Large and Small Solutions of a Class of Quasilinear Elliptic Eigenvalue Problems

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Existence and uniqueness results for large positive solutions are obtained for a class of quasilinear elliptic eigenvalue problems in general bounded smooth domains via a generalization of a sweeping principle of Serrin. The nonlinear terms of the problems can be negative in some intervals. The existence and structure of a mountain pass solution are also discussed. We show that this solution develops to a spike layer solution.

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1. INTRODUCTION

Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a bounded, connected, smooth domain, and let \( \Delta_p \) be the \( p \)-Laplacian defined by \( \Delta_p u = \text{div}(|Du|^{p-2}Du) \). We consider the existence of positive solutions of the quasilinear eigenvalue problem

\[
-\Delta_p u = \lambda f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]

(1.1)

for \( 1 < p < \infty, \lambda > 0 \), under appropriate smoothness conditions on \( f \).

By a positive solution of Eq. (1.1) we mean a pair \( (\lambda, u) \) in \( \mathbb{R}^+ \times C_0^1(\bar{\Omega}) \) satisfying Eq. (1.1) in the weak sense and with \( u > 0 \) in \( \Omega \).

This problem appears in the study of non-Newtonian fluids. The quantity \( p \) is a characteristic of the medium. Media with \( p > 2 \) are called dilatant.
fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids (see, for example, [12] and its bibliography.) Other applications of such problems are found when seeking soliton-like solutions of Lorentz invariant equations; see [1, 2].

In this paper we allow $f$ to change sign, in particular we assume $f$ satisfies the following conditions.

$$(F_1) \quad f(0) = 0; \text{ there are precisely two numbers } 0 < \rho_1 < \rho_2 \text{ such that } f(\rho_1) = f(\rho_2) = 0 \text{ and } f > 0 \text{ in } (\rho_1, \rho_2), \text{ } f < 0 \text{ in } (0, \rho_1), \lim_{s \to 0^+} f(s)/s^{p-1} = -m < 0, \lim_{s \to \rho_1^+} f(s)/(s - \rho_1)^{p-1} < 0, 0 < \lim_{s \to \rho_1^-} f(s)/(s - \rho_1)^{p-1} < \infty. $$

$$(F_2) \quad \int_{\rho}^{\rho_2} f(s) \, ds > 0 \text{ for every } \rho \in [0, \rho_2). \text{ We denote by } \hat{\mu} \in (\rho_1, \rho_2) \text{ the unique number such that } \int_0^{\hat{\mu}} f(s) \, ds = 0. $$

Such problems have been extensively studied by many authors; see, for example, [17–24, 27, 29, 33–35, 38, 39].

In [17], it was shown that $(F_2)$ is a necessary condition for the existence of a positive solution $u_i$ of (1.1) with $\max u_i \in (\rho_1, \rho_2]$. In the present paper we obtain a uniqueness result for positive solutions of (1.1) whose maximum is close to $\rho_2$ under the extra condition:

$$(F_3) \quad \text{There exists } \delta > 0 \text{ such that } f'(s) < 0 \text{ in } (\rho_2 - \delta, \rho_2) \text{ and there exists } M > 0 \text{ such that } f(s) \leq M(\rho_2 - s)^{p-1} \text{ for } 0 < s \leq \rho_2. $$

A typical example of $f$ satisfying $(F_1)$–$(F_3)$ is

$$f(s) = \begin{cases} -ms^{p-1}|a - s|^{p-2}(a - s)|1 - s|^{p-2}(1 - s), & \text{for } p \geq 2 \\ -ms^{p-1}(a-s)(1-s), & \text{for } 1 < p < 2, \end{cases}$$

where $0 < a < 1/4$.

Furthermore, we obtain, by the mountain pass lemma, the existence of a second positive solution $\bar{u}_2$ of Eq. (1.1) and we study the structure of $\bar{u}_2$.

The main results of this paper are the following theorems.

**Theorem A.** Let $f \in C^0([0, \infty)) \cap C^1((0, \infty))$ satisfy $(F_1)$–$(F_3)$. Then for each nonnegative function $\zeta \in C^0_\alpha(\Omega)$ with $\max \zeta \in (\rho_1, \rho_2)$, there is $\lambda_0 = \lambda_0(\zeta) > 0$ such that for all $\lambda > \lambda_0$, (1.1) possesses exactly one solution $\bar{u}_1$ satisfying $\zeta < \bar{u}_1 < \rho_2$ and $\lim_{s \to a^-} \max \bar{u}_1 = \rho_2$. Moreover, for any compact set $K \subset \bar{\Omega}$, $\bar{u}_1 \to \rho_2$ in $K$ as $\lambda \to \infty$.

**Theorem B.** Assume $p > 2$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a convex domain, and $f$ satisfies the conditions $(F_1)$–$(F_3)$. Then, for $\lambda$ sufficiently large, there exists a
positive solution $u_\lambda \neq \bar{u}_\lambda$, which is a mountain pass solution of (1.1) for the functional

$$J_\lambda(u) = \frac{1}{p} \int_\Omega |Du|^p - \lambda \int_\Omega F(u)$$

where $F(u) = \int_0^u f(s) \, ds$.

Moreover, $\rho_1 < \max_\Omega u_\lambda < \rho_2$, and $u_\lambda$ has the following properties:

(i) $C^* \lambda^{N/p} J_\lambda(u) \leq C_1 \lambda^{N/p}$ for $1 < p < N$, where $C^* > 0$, $C_1 > 0$ are independent of $\lambda$.

(ii) $C^* \lambda^{-\alpha + p/q} J_\lambda(u) \leq C_1 \lambda^{-N/p}$ for $p \geq N$ and any $q > 0$, where $C^* > 0$, $C_1 > 0$ are independent of $\lambda$.

(iii) $\int_\Omega u^p dx \leq C_2 \lambda^{-N/p}$ for some constant $C_2 > 0$ independent of $\lambda$.

Theorem C. Let $p > 2$, $\Omega$ be as in Theorem B, and suppose $f$ satisfies $(F_1)$–$(F_3)$ and

$$(F_4) \quad (s - \rho_1) f'(s) - (p - 1) f(s) < 0 \quad \text{for} \quad s \in (\rho_1, \rho_2).$$

Suppose that $u_\lambda$ is the solution obtained in Theorem B which is such that for a $\sigma^* > 0$ satisfying $\rho_1 + \sigma^* < \bar{u}$, the set $\Omega_{\rho_1 + \sigma^*} = \{x \in \Omega : u_\lambda > \rho_1 + \sigma^*\}$ is a connected convex set. Then, for $\lambda$ sufficiently large, $u_\lambda$ has only one local (hence global) maximum point $P_\lambda \in \Omega$, $\text{dist}(P_\lambda, \partial\Omega) \geq \theta > 0$: $u_\lambda \to 0$ outside any neighbourhood of $P_\lambda$ and $u_\lambda(P_\lambda) \to w(0)$, where $w$ is the unique positive (radial) solution of

$$-\Delta w + f(w) = 0 \quad \text{in} \quad \mathbb{R}^N, \quad w \to 0 \quad \text{as} \quad |x| \to \infty,$$

with $\rho_1 < w(0) < \rho_2$. More precisely, $u_\lambda(\lambda^{-1/p} \cdot + P_\lambda) \to w(\cdot)$ uniformly in $C^{1,\text{loc}}(\Omega)$ where $\Omega_\lambda = \{y : \lambda^{-1/p} y + P_\lambda \in \Omega\}$.

The existence and uniqueness of the positive radial solution $w$ with $w(0) > \bar{u}$, $w'(0) = 0$, $w'(r) < 0$ for $r > 0$ of Eq. (1.2) under the assumptions $(F_1)$–$(F_3)$ and $(F_4)$ with $p > 2$ have been obtained in [20]. It is also shown in [20] that

$$\lim_{r \to \infty} \sup_{\eta \in (0, m/(p-1))} \frac{w(r) e^{\left(\frac{\bar{u}}{p-1} - \eta\right) \frac{1}{p}}}{r} < \infty$$

for any $\eta \in (0, m/(p-1))$ and

$$\lim_{r \to \infty} \frac{w'(r)}{w(r)} = -\left(\frac{m}{p-1}\right)^{1/p}.$$
Such kinds of uniqueness results have also been obtained in [13] with some assumptions on $f$ different from those in [20]. The main results in [20] are closely related to those in [8, 36], for $p = 2$.

In [20], it was proved that when $\Omega = B$, the unit ball of $\mathbb{R}^N$, $f$ satisfies the conditions of Theorem C, (1.1) has precisely two positive solutions, and they are both radial. The following result, Theorem 2 from [4], was used in [20].

**Theorem D.** Let $p \in (1, \infty)$ and let $f = f(s)$ be continuous and bounded on $\mathbb{R}^+_0$ and satisfy:

(a) if $f(S) = 0$ for some $S > 0$, then there is a function $\beta \in A_p$ such that

$$f(s) \leq \beta(S-s) \quad \text{for} \quad 0 \leq s \leq S,$$

where

$$A_p = \left\{ \beta \in C(\mathbb{R}^+_0); \beta(0) = 0, \beta \text{ is nondecreasing and} \right. \int_0^1 (s\beta(s))^{-1/p} \, ds = +\infty \left. \right\}.$$  

Assume that $u \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ satisfies

$$-A_p u = f(u), \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N. \quad (1.3)$$

Suppose that $\lim_{|x| \to \infty} u(x) = 0$, $u > 0$, and there is some number $\tilde{\delta} > 0$ such that $f(s)$ is nonincreasing for $0 < s < \tilde{\delta}$ and $f(u(\cdot)) \in L^1(\mathbb{R}^N)$. Then $u$ is radially symmetric about some $x_0 \in \mathbb{R}^N$.

Theorem D was established by using a new rearrangement technique called continuous Steiner symmetrization (see [3, 4]) together with the maximum principle for the $p$-Laplacian. Note that $\beta(s) = Ms^{p-1} \in A_p$. For other radially symmetric results similar to Theorem D for $1 < p < 2$, we refer to [11, 41].

As in [20], we shall assume that $x_0 = 0$ and $0 \in \Omega$. Otherwise, we may use a simple transformation to make this true. Theorem D and the results in [20] imply that Eq. (1.2) has precisely one positive radial solution $w$ with $w'(r) < 0$ for $r > 0$ and $w(r) \to 0$ as $r \to \infty$.

We call $u_l$ in Theorem A a large solution of (1.1) and $u_{l1}$ in Theorem B a small solution of (1.1).

When $p = 2$, Theorem A has been obtained by Clement and Sweers [6] by using the strong maximum principle, the strong comparison principle,
and the sweeping principle of Serrin for the Laplacian \( \Delta \). When \( p \neq 2 \) such nice features seem to be lost or at least difficult to verify. Clement and Sweers [6] obtained uniqueness of solutions of \( (1.1) \) with \( p = 2 \) by use of Leray–Schauder degree theory. They first showed that any solution \( u_1 \) is isolated and then calculated the index of \( u_1 \). To prove the isolatedness of \( u_1 \), they used linearization of the equation at \( u_1 \). We cannot use such a method when \( p \neq 2 \) since the linearization of our equation seems to be very complicated. We have obtained some uniqueness results for problem \( (1.1) \) under the assumptions that \( f \) is increasing in \( (0, +\infty) \) (see [23]) or \( \Omega \) is a ball or an annulus (see [18, 20, 24]). The problem seems to be more difficult without such special assumptions.

It follows from a strong maximum principle in [34] and [44] that if \( f \) satisfies \( (F_3) \) and \( u_1 \) is a positive solution of \( (1.1) \), then \( u_1 < \rho_2 \) in \( \Omega \). If \( f(s) \sim C(\rho_2 - s)^k \) with \( 0 < k < p - 1 \) for \( s \) near \( \rho_2 \), a flat core of \( u_1 \) may occur. That is, \( E = \{ x \in \Omega : u_1(x) = \rho_2 \} \neq \emptyset \) (see [26, 29]). Notice that if \( p > 2 \), \( f \in C^0( [0, \infty) ) \cap C^1( (0, \infty) ) \) satisfies \( (F_1) \)–\( (F_2) \) and \( f'(\rho_2) < 0 \), then \( f \) does not satisfy \( (F_3) \). This case is difficult since we need information about the flat core of the large solution; we will leave the discussions to [22].

Theorems B and C have been studied in the case \( p = 2 \) by many authors; see, for example, [10, 25, 30–32]. We overcome many technical difficulties here for \( p > 2 \).

2. SWEEPING OUT RESULTS

In this section we give some results which are useful for the proofs of the main theorems.

**Lemma 2.1.** Assume that \( f \) satisfies \( (F_1) \) and \( (F_2) \). Let \( (\lambda, u) \) be a positive radial solution of the problem

\[
-\text{div}(|Du|^{p-2} Du) = \lambda f(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \tag{2.1}
\]

where \( B \) is the unit ball in \( \mathbb{R}^N \), satisfying \( \max u \leq \rho_2 \). Then

(i) if \( \max u < \rho_2 \), \( u'(r) < 0 \) for \( r \in (0, 1] \).

(ii) if \( \max u = \rho_2 \), there exists \( 0 \leq r_0 < 1 \) such that \( u \equiv \rho_2 \) in \( [0, r_0] \).

\( u'(r) < 0 \) for \( r \in (r_0, 1] \).

