lies in $U^u(v_0)$, for each $v_0 \in M$. Indeed, the invariance property (4.7) implies that $P^u(v_0)\,\Phi(S, -\tau) = \Phi(S, -\tau)\,P^u(S)$, which in turn, implies that $P^u(v_0)\,\Gamma(V, v_0) = \Gamma(V, v_0)$, for all $(V, v_0) \in \mathscr{D}(\Gamma)$. (The reader should verify that one has $V = \Gamma(V, v_0)$ if and only if $g(v) = P^u(v)\,n$.)

Since $w(t) = w(t, y_0(V)) = y(t, y_0(V)) - S(t)\,v_0$ is a solution of Eq. (4.9), it follows from (6.36) that

$$\hat{e}(V, t) = \int_0^t \Phi(S(s)\,v_0, t - s)\,H(S(s)\,v_0, w(s))\,ds.$$

Inequality (6.38) implies that

$$\|A^\beta \hat{e}(V, t)\| \leqslant (\varepsilon + \delta)\,b_0(\varepsilon, \delta) + C_1\delta, \qquad \text{for all} \quad (V, v_0) \in \mathscr{D}(\Gamma), \tag{6.71}$$

and $0 \leqslant t \leqslant \tau$. Since $w(s)$ is Lipschitz continuous in V, it follows from inequalities (4.4) and (6.13) that $\hat{e}(V, t)$ is Lipschitz continuous in V, as well. We claim that there is a $\beta_3 \in \Sigma$ such that

$$\|A^\beta(\hat{e}(V_1, t) - \hat{e}(V_2, t))\| \leqslant \beta_3(\varepsilon, \delta)\,\|A^\beta(V_1 - V_2)\|, \qquad \text{for} \quad 0 \leqslant t \leqslant \tau, \tag{6.72}$$

where (V_i, v_0) lie in $\mathscr{D}(\Gamma)$, for $i = 1, 2$. In order to prove inequality (6.72), we return to the formulation of $e(t)$ given in Eq. (6.36). For $(V_i, v_0) \in \mathscr{D}(\Gamma)$ we set $y_{i0} = v_0 + f(v_0) + V_i$, $w_{i0} = f(v_0) + V_i$, $w_i(t) = w(t, y_{i0})$, and

$$\hat{e}(V_i, t) = \hat{e}_i(t) = w_i(t) - \Phi(v_0, t)\,w_i(0), \qquad \text{for} \quad i = 1, 2.$$

Set $m = m(t) = \hat{e}_1(t) - \hat{e}_2(t)$. Then (6.37) implies that m satisfies

$$m(t) = \int_0^t e^{-A(t-s)} B(s)\,m(s)\,ds + \int_0^t e^{-A(t-s)}[E_1(s) - E_2(s)]\,ds$$
$$+ \int_0^t e^{-A(t-s)}[G_1(s) - G_2(s)]\,ds,$$

where $E_i(s) = E(S(s)\,v_0, w_i(s))$ and $G_i(s) = G(S(s)\,v_0 + w_i(s))$, for $i = 1, 2$. Next it follows from inequalities (2.6), (4.4), (6.13), and (6.71) that there is a constant $K_7 > 0$ and a $\beta_4 \in \Sigma$ such that

$$\|A^\beta m(t)\| \leqslant K_7 \int_0^t (t - s)^{-\beta}\,e^{-a(t-s)}\,\|A^\beta m(s)\|\,ds + \beta_4(\varepsilon, \delta)\,\|A^\beta(V_1 - V_2)\|.$$

This fact, together with the Gronwall–Henry inequality, then implies inequality (6.72).

The next step in the argument is to show that $\Gamma(V, v_0)$, the right side of Eq. (6.69), is a contraction in the V-variable under the conditions stated in this lemma. In particular, we will now show that for small ε and δ, the following two properties are valid:

(1) For any $(V, v_0) \in \mathcal{D}(\Gamma)$ one has $\|A^\beta \Gamma(V, v_0)\| \leqslant \frac{3}{4}\varepsilon$.

(2) One has $\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\| \leqslant \frac{1}{2}\|A^\beta(V_1 - V_2)\|$, for any $(V_i, v_0) \in \mathcal{D}(\Gamma)$, for $i = 1, 2$.

Since one has $g(v) \in U^u(v)$, it follows that $P^u(v)\, g(v) = g(v)$, and therefore

$$g(v) = P^u(S)\, g(v) + [P^u(v) - P^u(S)]\, g(v).$$

By applying $\Phi(S, -\tau)\, P^u(S)$ to the last equation and using the continuity of P^u and inequalities (3.10), (6.10), and (6.40), we find that there exists a $\beta_5 \in \Sigma$ such that

$$\begin{aligned}\|A^\beta \Phi(S, -\tau)\, P^u(S)\; g(v)\| &\leqslant K e^{-\lambda_4 \tau}\, \|A^\beta g(v)\| \\ &\quad + \|A^\beta \Phi(S, -\tau)\, P^u(S)[P^u(v) - P^u(S)]\, g(v)\| \\ &\leqslant (\tfrac{1}{4} + \beta_5(\varepsilon, \delta))\, \varepsilon.\end{aligned}$$

Since $\|A^\beta V\| \leqslant \varepsilon$, it follows from inequalities (3.10), (6.8), (6.10), and (6.40) that

$$\begin{aligned}\|A^\beta \Phi(S, -\tau)\, P^u(S)(v - S)\| &\leqslant K e^{-\lambda_4 \tau}\, \|A^\beta P^u(S)(v - S)\| \\ &\leqslant \tfrac{1}{4} K^2 L_0\, \|A^\beta (v - S)\|^2 \leqslant \tfrac{1}{4} K^2 L_0 C_0^2 (\varepsilon + \delta)^2.\end{aligned}$$

The continuity of P^u, Lemma 6.6 and inequality (6.71) imply that there is a constant $C_5 > 0$ and a $\beta_6 \in \Sigma$, such that

$$\|A^\beta \Phi(S, -\tau)\, P^u(S)[P^u(v) - P^u(S)]\, n\| \leqslant (\varepsilon + \delta)\, \beta_6(\varepsilon, \delta),$$

$$\|A^\beta \Phi(S, -\tau)\, P^u(S)\, \hat{e}(V, \tau)\| \leqslant (\varepsilon + \delta)\, \beta_6(\varepsilon, \delta) + C_5 \delta.$$

By putting these estimates together, one finds a $\beta_7 \in \Sigma$ such that

$$\|A^\beta \Gamma(V, v_0)\| \leqslant \tfrac{1}{4}\varepsilon + C_5 \delta + (\varepsilon + \delta)\, \beta_7(\varepsilon, \delta), \qquad \text{for all} \quad (V, v_0) \in \mathcal{D}(\Gamma).$$

Now choose $\bar{\varepsilon}$ so that $0 < \bar{\varepsilon} \leqslant \varepsilon_5$, where ε_5 is given in Lemma 6.8, so that $\beta_7(\bar{\varepsilon}, \bar{\varepsilon}) \leqslant \frac{1}{8}$, and set $\bar{\delta}(\varepsilon) = \min(\varepsilon, \varepsilon/(4C_5), \delta_5(\varepsilon))$, where δ_5 is given in Lemma 6.8. Then with $\delta \leqslant \bar{\delta}$ and $0 < \varepsilon \leqslant \bar{\varepsilon}$, one has $\|A^\beta \Gamma(V, v_0)\| \leqslant \frac{3}{4}\varepsilon$.

In order to prove the Lipschitz property for $\Gamma(V)$, we let $(V_i, v_0) \in \mathscr{D}(\Gamma)$, for $i=1, 2$. Next we define $S=S(\tau)\, v_0$, $y_i = y(\tau, y_0(V_i))$, $v_i = v(\tau, y_0(V_i))$, $\Gamma_i = \Gamma(V_i, v_0)$, and $g_i = g(v_i)$, for $i=1, 2$. In this case one has $\Delta y(0) = \Delta n(0) = V_1 - V_2$, $\Delta v(0)=0$, and $\Delta S(t)=0$. In order to estimate $\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\|$, we note that (6.70) implies that

$$\Gamma_1 - \Gamma_2 = \Phi(S, -\tau)\, P^u(S)[-(\hat{e}_1 - \hat{e}_1) + \Delta v + \Delta g + [P^u(S) - P^u(v_2)]\, \Delta n + [P^u(v_2) - P^u(v_1)]\, n_1],$$

where $\hat{e}_i = \hat{e}(V_i, \tau)$, for $i=1, 2$, $\Delta n = n_1 - n_2$, $\Delta v = v_1 - v_2$, and $\Delta g = g_1 - g_2$. From (6.72) there is a constant $c_5 > 0$ such that

$$\|A^\beta \Phi(S, -\tau)\, P^u(S)(\hat{e}_1 - \hat{e}_2)\| \leqslant c_5\, \beta_3(\varepsilon, \delta)\, \|A^\beta(V_1 - V_2)\|.$$

Also, there is a $\beta_4 \in \Sigma$ such that

$$\left\{ \begin{matrix} \|A^\beta \Phi(S, -\tau)\, P^u(S)[P^u(S) - P^u(v_2)]\, \Delta n\| \\ \|A^\beta \Phi(S, -\tau)\, P^u(S)[P^u(v_2) - P^u(v_1)]\, n_1\| \end{matrix} \right. \leqslant \beta_4(\varepsilon, \delta)\, \|A^\beta(V_1 - V_2)\|.$$

Since $\ell \leqslant 1$, one has $\|A^\beta\, \Delta g\| \leqslant \|A^\beta\, \Delta v\|$. From inequalities (3.10) and (6.10), one has

$$\|A^\beta \Phi(S, -\tau)\, P^u(S)[\Delta v + \Delta g]\| \leqslant 2K e^{-\lambda_4 \tau}\, \|A^\beta\, \Delta v(\tau)\| \leqslant \|A^\beta\, \Delta v(\tau)\|.$$

From (6.39) one has $\Delta v(t) = E_v(t)$, since $\Delta v(0)=0$. It then follows from inequality (6.43) that

$$\|A^\beta\, \Delta v(\tau)\| = \|A^\beta E_v(\tau)\| \leqslant (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta(V_1 - V_2)\|.$$

As a result, we find that there is a $\beta_5 \in \Sigma$ such that

$$\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\| \leqslant (b_4^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \delta))\, \|A^\beta(V_1 - V_2)\|.$$

Next we let ε_7 be chosen so that $0 < \varepsilon_7 \leqslant \bar{\varepsilon}$ and $\beta_5(\varepsilon_7, \varepsilon_7) \leqslant \frac{3}{8}$. Then set $\delta_7(\varepsilon) = \bar{\delta}(\varepsilon))$, for $0 < \varepsilon \leqslant \varepsilon_7$. Since inequality (6.60) implies that $b_4^F(\rho, 2\varepsilon) \leqslant \frac{1}{144}$, one obtains

$$\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\| \leqslant \tfrac{1}{2}\, \|A^\beta(V_1 - V_2)\|,$$

for all $(V_i, v_0) \in \mathscr{D}(\Gamma)$ and $i=1, 2$.

Finally the function $\bar{g}_\tau(v_0)$ is a continuous function of $v_0 \in M$, because: (1) the mapping $(V, v_0) \to \Gamma(V, v_0)$ given by Eq. (6.70) is continuous on $\mathscr{D}(\Gamma)$; and (2) the value $\bar{g}_\tau(v_0)$ is the unique fixed point of a contraction mapping. Similarly, one shows that $\bar{g}_\tau(v_0)$ is jointly continuous in v_0 and τ. ∎

In the next two lemmas, we show that $\bar{g}_\tau(v_0)$ is locally Lipschitz continuous in v_0 and that the mapping $(f, g) \to \bar{g}_\tau$ is contracting.

LEMMA 6.11. *Let the hypotheses of the Main Theorem be satisfied. Then there is an ε_8 with $0 < \varepsilon_8 \leqslant \varepsilon_7$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_8$, there exists a $\delta_8 = \delta_8(\varepsilon) \in \Sigma$, with $0 < \delta_8 \leqslant \delta_7$, where ε_7 and δ_7 are given in Lemma* 6.10, *and, for $0 < \rho \leqslant \rho_0$, there exists an $\ell_8 = \ell_8(\rho, \varepsilon) > 0$ with $16KC_2\ell_8(\rho_0, \sigma_0) \leqslant 1$ and $\ell_8(\rho, \varepsilon) < 1$, such that if $G \in C^1_{\mathrm{Lip}}$ satisfies $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta_8$ and if $f \in \mathscr{F} = \mathscr{F}(\varepsilon, \ell)$ and $g \in \mathscr{G} = \mathscr{G}(\varepsilon, \ell)$ with $\ell \leqslant 1$ and $\varepsilon \leqslant \varepsilon_8$, then the restriction of $\bar{g}_\tau$ to the disk $\mathscr{D}_\rho(v_0)$ is Lipschitz continuous with Lipschitz coefficient ℓ_8, for every $v_0 \in M$. Moreover, one has $\bar{g}_\tau \in \mathscr{G}$ and $\ell_8(\rho, \varepsilon) \in \Sigma$.*

Proof. Let $(f, g) \in \mathscr{F} \times \mathscr{G}$ be given as in the statement of the lemma, and let τ be fixed, where $T \leqslant \tau \leqslant 2T$. While the proof of the Lipschitz continuity of $\bar{g}_\tau(v_0)$ with respect to v_0 has some similarities to the argument of the Lipschitz continuity of $\bar{f}_\tau(v_0)$ used in Lemma 6.8, one encounters some new issues which arise when one needs to study the behavior of a semiflow for time $t < 0$.

For $i = 1, 2$, we let $\hat{v}_{i0} \in M$ be given, where $\|A^\beta \Delta\hat{v}(0)\| \leqslant \rho$, for some ρ with $0 < \rho \leqslant \rho_0$, and $\Delta\hat{v}(0) = \hat{v}_{10} - \hat{v}_{20}$. We define $\hat{y}_{i0}$ by $\hat{y}_{i0} = \hat{v}_{i0} + f(\hat{v}_{i0}) + V_i$, where $V_i = \bar{g}_\tau(\hat{v}_{i0})$ is the solution of Eq. (6.66) given by Lemma 6.10. We define $\Delta\hat{y}(0) = \hat{y}_{10} - \hat{y}_{20}$, where $y(t, \hat{y}_{i0})$ is the solution of Eq. (1.2) through $\hat{y}_{i0}$, and we set $y_{i0} = y(\tau, \hat{y}_{i0})$ and $y_{i0} = v_{i0} + n_{i0}$, see Eq. (6.24), for $i = 1, 2$. Like the situation described in (3.17), the solution of Eq. (1.2) passing through y_{i0} admits a (partial) negative continuation, for $-\tau \leqslant t \leqslant 0$. We will denote this negative continuation by $y_i = y_i(t) = y(t, y_{i0})$, for $-\tau \leqslant t \leqslant 0$, and one has

$$y_i(t) = y(t, y_{i0}) = y(\tau + t, \hat{y}_{i0}), \qquad \text{for} \quad -\tau \leqslant t \leqslant 0,$$

and $y_i(-\tau) = y(-\tau, y_{i0}) = \hat{y}_{i0}$. The decomposition $y(t, y_{i0}) = v(t, y_{i0}) + n(t, y_{i0})$ given by Eq. (6.24) is valid for $-\tau \leqslant t \leqslant 0$, and one has

$$v_i(t) = v(t, y_{i0}) = v(\tau + t, \hat{y}_{i0}), \qquad \text{and} \qquad n_i(t) = n(t, y_{i0}) = n(\tau + t, \hat{y}_{i0}),$$

for $-\tau \leqslant t \leqslant 0$. Similarly, we consider the solution $S_i(t) \stackrel{\text{def}}{=} S(\tau + t)\, \hat{v}_{i0}$ of Eq. (1.1), for $-\tau \leqslant t \leqslant 0$, and we set $\Delta S(t) = S_1(t) - S_2(t)$. It follows from inequality (6.31) that, at $t = 0$, one has

$$\|A^\beta \Delta S(0)\| = \|A^\beta (S(\tau)\, \hat{v}_{10} - S(\tau)\, \hat{v}_{20})\| \leqslant 4K^2 e^{\lambda_3 \tau} \|A^\beta \Delta\hat{v}(0)\|. \qquad (6.73)$$

We will use the Special Notation, see (6.39), where $\Delta n(t) = \Delta u(t) + \Delta s(t)$, $\Delta z(t) = \Delta v(t) - \Delta S(t)$, and $z_i = v_i - S_i$ are thus defined for $-\tau \leqslant t \leqslant 0$. Note that

$$\begin{aligned} \Delta u(-\tau) = V_1 - V_2, \qquad \Delta u(0) = g(v_{10}) - g(v_{20}), \qquad \text{and} \\ \Delta s(0) = \bar{f}(v_{10}) - \bar{f}(v_{20}), \end{aligned} \tag{6.74}$$

where $\bar{f}(v_{i0}) = \bar{f}_\tau(v_{i0})$, for $i = 1, 2$, is given by Lemma 6.7. Due to Lemma 6.8, one has

$$\|A^\beta \, \Delta n(0)\| \leqslant (\ell + \ell_5) \, \|A^\beta \, \Delta v(0)\|, \qquad \text{where} \quad \ell_5 = \ell_5(\rho, \varepsilon) < 1. \tag{6.75}$$

The remainder of the argument now follows the pattern used in the proof of Lemma 6.8. We define $E_u = E_u(t)$ by the equation

$$\Delta u = \Delta u(t) = \Phi(S(\tau) \, \hat{v}_{20}, t) \, P^u(S(\tau) \, \hat{v}_{20}) \, \Delta n(0) + E_u(t), \tag{6.76}$$

for $-\tau \leqslant t \leqslant 0$. As argued in Lemma 6.8, in the case of Δs and E_s, one has

$$\Delta u(t) = P^u(v_1(t)) \, n_1(t) - P^u(v_2(t)) \, n_2(t), \qquad \text{for} \quad -\tau \leqslant t \leqslant 0,$$

and

$$E_u(t) = E_{1u}(t) + P^u(S_2) \, \Phi(S(\tau) \, \hat{v}_{20}, t) \, \Delta v(0),$$

where $E_{1u} = E_{1u}(t)$ satisfies

$$\begin{aligned} E_{1u} = [P^u(v_1) - P^u(v_2)] \, n_1 + [P^u(v_2) - P^u(S_2)] \, \Delta n + P^u(S_2) \, E_y \\ + P^u(S_2)[Q^o(v_2) - Q^o(S_2)] \, \Delta y - P^u(S_2) \, e_3(y_2, \Delta y), \end{aligned}$$

for $-\tau \leqslant t \leqslant 0$. Furthermore, there is a $\beta_3 \in \Sigma$ such that

$$\|A^\beta E_{1u}(t)\| \leqslant (b_4^F(\rho, 2\varepsilon) + \beta_3(\varepsilon, \delta)) \, \|A^\beta \, \Delta \hat{y}(0)\|, \qquad \text{for} \quad -\tau \leqslant t \leqslant 0.$$

From Lemma 6.6 and inequalities (3.10), (4.7), and (6.8), one obtains

$$\begin{aligned} \|A^\beta P^u(S_2) \, \Phi(S(\tau) \, \hat{v}_{20}, t) \, \Delta v(0)\| &\leqslant K e^{\lambda_4 t} \, \|A^\beta P^u(S(\tau) \, \hat{v}_{20}) \, \Delta v(0)\| \\ &\leqslant K^3 L_0 \, \rho \, e^{\lambda_4 t} \, \|A^\beta \, \Delta v(0)\|, \end{aligned}$$

for $-\tau \leqslant t \leqslant 0$. It follows from inequality (3.10) and Eq. (6.75) that

$$\begin{aligned} &\|A^\beta \Phi(S(\tau) \, \hat{v}_{20}, t) \, P^u(S(\tau) \, \hat{v}_{20}) \, \Delta n(0)\| \\ &\quad \leqslant K e^{\lambda_4 t} (\ell + \ell_5) \, \|A^\beta \, \Delta v(0)\|, \qquad \text{for} \quad t \leqslant 0. \end{aligned}$$

Thus, for $-\tau \leqslant t \leqslant 0$, one has

$$\|A^\beta E_u\| \leqslant K^3 L_0\, \rho e^{\lambda_4 t}\, \|A^\beta\, \Delta v(0)\| + (b_4^F(\rho, 2\varepsilon) + \beta_3(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|, \tag{6.77}$$

and it follows from Eqs. (6.74) and (6.76) and the last three inequalities that at $t = -\tau$, one has

$$\|A^\beta(V_1 - V_2)\| \leqslant K e^{-\lambda_4 \tau}(\ell + \ell_5 + K^2 L_0\, \rho)\, \|A^\beta\, \Delta v(0)\| + (b_4^F(\rho, 2\varepsilon) + \beta_3(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|. \tag{6.77a}$$

Note that, since $\ell \leqslant 1$, one has

$$\|A^\beta\, \Delta \hat{y}(0)\| \leqslant 2\, \|A^\beta\, \Delta \hat{v}(0)\| + \|A^\beta(V_1 - V_2)\|. \tag{6.78}$$