**Proof.** Since \( u \) is a positive radial solution of Eq. (2.1), \( u \) satisfies

\[
-|u'|^{p-2} u' - \frac{N - 1}{r} |u'|^{p-2} u' = \lambda f(u). \tag{2.2}
\]
Let $\tau \in [0, 1)$ be a point such that $u'(\tau) = 0$. Multiplying both sides of Eq. (2.2) by $u'$ and integrating on $(\tau, 1)$, we obtain

$$(1 - 1/p) |u'(1)|^p + \int_{\tau}^{1} \frac{N-1}{\xi} |u'(\xi)|^p d\xi + \lambda \int_{u(\tau)}^{0} f(s) ds = 0.$$

This implies that $u(\tau) > \mu$, the unique number such that $\int_{\tau}^{0} f(s) ds = 0$. In particular $u(0) > \mu$.

If $u'(r_1) = 0$ and $u'(r_2) = 0$ for $0 \leq r_1 < r_2 \leq 1$ then multiplying by $u'$ on both sides of Eq. (2.2) and integrating on $(r_1, r_2)$, we have

$$\int_{r_1}^{r_2} \frac{N-1}{\xi} |u'(\xi)|^p d\xi + \lambda \int_{u(r_1)}^{u(r_2)} f(s) ds = 0. \tag{2.3}$$

This is impossible if $u(r_2) > u(r_1)$. Hence there cannot exist a local minimum since it would have to be followed by a local maximum. Also max $u = u(0)$ and so $u$ is decreasing for small $r > 0$.

Since the strong maximum principle need not hold for our operator (see [27]), there may exist $0 < r_0 < 1$ such that $u(r) = u(0)$ in $[0, r_0]$. This constant value must be $\rho_2$. This is impossible in case (i).

In case (i) we claim that $u'(r) < 0$ for all $r \in (0, 1)$. Indeed, if not, we let $r_1$ be the first point with $u'(r_1) = 0$, so that $u$ is decreasing on $(0, r_1)$ and $u(r_1) > \mu$. Therefore $f(u(r)) > 0$ for $r \in (0, r_1)$. Writing Eq. (2.1) in the form

$$-(r^{N-1} |u'|^{p-2} u')' = \lambda r^{N-1} f(u)$$

and integrating on $(0, r_3)$ we obtain

$$\int_{0}^{r_3} r^{N-1} f(u(r)) dr = 0.$$ 

This contradiction shows that no such $r_3$ exists. The rest of case (ii) is now clear.

**Lemma 2.2.** Assume that $f$ satisfies $(F_1)$. Then there is no positive radial solution $u(r)$ of Eq. (2.1) satisfying $u(1) = 0$, $u'(1) = 0$ for $u > 0$ in $(0, 1)$, and $u' < 0$ in $(1 - \hat{\delta}, 1)$, $\hat{\delta} > 0$.

**Proof.** Suppose that there exists a solution $u$ satisfying the listed properties. Then

$$u(r) = \lambda^{1/(p-1)} \int_{r}^{1} s^{1-N} \int_{r}^{1} \xi^{N-1} (-f(u(\xi))) d\xi \frac{1}{(p-1)} ds$$
for \( r \in (1-\delta, 1) \). Since \( f(u) \geq -(m+1) u^{p-1} \) for sufficiently small \( u > 0 \), we have

\[
u(r) \leq \lambda^{1/(p-1)}(m+1)/N [((1-r^N)/(r^{N-1})]^{1/(p-1)} (1-r) u(r),
\]

for \( r \in (1-\delta, 1) \). Thus, \( u \equiv 0 \) in \((1-\delta, 1)\) for \( \delta \) sufficiently small. This is a contradiction.

Let \( \varepsilon > 0 \) (which will be chosen to be sufficiently small below). We make an extension \( f_\varepsilon \) of \( f \) such that \( f_\varepsilon \) satisfies \( f_\varepsilon = f \) on \([0, \rho_2]\) and the following conditions which we refer to as (\( F_\varepsilon \)):

\[ f_\varepsilon \text{ is bounded}, \]
\[ f_\varepsilon(s) = \varepsilon \quad \text{for} \quad s \in (-\infty, -1] \]
\[ f_\varepsilon \in C^1(-1, 0), f_\varepsilon(s) \in (0, \varepsilon) \text{ and is decreasing} \quad \text{for} \quad s \in (-1, 0) \]
\[ \lim_{s \to 0^-} f_\varepsilon(s)/(|s|^{p-2}s) = -m \]
\[ f_\varepsilon(s) < 0 \quad \text{for} \quad s \in (\rho_2, \infty) \]
\[ \int_0^{\rho_2} f_\varepsilon(s) \, ds > 0 \quad \text{for} \quad \rho \in [-1, 0]. \]

Moreover,

\[
\lim_{s \to (-1)^+} (f_\varepsilon(s) - \varepsilon)/(s+1)^{p-1} = 0 \quad \text{for} \quad p > 1
\]

and for \( 1 < p < 2 \),

\[
\lim_{s \to (-1)^+} f_\varepsilon'(s) = 0.
\]

Since

\[
f(s)/s = (f(s)/s^{p-1}) s^{p-2}, \quad \frac{f_\varepsilon(s) - \varepsilon}{(s+1)^{p-1}} = \frac{f_\varepsilon(s) - \varepsilon}{(s+1)^{p-1}} (s+1)^{p-2},
\]

we have

\[
\lim_{s \to 0} f_\varepsilon'(s) = 0, \quad \lim_{s \to -1} f_\varepsilon'(s) = 0 \quad \text{for} \quad p > 2,
\]
\[
\lim_{s \to 0} f_\varepsilon'(s) = -m, \quad \lim_{s \to -1} f_\varepsilon'(s) = 0 \quad \text{for} \quad p = 2.
\]
and
\[
\lim_{s \to 0^+} f'_e(s) = -\infty, \quad \lim_{s \to -1^-} f'_e(s) = 0 \quad \text{for} \quad 1 < p < 2.
\]

Therefore, \( f_e \in C^1(-\infty, \infty) \) for \( p \geq 2 \) and \( f_e \in C^1((-\infty, \infty) \setminus \{0\}) \) for \( 1 < p < 2 \). Moreover, since \( \lim_{s \to 0} f_e(s)/|s|^{p-2} s = -m \), we have
\[
f'_e(s) \sim |s|^{p-2} \quad \text{for} \quad s \text{ near} \ 0.
\]

Thus, we can choose \( M > 0 \) sufficiently large such that \( f_e(s) + M |s|^{p-2} s \) is increasing on \((-\infty, \rho_2]\).

**Lemma 2.3.** Let \( f_e \) be defined as above. Then there exists \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \) and \( \mu > 0 \) sufficiently large (\( \mu \) is independent of \( \varepsilon \)), there exists \( v_{\mu, \varepsilon} \in C^1(\mathbb{R}^N) \), radially symmetric, which satisfies:
\[
-\text{div}(Dv_{\mu, \varepsilon}^{|\mu|} Dv_{\mu, \varepsilon}) = \mu (f_e(v_{\mu, \varepsilon}) - \varepsilon) \quad \text{in} \ \mathbb{R}^N,
\]
\[
v_{\mu, \varepsilon}(0) \in (\rho_1, \rho_2),
\]
\[
v_{\mu, \varepsilon}(1) = -1.
\]

Moreover, either \( v'_{\mu, \varepsilon}(r) < 0 \) for \( r > 0 \) or \( v'_{\mu, \varepsilon}(r) \equiv 0 \) in \([0, r_{\mu, \varepsilon}]\) with \( 0 \leq r_{\mu, \varepsilon} < 1 \), \( v'_{\mu, \varepsilon}(r) < 0 \) for \( r > r_{\mu, \varepsilon} \), \( \lim_{\varepsilon \to 0} \lim_{\mu \to \infty} \max_{\mathbb{R}^N} v_{\mu, \varepsilon} = \rho_2 \).

**Proof.** We first choose \( \varepsilon > 0 \) such that \( \tilde{f}_e(s) = f_e(s-1) - \varepsilon \) satisfies (F\(_1\)) and (F\(_2\)) (with \(-m \) in (F\(_1\)) being 0 here). In fact, since \( f_e \) satisfies (F\(_1\)), we see that \( \tilde{f}_e(0) = 0 \). For \( \varepsilon > 0 \) sufficiently small, there exists \( \theta_1(\varepsilon) > 0 \) \((i = 1, 2)\) satisfying \( \theta_i(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) such that \( \tilde{f}_e(p_i + 1 - \theta_i(\varepsilon)) = 0 \) and \( \tilde{f}_e(p_2 + 1 - \theta_2(\varepsilon)) = 0 \) and \( \tilde{f}_e(0) \) in \((0, p_1 + 1 - \theta_1(\varepsilon))\), \( \tilde{f}_e > 0 \) in \((p_1 + 1 - \theta_1(\varepsilon), p_2 - 1 - \theta_2(\varepsilon))\), and \( \tilde{f}_e < 0 \) in \((p_2 + 1 - \theta_2(\varepsilon), \infty)\). Moreover,
\[
\int_{0}^{p_2 + 1 - \theta_2(\varepsilon)} \tilde{f}_e(s) \, ds = \int_{0}^{p_2 + 1 - \theta_2(\varepsilon)} f_e(s) \, ds - \varepsilon(p_2 - 1 - \theta_2(\varepsilon))
\]
\[
= \int_{-1}^{0} f_e(s) \, ds + \int_{0}^{p_2 - \theta_2(\varepsilon)} f_e(s) \, ds - \varepsilon(p_2 + 1 - \theta_2(\varepsilon)).
\]

Since \( \lim_{\varepsilon \to 0} \theta_2(\varepsilon) = 0 \), it follows that
\[
\lim_{\varepsilon \to 0^+} \left[ \int_{-1}^{0} f_e(s) \, ds + \int_{0}^{p_2 - \theta_2(\varepsilon)} f_e(s) \, ds - \varepsilon(p_2 + 1 - \theta_2(\varepsilon)) \right] > 0.
\]
as $\int_0^\varepsilon f_e(s)\,ds > 0$ and $\int_0^\varepsilon f_e(s)\,ds \leq \varepsilon$. Thus there exists $\varepsilon_0 > 0$ such that

$$\int_0^{\rho_2+1-\theta_2(e_0)} \tilde{f}_e(s)\,ds = \int_0^{\rho_2+1-\theta_2(e_0)} f_e(\xi-1)\,d\xi - \varepsilon_0(\rho_2 + 1 - \theta_2(\varepsilon_0)) > 0.$$ 

Therefore, $\int_0^{\rho_2+1-\theta_2(e_0)} \tilde{f}_e(s)\,ds > 0$ for $0 < \varepsilon < \varepsilon_0$. Similarly we obtain

$$\int_0^{\rho_2+1-\theta_2(e)} \tilde{f}_e(s)\,ds > 0 \text{ for } \rho \in [0, \rho_2 + 1 - \theta_2(e)).$$ 

Here we use the facts that $\int_0^{\rho_2-\theta_2(e)} f_e(s)\,ds \geq \int_0^{\rho_2-\theta_2(e)} f_e(s)\,ds$ for $\rho \geq 0$ and that $\int_0^{\rho} f_e(s)\,ds > 0$ if $\rho \in [-1, 0)$. Therefore, as in the proof of Lemma 5.1 below, there exists a positive radial solution $w_{\mu, e} \in C_0^1(B)$ of

$$-\Delta u = \mu \tilde{f}_e(u) \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

where $B$ is the unit ball in $\mathbb{R}^N$, $w_{\mu, e}$ is a global minimizer of the functional

$$I_e(u) = \frac{1}{p} \int_0^1 r^{N-1} |u'|^p \,dr - \mu \int_0^1 r^{N-1} \tilde{F}_e(u(r)) \,dr,$$

where $\tilde{F}_e(u) = \int_0^u \tilde{f}_e(s)\,ds$, and satisfies

$$\max w_{\mu, e} \in (\rho_1+1+\theta_1(e), \rho_2+1-\theta_2(e)) \subset (\rho_1+1, \rho_2+1)$$

and

$$\max w_{\mu, e} \to \rho_2 + 1 - \theta_2(e) \text{ as } \mu \to +\infty. \text{ (Note that } \lim_{\mu \to 0^+} \theta_i(e) = 0 \text{ for } i = 1, 2. \text{)}$$

The limit of $\max w_{\mu, e}$ follows from an argument similar to that in the proof of Lemma 5.1 below.

From Lemmas 2.1 and 2.2 we have the following: $w_{\mu, e}'(r) < 0$ for $r \in (0, 1]$ if $\max w_{\mu, e} < \rho_2 + 1 - \theta_2(e)$; there exists $0 \leq r_{\mu, e} < 1$ such that $w_{\mu, e} \equiv \rho_2 + 1 - \theta_2(e)$ in $[0, r_{\mu, e}]$ and $w_{\mu, e}'(r) < 0$ for $r_{\mu, e} < r \leq 1$ if $\max w_{\mu, e} = \rho_2 + 1 - \theta_2(e)$. Here we use the facts that

$$\lim_{s \to 0^+} \frac{\tilde{f}_e(s)}{s^{p-1}} = 0 \quad \text{for} \quad p > 1$$

and that Lemma 2.2 still holds when $-m$ in (F) is equal to 0.

Set $v_{\mu, e}(r) = w_{\mu, e}(r) - 1$ for $r \in [0, 1]$ and

$$v_{\mu, e}(r) = \begin{cases} -1 + (r^d - 1) \cdot d^{-1} \cdot w_{\mu, e}'(1) & \text{for} \quad r \in (1, \infty) \quad \text{if} \quad p \neq N, \\ -1 + \log r \cdot w_{\mu, e}'(1) & \text{for} \quad r \in (1, \infty) \quad \text{if} \quad p = N, \end{cases}$$

where $d = (p-N)/(p-1)$. Since $f_e - \varepsilon = 0$ on $(-\infty, -1]$, one verifies that $v_{\mu, e}$ is the required function. This completes the proof.  \[\square\]
**Definition 1.** We call a function \( v \) a subsolution (supersolution) of the problem

\[
-D_p u = \lambda g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]

if

(i) \( v \in W^{1, p}(\Omega) \cap C^0(\overline{\Omega}) \),

(ii) \( v \leq (\geq) 0 \) on \( \partial \Omega \), and

(iii) \( \int_\Omega (|Dv|^{p-2} Dv \cdot \partial g(v) \phi) \, dx \leq 0 \) (\( \geq 0 \)) for every \( \phi \in D^+ (\Omega) \), where \( D^+ (\Omega) \) consists of all nonnegative functions in \( C^\infty_0 (\Omega) \).

We write (1.1) to denote the problem (1.1) with \( f \) replaced by \( f_\varepsilon \).

**Corollary 2.4.** Let \( (\mu, v_{\mu, \varepsilon}) \) be as in Lemma 2.3, and let \( a_{\mu, \varepsilon} \in (0, 1) \) be the unique zero of \( v_{\mu, \varepsilon} \). Then for \( y \in \Omega \) and \( \lambda > \mu \cdot a_{\mu, \varepsilon} \cdot \text{dist}(y, \partial \Omega)^{-p} \),

\[
w_{\mu, \varepsilon}(\lambda, y; x) := v_{\mu, \varepsilon}((\lambda / \mu)^{1/p} \cdot (x - y)), \quad x \in \Omega \quad (2.4)
\]

is a subsolution of (1.1).  

**Proof.** We omit the subscripts \( \mu \) and \( \varepsilon \) on \( w \) and \( v \) in the following for simplicity. The function \( w(\lambda, y) \in C^1(\Omega) \) satisfies

\[
-D_p w = \lambda (f_\varepsilon (w) - \varepsilon) \quad \text{in } \Omega;
\]

hence \( \int_\Omega (|Dw|^{p-2} Dw \cdot \partial f_\varepsilon (w) \phi) \, dx \leq 0 \) for all \( \phi \in D^+ (\Omega) \). Since \( w(\lambda, y) < 0 \) on \( \partial \Omega \) for \( \lambda > \mu a_{\mu, \varepsilon} \cdot \text{dist}(y, \partial \Omega)^{-p} \), \( w(\lambda, y) \) satisfies the definition of subsolution. This proves the corollary.

Next we prove an appropriate version of the sweeping principle of Serrin.

**Proposition 2.5.** Let \( \varepsilon > 0 \) and \( f_\varepsilon \) be as above, let \( u \) with \( \sup_\Omega u < \rho_2 \) be a supersolution of the problem \(-D_p u = f_\varepsilon (u), u = 0 \) on \( \partial \Omega \), and let

\[
A = \{ v_t : \sup_\Omega v_t < \rho_2, \ t \in [0, 1] \}
\]

be a family of subsolutions of

\[
-D_p v = \lambda (f_\varepsilon (v) - \varepsilon) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega
\]
satisfying $v_t < 0$ on $\partial \Omega$ for all $t \in [0, 1]$. If

(i) $t \mapsto v_t$ is continuous relative to the $\| \cdot \|_0$-norm,
(ii) $u \geq v_0$ in $\bar{\Omega}$, and
(iii) $u \not \equiv v_t$, for all $t \in [0, 1],

then $u \geq v_t$ in $\bar{\Omega}$ for all $t \in [0, 1]$.

Proof. Set $E = \{ t \in [0, 1]; u \geq v_t \}$ in $\bar{\Omega}$. By (ii), $E$ is nonempty. Moreover, $E$ is closed. By the conditions on $f_e$, there exists $M > 0$ such that

$$g_e(s) := f_e(s) + M |s|^{p-2} s$$

is strictly increasing on $[0, \rho_2]$ since $\lim_{s \to 0} |f_e(s)|/(|s|^{p-2} s) < \infty$.

As $u > v_t$ on $\partial \Omega$ for all $t \in [0, 1]$, it follows that there exists $\tau > 0$ independent of $t$ such that $u \geq v_t + \tau$ on $\partial \Omega$. Let $w = v_t + \tau$. We choose $\tau > 0$ such that

$$[M(|v_t + \tau|^{p-2}(v_t + \tau) - |v_t|^{p-2} v_t) - \varepsilon] < 0$$

in $\Omega$ for all $t \in [0, 1]$. Then for $t \in E$, we have

$$-A_p u + \lambda M |u|^{p-2} u \geq \lambda f_e(u) + \lambda M |u|^{p-2} u$$

$$\geq \lambda f_e(v_t) + \lambda M |v_t|^{p-2} v_t$$

$$\geq -A_p v_t + \lambda e + \lambda M |v_t|^{p-2} v_t$$

$$\geq -A_p w + \lambda M |w|^{p-2} w.$$

The weak comparison principle [5] implies that $u \geq v_t + \tau$ in $\Omega$. Thus, $u \geq v_t + \tau$ in $\bar{\Omega}$. Since $t \mapsto v_t$ is continuous with respect to the $\| \cdot \|_0$-norm, this shows that $E$ is also open. Hence $E = [0, 1]$.

Remark 2.6. (1) It is easily seen that Proposition 2.5 is valid for every small $\varepsilon > 0$.

(2) We can obtain a similar sweeping principle when $u$ with $\sup u < \rho_2$ is a subsolution of (1.1) and $A = \{ v_t; \sup u \leq \rho_2$, $t \in [0, 1] \}$ is a family of supersolutions of

$$-A_p v = \lambda (f(v) + e) \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial \Omega$$

satisfying $v_t > 0$ on $\partial \Omega$ by reversing the remaining inequalities.
3. ASYMPTOTIC BEHAVIOUR OF POSITIVE SOLUTIONS OF (1.1) WHEN $\lambda$ IS LARGE

In this section we shall study the asymptotic behaviour of the positive solutions of (1.1) when the parameter $\lambda$ is large.

In the following we always assume that $\Omega$ is a bounded connected smooth domain in $\mathbb{R}^N$. We denote by $r(x)$ the distance from $x \in \Omega$ to the boundary $\partial \Omega$ and by $s(x)$ the point of $\partial \Omega$ (s(x) = x if $x \in \partial \Omega$) which is closest to $x$ (which is uniquely defined if $x$ is close enough to $\partial \Omega$). We choose $\delta_0 > 0$ so small that the boundary strip $\{x \in \Omega : 0 < r(x) < \delta_0\}$ is covered (and only covered) by the straight lines in the inner normal direction $n(x)$ and emanating from $s(x)$.

Let $x^* \in \Omega$ and define $\lambda^* = \mu \text{ dist}(x^*, \partial \Omega)^{-\frac{1}{p}}$ and for $\lambda > \lambda^*$ let $z_l = w(\lambda, x^*)$, where $\mu$ and $w$ are as defined in Corollary 2.4.

**Lemma 3.1.** Let $f_\tau$ satisfy (F1)–(F3) and (F_\tau). Then

(i) for $\lambda > \lambda^*$, (1.1) possesses a maximal solution $u_l \in [z_l, \rho_2]$.

(ii) there exist $\lambda^* > \lambda^*, c > 0$, and $\tau \in (\rho_1, \rho_2)$ such that for $\lambda > \lambda^*$ every solution $u_l \in [z_l, \rho_2]$ of (1.1) satisfies

$$u_l(x) > \min \left( c \lambda^{1/p} \text{ dist}(x, \partial \Omega), \tau \right) \quad \text{for all} \quad x \in \Omega. \quad (3.1)$$

**Proof.** By Corollary 2.4, for $\lambda > \lambda^*$ we have $z_l$ is a subsolution of (1.1), and $z_l < \rho_2$. Since $\rho_2$ is a supersolution of (1.1), and there exists $M > 0$ such that $f_\tau(s) + M |s|^{p-2} s$ is strictly increasing in $(\min_a z_l, \rho_2)$, by a monotone method as in [5, 17, 40], there is a maximal solution $u_l \in [z_l, \rho_2]$ for $\lambda > \lambda^*$. This proves (i).

Since $\Omega$ satisfies a uniform interior sphere condition, there exists $\eta_0 > 0$ such that $\Omega = \bigcup \{B(x, \eta); x \in \Omega_\eta\}$ for $\eta \in (0, \eta_0]$, where $\Omega_\eta = \{x \in \Omega; \text{ dist}(x, \partial \Omega) > \eta\}$. Set

$$\lambda^{**} = \max(\lambda^*, \mu \alpha^* \eta_0^{-\frac{1}{p}}), \quad c = \mu^{-\frac{1}{p}} \inf \{(\alpha - r)^{-1} v(r); r \in [0, \alpha]\}, \quad \tau = v(0)$$

with $\mu$, $v$, and $\alpha$ as in Corollary 2.4. Note that $c > 0$, since $v > 0$ on $[0, \alpha]$ and $v'(\alpha) < 0$.

Let $u_l$ be a solution of (1.1), $\lambda > \lambda^{**}$, and $u_l \in [z_l, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\mu^{**} \eta_0^{1/p}}$ is arwise connected (note that $x^* \in \Omega_{\mu^{**} \eta_0^{1/p}}$) and since $w(\lambda, y)$ is a subsolution for $y \in \Omega_{\mu^{**} \eta_0^{1/p}}$, with $w(\lambda, y) < 0$ on $\partial \Omega$, by Proposition 2.5 we obtain

$$u_l > w(\lambda, y) \quad \text{in} \quad \Omega \quad \text{for all} \quad y \in \Omega_{\mu^{**} \eta_0^{1/p}}.$$
(Note that \( w(\lambda, y) \) is a subsolution of the problem

\[-D_p v = \lambda (f_v(v) - \varepsilon) \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega. \]

Hence, an argument similar to that in [6] implies

\[
\begin{align*}
  u_s(x) &> c \lambda^{1/p} \text{ dist}(x, \partial \Omega) \quad \text{for all } x \in \Omega \setminus \Omega_{a(\mu)^{1/p}}, \\
  u_s(x) &> \tau \quad \text{for all } x \in \Omega_{a(\mu)^{1/p}},
\end{align*}
\]

which completes the proof.

Remark 3.2. It follows from Eq. (3.1) that the maximal solution \( u_s \) is positive for \( \lambda > \lambda^{**} \), and that max \( u_s \in (\rho_1, \rho_2) \) for \( \lambda \) sufficiently large. This implies that \( u_s \) is a positive solution of (1.1).

Let \( \psi \) be the eigenfunction corresponding to the first eigenvalue \( v_1 \) of

\[-D_p v = v |v|^{p-2} v \text{ in } B, \quad v = 0 \text{ on } \partial B,
\]

where \( B \) denotes the unit ball in \( \mathbb{R}^N \). Let \( \psi \) be normalized so that \( \max \psi = 1 \). It is well known that \( \psi > 0 \) in \( B \), \( \psi \) is radially symmetric, and \( \psi(0) = 1 \).

Lemma 3.3. Let \( u \) satisfy \(-D_p u = \lambda f_v(u) \) in an open set \( \Omega' \subset \Omega \). Let \( \sigma, \varepsilon > 0 \) (with \( \varepsilon \) sufficiently small) be such that \( f_v(s) - \varepsilon \geq \sigma(s - a)^{p-1} \) for \( s \in [a, b] \). Suppose that \( u(x) > a \) for \( x \in \Omega' \). If \( x_0 \in (\Omega')_{(N/(p+1))^{1/p}} \), then \( u(x_0) > b \).

Proof. Set \( \theta(x_0, \lambda, t; x) = a + \psi((\sigma \lambda) / v_1) (x - x_0) \) for \( x \in \tilde{B} \) and \( t \in [0, b-a] \), where \( \tilde{B} \) is the ball \( B(x_0, v_1 / \sigma \lambda)^{1/p} \). We claim that the set \( \{ \theta(x_0, \lambda, t; t \in [0, b-a] \} \) is a family of subsolutions of the problem

\[-\text{div}(|Dv|^{p-2} Dv) = \lambda (f_v(v) - \varepsilon) \text{ in } \tilde{B}, \quad v = u \text{ on } \partial \tilde{B} \quad (3.2)\]

and the closure of \( \tilde{B} \) is contained in \( \Omega' \). Then, by a method similar to that in the proof of Proposition 2.5, we would obtain \( u(x_0) > b \). So it remains to show that \( \theta(x_0, \lambda, t) \) is a subsolution of Eq. (3.2). This can be seen from a routine calculation and the fact that \( u > a = \theta(x_0, \lambda, t) \) on \( \partial \tilde{B} \).