In Lemma 6.6 we derived inequality (6.52) under the assumption that inequality (6.51) is valid. Our next goal is to study the analogue of inequality (6.52), for $-\tau \leqslant t \leqslant 0$, where (6.78) now holds in place of (6.51). Since $\Delta z(-\tau) = \Delta v(-\tau) - \Delta S(-\tau) = 0$, one has $\Delta z(t) = E_v(t) - E_S(t)$, for $-\tau \leqslant t \leqslant 0$. It then follows from inequality (6.43) that

$$\begin{aligned}\|A^\beta\, \Delta z(t)\| &\leqslant \|A^\beta E_v(t)\| + \|A^\beta E_S(t)\| \\ &\leqslant b_3^F(\rho)\, \|A^\beta\, \Delta \hat{v}(0)\| + (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|,\end{aligned}$$

for $-\tau \leqslant t \leqslant 0$. It then follows from (6.34) and (6.78) that

$$\begin{aligned}\|A^\beta\, \Delta z(t)\| \leqslant{}& (b_5^F(\rho, 2\varepsilon) + 2\beta_2(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{v}(0)\| \\ &+ (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta(V_1 - V_2)\|,\end{aligned}$$

for $-\tau \leqslant t \leqslant 0$. By using the last inequality with inequality (6.73) and the definition of $\Delta z(t)$ at $t = 0$, one finds that

$$\begin{aligned}\|A^\beta\, \Delta v(0)\| &\leqslant \|A^\beta\, \Delta S(0)\| + \|A^\beta\, \Delta z(0)\| \\ &\leqslant (4K^2 e^{\lambda_3 \tau} + b_5^F(\rho, 2\varepsilon) + 2\beta_2(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{v}(0)\| \\ &\quad + (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta(V_1 - V_2)\|.\end{aligned} \tag{6.79}$$

Next we note that $K e^{-\lambda_4 \tau}(\ell + \ell_5 + K^2 L_0\, \rho) \leqslant \frac{9}{64}$, due to inequalities (6.10), $\ell + \ell_5 < 2$, and $4K^2 L_0\, \rho \leqslant 1$ (see Lemma 6.1). As a result, inequalities (6.77a), (6.78), and (6.79) imply that, for some $\beta_4 \in \Sigma$, one has

$$\begin{aligned}\|A^\beta(V_1 - V_2)\| \leqslant{}& K e^{-\lambda_4 \tau}(\ell + \ell_5 + K^2 L_0\, \rho)(4K^2 e^{\lambda_3 \tau} + b_5^F(\rho, 2\varepsilon) + 2\beta_2(\varepsilon, \delta)) \\ &\times \|A^\beta\, \Delta \hat{v}(0)\| + (2b_4^F(\rho, 2\varepsilon) + 2\beta_3(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{v}(0)\| \\ &+ (2b_4^F(\rho, 2\varepsilon) + \beta_4(\varepsilon, \delta))\, \|A^\beta(V_1 - V_2)\|.\end{aligned}$$

Next we impose the first of two conditions on ε_8 and $\delta_8(\varepsilon)$ by requiring that $0 < \varepsilon_8 \leqslant \varepsilon_7$, $\beta_4(\varepsilon_8, \varepsilon_8) \leqslant \frac{1}{8}$, and $\delta_8(\varepsilon) = \min(\varepsilon, \delta_7(\varepsilon))$, for $0 < \varepsilon \leqslant \varepsilon_8$. From inequality (6.53), one then has $2b_4^F(\rho, 2\varepsilon) + \beta_4(\varepsilon, \delta) \leqslant \frac{1}{2}$, for $0 < \delta \leqslant \delta_8$, which in turn implies that, for some $\beta_5 \in \Sigma$, one has

$$\|A^\beta(V_1 - V_2)\| \leqslant \ell_8(\rho, \varepsilon)\ \|A^\beta\ \Delta\hat{v}(0)\|,$$

where

$$\ell_8(\rho, \varepsilon) = 8K^3 e^{-(\lambda_4 - \lambda_3)\tau}(\ell + \ell_5(\rho, \varepsilon) + K^2 L_0 \rho) + 2b_5^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \delta), \tag{6.80}$$

see Eq. (6.34). From inequality (6.53), one has $2b_5^F(\rho, 2\varepsilon) \leqslant \frac{1}{8}$. Since $(\ell + \ell_5 + K^2 L_0 \rho) < 3$, inequality (6.10) yields $8K^3 e^{-(\lambda_4 - \lambda_3)\tau}(\ell + \ell_5 + K^2 L_0 \rho) < \frac{1}{2}$. The second condition on ε_8 and $\delta_8(\varepsilon)$ is that we require that $\beta_5(\varepsilon_8, \varepsilon_8) \leqslant \frac{3}{8}$. One then obtains $\ell_8(\rho, \varepsilon) < 1$. Finally, one can treat $\ell = \ell_8$ as the (unique) fixed point for Eq. (6.80), in which case, we conclude that $\ell_8 \in \Sigma$. Since $\ell_8 \in \Sigma$, we obtain $16KC_2\, \ell_8(\rho_0, \sigma_0) \leqslant 1$, by choosing smaller values of ρ_0 and σ_0, if necessary. ∎

Lemma 6.12. *Let the hypotheses of the Main Theorem be satisfied. Then there is an* ε_9 *with* $0 < \varepsilon_9 \leqslant \varepsilon_8$ *and for every* ε *with* $0 < \varepsilon \leqslant \varepsilon_9$ *there is a* $\delta_9 = \delta_9(\varepsilon) \leqslant \delta_8(\varepsilon)$, *where* ε_8 *and* δ_8 *are given in Lemma* 6.11, *such that if* $G \in C^1_{\mathrm{Lip}}$ *satisfies* $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta_9$, *then for all* $(f_i, g_i) \in \mathscr{F}(\varepsilon, \ell) \times \mathscr{G}(\varepsilon, \ell)$, *where* $\ell \leqslant 1$, *and* $\bar{g}_{\tau, i}$ *is given by* (6.66), *for* $i = 1, 2$, *one then has*

$$\|A^\beta(\bar{g}_{\tau,1}(u) - \bar{g}_{\tau,2}(u))\| \leqslant \tfrac{1}{2}\, \|(f_1, g_1) - (f_2, g_2)\|_\infty \qquad \textit{for all} \quad u \in M.$$

Proof. Once again we will use the Special Notation (6.39), as well as the notation of Lemma 6.11. The reader should notice that, with some minor modifications, which arise due to the perennial issue of the time-reversibility of semiflows, the argument follows the methodology of Lemma 6.9.

Let $(f_i, g_i) \in \mathscr{F} \times \mathscr{G}$ be given, for $i = 1, 2$, and fix τ so that $T \leqslant \tau \leqslant 2T$. Let $v_0 \in M$ be fixed and set $\hat{v}_{i0} = v_0$, $\hat{y}_{i0} = v_0 + f_i(v_0) + V_i$, where $V_i = \bar{g}_{\tau, i}(v_0)$ is the solution of Eq. (6.66), with (f, g) replaced by (f_i, g_i), given by Lemma 6.10, so that one has

$$g(v(\tau, \hat{y}_{i0})) = P^u(v(\tau, \hat{y}_{i0}))\, n(\tau, \hat{y}_{i0}), \qquad \text{for} \quad i = 1, 2.$$

In the notation of Lemma 6.11, one now has

$$\Delta\hat{v}(0) = 0 \qquad \text{and} \qquad \Delta\hat{y}(0) = \Delta\hat{n}(0) = f_1(v_0) - f_2(v_0) + V_1 - V_2.$$

As in Lemma 6.11, we study the various functions $y_i(t)$, $v_i(t)$, ..., for $-\tau \leqslant t \leqslant 0$ and $i = 1, 2$. Since $\Delta\hat{v}(0) = 0$, one has $S(\tau)\, \hat{v}_{10} = S(\tau)\, \hat{v}_{20}$. Thus we set $S_1(t) = S_2(t) = S(\tau + t)\, v_0$, for $-\tau \leqslant t \leqslant 0$. We note that $E_S(t) = 0$, for $-\tau \leqslant t \leqslant 0$. From Lemmas 6.7 and 6.10, we see that, at $t = 0$, $\Delta n(0) = \Delta u(0) + \Delta s(0)$ satisfies

$$\Delta u(0) = g_1(v_{10}) - g_2(v_{20}) \qquad \text{and} \qquad \Delta s(0) = \bar{f}_1(v_{10}) - \bar{f}_2(v_{20}). \tag{6.81}$$

Since $g_i(v_{i0}) = P^u(v_{i0})\, n_{i0}$ and $\bar{f}_i(v_{i0}) = P^s(v_{i0})\, n_{i0}$, where $y_{i0} = y(\tau, \hat{y}_{i0}) = v_{i0} + n_{i0}$ and $n_{i0} = s_{i0} + u_{i0}$, one has $g_i(v_{i0}) = P^u(v_{i0})\, g_i(v_{i0})$ and $\bar{f}_i(v_{i0}) = P^s(v_{i0})\, \bar{f}_i(v_{i0})$, for $i = 1, 2$. After a lengthly calculation, which uses (6.81), along with (6.4), (6.5), and Lemmas 6.7 and 6.10, one finds that

$$\begin{aligned} \|A^\beta P^u(S(\tau)\, \hat{v}_{20})\, \Delta n(0)\| \leqslant{}& \|A^\beta\, \Delta g(v_{20})\| + K(\ell + L_0 \varepsilon)\, \|A^\beta\, \Delta v(0)\| \\ &+ \beta_4(\varepsilon, \delta)\, \|A^\beta\, \Delta \hat{y}(0)\|, \end{aligned} \tag{6.82}$$

for some $\beta_4 \in \Sigma$. We also note that inequality (3.10) implies that

$$\|A^\beta \Phi(S(\tau)\, \hat{v}_{20}, -\tau)\, P^u(S(\tau)\, \hat{v}_{20})\, \Delta n(0)\| \leqslant K e^{-\lambda_4 \tau}\, \|A^\beta P^u(s(\tau)\, \hat{v}_{20})\, \Delta n(0)\|. \tag{6.83}$$

Since $\Delta \hat{v}(0) = 0$, it follows from (6.39) that $\Delta v(t) = E_v(t)$, for $-\tau \leqslant t \leqslant 0$, and inequality (6.43) implies that, at $t = 0$, one has

$$\|A^\beta \Delta v(0)\| = \|A^\beta E_v(0)\| \leqslant (b_4^F(\rho, 2\varepsilon) + \beta_2(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|. \tag{6.84}$$

Now $\|A^\beta E_u(t)\|$ satisfies (6.77), for $-\tau \leqslant t \leqslant 0$. By using inequality (6.84), with the fact that $K^3 L_0\, \rho\, e^{-\lambda_4 \tau} \leqslant \frac{1}{16}$ (see Lemma 6.1 and (6.10)), one obtains

$$\|A^\beta E_u(-\tau)\| \leqslant (2 b_4^F(\rho, 2\varepsilon) + \beta_5(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|, \tag{6.85}$$

for some $\beta_5 \in \Sigma$. It then follows from inequalities (6.10), (6.76), (6.82), (6.83), (6.84), and (6.85) that, at $t = -\tau$, where $\Delta u(-\tau) = V_1 - V_2 = \Delta \bar{g}(v_0)$, one finds a $\beta_6 \in \Sigma$ such that

$$\begin{aligned} \|A^\beta\, \Delta \bar{g}(v_0)\| \leqslant{}& K e^{-\lambda_4 \tau} (\|A^\beta\, \Delta g(v_0)\| + K b_4^F(\rho, 2\varepsilon)\, \|A^\beta\, \Delta \hat{y}(0)\|) \\ &+ (2 b_4^F(\rho, 2\varepsilon) + \beta_6(\varepsilon, \delta))\, \|A^\beta\, \Delta \hat{y}(0)\|. \end{aligned} \tag{6.86}$$

Note that $(2 + K^2 e^{-\lambda_4 \tau})\, b_4^F(\rho, 2\varepsilon) \leqslant \frac{1}{36}$, due to inequalities (6.60), $K \geqslant 1$ and $\lambda_2 \leqslant 0 < \lambda_4$. By substituting $\Delta \hat{y}(0) = \Delta f(v_0) + \Delta \bar{g}(v_0)$ into inequality (6.86), and using (6.10), one then obtains

$$\begin{aligned} \|A^\beta\, \Delta \bar{g}(v_0)\| \leqslant{}& (\tfrac{1}{36} + \beta_6(\varepsilon, \delta))\, \|A^\beta\, \Delta f(v_0)\| + \tfrac{1}{4}\, \|A^\beta\, \Delta g(v_0)\| \\ &+ (\tfrac{1}{36} + \beta_6(\varepsilon, \delta))\, \|A^\beta\, \Delta \bar{g}(v_0)\|. \end{aligned} \tag{6.87}$$

Finally, we fix ε_9 so that $0<\varepsilon_9 \leqslant \varepsilon_8$ and $\beta_6(\varepsilon_9, \varepsilon_9) \leqslant \frac{1}{8}$, and we set $\delta_9(\varepsilon) = \min(\varepsilon, \delta_8(\varepsilon))$, for $0<\varepsilon \leqslant \varepsilon_9$. One then has $\beta_6(\varepsilon, \delta) \leqslant \frac{1}{8}$, for $0<\delta \leqslant \delta_9(\varepsilon)$, and lemma now follows from inequality (6.87). ▮

The objective of the next result is to give a more precise formulation of a portion of the Main Theorem.

THEOREM 6.13. *Let the hypotheses of the Main Theorem be satisfied. Then for each* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *and, for* $0<\rho \leqslant \rho_0$, *there is an* $\ell = \ell(\rho, \varepsilon) < 1$, *where* $\delta(\varepsilon)$, $\ell(\rho, \varepsilon) \in \Sigma$, *such that if* $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, *then there is a continuous mapping* $h\colon M \to V^{2\beta}$ *such that the following properties hold*:

(1) *The image* $M^G = h(M)$ *is a compact invariant set for the perturbed Eq.* (1.2).

(2) *For each* $v \in M$, *one has* $\phi(v) \in U^s(v) \oplus U^u(v)$, *where* $h(v) = v + \phi(v)$;

(3) *the restriction of h to any disk* $\mathscr{D}_\rho(v_0)$, *where* $v_0 \in M$ *and* $0<\rho \leqslant \rho_0$, *is Lipschitz continuous with*

$$\|A^\beta(\phi(v_1) - \phi(v_2))\| \leqslant 2\ell\, \|A^\beta(v_1 - v_2)\|, \qquad \textit{for all} \quad v_1, v_2 \in \mathscr{D}_{\rho_0}(v_0). \tag{6.88}$$

(4) *One has* $\|A^\beta \phi(v)\| = \|A^\beta(h(v) - v)\| \leqslant 2\varepsilon$, *for all* $v \in M$.

(5) *The mapping* $h\colon M \to M^G$ *is a* (*local*) *Lipschitz continuous homeomorphism with a* (*local*) *Lipschitz continuous inverse, and consequently,* M^G *is a Lipschitz manifold.*

Proof of Theorem 6.13 *and the Shadow Theorem.* Let T be given by (6.10), and for $T \leqslant \tau \leqslant 2T$, let A_τ be the mapping on $\mathscr{F} \times \mathscr{G} = \mathscr{F}(\varepsilon, \ell_1) \times \mathscr{G}(\varepsilon, \ell_1)$ defined by

$$A_\tau\colon (f, g) \to (\bar{f}_\tau, \bar{g}_\tau),$$

where $\ell_1 = 1$, $\bar{f}_\tau$ is given by Eq. (6.56), and $V = \bar{g}_\tau(v_0)$ is given by Lemma 6.10, see Eqs. (6.66) and (6.69). Let ε_9 and δ_9 be given by Lemma 6.12, and let ℓ_5 and ℓ_8 be given by Lemmas 6.8 and 6.11. Define $\ell = \ell_9(\rho, \varepsilon)$ by

$$\ell = \ell_9(\rho, \varepsilon) = \max(\ell_5(\rho, \varepsilon), \ell_8(\rho, \varepsilon)), \qquad \text{for} \quad 0<\varepsilon \leqslant \varepsilon_9 \quad \text{and} \quad 0<\rho \leqslant \rho_0.$$

From the construction of ℓ_5 and ℓ_8 one has $\ell_9(\rho, \varepsilon) < 1$ and $\ell_9(\rho, \varepsilon) \in \Sigma$. Let ε be fixed with $0<\varepsilon \leqslant \varepsilon_9$, and set $\delta = \delta_9(\varepsilon)$.

Assume that $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$. From Lemmas 6.8 and 6.11, we see that A_τ maps $\mathscr{F} \times \mathscr{G}$ into itself, for each τ with $T \leqslant \tau \leqslant 2T$. Also Lemmas 6.9 and 6.12 imply that A_τ is a strict contraction on $\mathscr{F} \times \mathscr{G}$. Since $\mathscr{F} \times \mathscr{G}$ is a complete metric space, the mapping A_τ has a unique fixed point. Note that, due to Eqs. (6.56) and (6.66), the pair (f, g) is a fixed point of A_τ, where $T \leqslant \tau \leqslant 2T$, if and only if the mapping h given by

$$h(u) = u + f(u) + g(u), \qquad \text{for} \quad u \in M, \tag{6.89}$$

satisfies

$$\begin{aligned} f(v(\tau, h(u))) &= P^s(v(\tau, h(u)))(y(\tau, h(u)) - v(\tau, h(u))) = s(\tau, h(u)) \\ g(v(\tau, h(u))) &= P^u(v(\tau, h(u)))(y(\tau, h(u)) - v(\tau, h(u))) = u(\tau, h(u)), \end{aligned} \tag{6.90}$$

for $u \in M$. We now fix $\tau = T$ and we let (f_0, g_0) denote the fixed point of A_T. Let $h = h_0$ satisfy (6.89), where (f, g) are replaced by (f_0, g_0), and define $M^G = h_0(M)$. Note that Eqs. (6.90), with $\tau = T$, can be rewritten in the form

$$y(T, h_0(u)) = h_0(v(T, h_0(u))), \qquad \text{for} \quad u \in M,$$

or equivalently,

$$S_2^G(T)\, h_0(u) = h_0(v(T, h_0(u))), \qquad \text{for} \quad u \in M. \tag{6.91}$$

It follows that $S_2^G(nT)\, M^G = M^G$, for all integers $n \geqslant 0$.

We define $S_1^G(t)$ on M by

$$S_1^G(t)\, v_0 \overset{\text{def}}{=} \psi^e(S_2^G(t)\, h(v_0)) = v(t, h(v_0)), \qquad \text{for} \quad v_0 \in M, \tag{6.92}$$

and $t \in [0, \infty)$. (Recall that ψ^e is an extension of ψ, see Eq. (6.54).) Since $S_2^G(nT)\, M^G = M^G$, for all integers $n \geqslant 0$, it follows that for each $y_0 \in M^G$, there is a global solution, which we denote by $S_2^G(t)\, y_0$ for $t \in R$, with $S_2^G(nT)\, y_0 \in M^G$, for all $n \in Z$. Consequently, Eq. (6.92) defines $S_1^G(t)\, v_0$, for all $t \in R$. Since ψ^e and S_2^G are continuous, Eq. (6.92) implies that $S_1^G(t)$ depends continuously on G, as $G \to 0$, in the topology $\mathscr{T}_A^1$.

For $0 \leqslant t \leqslant T$ we define $h_t \colon M \to V^{2\beta}$ by

$$h_t(S_1^G(t)\, v_0) = S_2^G(t)\, h(v_0) = y(t, h(v_0)), \qquad \text{for} \quad v_0 \in M. \tag{6.93}$$

Because of (6.55), we see that h_t is well-defined for all $v_0 \in M$, and the definition extends to $0 \leqslant t \leqslant T$. From (6.91) we see that $h_T = h_0$. Also from (6.23) we see that one has the local coordinate representation

$$h_t(v(t, h(v_0))) = y(t, h(v_0)) = v(t, h(v_0)) + s(t, h(v_0)) + u(t, h(v_0)),$$

for $0 \leqslant t \leqslant T$, where $s(t, h(v_0)) \in U^s(v(t, h(v_0)))$ and $u(t, h(v_0)) \in U^u(v(t, h(v_0)))$. Now define (f_t, g_t), for $0 \leqslant t \leqslant T$, by

$$f_t(v(t, h(v_0))) = s(t, h(v_0)) \qquad \text{and} \qquad g_t(v(t, h(v_0))) = u(t, h(v_0)). \tag{6.94}$$

Note that one has $h_T = h = h_0$, $f_T = f_0$, and $g_T = g_0$. From (6.54) one obtains the commutivity relation $S_1^G(T)\, S_1^G(t) = S_1^G(t)\, S_1^G(T)$, for $0 \leqslant t \leqslant T$. In addition, the semiflow $S_2^G(t)$ satisfies $S_2^G(T)\, S_2^G(t) = S_2^G(t)\, S_2^G(T)$, as well. By using these commutivity relations and (6.93), one then obtains

$$\begin{aligned} S_2^G(T)\, h_t(S_1^G(t)\, v_0) &= S_2^G(T)\, S_2^G(t)\, h(v_0) = S_2^G(t)\, S_2^G(T)\, h(v_0) \\ &= S_2^G(t)\, h_T(S_1^G(T)\, v_0) = S_2^G(t)\, h(S_1^G(T)\, v_0) \\ &= h_t(S_1^G(t)\, S_1^G(T)\, v_0) = h_t(S_1^G(T)\, S_1^G(t)\, v_0), \end{aligned} \tag{6.95}$$

for all $v_0 \in M$ and $0 \leqslant t \leqslant T$. Now (6.91) and (6.95) imply that (f_t, g_t), where (f_t, g_t) is given by (6.94), is a fixed point of A_T. The next question is: which space does (f_t, g_t) reside in? Is (f_t, g_t) in the space $\mathscr{F} \times \mathscr{G}$? If so, then the uniqueness of the fixed point for A_T implies that $h = h_T = h_t$ and $(f_t, g_t) = (f_T, g_T)$. We show next that (f_t, g_t) does indeed lie in $\mathscr{F} \times \mathscr{G}$.