Lemma 3.4. Let \( f_v \) satisfy (F1)–(F5) and (F7). For every \( \tilde{\delta} > 0 \) there is a \( c(\tilde{\delta}) > 0 \) such that for all solutions \( u_s \) of (1.1), \( \lambda > \lambda^{**} \), and \( u_s \in [z_1, \rho_2] \), the following inequality holds

\[
u_s(x) > \min(c(\tilde{\delta}) \lambda^{1/p} \text{ dist}(x, \partial \Omega), \rho_2 - \tilde{\delta}) \quad \text{for all } x \in \Omega,
\]

where \( \lambda^{**} \) and \( z_1 \) are as in Lemma 3.1.
Proof. If \( \rho_2 - \tilde{\delta} < \tau \), we are done with \( c(\tilde{\delta}) = c \) as in Lemma 3.1. Otherwise, by (F), there exist \( \sigma, \epsilon > 0 \) (\( \epsilon \) sufficiently small) such that \( f_s(s) = f_s(s-\delta) + \delta > \sigma(s-\tau)^{\rho-1} \) for all \( s \in [\tau, \rho_2 - \tilde{\delta}] \) since \( f_s(s) > 0 \) in \( [\tau, \rho_2 - \tilde{\delta}] \). (Note that \( \sigma \) depends on \( \epsilon \).) Let \( v_1 \) be as in Lemma 3.3. Using Lemma 3.3 with \( \Omega' = \Omega_{2\delta}^{-1/\rho}, k = c^{-1} \), and Lemma 3.1, since \( (\Omega')_{x/(\rho x)}^{1/\rho} = \Omega_{(x/\rho x)^{1/\rho} + k}^{-1/\rho} \), we obtain

\[
u(x) > \rho_2 - \tilde{\delta} \quad \text{for all} \quad x \in \Omega_{x/(\rho x)^{1/\rho} + k}^{-1/\rho}.
\]

By (3.1) we have

\[
u(x) > c(\tilde{\delta}) \lambda^{1/\rho} \text{dist}(x, \partial \Omega) \quad \text{for all} \quad x \in \Omega \Omega_{(x/\rho x)^{1/\rho} + k}^{-1/\rho} \quad (3.3)
\]

with \( c(\tilde{\delta}) = \min\{c, \tau((v_1/\sigma)^{-1/\rho} + k)^{-1}\} \). This completes the proof. \( \blacksquare \)

Now we consider the problem

\[-((|y'|^{p-2} y')' = f(y), \quad y(0) = 0, \quad y(\infty) = \rho_2. \quad (3.4)\]

By arguments similar to those in the proof of Lemma 3.1 in [24], we see that Eq. (3.4) has a unique solution \( z_0(t) \) when \( f \) satisfies (F)-(F). Moreover, \( z_0 \) satisfies either

(A) \( z_0 > 0, \ z'_0 > 0 \) in \( (0, \infty) \) and \( \lim_{t \to \infty} z_0(t) = \rho_2, \) or

(B) \( z_0 > 0, \) there exists \( \bar{t} > 0 \) such that \( z'_0 > 0 \) in \( (0, \bar{t}), \ z'_0(\bar{t}) = 0, \) and \( z_0 \equiv \rho_2 \) in \( [\bar{t}, \infty). \)

Remark 3.5. Case (B) cannot occur when \( f \) satisfies (F). In fact, from the first integral of Eq. (3.4), we have

\[|z'_0(t)|^p + |F(z_0(t))| = C, \quad t \in (0, \infty),\]

where \( 1/p + 1/p' = 1. \) Therefore,

\[|z'_0|^p = p'(F(\rho_2) - F(z_0)).\]

Since \( F(\rho_2) > F(s) \) for \( 0 < s < \rho_2, \) we have

\[\int_0^{\rho_2} (F(\rho_2) - F(s))^{-1/p} = (p')^{1/p} t.\]

Since \( \int_0^{\rho_2} (F(\rho_2) - F(s))^{-1/p} ds = \infty \) when \( f \) satisfies (F), (B) cannot occur.

If \( x \in \Omega \) and \( x \) is near \( \partial \Omega, \) \( x \) can be uniquely written in the form \( x = s + t_n, \) where \( s = s(x) \in \partial \Omega, \ n = n_{s(x)} \) denotes the inward unit normal
vector to \( \partial \Omega \) at \( s(x) \), and \( t = t(x) \) is small and positive. We will make frequent use of these coordinates. If \( \lambda > 0 \), define \( \eta_{i}(x) = z_{i}(\lambda^{1/p} t) \) if \( x \) is near \( \partial \Omega \) and \( \eta_{i}(x) = \rho_{2} \) otherwise.

**Proposition 3.6.** Let \( f \) satisfy \((F_{1})-(F_{3})\). For every small \( \varepsilon > 0 \), there is \( \bar{\lambda} = \bar{\lambda}(\varepsilon) > \lambda^{**} \) such that if \( \lambda \geq \bar{\lambda} \) and \( u_{i} \in [z_{i}, \rho_{2}] \) is a positive solution of (1.1), then

\[
(1 - \varepsilon) \eta_{i} \leq u_{i} \leq (1 + \varepsilon) \eta_{i}.
\]

**Proof.** It is clear that if \( u_{i} \) is a positive solution of (1.1) with \( \|u_{i}\|_{\infty} \leq \rho_{2} \), then \( u_{i} \) is a solution of (1.1), for any small \( \varepsilon > 0 \). We also know that \( \|u_{i}\|_{\infty} < \rho_{2} \) by the strong maximum principle in [44] (this can be obtained by the argument similar to that in the proof of Proposition 4.2). For convenience, we omit the subscript \( \lambda \) on \( u_{i} \) below.

By Lemma 3.4, it remains to prove the result for points whose distance from \( \partial \Omega \) is of order \( \lambda^{-1/p} \). To prove this, we construct sub- and supersolutions. The key step in the proof below is to establish the sweeping out results.

Near \( \partial \Omega \), we use the \( s, t \) coordinates. In these variables,

\[
\Delta_{s} u = (|u'|^{p-2} u')' + b(s, t) |u'|^{p-2} u' + [(|u'| + u'|^{p-2} u')' - (|u'|^{p-2} u')']
\]

\[
+ b(s, t)[(|u'| + u'|^{p-2} u' - |u'|^{p-2} u')'] + \text{terms involving } s \text{ derivatives},
\]

where \( u'_{i} = \frac{u_{i}}{\rho_{2}} \), \( u'_{i} = \frac{u_{i}}{\rho_{2}} \). By the conditions imposed on \( f \), there exists \( M > 0 \) such that \( g(s) := f(s) + Ms^{p-1} \) is increasing in \( (0, \rho_{2}) \).

If \( \bar{z} < z_{0}(0) \) but is close, using the first integral of Eq. (3.4) we easily prove that the solution \( \bar{z} \) of Eq. (3.4) with \( \bar{z}(0) = 0, \bar{z}'(0) = \bar{a} \), first increases to a number near \( \rho_{2} \) but less than \( \rho_{2} \) and then decreases to zero (see [24]). Hence there is \( \bar{t} \) near \( \rho_{2} \) and \( \bar{t} > 0 \) such that \( \bar{z}(\bar{t}) = \bar{t}, \bar{z}'(\bar{t}) = 0 \). Since \((|z'|^{p-2} z')' = -f(z(t)) \neq 0, z'(t) \) changes sign at \( \bar{t} \). Hence if \( \mu \) is close to 1 and \( \beta \) is small, the solution \( z \) of

\[
-(|x'|^{p-2} x')' - \beta |x'|^{p-2} x' + Mx^{p-1} = \mu g(x(t)), \quad x(0) = 0, x'(0) = \bar{a}
\]

increases until \( \bar{t} \) where \( z'(\bar{t}) = 0 \) and \( z(\bar{t}) \) is close to \( \rho_{2} \) but less than \( \rho_{2} \).

Define

\[
\bar{\eta}_{i}(x) = \begin{cases} \frac{\pi}{\pi}(\lambda^{1/p} t), & \text{if } x \text{ is close to } \partial \Omega \text{ and } 0 \leq t \leq \lambda^{-1/p} \bar{t}, \\ z(\bar{t}), & \text{otherwise}, \end{cases}
\]
where \( x = s + tn \) if \( x \) is near \( \partial \Omega \). (Thus \( \tilde{\eta}_j \) is constant except near \( \partial \Omega \).) Suppose we can show that, if \( \lambda \) is large and \( u \in [z_1, \rho_2] \) is a positive solution of (1.1), then \( u \gtrsim \tilde{\eta}_j \). Since \( \tilde{z} \) is close to \( z_0 \) on compact intervals if \( \tilde{z} \) is near \( z_0^n(0) \), \( \mu \) is near 1, and \( \beta \) is small, then we obtain that \( u \gtrsim (1 - \tilde{\eta}_j) \) for \( \lambda \) large. This will prove half of Proposition 3.6.

By choosing \( \beta < 0 \) and \( \mu < 1 \), we have \( \tilde{\eta}_j \in C^1 \). Moreover, there is \( c \in (0, 1) \) such that \( u \gtrsim \tilde{\eta}_j \) for \( \lambda \) large by the proof of Lemma 3.4. Now we show that

\[
u \gtrsim \tilde{\eta}_j \quad \text{for} \quad j \in [c, 1]. \tag{3.7}
\]

A sweeping principle similar to that of Proposition 2.5 will give (3.7). We only need to show that if \( j \in [c, 1] \) and \( u \gtrsim \tilde{\eta}_j \) in \( \Omega \), there exists \( \xi > 0 \) such that

\[
u - \tilde{\eta}_j \gtrsim \xi_{e_0} \quad \text{in} \quad \Omega, \tag{3.8}
\]

where \( e_o(x) \) is the unique positive solution of the problem

\[-A_\rho e_0 = 1 \quad \text{in} \quad \Omega, \quad e_0 = 0 \quad \text{on} \quad \partial \Omega.
\]

Our conclusion (3.7) will be obtained from the continuity of \( \tilde{\eta}_j \) and \( \partial \tilde{\eta}_j / \partial v \) under the norms \( \| \cdot \|_{L^p(\partial \Omega)} \), \( \| \cdot \|_{L^p(\partial \Omega)} \) (respectively) with respect to \( j \in [c, 1] \) where \( v(x) = -n_{s(x)} \) is the outward normal vector to \( \partial \Omega \) at \( x \) (note that \( s(x) = x \)).

In fact, since \( u(x) = 0, \tilde{\eta}_j(x) = 0 \) for \( x \in \partial \Omega \), and \( g(s) \) is increasing for \( s \in (0, \rho_z] \), it follows from the strong maximum principle in [44] that

\[
\frac{\partial u}{\partial v} < 0, \quad \frac{\partial \tilde{\eta}_j}{\partial v} \leq -(c\lambda)^{1/p} z'(0) < 0 \quad \text{on} \quad \partial \Omega.
\]

Thus, there exists a one-sided neighbourhood \( A_t \) of \( \partial \Omega \) contained in \( \Omega \) (we may choose \( A_t \subset \{ x \in \Omega : 0 < t < \lambda_\lambda^{-1/p} \} \subset \{ x \in \Omega : 0 < t < \tilde{\delta}_j \} \) such that \( \partial u / \partial (n_{s(x)}) < 0 \) and \( \partial \tilde{\eta}_j / \partial (n_{s(x)}) < 0 \) for \( x \in A_t \). For \( x \in A_t \),

\[
- \text{div}(|D\eta_j|^{p-2} D\eta_j) - \{ - \text{div}(|D\tilde{\eta}_j|^{p-2} D\tilde{\eta}_j) \}
= \lambda (f(u) - j\mu f(\tilde{\eta}_j)) + j\lambda (b(s, t)(j\lambda)^{-1/p} - \beta) (z''((j\lambda)^{1/p} t))^{p-1} + j\lambda (1 - \mu) M\tilde{\eta}_j^{p-1}.
\]
Since the first term on the right hand side of the above identity is 0 on \(\partial\Omega\) and the second term is positive (for \(\lambda\) sufficiently large), there is a one-sided neighbourhood \(A^*_i \subset A_i\) of \(\partial\Omega\) such that the right hand side of the above identity is nonnegative for \(x \in A^*_i\). On the other hand, since \(u - \bar{\eta}_{ji} = 0\) on \(\partial\Omega\) and

\[
-\text{div}(|Du|^{p-2} Du) - \{-\text{div}(|D\bar{\eta}_{ji}|^{p-2} D\bar{\eta}_{ji})\} = -L(u - \bar{\eta}_{ji})
\]

(see [23]) where \(L\) is a uniformly elliptic operator in \(A^*_i\), by the maximum principle for \(L\), we have \((\partial/\partial v)(u - \bar{\eta}_{ji}) < 0\) on \(\partial\Omega\). This implies that there exist \(\zeta_i > 0\) and \(A_{ji}^{**} \subset A^*_i\) such that

\[
u(x) - \bar{\eta}_{ji}(x) \geq \zeta_i e_0(x) \quad \text{for} \quad x \in A_{ji}^{**}. \tag{3.9}
\]

Choose a smooth domain \(\Omega_j \subset \Omega\) with \(\partial\Omega_j \subset A_{ji}^{**}\). Then, there exists \(\tau_j > 0\) such that \(u \geq \bar{\eta}_{ji} + \tau_j\) on \(\partial\Omega_j\). Defining \(\Omega_j = \{x = s + m, s \in \Omega; \ t > (j\lambda)^{-3/2} t_j\}\), without loss of generality we may assume \(\Omega_j \subset \subset \Omega_j\).

We claim that

\[
u(x) > \bar{\eta}_{ji}(x) \quad \text{for} \quad x \in \overline{\Omega}_j. \tag{3.10}
\]

In fact, we know that \(\bar{\eta}_{ji} \equiv \bar{z}(t)\) in \(\overline{\Omega}_j\). Suppose that there exists \(x_0 \in \overline{\Omega}_j\) such that \(u(x_0) = \bar{z}(t)\). We consider two cases: (i) \(x_0 \in \Omega_j\) and (ii) \(x_0 \in \partial\Omega_j\).