We claim that there is an ε_{10}, with $0 < \varepsilon_{10} \leqslant \varepsilon_9$, such that, for $0 < \varepsilon \leqslant \varepsilon_{10}$, there is a $t_0 > 0$ where $(f_t, g_t) \in \mathscr{F} \times \mathscr{G}$, for $0 \leqslant t \leqslant t_0$. Indeed, from Lemmas 6.7, 6.8, 6.10, and 6.11, and since the fixed point (f_0, g_0) is in the range of A_T, one has $\|A^\beta f_0(u)\| \leqslant \frac{3}{4}\varepsilon$ and $\|A^\beta g_0(u)\| \leqslant \frac{3}{4}\varepsilon$, for all $u \in M$. By continuity, there is a $t_1 > 0$ such that $\|A^\beta f_t(u)\| \leqslant \varepsilon$ and $\|A^\beta g_t(u)\| \leqslant \varepsilon$, for all $u \in M$ and $0 \leqslant t \leqslant t_1$. Since $0 \leqslant \ell < 1$, it follows from Lemmas 6.8 and 6.11 that there is a $t_2 > 0$ such that both f_t and g_t are Lipschitz continuous on each disk $\mathscr{D}_\rho(v_0)$, for $0 < \rho \leqslant \rho_0$, with Lipschitz coefficient 1, for $0 \leqslant t \leqslant t_2$. By setting $t_0 = \min(t_1, t_2)$, we conclude that $(f_t, g_t) \in \mathscr{F} \times \mathscr{G}$, for $0 \leqslant t \leqslant t_0$.

Since the fixed point of A_T is unique, we have $h_t = h$, for all t, with $0 \leqslant t \leqslant t_0$. By iteration of this argument, we conclude that $h_t = h$, for all $t \geqslant 0$. This implies that (4.11) holds, i.e., $S_1^G(t)$ is a G-continuous shadow semiflow on M. From (6.92) and (6.93) we see that ψ^e is the inverse of h. Since h and ψ^e are both (locally) Lipschitz continuous, we see that $h: M \to M^G$ is a Lipschitz continuous homeomorphism and that M^G is a Lipschitz manifold. Also, $S_1^G(t): M \to M$ is a semiflow in the sense that $S_1^G(t)\, v_0$ is jointly continuous in $(v_0, t) \in M \times [0, \infty)$; $S_1^G(0)\, v_0 = v_0$; and the semigroup property $S_1^G(t)\, S_1^G(s) = S_1^G(t+s)$, for all $s, t \in R$, is valid. In addition, one has $S_1^G(t)\, M = M$, for all $t \geqslant 0$, by (6.55). This completes the proof of Theorem 6.13 and the Shadow Theorem. ∎

One can readily verify that $v(t)=v(t, h(v_0))=S_1^G(t)\, v_0$ is the unique mild solution of the differential equation

$$\partial_t v + Av = P^o(v)[F(v+\phi(v)) + G(v+\phi(v))] = P^o(v)[F(h(v)) + G(h(v))],$$

with $v(0)=v_0 \in M$. As noted in Section 2, $v(t)$ is a strong solution of this equation and that $M^G \subset \mathscr{D}(A)$. Since $h_t = h = h_T$, for all $t \in R$, the equalities in (6.95) remain valid when T is replaced by an arbitrary $\sigma \in R$. As a result, one then obtains

$$S_2^G(t+\sigma)\, h(v_0) = S_2^G(t)\, h(S_1^G(\sigma)\, v_0) = S_2^G(t)\, h(v(\sigma, h(v_0))), \tag{6.96}$$

for all $(v_0, \sigma, t) \in M \times R \times R$.

Theorem 6.14. *Let the hypotheses of the Main Theorem be satisfied. Then for each* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *and, for* $0 < \rho \leqslant \rho_0$, *there is an* $\ell = \ell(\rho, \varepsilon) < 1$, *where* $\delta(\varepsilon), \ell(\rho, \varepsilon) \in \Sigma$, *such that if* $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta$, *then the conclusions of Theorem* 6.13 *are valid and the following properties hold*:

(5) *The linear skew product semiflow generated by the linearization of Eq.* (1.2) *on the invariant manifold* M^G *has an exponential trichotomy over* M^G.

(6) *The manifold* M^G *is of class* C^1, *and it normally hyperbolic for the nonlinear dynamics generated by the mild solutions of Eq.* (1.2).

Proof of Theorem 6.14 *and the Main Theorem.* Since the Main Theorem is included in Theorems 6.13 and 6.14, it suffices to verify Items (5) and (6). We let h, $S_1(t)$, $S_1^G(t)$, and $S_2^G(t)$ be given as in the proof of Theorem 6.13. In order to show that the compact manifold M^G is normally hyperbolic for the solutions of the perturbed equation (1.2), when δ is sufficiently small, we will use the Robustness of Trichotomies Theorem in Pliss and Sell (1999). For this purpose, we set $\mathscr{W} = \mathscr{M}^\infty(R; \mathscr{L})$, and we let d denote the translation invariant metric on the Fréchet space $\mathscr{W}$ described in Section 2. By hypotheses, the linear skew product semiflow $\pi(v_0, B; \tau) = (\Phi(B, \tau)\, v_0, B_\tau)$ on $V^{2\beta} \times \mathscr{M}^\infty$ has an exponential trichotomy over $\mathscr{K} \subset \mathscr{W} = \mathscr{M}^\infty(R; \mathscr{L})$, where $\mathscr{K} = \{DF(S(\cdot)\, v_0 \colon v_0 \in M\}$. With G given and satisfying inequality (4.2), we now consider the collection of those $B \in \mathscr{M}^\infty$ given by

$$B(t) = B(v_0, t) \stackrel{\text{def}}{=} [DF(S_2^G(t)\, h(v_0)) + DG(S_2^G(t)\, h(v_0))],$$

for $(v_0, t) \in M \times R$. It follows from Eq. (6.96) and the definition of B that

$$B_\sigma(v_0, t) = B(v_0, \sigma + t) = B(S_1^G(\sigma)\, v_0, t), \qquad \text{for all} \quad (v_0, \sigma, t) \in M \times R \times R. \tag{6.97}$$

Since both $S_2^G(t)$ and $S_1^G(t)$ are continuous in G, it follows that

$$S_1^G(t)\, v_0 \to S_1(t)\, v_0 \qquad \text{and} \qquad S_2^G(t)\, h(v_0) \to S_1(t)\, v_0, \qquad \text{as} \quad \delta \to 0$$

(and as $G \to 0$ in the space $(C^1_{\mathrm{Lip}}, \mathscr{T}^1_A)$), uniformly for (v_0, t) in compact subsets of $M \times [0, \infty)$. Since $h(v) \to v$, as $\delta \to 0$, it follows from the continuity of the Fréchet derivatives DF and DG, that

$$DF(S_2^G(t)\, v_0) \to DF(S_1(t)\, v_0) \qquad \text{and} \qquad DG(S_2^G(t)\, h(v_0)) \to 0, \qquad \text{as} \quad \delta \to 0,$$

in the space $\mathscr{L} = \mathscr{L}(V^{2\beta}, W)$, uniformly for (v_0, t) in compact subsets of $M \times [0, \infty)$.

Now let $\varepsilon_0 > 0$ be given by the Robustness of Trichotomies Theorem, see Pliss and Sell (1999), for the exponential trichotomy for π over the unperturbed manifold M. Let $N_{\varepsilon_0}(\mathscr{K})$ denote the ε_0-neighborhood of $\mathscr{K}$ in the space $(\mathscr{W}, \mathscr{T}^1_A)$. Let $\delta(\varepsilon)$ and $\ell(\rho, \varepsilon)$ be given by Theorem 6.13, for $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < \rho \leqslant \rho_0$. Let $N = N(\varepsilon_0)$ be an integer with $\sum_{n=N+1}^{\infty} 2^{-n} \leqslant \frac{1}{2}\varepsilon_0$. Next we fix $\delta > 0$ so that $0 < \delta \leqslant \delta(\varepsilon_0)$ *and*

$$\|B(v_0, t) - DF(S_1(t)\, v_0)\|_{\mathscr{L}} \leqslant \tfrac{1}{2}\varepsilon_0, \qquad \text{for all} \quad (v_0, t) \in M \times [-N, N].$$

As a result, it follows from Eq. (6.97) that

$$\|B_\sigma(v_0, t) - DF(S_1(t)\, S_1^G(\sigma)\, v_0)\|_{\mathscr{L}} \leqslant \tfrac{1}{2}\varepsilon_0,$$

for all $(v_0, \sigma, t) \in M \times R \times [-N, N]$. It follows then that $B_\sigma \in N_{\varepsilon_0}(\mathscr{K})$, for all $\sigma \in R$. Item (5) now follows from the Robustness of Trichotomies Theorem.

In order to prove Item (6) it suffices to verify the validity of Eq. (4.10) for the linear skew product semiflow generated by the linearization of Eq. (1.2) over the manifold M^G. Indeed, the projections $R_G(v) = P^o_G(v)$ vary continuously for $v \in M^G$, and Eq. (4.10) implies that the manifold M^G is of class C^1.

We will denote the linear skew product semiflow π^G over M^G in the form

$$\pi^G(v_0, y_0, t) = (\Phi^G(y_0, t)\, v_0, y(t, y_0)),$$

for $(v_0, y_0, t) \in V^{2\beta} \times M^G \times [0, \infty)$, where $y(t, y_0) = S_2^G(t)\, y_0$ and $y_0 = h(v_0)$. Since the manifold M^G is invariant under the nonlinear dynamics $S_2^G(t)$, for each $y_0 \in M^G$, there is a global solution, which we denote by $y(t) = y(t, y_0)$, through y_0, and one has $y(t, y_0) \in M^G$, for all $t \in R$. The linear operator $\Phi^G(y_0, t)$ is the solution operator generated by the mild solutions of the linear variational equation

$$\partial_t v + Av = [DF(y(t, y_0)) + DG(y(t, y_0))] v.$$

Since F and G are in C^1_{Lip}, one has

$$F(y+w) + G(y+w) = F(y) + G(y) + [DF(y) + DG(y)] w + H(y, w),$$

where the last equation redefines the term $H = H(y, w)$. We note that there is a $\gamma \in \Sigma$, such that for any $y \in M^G$, the error term H satisfies

$$\|H(y, w_1) - H(y, w_2)\| \leqslant \gamma(\sigma) \|A^\beta(w_1 - w_2)\|,$$

whenever $\|A^\beta w_1\|, \|A^\beta w_2\| \leqslant \sigma$, and

$$\|H(y, w)\| \leqslant \gamma(\sigma) \|A^\beta w\| \leqslant \sigma\gamma(\sigma), \qquad \text{for} \quad \|A^\beta w\| \leqslant \sigma.$$

Let $y_i = y_i(t) = y(t, y_{i0})$, for $i = 1, 2$, denote two solutions of the perturbed Eq. (1.2), where $y_{20} \in M^G$, and set $w = w(t) = y_1 - y_2$. From the alternate Variation of Constants formula, see Eq. (2.33) (as applied to Eq. (1.2)), we see that w satisfies

$$w(t) = \Phi^G(y_{20}, t) w(0) + \int_0^t \Phi^G(y(s, y_{20}), t-s) H(y(s, y_{20}), w(s)) \, ds,$$

where $w(0) = y_{10} - y_{20}$. For G satisfying inequality (4.2), we let (P^s_G, P^o_G, P^u_G) denote the invariant projectors on M^G given by Item (5). Set $Q^o_G = P^s_G + P^u_G$. We will let K and λ_i, for $1 \leqslant i \leqslant 4$, denote the charactersitics of the exponential trichotomy over M^G. While these characteristics depend on G, we do not include this dependence explicitly in this notation.

The proof of the validity of the relation (4.10) for the perturbed dynamics on M^G is via an argument by contradiction involving two steps: (1) a basic inequality for the dynamics on M^G, and (2) a comparison of this inequality in terms of the Lipschitz property for the mapping $h: M \to M^G$.

Step 1. We let $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ be given so that Item (5) and Theorem 6.13 hold. We will now show that there is a δ_2, with $0 < \delta_2 \leqslant \delta_1$, such that if $\|G\|_{\{A;\, C^1(\Omega)\}} \leqslant \delta_2$, then for every $v_0 \in M$ the space $U^o_G(y_0)$ is tangent to the manifold M^G at the point y_0, where $y_0 = h(v_0)$. Observe that $U^o_G(y_0)$ is tangent at $y_0 \in M^G$ if and only if, for every convergent sequence $y_i \in M^G$, with $y_i \to y_0$ in $V^{2\beta}$, one has

$$\lim_{i \to \infty} \|A^\beta Q^o_G(y_0)(y_i - y_0)\| \, \|A^\beta(y_i - y_0)\|^{-1} = 0.$$

If on the contrary, the space $U^o_G(y_0)$ is not tangent to M^G at the point y_0, where $y_0 = h(v_0)$ and $v_0 \in M$, then there exist an $\eta > 0$ and a sequence of points $v_i \in \mathscr{D}_{\rho_0}(v_0)$, for $i = 1, 2, ...$, such that $v_i \to v_0$ in $M \subset V^{2\beta}$, as $i \to \infty$, and $y_i = h(v_i)$ satisfies

$$\|A^\beta Q^o_G(y_0)(y_i - y_0)\| \geqslant \eta \|A^\beta(y_i - y_0)\| > 0, \qquad \text{for all} \quad i \geqslant 1. \tag{6.98}$$

By taking a subsequence, if necessary, we can assume that, one has, either

$$\|A^\beta P^u_G(y_0)(y_i - y_0)\| \geqslant \tfrac{1}{2} \|A^\beta Q^o_G(y_0)(y_i - y_0)\|, \qquad \text{for all} \quad i \geqslant 1, \tag{6.99}$$

or

$$\|A^\beta P^s_G(y_0)(y_i - y_0)\| \geqslant \tfrac{1}{2} \|A^\beta Q^o_G(y_0)(y_i - y_0)\|, \qquad \text{for all} \quad i \geqslant 1. \tag{6.100}$$

Next define the two cone bundles $\mathscr{V}_1$ and $\mathscr{V}_2$ over M^G by

$$\mathscr{V}_1 \stackrel{\text{def}}{=} \{(w, y) \in V^{2\beta} \times M^G : \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta Q^o_G(y) w\| \leqslant 2 \|A^\beta w\|\},$$

and for $C > 0$,

$$\mathscr{V}_2(C) \stackrel{\text{def}}{=} \{(w, y) \in V^{2\beta} \times M^G : C \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta Q^o_G(y) w\|\}.$$

Let $(w, y) \in \mathscr{V}_2(C)$, where $C \geqslant 2$. Then one has $\|A^\beta P^o_G(y) w\| \leqslant C \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta Q^o_G(y) w\| \leqslant \|A^\beta w\| + \|A^\beta P^o_G(y) w\|$, which implies that $(C-1) \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta w\|$. If $(w, y) \notin \mathscr{V}_1$, then one has $2 \|A^\beta w\| < \|A^\beta Q^o_G(y) w\| \leqslant \|A^\beta w\| + \|A^\beta P^o_G(y) w\|$, which implies that $\|A^\beta w\| < \|A^\beta P^o_G(y) w\|$. Since $C \geqslant 2$, one finds that $\|A^\beta P^o_G(y) w\| \leqslant (C-1) \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta w\| < \|A^\beta P^o_G(y) w\|$, which is a contradiction. We have thus shown that $\mathscr{V}_2(C) \subset \mathscr{V}_1$, whenever $C \geqslant 2$. We now fix $C = C_4 \geqslant 20$ and set $\mathscr{V}_2 = \mathscr{V}_2(C_4) \subset \mathscr{V}_1$. To put it another way, the implication

$$C_4 \|A^\beta P^o_G(y) w\| \leqslant \|A^\beta Q^o_G(y) w\| \Rightarrow \|A^\beta Q^o_G(y) w\| \leqslant 2 \|A^\beta w\| \tag{6.101}$$

is valid, with $C_4 \geqslant 20$.

We now return to the inequalities (6.99) and (6.100). Since M^G is invariant under $S^G_2(t)$, it follows that, for $y_1, y_2 \in M^G$, the two solutions through y_1 and y_2 have negative continuations, which we will denote by $y(t, y_1)$ and $y(t, y_2)$, for $t \in R$. Our next objective is to show that, if either (6.99) or (6.100) hold, then there is a time $\tau \in R$ and an integer $i_0 \geqslant 1$, with the property that

$$\begin{aligned} &\|A^\beta Q^o_G(y(\tau, y_0))(y(\tau, y_i) - y(\tau, y_0))\| \\ &\qquad \geqslant C_4 \|A^\beta P^o_G(y(\tau, y_0))(y(\tau, y_i) - y(\tau, y_0))\| \end{aligned} \tag{6.102}$$

for $i \geqslant i_0$, i.e., $(y(\tau, y_i) - y(\tau, y_0), y(\tau, y_0)) \in \mathscr{V}_2$. By combining (6.101) and (6.102), we find that, for $i \geqslant i_0$, one then has

$$\|A^\beta(y(\tau, y_i) - y(\tau, y_0))\| \geqslant \tfrac{1}{2} \|A^\beta Q^o_G(y(\tau, y_0))(y(\tau, y_i) - y(\tau, y_0))\|. \tag{6.103}$$

Since the manifold M^G is of Lipschitz class, we can use Lemma 6.4 and Eq. (6.28) to define $H_2(t)$ by

$$y(t, y_i) - y(t, y_0) = \Phi^G(y_0, t)(y_i - y_0) + H_2(t), \qquad \text{for} \quad t \geqslant 0.$$

It follows from Lemma 6.4, as applied here,[4] that there is a $b_3 \in \Sigma$ such that, for any two points $y_1, y_2 \in M^G$ with $\|A^\beta(y_1 - y_2)\| \leqslant \rho$, one has

$$\|A^\beta H_2(t)\| \leqslant b_3(\rho) \|A^\beta(u_1 - u_2)\|, \qquad \text{for all} \quad t \in [0, 2T]. \tag{6.104}$$

For the proof of inequality (6.102), we first treat the case where inequality (6.99) holds. Choose, $i_0 \geqslant 1$ so that $\|A^\beta(y_i - y_0)\| \leqslant \rho$, for $i \geqslant i_0$. Hence $H_2(t)$ satisfies inequality (6.104), for $i \geqslant i_0$. We now fix $\tau > 0$ and after that fix $\rho > 0$ so that

$$(C_4 + 1) K b_3(\rho) \leqslant (C_4 + 1) K e^{\lambda_3 \tau} \leqslant \frac{\eta}{4K} e^{\lambda_4 \tau}, \tag{6.105}$$

where C_4 is given above. For an estimate of $\|A^\beta \Phi^G(y_0, t)(y_i - y_0)\|$, for $0 \leqslant t \leqslant T$, we will use the identity

$$y_i - y_0 = P^s_G(y_0)(y_i - y_0) + P^o_G(y_0)(y_i - y_0) + P^u_G(y_0)(y_i - y_0).$$

By using inequalities (6.98) and (6.99), one obtains

$$\|A^\beta P^u_G(y_0)(y_i - y_0)\| \geqslant \frac{\eta}{2} \|A^\beta(y_i - y_0)\|, \qquad \text{for} \quad i \geqslant i_0,$$

and by combining this with inequality (3.14), we find that, for all $t \geqslant 0$, one has

$$\begin{aligned} \|A^\beta \Phi^G(y_0, t) P^u_G(y_0)(y_i - y_0)\| &\geqslant K^{-1} e^{\lambda_4 t} \|A^\beta P^u_G(y_0)(y_i - y_0)\| \\ &\geqslant \frac{\eta}{2K} e^{\lambda_4 t} \|A^\beta(y_i - y_0)\|. \end{aligned} \tag{6.106}$$

[4] We drop the superscript F here because the term b_3 now depends on both F and G.