In case (i), letting \(w = u - \bar{z}(t)\), we have \(w \geq 0\) in \(\Omega_j\) and \(w\) satisfies

\[-A_p w = \lambda f(u) > 0 \quad \text{in} \quad \Omega_j\]

since \(u \geq \bar{z}(t) > \rho_j\) in \(\Omega_j\). The strong maximum principle in [17] implies

\[w \equiv 0, \quad \text{that is} \quad u \equiv \bar{z}(t) \text{ in } \Omega_j,\]

This is a contradiction since \(f(\bar{z}(t)) \neq 0\). In case (ii), we know \(V(u - \bar{\eta}_{ji})(x_0) = 0\) since \(x_0\) is a minimum point of \(u - \bar{\eta}_{ji}\). But also \(u - \bar{\eta}_{ji} = w\) in \(\Omega_j\) and the Hopf type of maximum principle (see [17]) implies \(w_t (x_0) < 0\), where \(v\) is the outward normal to \(\partial\Omega_j, (\partial\Omega_j\) is smooth for \(\lambda\) large since \(\partial\Omega\) is smooth). This is clearly impossible. Thus, our claim (3.10) holds.

Now we consider the domain \(\Omega_j \setminus \overline{\Omega}_j\). It is clear that there exists \(\tau_j^{(2)} > 0\) such that

\[u - \bar{\eta}_{ji} \geq \tau_j^{(2)} \quad \text{on} \quad \partial(\Omega_j \setminus \overline{\Omega}_j).\]
We also have, for $x \in \Omega_j \setminus \bar{\Omega}_j$,
\begin{align*}
\text{div}(\nabla |u|^{p-2} u Du) &+ \lambda Mu^{p-1} - \{ - \text{div}(\nabla |\nabla \tilde{g}_{\mu}|^{p-2} \nabla \tilde{g}_{\mu}) + \lambda M |\nabla \tilde{g}_{\mu}|^{p-1} \\
&= \lambda (g(u) - g(\tilde{g}_{\mu})) + \lambda (1 - j\mu) f(\tilde{g}_{\mu}) \\
&\quad + j\lambda [b(s, t)(j\lambda)^{-1/p} - \beta] (\xi'((j\lambda)^{1/p} t))^{p-1} + \lambda \mu (1 - \mu) M |\nabla \tilde{g}_{\mu}|^{p-1} \\
&= \lambda (g(u) - g(\tilde{g}_{\mu})) + \lambda (1 - j\mu) \left[ f(\tilde{g}_{\mu}) + \frac{j(1 - \mu)}{1 - j\mu} M |\nabla \tilde{g}_{\mu}|^{p-1} \right] \\
&\quad + j\lambda [b(s, t)(j\lambda)^{-1/p} - \beta] (\xi'((j\lambda)^{1/p} t))^{p-1}.
\end{align*}

Define $m_j(s) = f(s) + \frac{(1 - j\mu)}{1 - j\mu} M s^{p-1}$. Since $\frac{(1 - j\mu)}{1 - j\mu} \cdot \tilde{g} > 0$ for $j \in [c, 1]$, if we choose $M$ sufficiently large, we have that $m_j$ is also strictly increasing in $(0, \rho_2]$. Thus, $m_j(\tilde{g}_{\mu}) > 0$ in $\Omega$. Since $g$ is strictly increasing in $(0, \rho_2]$, we see that $g(u) \geq g(\tilde{g}_{\mu})$ in $\Omega$. On the other hand, since $0 < \mu < 1$, $c \leq j \leq 1$, and $\beta < 0$, there exists $\xi > 0$ (depending upon $\mu, \lambda, M$) such that
\[
\lambda (1 - j\mu) m_j(\tilde{g}_{\mu}) + j\lambda [b(s, t)(j\lambda)^{-1/p} - \beta] (\xi'((j\lambda)^{1/p} t))^{p-1} \geq \xi
\] in $\Omega \setminus \bar{\Omega}_j$.

Similar arguments to those in the proof of Proposition 2.5 imply that we can choose $\tau_j > 0$ so that $u \geq \tilde{g}_{\mu} + \tau_j$ in $\Omega \setminus \bar{\Omega}_j$. This implies that there exists $\xi > 0$ such that
\[
u(x) - \tilde{g}_{\mu}(x) \geq \xi_2 \nu_0(x) \quad \text{for} \quad x \in \bar{\Omega}_j.
\] (3.11)

Our claim (3.8) is obtained by choosing $\xi = \min\{\xi_1, \xi_2\}$. Hence, (3.7) holds.

To prove the estimate in the opposite direction, we use another family of functions. If $\tilde{z}_i > z_i'(0)$, it is easy to show from the first integral that the solution $\tilde{z}_i$ of Eq. (3.4) such that $\tilde{z}_i(0) = 0$, $\tilde{z}_i'(0) = \tilde{z}_i$, increases till it hits $y = \rho_2$. Once again, by continuous dependence, the solution $\tilde{z}_i$ of Eq. (3.6) such that $\tilde{z}_i(0) = 0$, $\tilde{z}_i'(0) = \tilde{z}_i$, increases till it hits $y = \rho_2$ at $t = \bar{t}_i$, provided $\mu$ is near 1 and $\beta$ is small. We define
\[
\tilde{g}_{\lambda}(x) = \begin{cases} 
\tilde{z}_i(\lambda^{1/p} t), & \text{if} \quad 0 \leq t \leq \lambda^{-1/p} \bar{t}_i, \\
\rho_2, & \text{otherwise}.
\end{cases}
\]

By choosing $\mu > 1$ and $\beta > 0$ and using arguments similar to the above, we obtain
\[
u \leq \tilde{g}_{\mu} \quad \text{in} \quad \Omega, \quad \text{for} \quad j \in [1, c]
\] (3.12)
provided it is possible to choose $c > 1$ such that $u \leq \bar{\eta}_{\lambda}$ for $\lambda$ large and all positive solutions $u \in [\bar{z}_1, \rho_2]$ of (1.1) with $\max_\Omega u < \rho_2$. (Note that $\bar{\eta}_{\lambda} \in W_0^{1, r}(\Omega) \cap C^0(\overline{\Omega})$.)

We only need to show that if $j \in [1, c]$ and $\bar{\eta}_{\lambda} \geq u$ in $\Omega$, there exists $\xi > 0$ such that

$$
\bar{\eta}_{\lambda} - u \geq \xi e_0 \quad \text{in } \Omega,
$$

(3.13)

where $e_0$ is as above. In fact, since $\max_\Omega u < \rho_2$, it follows that $\bar{\eta}_{\lambda} > u$ in $\overline{\Omega}$, where $\Omega_j = \{ x = s + t_\lambda \in \Omega : t > (j\lambda)^{-1/r} \bar{T}_j \}$. Also, by arguments similar to those in the proof of the left inequality of (3.6), there is a one-sided neighbourhood $A_\lambda$ of $\partial \Omega$ contained in $\Omega$ such that

$$
\bar{\eta}_{\lambda} - u \geq \xi_1 e_0 \quad \text{for } x \in A_\lambda, \text{ where } \xi_1 > 0.
$$

Choose a smooth domain $\Omega_j \subset \Omega$ such that $\partial \Omega_j \subset A_\lambda$. Then, there exists $\tau_j > 0$ such that $\bar{\eta}_{\lambda} \geq u + \tau_j$ on $\partial \Omega_j$. Without loss of generality, we assume that $\Omega_j \subset \subset \Omega_j$. It is clear that $\bar{\eta}_{\lambda} > u$ on $\partial \Omega_j$. Now, for $x \in \Omega_j \setminus \overline{\Omega}_j$,

$$
- \text{div}(|Du|^p - 2 Du) + \lambda M \bar{\eta}_{\lambda}^{p-1} - \text{div}(|Du'|^{p-2} Du) + \lambda M u^{p-1})
$$

$$
= \lambda(g(\bar{\eta}_{\lambda}) - g(u)) + \lambda(j \mu - 1) \left( f(\bar{\eta}_{\lambda}) + \frac{j(\mu - 1)}{j\mu - 1} M \bar{\eta}_{\lambda} \right)
$$

$$
+ j\lambda[\beta - b(s, t)(j\lambda)^{-1/p}((z_1^{1/p} t))^{p-1}
$$

provided $\mu > 1$ and $\beta > 0$. Let $m_j(s)$ be as above. Since $\frac{\bar{\eta}_{\lambda}^{p-1}}{p-1} > \frac{\bar{\eta}_{\lambda}^{p-1}}{p-1}$ for $j \in [1, c]$, for $M$ sufficiently large, we have that $m_j$ is also strictly increasing on $(0, \rho_2)$ for all $j \in [1, c]$. Thus, $m_j(\bar{\eta}_{\lambda}) > 0$ in $\bar{\Omega}_j$. Since $g$ is strictly increasing on $(0, \rho_2)$, we see that $g(\bar{\eta}_{\lambda}) \geq g(u)$ in $\bar{\Omega}_j$. Since $\mu > 1$, $1 \leq j \leq c$, and $\beta > 0$, there exists $\tilde{\xi} > 0$ (depending on $\mu$, $\lambda$, $M$) such that

$$
\lambda(j \mu - 1) m_j(\bar{\eta}_{\lambda}) + j\lambda[\beta - b(s, t)(j\lambda)^{-1/p}((z_1^{1/p} t))^{p-1} \geq \tilde{\xi} \quad \text{in } \bar{\Omega}_j \setminus \overline{\Omega}_j.
$$

(Note that $\bar{\eta}_{\lambda}^{p-1}(t^{1/p} t)$ has a discontinuity when $t = (j\lambda)^{-1/p} \bar{T}_j$; however, the presence of a jump of $\bar{\eta}_{\lambda}^{p-1}(t^{1/p} t)$ here is not a difficulty since $\bar{\eta}_{\lambda}^{p-1}(t^{1/p} t) \geq \tilde{\xi} > 0$ in $\bar{\Omega}_j \setminus \overline{\Omega}_j$.) Similar arguments to those in the proof of Proposition 2.5 imply that we may choose $\tau_j > 0$ so that $\bar{\eta}_{\lambda} \geq u + \tau_j$ in $\Omega_j \setminus \overline{\Omega}_j$. This implies that there exists $\bar{\xi}_j > 0$ such that

$$
\bar{\eta}_{\lambda}(x) - u(x) \geq \bar{\xi}_j e_0(x) \quad \text{for } x \in \Omega_j.
$$

Thus, (3.13) is obtained by choosing $\xi = \min\{\bar{\xi}_1, \bar{\xi}_2\}$. This also implies (3.12).
Now we show that it is possible to choose $c > 1$ such that $u \leq \tilde{u}_j$ for $\lambda$ large and all positive solutions $u$ in $(0, \rho_2)$ of (1.1). It is easy to see that this reduces to showing that there is $K_0 > 0$ such that $u(x) \leq K_0 \lambda^{1/p} t$ if $u$ is a positive solution of (1.1), $x$ is near $\partial \Omega$, and $\lambda$ is large. Obviously, it suffices to prove the result for $t \leq K_1 \lambda^{-1/p}$. For arbitrary $x_0 \in \partial \Omega$, let $X = \lambda^{1/p}(x - x_0)$ and $\tilde{u}(X) = u(x)$; then

$$-\text{div}(|Du\tilde{u}|^{p-2} Du\tilde{u}) = f(u\tilde{u}) \text{ in } \tilde{\Omega}_j, \quad \tilde{u} = 0 \text{ on } \partial \tilde{\Omega}_j,$$

where $\tilde{\Omega}_j = \{X: \lambda^{-1/p} X + x_0 \in \Omega\}$. By a blow-up argument as in [7], the stretching only flattens the boundary as $\lambda \to \infty$. Since $0 \in \partial \tilde{\Omega}_j$ and $\|\tilde{u}\|_{\infty} \leq \rho_2$, we apply the regularity result of Proposition 2.2 of [17] to see that $\nabla \tilde{u}$ is bounded on the bounded subsets of $\tilde{\Omega}_j$ which contain neighborhoods of $0$ on $\partial \tilde{\Omega}_j$. Hence, in the original variables, $\|\nabla u\|_{\infty} \leq K_0 \lambda^{1/p}$ on the subsets of $\Omega$ which contain neighborhoods of $x_0$ on $\partial \Omega$. The required estimate for $u$ near $\partial \Omega$ now follows since $\partial \Omega$ is compact. This completes the proof.

4. PROOF OF THEOREM A

We first show the existence of a positive solution $\zeta < u_\lambda < \rho_2$ of (1.1) for each $\zeta$ satisfying the conditions in Theorem A. From the definition of $\zeta$, there exist $\xi \in (\rho_1, \rho_2)$ and a ball $B(x_0, r) \subset \Omega$ such that $\zeta > \xi$ in $B(x_0, r)$ and $\zeta(x_0) = \max_{B(x_0, r)} \zeta(x) \in (\rho_1, \rho_2)$ and $w(\lambda, x_0)$ be as in (2.4). We know that $u(\lambda, x_0)$ is a subsolution of Eq. (1.1), $\lambda > \mu \xi^p \text{ dist}(x_0, \partial \Omega)^{-p}$. Therefore, it follows from the sub- and supersolution argument, Lemma 3.1, and Remark 3.2 that for $\lambda > \lambda_{s_2}^{**}$ (with $\mu^*$ replaced by $x_0$) there exists a maximal positive solution $u_\lambda$ of Eq. (1.1) in $[w(\lambda, x_0), \rho_2]$ such that

$$u_\lambda(x) > \min(\xi, \lambda^{1/p} \text{ dist}(x, \partial \Omega), \tau) \text{ for } x \in \Omega.$$ 

Hence there exists $\lambda_1(\zeta) > \lambda_{s_2}^{**}$ such that

$$\left| \frac{\partial \xi}{\partial y} \right| < \left| \frac{\partial u_\lambda}{\partial y} \right| \text{ on } \partial \Omega,$$

and thus,

$$u_\lambda(x) > \zeta(x) \text{ for } x \in \Omega \setminus \Omega_{d(\lambda^{1/p})^{1/r}}.$$

Since $\tau > \zeta(x_0)$, we see that for $\lambda > \lambda_1(\zeta)$, $u_\lambda(x) > \zeta(x)$ in $\Omega$. This implies existence.
Now we show that for any \( \varepsilon > 0 \) we can choose \( \lambda(\zeta, \varepsilon) \) such that when \( \lambda > \lambda(\zeta, \varepsilon) \), any solution \( u \in (\zeta, \rho_2) \) (we omit the subscript \( \lambda \)) of Eq. (1.1) has the behaviour of Proposition 3.6; that is, (3.5) holds.