In addition, inequalities (3.9) and (3.11) imply that

$$\|A^\beta \Phi^G(y_0, t)\, P^s_G(y_0)(y_i - y_0)\| \leqslant K e^{\lambda_1 t}\, \|A^\beta (y_i - y_0)\|$$
$$\|A^\beta \Phi^G(y_0, t)\, P^o_G(y_0)(y_i - y_0)\| \leqslant K e^{\lambda_3 t}\, \|A^\beta (y_i - y_0)\|.$$

Next we use Lemma 6.4 to estimate both sides of inequality (6.102). Now for $i \geqslant i_0$ and $0 \leqslant t \leqslant \tau$, and since $\lambda_1 < \lambda_3$, one has

$$\begin{aligned}
&\|A^\beta Q^o_G(y(t, y_0))(y(t, y_i) - y(t, y_0))\| \\
&\quad \geqslant \|A^\beta Q^o_G(y(t, y_0))\, \Phi^G(y_0, t)(y_i - y_0)\| - \|A^\beta Q^o_G(y(t, y_0))\, H_2(t)\| \\
&\qquad\qquad \text{by the invariance of } Q^o_G \\
&\quad \geqslant \|A^\beta \Phi^G(y_0, t)\, P^u_G(y_0)(y_i - y_0)\| - \|A^\beta \Phi^G(y_0, t)\, P^s_G(y_0)(y_i - y_0)\| \\
&\qquad - \|A^\beta Q^o_G(y(t, y_0))\, H_2(t)\| \qquad \text{from the inequalities above} \\
&\quad \geqslant \left(\frac{\eta}{2K} e^{\lambda_4 t} - K e^{\lambda_3 t} - K b_3(\rho)\right) \|A^\beta (y_i - y_0)\|.
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&\|A^\beta P^o_G(y_0(t))(y(t, y_i) - y(t, y_0))\| \\
&\quad \leqslant \|A^\beta P^o_G(y_0(t))\, \Phi^G(y_0, t)(y_i - y_0)\| + \|A^\beta P^o_G(y_0(t))\, H_2(t)\| \\
&\quad \leqslant \|A^\beta \Phi^G(y_0, t)\, P^o_G(y_0)(y_i - y_0)\| + \|A^\beta P^o_G(y_0(t))\, H_2(t)\| \\
&\quad \leqslant (K e^{\lambda_3 t} + K b_3(\rho))\, \|A^\beta (y_i - y_0)\|.
\end{aligned}$$

Observe that inequality (6.102) is now a consequence of inequality (6.105).

Next we assume that inequality (6.100) is valid. While this case appears to be similar to the case where inequality (6.99) holds, there is a major difference in connection with the negative continuations of solutions of the linear problem, $\Phi^G(y_2, t)\, w$, for $t \leqslant 0$ and $w \in \mathcal{U}^s_G(y_2)$. When w is in $\mathcal{U}^u_G(y_2)$, or in $\mathcal{U}^o_G(y_2)$, then one can use the exponential trichotomy property to define $\Phi^G(y_2, t)\, w$, for $t \leqslant 0$, and inequalities (3.14) and (3.15) lead to good information about the growth of $\Phi^G(y_2, t)\, w$, for $t \geqslant 0$, see inequality (6.106), for example. On the other hand, if $w \in \mathcal{U}^s_G(y_2)$, then in general, one cannot extend the linear solution for $t \leqslant 0$. This is a problem even when $w = P^s_G(y_2)(y_1 - y_2)$, where $y_1, y_2 \in M^G$ and the nonlinear solutions $y(t, y_1)$ and $y(t, y_2)$ are defined for all $t \in R$. We must use a different approach, an approach which uses the partial extension given in (3.17) and (3.19).

Assume now that inequality (6.100) holds and let $\tau < 0$ be given. The value of τ will be fixed later. Recall that $S^G_2(t)\, y_i = y(t, y_i)$ and $S^G_2(t)\, y_0 = y(t, y_0)$ are the global solutions of Eq. (1.2) passing through y_i and y_0, respectively. We define $\hat{y}_i = y(\tau, y_i)$, $\hat{y}_0 = y(\tau, y_0)$, $\hat{z}_i = \hat{y}_i - \hat{y}_0$, and $z_i = \Phi^G(\hat{y}_0, -\tau)\,\hat{z}_i$, for $i \geqslant 1$. For the purpose of using Lemma 6.4 in this case, we define $H_2(t)$ by

$$S_2^G(t)\,\hat{y}_i - S_2^G(t)\,\hat{y}_0 = \Phi^G(\hat{y}_0, t)(\hat{y}_i - \hat{y}_0) + H_2(t), \qquad \text{for} \quad t \geqslant 0.$$

Let i_0 be chosen so that $\|A^\beta \hat{z}_i\| \leqslant \rho$, for $i \geqslant i_0$. From inequality (6.104) one obtains $\|A^\beta H_2(t)\| \leqslant b_3(\rho)\,\|A^\beta \hat{z}_i\|$, for $0 \leqslant t \leqslant -\tau$, By setting $t = -\tau$ in the previous equation, we obtain

$$y_i - y_0 = \Phi^G(\hat{y}_0, -\tau)\,\hat{z}_i + H_2(-\tau) = z_i + H_2(-\tau). \tag{6.107}$$

From this equation and inequality (6.104), we obtain

$$\|A^\beta z_i\| \leqslant (1 + b_3(\rho))\,\|A^\beta(y_i - y_0)\|, \tag{6.108}$$

and inequalities (6.29), (6.98), and (6.100) yield

$$\begin{aligned}\|A^\beta P_G^s(y_0)\,z_i\| &\geqslant \|A^\beta P_G^s(y_0)(y_i - y_0)\| - \|A^\beta P_G^s(y_0)\,H_2(-\tau)\| \\ &\geqslant \left(\frac{\eta}{2} - Kb_3(\rho)\right)\|A^\beta(y_i - y_0)\|.\end{aligned}$$

Among other things, we will require that τ and $\rho > 0$ satisfy $Kb_3(\rho) \leqslant \frac{\eta}{4}$, which then implies that

$$\|A^\beta P_G^s(y_0)\,z_i\| \geqslant \frac{\eta}{4}\,\|A^\beta(y_i - y_0)\|. \tag{6.109}$$

We now use Eq. (3.17) to define $\Phi^G(y_0, t)\,z_i = \Phi^G(\hat{y}_0, -\tau + t)\,\hat{z}_i$, for $\tau \leqslant t \leqslant 0$, which implies that

$$\hat{y}_i - \hat{y}_0 = \hat{z}_i = \Phi^G(y_0, \tau)\,z_i.$$

Now inequalities (3.19), (6.100), (6.109), and Eq. (6.107) imply that

$$\begin{aligned}\|A^\beta \Phi^G(y_0, \tau)\,P_G^s(y_0)\,z_i\| &\geqslant K^{-1} e^{\lambda_1 \tau}\,\|A^\beta P_G^s(y_0)\,z_i\| \\ &\geqslant \frac{\eta}{4K}\,e^{\lambda_1 \tau}\,\|A^\beta(y_i - y_0)\|.\end{aligned} \tag{6.110}$$

Next we observe that inequalties (3.10), (3.12), and (6.108) imply that

$$\|A^\beta \Phi^G(y_0, \tau)\,P_G^u(y_0)\,z_i\| \leqslant K(1 + b_3(\rho))\,e^{\lambda_4 \tau}\,\|A^\beta(y_i - y_0)\|, \tag{6.111}$$

and

$$\|A^\beta \Phi^G(y_0, \tau)\, P_G^o(y_0)\, z_i\| \leqslant K e^{\lambda_2 \tau}\, \|A^\beta z_i\| \leqslant K(1+b_3(\rho))\, e^{\lambda_2 \tau}\, \|A^\beta(y_i - y_0)\|.$$

From inequalities (6.110) and (6.111) and $\lambda_2 < \lambda_4$, we get

$$\begin{aligned}\|A^\beta \Phi^G(y_0, \tau)\, Q_G^o(y_0)\, z_i\| &\geqslant \|A^\beta \Phi^G(y_0, \tau)\, P_G^s(y_0)\, z_i\| \\ &\quad - \|A^\beta \Phi^G(y_0, \tau)\, P_G^u(y_0)\, z_i\| \\ &\geqslant \left(\frac{\eta}{4K} e^{\lambda_1 \tau} - K(1+b_3(\rho))\, e^{\lambda_2 \tau}\right) \|A^\beta(y_1 - y_0)\|.\end{aligned}$$

We now fix $\tau < 0$ and afterwards fix $\rho > 0$ so that

$$(C_4 + 1)\, e^{\lambda_2 \tau} \leqslant \frac{\eta}{8K^2}\, e^{\lambda_1 \tau}, \qquad \text{and} \qquad b_3(\rho) \leqslant \min\left(\frac{\eta}{4K}, 1\right).$$

With this choice of τ and ρ, we see that

$$C_4 K(1+b_3(\rho))\, e^{\lambda_2 \tau} \leqslant \frac{\eta}{4K}\, e^{\lambda_1 \tau} - K(1+b_3(\rho))\, e^{\lambda_2 \tau},$$

which implies that inequality (6.102) is valid for $i \geqslant i_0$.

Step 2. Let us next return to the shadow semiflow $S_1^G(t)\, v_i = v(t, y_i)$ on the unperturbed manifold M. Recall that $y(t, y_i) = h(v(t, y_i))$ and $y(t, y_0) = h(v(t, y_0))$, for all $t \in R$. Since $y_i \to y_0$ in $V^{2\beta}$, it follows from inequality (6.41) that there is an $i_1 \geqslant i_0$ such that

$$\|A^\beta(y(t, y_i) - y(t, y_0))\| \leqslant \varepsilon, \qquad \text{and} \qquad \|A^\beta(v(t, y_i) - v(t, y_0))\| \leqslant \varepsilon,$$

for $i \geqslant i_1$, and $|t| \leqslant |\tau|$. Let $\bar{v}_i = v(\tau, y_i)$, and $\bar{y}_i = y(\tau, y_i) = h(\bar{v}_i)$, for $i \geqslant 1$, and set $\bar{v}_0 = v(\tau, y_0)$ and $\bar{y}_0 = y(\tau, y_0) = h(\bar{v}_0)$. By choosing a subsequence, which we relabel as $\bar{v}_i$, one has $\bar{v}_i \in \mathscr{D}_{\rho_0}(\bar{v}_0)$, $\bar{v}_i \to \bar{v}_0$, and $\bar{y}_i \to \bar{y}_0$, as $i \to \infty$. Since $\ell \leqslant 1$, it follows from inequality (6.88) that

$$\|A^\beta(\bar{y}_i - \bar{y}_0)\| = \|A^\beta(h(\bar{v}_i) - h(\bar{v}_0))\| \leqslant 3\, \|A^\beta(\bar{v}_i - \bar{v}_0)\|. \tag{6.112}$$