First note that \( \zeta > \xi \) in \( B(x_0, r) \). Let \( \sigma, \varepsilon > 0 \) (\( \varepsilon \) sufficiently small) be such that \( f(s) - \varepsilon > \sigma(s - \xi)^{p-1} \) for \( s \in [\xi, \tau] \), where \( \tau \) is as in Lemma 3.1. For \( \lambda > \lambda_2(\xi) := ((v_1 / \sigma)^{1 / p} + \mu^{1 / p}) r^{-p} \),

with \( \mu \) defined in Lemma 2.3 and \( \alpha \) defined in Lemma 3.1, Lemma 3.3 shows that for any \( u \in (\zeta, \rho_2) \),

\[
u(x) > \tau \quad \text{for} \quad x \in B(x_0, \alpha(\mu / \lambda)^{1 / p}) \subset B(x_0, r - (v_1 / \sigma)^{1 / p} \lambda^{-1 / p}).
\]

Observe that \( w(\lambda, x_0) < u \) in \( \Omega \) for \( \lambda > \lambda_2(\xi) \). In fact, we know that \( w(\lambda, x_0) \leq \tau \) in \( B(x_0, \alpha(\mu / \lambda)^{1 / p}) \) and \( w(\lambda, x_0) \leq 0 \) for \( \Omega \setminus B(x_0, \alpha(\mu / \lambda)^{1 / p}) \). The proof of Lemma 3.1 implies that the conclusion of Lemma 3.1 holds for \( u \) if \( \lambda > \lambda_2(\xi) = \max \{ \lambda_2(\zeta), \mu^{2 / p} \xi^{p / r} \} \). The proof of Proposition 3.6 implies that for any \( \varepsilon > 0 \), there exists \( \lambda(\zeta, \varepsilon) > \lambda_2(\zeta) \), such that for \( \lambda > \lambda(\zeta, \varepsilon) \), the behaviour of Proposition 3.6 holds for \( u \).

We now prove uniqueness of \( u_n \). We shall assume that \( p \neq 2 \), the uniqueness result for \( p = 2 \) is known from Theorem 2 of [7].

Suppose there are sequences \( \{ \lambda_n \} \) with \( \lambda_n \rightarrow \infty \) as \( n \rightarrow \infty \) and \( \{ u_{n_1} \} \equiv \{ u_n \} \), \( \{ u_{n_2}^* \} \equiv \{ u_n^* \} \) which are solutions of (1.1) with \( \lambda = \lambda_n \) and with \( u_{n_1} \neq u_{n_2}^* \) in \( \Omega \) and \( u_{n_1}, u_{n_2}^* \in (\zeta, \rho_2) \). The maximum principle implies that \( \max_{\Omega} u_{n_1} < \rho_2 \) and \( \max_{\Omega} u_{n_2}^* < \rho_2 \). Without loss of generality, we assume that \( u_{n_1} \) is the maximal solution; thus, \( u_{n_2}^* < u_{n_1} \). Let \( w_\alpha = (u_{n_1} - u_{n_2}^*) / \| u_{n_1} - u_{n_2}^* \|_\infty \). Then, \( w_\alpha \geq 0 \) in \( \Omega \) and \( \| w_\alpha \|_\infty = 1 \). Therefore, \( w_\alpha \) satisfies the problem

\[
- L(w_\alpha) := - \left[ a_{ij}^{(\lambda_n)} \frac{\partial}{\partial x_j} w_\alpha \right]_{i=1} = \lambda_n u_n f'(\xi_n) w_\alpha \quad \text{in} \quad \Omega, \quad w_\alpha = 0 \quad \text{on} \quad \partial \Omega,
\]

(4.1)

where \( a_{ij}^{(\lambda_n)}(x) = \int_0^1 \left( \frac{\partial a_i^{(\lambda_n)} / \partial q_j}{\partial q_i} \right) [sD_u + (1 - s) Du_n^*] ds \) and \( a'(q) = |q|^{p-2} q \) \( (i = 1, 2, \ldots, N) \) for \( q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N, \xi_n \in [u_{n_1}^*, u_{n_1}], L \) is a degenerate elliptic operator, and

\[
\int_\Omega \sum_{ij} a_{ij}^{(\lambda_n)}(x) \frac{\partial w_\alpha}{\partial x_i} \frac{\partial w_\alpha}{\partial x_j} dx \geq 0.
\]

Now we show that if \( \eta_n \in \Omega \) is such that \( w_\alpha(\eta_n) = 1 \), then

\[
\text{dist}(\eta_n, \partial \Omega) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

(4.2)
In the contrary case, there exists a compact set $K \subset \Omega$ such that $\eta_n \in K$ for all $n$ large (choose a subsequence if necessary). Since $u_n \to \rho_2$ and $u_n^* \to \rho_2$ in $K$ (by Proposition 3.6), we have $f'(z_n^*) < 0$ in $K$ as $n \to \infty$ (by (F3)). Also, for $n$ sufficiently large, we can find a small neighbourhood $D$ of $K$ in $\Omega$ such that $f'(z_n^*) < 0$ on $D$ and $\max_{x \in D} w_n < 1$. Let $\tilde{w}_n(x) = (w_n(x) - \max_{x \in D} w_n)^+$. We have $\tilde{w}_n \in W^{1,p}_0(D) \cap C^0(D)$ (for any $1 < q \leq \infty$) and there exists $A_n \subset D$ with $\text{meas}(A_n) \neq 0$ and $\tilde{w}_n(x) > 0$ for $x \in A_n$. Multiplying both sides of Eq. (4.1) by $\tilde{w}_n$ and integrating on $D$, we derive a contradiction.

Now we use a blow-up argument as in [7] to deduce that (4.2) cannot occur and thus we have uniqueness. Let $I_n = \{x \in \Omega : w_n(x) = 1\}$. $I_n$ are closed nonempty sets. Let $\eta_n \in I_n$ be such that

$$\text{dist}(\eta_n, \partial \Omega) = \text{dist}(I_n, \partial \Omega),$$

and let $\bar{\eta}_n$ be the point of $\partial \Omega$ closest to $\eta_n$. Suppose $\bar{\eta}_n \to \bar{\eta} \in \partial \Omega$ (for a subsequence). Choose coordinates such that $T_{\bar{\eta}}(\partial \Omega) = \{x \in \mathbb{R}^N : x_1 = 0\}$ and $n_{(0)} = n_q = e_1 = (1, 0, \ldots, 0)$. By choosing subsequences if necessary, there are two cases to be considered:

1. $\lambda_n^{1/p} d(\eta_n, \bar{\eta}_n) \to \infty$ as $n \to \infty$,
2. $\lambda_n^{1/p} d(\eta_n, \bar{\eta}_n) \leq Z$, $0 < Z < \infty$ for $n$ sufficiently large.

In case (i), we have from Proposition 3.6 that $u_n(\eta_n) \geq \rho_2 - \delta/4$, $u_n^*(\eta_n) \geq \rho_2 - \delta/4$ when $n$ is sufficiently large, where $\delta$ is as in (F3). This and Proposition 3.6 imply that we can choose open sets $\Omega_n \subset \subset \Omega$ such that

$$L_n \subset \Omega_n, \max_{x \in \Omega_n} w_n < 1$$

and $n$ sufficiently large. Since $f'(s) < 0$ for $s \in (\rho_2 - \delta/2, \rho_2)$, we derive a contradiction by the arguments similar to those in the proof of (4.2).

For case (ii), we change variables setting $X^* = \lambda_n^{1/p}(x - \bar{\eta}_n)$. Note that this change depends on $n$. Let $\bar{u}_n(X^*) = u_n(x)$, $\bar{u}_n^*(X^*) = u_n^*(x)$, $\bar{w}_n(X^*) = w_n(x)$, and $\bar{z}_n(X^*) = z_n(x)$. We have that $\bar{w}_n$ satisfies the problem

$$-\tilde{L}_n(\bar{w}_n) := -a_n^\| \frac{\partial \bar{w}_n}{\partial X^*} \| = f'(\bar{z}_n) \bar{w}_n, \quad \bar{w}_n = 0 \text{ on } \partial \tilde{\Omega}_n,$$

where

$$\tilde{\Omega}_n \equiv \{X^* = \lambda_n^{1/p}(x - \bar{\eta}_n) : x \in \Omega\},$$

$$a_n^\| = \left[ \int_0^1 \frac{\partial a}{\partial q} \left[ \xi D_x \bar{u}_n + (1 - \xi) D_x \bar{u}_n^* \right] d\xi \right]$$
and \( q, a'(q) \) are as above. Note that, in the new coordinates, \( \hat{\nu}_n(Z_n) = 1 \), where \( Z_n = \lambda_n^{1/p}(\eta_n - \bar{\eta}_n) \) is at distance at most \( Z \) from 0. Using Proposition 3.6 and a geometric argument similar to that in the proof of Theorem 2 of [7], we now show that \( \hat{\nu}_n \to \hat{\nu}_0 \), \( \hat{u}_n^* \to z_0 \in C^1_{loc}(T_j) \) as \( n \to \infty \), where \( T_j = \{ x \in \mathbb{R}^N : x_j > 0 \} \) and \( z_0 \) is the unique positive solution of (3.4).

In fact, if \( q \) is in \( T_j \), then, for large \( n \), \( x^*(q) = \bar{\eta}_n + \lambda_n^{1/p} \bar{q} \in \Omega \) and is close to \( \partial \Omega \). By elementary geometry, \( x^*(q) = s(x^*(q)) + t_n n_{x^*(q)} \), where \( s(x^*(q)) \) is near \( \bar{\eta} \) and \( n_{x^*(q)} = e_1 \). By Proposition 3.6 and the definition of \( \hat{\nu}_n \), we have \( \nu_n(\bar{\eta}_n + \lambda_n^{1/p} \bar{q}) = z_0(\bar{q}_1) + o(1) \), and this holds locally uniformly in \( q \) on \( \text{int} T_j \). This implies \( \hat{\nu}_n(X^*) \to z_0 \) in \( C^0_{loc}(T_j) \) as \( n \to \infty \) and similarly for \( \hat{u}_n^* \). The equations satisfied by \( \hat{\nu}_n \) and \( \hat{u}_n^* \) and the regularity of the \( p \)-Laplacian (see [17]) imply that \( \hat{\nu}_n \to \hat{\nu}_0 \), \( \hat{u}_n^* \to z_0 \) in \( C^1_{loc}(T_j) \) as \( n \to \infty \).

Moreover, we claim that \( \hat{\nu}_n \) converges in \( C^0_{loc}(T_j) \) to a nontrivial non-negative bounded solution \( \tilde{\nu} \) of

\[
-\hat{L}(\tilde{\nu}) = \hat{\nu}^{1/p} \frac{\partial \hat{\nu}}{\partial x_j} \hat{v} \quad \text{in} \ T_1, \quad \hat{v} = 0 \quad \text{on} \ \partial T_1.
\]

In fact, it follows easily that \( \hat{\nu}_n^{1/p} \to \hat{\nu}^{1/p} \) in \( C^0_{loc}(T_j) \) as \( n \to \infty \), where \( \hat{\nu}_n^{1/p}(x) = \hat{\nu}_n^{1/p} \hat{a}(\hat{\nu}_n^{1/p} ; \hat{\nu}_n^{1/p}, \hat{\nu}_n^{1/p}, \hat{\nu}_n^{1/p}) \) and thus, \( \hat{\nu}_n^{1/p} = 0 \) if \( i \neq j \) and \( \hat{\nu}_n^{1/p} = (\hat{\nu})^{1/p} \hat{a}[\hat{\nu}] \hat{\nu}_n^{1/p} \). Since \( \hat{\nu}_n^{1/p}(x_j) > 0 \) for all \( 0 \leq x_j < \infty \), we have that \( \hat{L} \) is uniformly elliptic on any compact subset of \( T_j \) and so is \( L \) for \( n \) sufficiently large. Thus our claim can be obtained from the regularity of uniformly elliptic operators and a blow-up argument similar to that in the proof of Theorem 1.1 of [15] or that in the proof of Theorem 2 of [7]. Here \( \tilde{\nu} \) is nontrivial because \( \hat{\nu}_n(Z_n) = 1 \) and \( d(0, Z_n) \ll Z \). It is easily shown that \( f'(\hat{\nu}_n(x_j)) \to f'(z_0(x_j)) \).

Now we show that \( \tilde{\nu} \) does not exist. The proof is divided into three steps. These steps are closely related to those in the proof of Proposition 2 in [7].

**Step 1.** We first find a solution \( \hat{u} \) of

\[
-(p-1)(z_0'|^{p-2} u')' = f'(z_0) u
\]

which is positive on \([0, \infty)\) and is not bounded as \( x_1 \to \infty \).

By differentiating the equation satisfied by \( z_0 \) with respect to \( x_1 \), we see that \( z_0' \) is a solution of Eq. (4.5). We know that \( z_0'(x_1) > 0 \) for \( x_1 \in [0, \infty) \) and \( z_0'(x_1) \to 0 \) as \( x_1 \to \infty \). Let \( w \) denote the solution of Eq. (4.5) satisfying \( w(0) = 0, w'(0) = 1 \). Since \( z_0'(x_1) > 0 \) on \([0, \infty)\), the Sturm comparison theorem implies that \( w(x_1) > 0 \) for \( x_1 > 0 \). Hence, if we can show that \( w(x_1) \to \infty \) as \( x_1 \to \infty \), we can define \( \hat{u} = z_0' + w \) and this step will be...
proved. The fact that \( w \) is not bounded as \( x_1 \to \infty \) can be obtained from the facts that

\[
[(p-1)|z_0'|^{p-2}(z_0'w'-wz_0')]' \equiv 0 \quad \text{in } [0, \infty),
\]

\[(p-1)|z_0'|^{p-2}(z_0'w'-wz_0') \equiv (p-1)(z_0'(0))^{p-1} > 0 \]

and \( |z'_0(x_1)|^{p-2}z_0'(x_1) \to 0, |z'_0(x_1)|^{p-2}z_0'(x_1) \to 0 \) as \( x_1 \to \infty \).

**Step 2.** If Eq. (4.4) has a nontrivial bounded nonnegative solution \( v \) and \( x_1 > 0 \), then \( v \) can be chosen so that \( T(x_1) \equiv \sup_{y \in \mathbb{R}^{N-1}} v(x_1, y) \) is achieved.

Obviously, there exist \( y_n \in \mathbb{R}^{N-1} \) such that \( v(x_1, y_n) \to T(x_1) \) as \( n \to \infty \). Let \( \tilde{v}_n(x_1, y) = T(x_1) \). It is easy to see that \( \tilde{v}_n \) is a solution of Eq. (4.4) and that

\[ \tilde{v}_n(x_1, 0) = \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}_n(x_1, y) \quad \text{as } n \to \infty. \]

We now use an argument similar to that in our blow-up constructions to choose a subsequence of \( \tilde{v}_n \) converging on compact subsets of \( T_1 \) to a nonnegative bounded solution \( \tilde{v} \) of Eq. (4.4). Moreover, \( \tilde{v}(x_1, 0) = T(x_1) \) by our choice of \( \tilde{v}_n \). Since it is easy to see that

\[ \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}_n(x_1, y) \leq \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}_n(x_1, y) = T(x_1), \]

we see that \( \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}(x_1, y) = \tilde{v}(x_1, 0) \). This proves Step 2. Note that our argument shows that \( \sup \{ \tilde{v}(x_1, y) : y \in \mathbb{R}^{N-1} \} \leq T(x_1) \) for all \( x_1 \geq 0 \). This will be useful later.

**Step 3.** We show that \( \tilde{w} \) does not exist. If \( \tilde{w} \) exists, using the notation of Step 2, we consider \( r(x) = \tilde{w}(x)/\tilde{u}(x_1) \), where \( \tilde{u} \) is the function defined in Step 1. Applying standard elliptic estimates on balls of radius 1/2 and half balls with centres at points where \( x_1 = 0 \) and of radius 1, we see that \( \nabla \tilde{w} \) is bounded on \( T_1 \). (Recall that \( \tilde{w} \) is bounded on \( T_1 \) and \( z'_0(x_1) > 0 \) for \( x_1 \in [0, \infty) \); hence \( \tilde{L} \) is uniformly elliptic on any compact subset of \( T_1 \).) Thus \( \tilde{w} \) is uniformly continuous on \( T_1 \) and hence \( T(x_1) \) is continuous. By Step 1 and the boundedness of \( \tilde{w} \), it follows that \( \lim_{x_1 \to 0} T(x_1)/\tilde{u}(x_1) = 0 \). Thus, since \( T(0) = 0 \), we can find \( 0 < \tilde{x}_1 < x_1 \) such that

\[ \sup \{ T(x_1)/\tilde{u}(x_1) : 0 \leq x_1 \leq \tilde{x}_1 \} = T(\tilde{x}_1)/\tilde{u}(\tilde{x}_1). \]

By Step 2, \( \tilde{w} \) can be chosen so that \( \tilde{w}(\tilde{x}_1, y) \) achieves its maximum on \( \mathbb{R}^{N-1} \) at 0. (Our construction of the new \( \tilde{w} \) may decrease \( T(x_1) \) for \( x_1 \neq \tilde{x}_1 \) but the maximum will still be attained at \( \tilde{x}_1 \).) By our construction, \( r(x) \) achieves its
maximum on \( \{(x_1, y): 0 \leq x_1 \leq \bar{x}_1, y \in \mathbb{R}^{N-1}\} \) at the interior point \((\bar{x}_1, 0)\).

However, since \( \bar{u} \) satisfies Eq. (4.5), a simple calculation shows that \( r \) satisfies an elliptic equation

\[
(p-1)(|z|^p \cdot r') + 2(p-1) |z|^p (u'/\bar{u}) r' + (p-1) |z|^p \Delta u \geq 0,
\]

where \( \Delta u \) denotes the Laplacian in the \( y \) variables. Hence, by applying the maximum principle on compact sets, we see that \( r(x_1, y) \) is constant if \( 0 \leq x_1 \leq \bar{x}_1, y \in \mathbb{R}^{N-1} \). This is impossible since \( r = 0 \) when \( x_1 = 0 \). This completes the proof.

\textbf{Remark 4.1.} From the proof of Theorem A we see that if \( \zeta_1, \zeta_2 \in C^0_c(\Omega) \) with \( \max \zeta_i \in (p_1, p_2) \) for \( i = 1, 2 \) and there exist \( x_i \in \Omega \) and some \( r > 0 \) such that \( \zeta_i > \zeta > p_1 \) in \( B(x_i, r) \subset \Omega \), then we can choose \( \lambda_0 := \lambda_0(\zeta_1) = \lambda_0(\zeta_2) \) such that for \( \lambda > \lambda_0 \), (1.1) has exactly one large positive solution \( \zeta_i \eta_i < \eta_i < p_2 \) and \( \zeta_2 \eta_i < \eta_i < p_2 \) in \( \Omega \). (Note that the relation between \( \lambda_0(\zeta) \) and \( \zeta \) in the proof of Theorem A depends only on \( r \). Then we can choose \( \lambda(\zeta_1, \delta) = \lambda(\zeta_2, \delta) \) so that \( \lambda_0(\zeta_1) = \lambda_0(\zeta_2) \).

Now we obtain the following proposition.

\textbf{Proposition 4.2.} Let \( f \) satisfy \((F_1)-(F_3)\). Then there exists \( \sigma^* > 0 \) independent of \( \lambda \) (sufficiently large) such that if \( u_\lambda \) is a positive solution of

\[ (1.1) \]

with \( \lambda_0 \) \( u_\lambda \in (p_2 - \sigma^*, p_2) \) then \( u_\lambda \equiv u_\lambda \).

\textbf{Proof.} Suppose there exists sequences \( \lambda_n \to \infty \) and \( \{u_{\lambda_n}\} \) such that \( u_{\lambda_n} \neq \bar{u}_{\lambda_n} \) is a positive solution of (1.1) for each \( n = 1, 2, \ldots \) and \( \max_{\Omega} u_{\lambda_n} \to p_2 \) as \( n \to \infty \).

Let \( \lambda_n \in \Omega \) be a point at which \( \max_{\Omega} u_{\lambda_n} \) is attained. For convenience, we divide the proof into two cases.

\textbf{Case 1.} \( \{\lambda_n\} \) is bounded away from \( \partial \Omega \).

We define the functions \( U_n(x) = u_\lambda(\lambda_n^{-1/p} x + x_0) \) in \( B(0, R_n) \), where \( R_n = \lambda_n^{-1/p} \text{dist}(x_n, \partial \Omega) \). Since \( R_n \to \infty \) as \( n \to \infty \), \( U_n \) is well defined in \( B(0, R) \) for any \( R > 0 \) if \( n \) is sufficiently large. By assumption \( 0 < U_n < p_2 \), \( U_n(0) = \max_{\Omega} u_{\lambda_n} \to p_2 \) as \( n \to \infty \) and \( U_n \) satisfies \( \Delta U_n + f(U_n) = 0 \) in \( B(0, R) \) for all sufficiently large \( n \). Note that \( \{f(U_n)\} \) is bounded in the \( L^\infty \)-norm; thus by the regularity of the \( p \)-Laplacian (see [17]) we obtain (by choosing a subsequence) that \( U_n \to U \) in \( C^1(B(0, R)) \) as \( n \to \infty \) where \( U \), with \( 0 \leq U \leq p_2 \), satisfies \( \Delta U + f(U) = 0 \) in \( B(0, R) \) and \( U(0) = p_2 \).

Since

\[ -\Delta(U)^{p_2 - U} + M(p_2 - U)^{p - 1} = -f(U) + M(p_2 - U)^{p - 1}, \]

it follows from \((F_3)\) that the right hand side of (4.6) is nonnegative. The strong maximum principle in [44] implies that \( U \equiv p_2 \) in \( B(0, R) \).
On the other hand, if $n$ is sufficiently large, $\bar{u}_n$ is the unique solution in the order interval $[w(\lambda_n, x_n), \rho_2]$. (This can be seen from the proof of Theorem A. In fact, since $\lambda_n^{1/p} \dist(x_n, \partial \Omega) \to \infty$ and $U_n \to \rho_2$ in $B(0, 2\mu^{1/\ell})$, then $\bar{u}_n > \tau$ in $B(x_n, \alpha(\mu/\lambda_n)^{1/p})$, where $\tau = v_{\mu, \ell}(0) < \rho_2$, and $\mu, \alpha$, and $v_{\mu, \ell}$ are defined in Corollary 2.4. This implies that $\bar{u}_n > w(\lambda_n, x_n)$ for $n$ sufficiently large.) Thus, $u_n(x) < w(\lambda_n, x_n)(x) < \tau$ at some point $x \in B(x_n, \alpha(\mu/\lambda_n)^{1/p})$. But this implies that $U_n(x) < \tau$ at some point $x \in B(0, 2\mu^{1/\ell})$ and therefore, $\{U_n\}$ cannot possess a subsequence which converges to $\rho_2$ uniformly in $B(0, \mu^{1/\ell})$. This yields a contradiction and completes the proof of Case 1.

**Case 2.** $x_n \to \bar{x} \in \partial \Omega$ as $n \to \infty$.

To make use of the same argument as in Case 1, it suffices to show that

$$\lambda_n^{1/p} \dist(x_n, \partial \Omega) \to \infty \quad \text{as} \quad n \to \infty$$

(for a subsequence). If not, we have that there exists $0 < Z < \infty$ such that

$$\lambda_n^{1/p} \dist(x_n, \partial \Omega) \leq Z \quad (4.7)$$

(choosing a subsequence if necessary). Define $\tilde{u}_n(x) = u(\lambda_n^{-1/p}x + \tilde{x}_n)$ with $\tilde{x}_n = s(x_n) \in \partial \Omega$. By a blow-up argument similar to that in the proof of Theorem A, we have that there exists $\tilde{U}$ such that $\tilde{u}_n \to \tilde{U}$ in $C^{1, 1}_b(T_1)$ as $n \to \infty$ and $\tilde{U}$ satisfies the problem

$$A_p \tilde{U} + f(\tilde{U}) = 0 \quad \text{in} \quad T_1, \quad \tilde{U} = 0 \quad \text{on} \quad \partial T_1,$$

where $T_1 = \{x : x_1 > 0\}$, as in the proof of Theorem A. By our assumption, there exists $x_0 \in T_1$ with $\dist(x_0, \partial T_1) \leq Z$ such that $\tilde{U}(x_0) = \rho_2$. Now arguments similar to those in Case 1 imply that $\tilde{U} \equiv \rho_2$ in $T_1$. This contradiction implies $\lambda_n^{1/p} \dist(x_n, \partial \Omega) \to \infty$ as $n \to \infty$. 

### 5. Existence of $u_\lambda$

In this section we will use the mountain pass lemma to obtain the solution mentioned in Theorem B. For convenience, we change (1.1) to the singularly perturbed form. Let $\varepsilon = 1/\lambda$. Then (1.1) becomes

$$-\varepsilon A_p u = f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega. \quad (5.1)$$

Denote the positive solution $\bar{u}_\varepsilon$ obtained in Theorem A by $\bar{u}_\varepsilon$ so that $\bar{u}_\varepsilon$ is a positive solution of Eq. (5.1).
Define $J_{e}: W_{0}^{1,p}(\Omega) \rightarrow \mathbb{R}$ by

$$J_{e}(u) = \frac{e}{p} \int_{\Omega} |Du|^{p} - \int_{\Omega} F(u),$$

where $F(u) = \int_{0}^{u} f(s) \, ds$. We first show that $\bar{u}_{e}$ is a global minimizer of $J_{e}(u)$. Since we only consider positive solutions $u_{e}$ of Eq. (5.1) with $\|u_{e}\|_{\infty} \leq \rho_{2}$, we assume that $f = 0$ for $s < 0$ and $s > \rho_{2}$.

**Lemma 5.1.** For $\varepsilon > 0$ sufficiently small, $\bar{u}_{e}$ is the global minimizer of $J_{e}(u)$. Moreover, $J_{e}(\bar{u}_{e}) < 0$.