By applying $P^o(\bar{v}_0)$ to $\bar{y}_i - \bar{y}_0 = (\bar{y}_i - \bar{v}_i) - (\bar{y}_0 - \bar{v}_0) + (\bar{v}_i - \bar{v}_0)$, we obtain

$$P^o(\bar{v}_0)(\bar{y}_i - \bar{y}_0) = P^o(\bar{v}_0)(\bar{y}_i - \bar{v}_i) - P^o(\bar{v}_0)(\bar{y}_0 - \bar{v}_0) + P^o(\bar{v}_0)(\bar{v}_i - \bar{v}_0). \tag{6.113}$$

Since $(\bar{y}_0 - \bar{v}_0) \in U^s(\bar{v}_0) \oplus U^u(\bar{v}_0)$ and $(\bar{y}_i - \bar{v}_i) \in U^s(\bar{v}_i) \oplus U^u(\bar{v}_i)$, one has

$$P^o(\bar{v}_0)(\bar{y}_0 - \bar{v}_0) = 0 \qquad \text{and} \qquad P^o(\bar{v}_i)(\bar{y}_i - \bar{v}_i) = 0, \qquad \text{for} \quad i \geqslant 1. \tag{6.114}$$

Consequently, the term $P^o(\bar{v}_0)(\bar{y}_i - \bar{v}_i)$ assumes the form

$$P^o(\bar{v}_0)(\bar{y}_i - \bar{v}_i) = (P^o(\bar{v}_0) - P^o(\bar{v}_i))(\bar{y}_i - \bar{v}_i). \tag{6.115}$$

From inequality (6.4) we obtain

$$\|A^\beta(P^o(\bar{v}_i) - P^o(\bar{v}_0))(\bar{y}_i - \bar{v}_i)\| \leqslant L_0 \, \|A^\beta(\bar{v}_i - \bar{v}_0)\| \, \|A^\beta(\bar{y}_i - \bar{v}_i)\|.$$

By combining this inequality with (6.115) and using Item (4) of Theorem 6.13, one obtains

$$\|A^\beta P^o(\bar{v}_0)(\bar{y}_i - \bar{v}_i)\| \leqslant 2L_0\varepsilon \, \|A^\beta(\bar{v}_i - \bar{v}_0)\|.$$

From Lemma 6.1 one finds that

$$\|A^\beta P^o(\bar{v}_0)(\bar{v}_i - \bar{v}_0)\| \geqslant \tfrac{4}{5} \, \|A^\beta(\bar{v}_i - \bar{v}_0)\|.$$

By combining the last two inequalities with (6.114) and (6.113), it follows that

$$\|A^\beta P^o(\bar{v}_0)(\bar{y}_i - \bar{y}_0)\| \geqslant \tfrac{1}{5}(4 - 10L_0\varepsilon) \, \|A^\beta(\bar{v}_i - \bar{v}_0)\|. \tag{6.116}$$

Let us next estimate the norm $\|A^\beta P^o_G(\bar{y}_0)(\bar{y}_i - \bar{y}_0)\|$. First note that

$$P^o_G(\bar{y}_0)(\bar{y}_i - \bar{y}_0) = P^o(\bar{v}_0)(\bar{y}_i - \bar{y}_0) + (P^o_G(\bar{y}_0) - P^o(\bar{v}_0))(\bar{y}_i - \bar{y}_0). \tag{6.117}$$

Since one has $y_0 = h(v_0)$ and

$$\lim_{(\varepsilon, \delta) \to (0, 0)} P^o_G(y_0) = P^o(v_0), \qquad \text{in } \mathscr{L}(V^{2\beta}, W),$$

uniformly for $v_0 \in M$, there is a $\beta_3 \in \Sigma$ such that

$$\|A^\beta(P^o_G(\bar{y}_0) - P^o(\bar{v}_0))(\bar{y}_i - \bar{y}_0)\| \leqslant \beta_3(\varepsilon, \delta) \, \|A^\beta(\bar{y}_i - \bar{y}_0)\|,$$

for small ε and δ. By combining this inequality with (6.116) and (6.117) we find that

$$\|A^\beta P^o_G(\bar{y}_0)(\bar{y}_i - \bar{y}_0)\| \geqslant \tfrac{1}{5}(4 - 10L_0\varepsilon) \, \|A^\beta(\bar{v}_i - \bar{v}_0)\| - \beta_3(\varepsilon, \delta) \, \|A^\beta(\bar{y}_i - \bar{y}_0)\|. \tag{6.118}$$

Now inequalities (6.102) and (6.103) imply that

$$\|A^\beta(\bar{y}_i-\bar{y}_0)\| \geqslant \frac{C_4}{2}\,\|A^\beta P_G^o(\bar{y}_0)(\bar{y}_i-\bar{y}_0)\|.$$

By combining this with (6.118) we conclude that

$$\|A^\beta(\bar{y}_i-\bar{y}_0)\| \geqslant \frac{C_4(2-5L_0\varepsilon)}{5}\,\|A^\beta(\bar{v}_i-\bar{v}_0)\| - \frac{C_4\,\beta_3(\varepsilon,\delta)}{2}\,\|A^\beta(\bar{y}_i-\bar{y}_0)\|,$$

which implies that

$$\|A^\beta(\bar{y}_i-\bar{y}_0)\| \geqslant \frac{C_4(4-10L_0\varepsilon)}{(10+5C_4\beta_3(\varepsilon,\delta))}\,\|A^\beta(\bar{v}_i-\bar{v}_0)\|.$$

Let ε_9 and δ_9 be given by Lemma 6.12. Now choose ε_{10} so that $0<\varepsilon_{10}\leqslant\varepsilon_9$, $L_0\varepsilon_{10}\leqslant\frac{1}{10}$, and $C_4\,\beta_3(\varepsilon_{10},\varepsilon_{10})\leqslant 1$. Set $\delta_{10}(\varepsilon)=\min(\varepsilon,\delta_9(\varepsilon))$, for $0<\varepsilon\leqslant\varepsilon_{10}$. Since $C_4\geqslant 20$, one finds that

$$\|A^\beta(\bar{y}_i-\bar{y}_0)\| \geqslant 4\,\|A^\beta(\bar{v}_i-\bar{v}_0)\|,$$

which contradicts inequality (6.112). By setting $\delta_2=\delta_{10}(\varepsilon_{10})$, we thereby complete the proof of Theorem 6.14 and the Main Theorem. ∎

Remark. The methodology used here to prove that the perturbed manifold M^G is of class C^1 and that it is normally hyperbolic, i.e., Eq. (4.10) is valid, can be used in other circumstances with only minor modifications. For example, assume that $F\in C^1_{\mathrm{Lip}}$, and that M is a compact, connected, finite dimensional, invariant Lipschitz-manifold in $V^{2\beta}$ for the Eq. (1.1). Assume further that the linear skew product semiflow π has an exponential trichotomy over M, and the linear space $\mathscr{R}(R(v))$ has the property that

$$\dim \mathscr{R}(R(v)) = \dim M, \qquad \text{for each} \quad v\in M. \tag{6.119}$$

Then the argument used above can be easily adapted to show that Eq. (4.10) is valid for all $v\in M$, and the manifold M is of class C^1, and it is normally hyperbolic for Eq. (1.1). In short, one has

$$\text{Lipschitz Manifold} + \text{Exponential Trichotomy} + (6.119)$$
$$\Rightarrow C^1 + \text{Normal Hyperbolicity}.$$

The same methodology can be used for establishing the smoothness of inertial manifolds. In this setting, one typically assumes that the nonlinearity F has bounded support in $V^{2\beta}$, see Foias, Sell, and Temam (1988).

As noted above, the exponential trichotomy arising in the inertial manifold theory has the property that $P \equiv 0$. One should compare the argument for smoothness used in this paper, as it might apply to inertial manifolds, with Chow, Lu, and Sell (1992).

APPENDIX: THE GRONWALL–HENRY INEQUALITY

In the following result, we present the Gronwall–Henry inequality. This important inequality is an infinite dimensional variation of the classical Gronwall inequality, and it arises in the study of the dynamics of solutions of nonlinear partial differential equations. For this purpose, we define

$$E_{r,c}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(nr+c)} z^{nr} \qquad \text{and}$$

$$E'_{r,c}(z) = \frac{d}{dz} E_{r,c}(z) = \sum_{n=1}^{\infty} \frac{nr\,\Gamma(c)}{\Gamma(nr+c)} z^{nr-1},$$

where r and c are positive real numbers.

Gronwall–Henry Inequality. Let $v(t)$ be a nonnegative function in $L^{\infty}_{\text{loc}}[0, \tau; R)$ and $h(\cdot) \in L^{1}_{\text{loc}}[0, \tau; R)$ satisfy

$$v(t) \leqslant h(t) + M \int_0^t (t-s)^{r-1} v(s)\, ds, \qquad t \in (0, \tau),$$

where $0 < \tau \leqslant \infty$ and $r > 0$. Then one has

$$v(t) \leqslant h(t) + \mu \int_0^t E'_{r,1}(\mu(t-s))\, h(s)\, ds, \qquad t \in (0, \tau), \tag{7.1}$$

where $\mu^r = M\Gamma(r)$. If in addition, one has $h(t) \equiv at^{c-1}$, where a and c are positive constants, then

$$v(t) \leqslant at^{c-1} E_{r,c}(\mu t), \qquad t \in (0, \tau). \tag{7.2}$$

Moreover, if $h(t) = ae^{\lambda t}$, where $\lambda > \mu$, then

$$v(t) \leqslant a(1-\theta)^{-1} e^{\lambda t}, \qquad \text{for} \quad t \in (0, \tau), \tag{7.3}$$

where $\theta = \mu^r \lambda^{-r}$. Finally, one has

$$\lim_{t \to \infty} = \frac{1}{t} \log(E_{r,c}(\mu t)) = \mu, \qquad \text{whenever} \quad r > 0 \quad \text{and} \quad c > 0. \tag{7.4}$$

The proof of (7.1) and (7.4) can be found in Henry (1981, p. 188). The proofs of (7.2) and (7.3) are based on the following elementary properties of the Beta and Gamma functions,

$$B(r, c) = \int_0^1 (1-x)^{r-1} x^{c-1} \, dx \qquad \text{and} \qquad \Gamma(r) = \int_0^\infty x^{r-1} e^{-x} \, dx,$$

where r and c are positive real numbers: One has $\Gamma(p) = (p-1)!$, for any integer $p \geqslant 1$, $\Gamma(r)$ is everywhere positive with a unique minimum at a point r_0, where $1 < r_0 < 2$, and $\Gamma(r)$ is strictly increasing, for $r > r_0$. Also one has $\Gamma(r+1) = r\Gamma(r)$,

$$B(r, c) = \frac{\Gamma(r)\,\Gamma(c)}{\Gamma(r+c)}, \qquad \text{and} \qquad \int_0^t (t-s)^{r-1} s^{c-1} \, ds = t^{r+c-1} B(r, c).$$

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