**Proof.** Since $f$ is bounded in $[0, \rho_{2}]$, $J_{e}$ is sequentially weakly lower semicontinuous and coercive on $W_{0}^{1,p}(\Omega)$ and so $J_{e}$ possesses a global minimizer, which we denote by $u_{e}$. From the regularity of the $p$-Laplacian (see [17]), $u_{e} \in C^{1}_{0}(\Omega)$. We claim that $u_{e} = \bar{u}_{e}$ for $\varepsilon$ sufficiently small. If not, there exists a sequence $\{e_{m}\}$ such that $e_{m} \rightarrow 0$ as $m \rightarrow \infty$ and $u_{e_{m}} \neq \bar{u}_{e_{m}}$. Then by Proposition 4.2, there exists $\sigma^{*} > 0$ (small) such that $\max_{\Omega} u_{e_{m}} < \rho_{2} - \sigma^{*}$ for $m = 1, 2, \ldots$.

For $\sigma > 0$, define $\Omega^{\sigma} = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \sigma\}$ and choose $a > 0$ such that $|\Omega^{\sigma}| \int_{\rho_{2}}^{\rho_{1}} f(s) \, ds < |\Omega| \int_{\rho_{2} - \sigma^{*}}^{\rho_{1} - \sigma^{*}} f(s) \, ds$. This is possible since $\int_{\rho_{2} - \sigma^{*}}^{\rho_{1} - \sigma^{*}} f(s) \, ds > 0$ and $|\Omega^{\sigma}| \rightarrow 0$ as $a \rightarrow 0$. Next we choose $w \in C^{1}_{0}(\Omega)$ such that $w = \rho_{2}$ in $\Omega \setminus \Omega^{\sigma}$. Then we have

$$J_{e_{m}}(w) - J_{e_{m}}(u_{e_{m}})$$

$$= \frac{e_{m}}{p} \left( \int_{\Omega} |Dw|^{p} \, dx - \int_{\Omega} |Du_{e_{m}}|^{p} \, dx \right) - \left( \int_{\Omega} F(w) \, dx - \int_{\Omega} F(u_{e_{m}}) \, dx \right)$$

$$= \frac{e_{m}}{p} \int_{\Omega} |Du_{e_{m}}|^{p} \, dx - \left( \int_{\Omega \setminus \Omega^{\sigma}} F(p_{2}) + \int_{\Omega^{\sigma}} F(w) \, dx - \int_{\Omega} F(u_{e_{m}}) \, dx \right)$$

$$= \frac{e_{m}}{p} \int_{\Omega} |Dw|^{p} \, dx - \left( \int_{\Omega \setminus \Omega^{\sigma}} F(p_{2}) - F(u_{e_{m}}) \right) \, dx + \int_{\Omega^{\sigma}} (F(w) - F(p_{2}) \, dx$$

$$\leq \frac{e_{m}}{p} \int_{\Omega} |Dw|^{p} \, dx - |\Omega| \int_{\rho_{2} - \sigma^{*}}^{\rho_{1}} f(s) \, ds + |\Omega^{\sigma}| \int_{\rho_{2}}^{\rho_{1}} f(s) \, ds < 0$$

for $m$ sufficiently large. Thus $J_{e_{m}}(w) < J_{e_{m}}(u_{e_{m}})$, contradicting the fact that $u_{e_{m}}$ is a global minimizer.

Now we claim that

$$J_{e}(\bar{u}_{e}) \rightarrow - F(p_{2}) |\Omega| < 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$
Since \( f(\rho_2) = 0 \), for every \( \epsilon > 0 \) sufficiently small, there is \( \delta > 0 \) such that for \( s \in (\rho_2 - \delta, \rho_2) \), \( f(s) < \epsilon \). Also, from Lemma 3.4, there is \( \hat{c}(\hat{\delta}) > 0 \) such that for \( \epsilon \) sufficiently small, \[
abla_\epsilon > \rho_2 - \hat{\delta} \quad \text{in} \quad \Omega \setminus \Omega^{\hat{\delta}, \epsilon^{1/p}}.
\]

We easily see from a blow-up argument that
\[
\max_{\Omega^{\hat{\delta}, \epsilon^{1/p}}} \epsilon^{1/p} |D\nabla_\epsilon| \leq C
\]
for any \( C > \max_{\epsilon_1 \in (0, \infty)} \zeta_0''(x_1) \). In fact, we will show that
\[
\max_{\Omega^{\hat{\delta}, \epsilon^{1/p}}} \epsilon^{1/p} |D\nabla_\epsilon| \leq C
\]
for \( C = \max_{\epsilon_1 \in (0, \infty)} \zeta_0'(x_1) + 1 \) (for example) for every \( \epsilon > \hat{\epsilon}(\hat{\delta}) \). If not, there exist \( \epsilon > \hat{\epsilon}(\hat{\delta}) \) and sequences \( \{\epsilon_n\} \) and \( \{\nabla_\epsilon_n\} \) such that, if \[
\epsilon_n^{1/p} |D\nabla_\epsilon_n(x_n)| = \max_{\Omega^{\hat{\delta}, \epsilon^{1/p}}} \epsilon_n^{1/p} |D\nabla_\epsilon(x)|,
\]
then \( \epsilon_n^{1/p} |D\nabla_\epsilon_n(x_n)| > C \). Define \( X^n = \epsilon_n^{-1/p}(x - \tilde{x}_n) \) and \( \nabla_\epsilon(X^n) = \nabla_\epsilon(x) \), where \( \tilde{x}_n = s(x_n) \in \partial \Omega \). We know from the proof of Theorem A that \( \nabla_\epsilon \to z_0 \) in \( C^0_{\text{loc}}(T_{\epsilon}) \) as \( n \to \infty \) (we can choose subsequences if necessary), where \( T_{\epsilon} = \{x \in T_1; 0 < x_1 < \epsilon\} \). Since \( \epsilon_n^{1/p} |D\nabla_\epsilon_n| = |D\nabla_\epsilon_n| \), we have that \[
\epsilon_n^{1/p} |D\nabla_\epsilon_n| \to z_0'(x_1) \quad \text{in} \quad C^0_{\text{loc}}(T_{\epsilon}) \quad \text{as} \quad n \to \infty.
\]
This also implies
\[
\max_{\Omega^{\hat{\delta}, \epsilon^{1/p}}} \epsilon_n^{1/p} |D\nabla_\epsilon| \leq C.
\]
This contradicts our assumption above.

Multiplying both sides of Eq. (5.1) by \( \nabla_\epsilon \) and integrating on \( E_\epsilon := \Omega \setminus \Omega^{\hat{\delta}, \epsilon^{1/p}} \), we have
\[
\epsilon \int_{E_\epsilon} |D\nabla_\epsilon|^p = \int_{E_\epsilon} f(\nabla_\epsilon) \nabla_\epsilon + \epsilon \int_{\Omega^{\hat{\delta}, \epsilon^{1/p}} \setminus \Omega} \nabla_\epsilon |D\nabla_\epsilon|^{p-2} (D\nabla_\epsilon, v)
\]
\[
< \epsilon \rho_2 |\Omega| + \rho_2 \epsilon^{1/p} C.
\]
Moreover,
\[
\epsilon \int_{\Omega^{\hat{\delta}, \epsilon^{1/p}} |D\nabla_\epsilon|^{p} \leq C |\Omega^{\hat{\delta}}|^{1/p} = o(\epsilon^{1/p}).
\]
This implies that
\[
e \int_{\Omega} |D\bar{u}_e|^p = \left( \int_{\Omega} + \int_{B(\bar{u}_e)^{1/p}} \right) |D\bar{u}_e|^p \leq e \rho_2 |\Omega| + O(e^{1/p}).
\]

Thus,
\[
e \int_{\Omega} |D\bar{u}_e|^p \to 0 \quad \text{as } e \to 0.
\]

From Proposition 3.6 we have
\[
\int_{\Omega} F(\bar{u}_e) \to F(\rho_2) |\Omega| \quad \text{as } e \to 0,
\]

as claimed.

**Theorem 5.2.** Assume that \( N \geq 3, p > 2, \) and \( f \) satisfies (F1)-(F3). Then there exists a positive solution \( u_e \) of (5.1) with \( u_e \neq \bar{u}_e \) and \( \|u_e\|_{\infty} > \rho_1. \)

**Proof.** Let \( E = W^{1,p}_0(\Omega) \). For \( \rho \in [0, 1/2) \), we consider the functional \( J_{\varepsilon, \rho}: E \to \mathbb{R} \) with
\[
J_{\varepsilon, \rho}(u) = \frac{\varepsilon}{2} \int_{\Omega} \rho |Du|^2 + J(u).
\]

Clearly, for any \( \varepsilon \) and \( \rho \), \( J_{\varepsilon, \rho} \) is of class \( C^1 \) and \( J_{\varepsilon, \rho}(0) = 0. \)

We prove that for any \( \varepsilon > 0, 0 < \rho < 1/2, \) \( J_{\varepsilon, \rho}(u) \) satisfies the Palais–Smale condition (P-S) in \( E \) [37]. In fact, let \( \{u_m\} \subset E \) be a sequence such that
\[
|J_{\varepsilon, \rho}(u_m)| \leq C \text{ and } J_{\varepsilon, \rho}'(u_m) \to 0 \quad \text{as } m \to \infty \text{ for some constant } C.
\]

From the inequality
\[
\frac{\varepsilon}{2} \int_{\Omega} \rho |Du_m|^2 + \frac{\varepsilon}{p} \int_{\Omega} |Du_m|^p \, dx = J_{\varepsilon, \rho}(u_m) + \int_{\Omega} F(u_m) \, dx
\]
\[
\leq |J_{\varepsilon, \rho}(u_m)| + F(\rho_2) |\Omega|,
\]
we immediately see that \( \{u_m\} \) is bounded in \( E. \)
We now show that \( \{u_m\} \) has a convergent subsequence in \( E \). Since \( f \equiv 0 \) for \( s \leq 0 \) and \( s > \rho_2 \), \( \{f(u_m)\} \) is bounded in \( L^\infty(\Omega) \). By arguments similar to those in the proof of Proposition 2.2 of [17], we have that

\[
A_{-1,e,p,r}^{-1} : L^\infty(\Omega) \to C_0^1(\Omega)
\]

is compact, where \( A_{-1,e,p,r}^{-1} \) is the inverse operator of \( -\varepsilon \div((\rho + |D \cdot |^{r-2}) D \cdot) \) under the Dirichlet boundary condition. (Notice that Proposition 2.2 of [17] deals with the regularity of \( A_{-1,e,0}^{-1} \) (i.e., \( \rho = 0 \)), but the conclusions remain valid for \( \rho \in [0, 1/2) \). Since in the proof of Proposition 2.2 of [17], the author used the regularity results from [28, 42, 43], one can easily check that the results are also true for \( \rho \in [0, 1/2) \). This implies that there exists a subsequence \( \{f(u_m)\} \) of \( \{f(u_m)\} \) in \( L^\infty(\Omega) \) such that \( \{A_{-1,e,p,r}^{-1} f(u_m)\} \) is a convergent sequence in \( E \). Therefore, the fact that \( J_{e,r}(u_m) \to 0 \) implies \( u_m - A_{-1,e,p,r}^{-1} f(u_m) \to 0 \) in \( E \). Thus, \( \{u_m\} \) is a convergent sequence in \( E \). This implies that \( J_{e,r} \) satisfies the (P-S) condition.

For \( 2 < p < N \), by the conditions that \( \lim_{s \to 0^+} f(s)/s^{p-1} = -m < 0 \), and \( f(0) = 0 \), we can choose constants \( C > 0 \) and \( q = p^*/(N - p) \) such that

\[
J_{e,r}(u) \geq (e/p) \int_\Omega |Du|^p dx - \int_\Omega F(u) \, dx \geq (e/p) \|u\|^p - C \|u\|^{q+p} dx.
\]

(5.2)

Since \( E \) is continuously embedded in \( L^{q+p}(\Omega) \), we have

\[
J_{e,r}(u) \geq (e/p) \|u\|^p - \int_\Omega C \|u\|^{q+p} dx \geq (e/p - C') \|u\|^q \|u\|^p.
\]

Taking \( \eta = (\theta e)^{1/q} \) where \( 0 < \theta < 1/(2pC') \) and \( \alpha = (e/p - C' \eta^q) \eta^p \geq C' e^{N/q} \), we obtain \( J_{e,r}(u) > 0 \) if \( 0 < \|u\| < \eta \) and \( J_{e,r}(u) \geq \alpha \) if \( \|u\| = \eta \).

For \( p \geq N \), and for every \( q > 0 \), (5.2) still holds. Since, by the Sobolev embedding theorem, \( E \) is continuously embedded in \( C^0(\bar{\Omega}) \), we have that

\[
J_{e,r}(u) \geq (e/p) \|u\|^p - \int_\Omega C \|u\|^{q+p} dx
\]

\[
\geq (e/p) \|u\|^p - C'(\Omega) \|u\|^{q+p}
\]

\[
\geq (e/p - C'' \|u\|^q) \|u\|^p.
\]

Choosing \( \eta = (\theta e)^{1/q} \) where \( 0 < \theta < 1/(2pC'') \) and \( \alpha = (e/p - C'' \eta^q) \eta^p \geq C' e^{N/q} \), we obtain \( J_{e,r}(u) > 0 \) if \( 0 < \|u\| < \eta \) and \( J_{e,r}(u) \geq \alpha \) if \( \|u\| = \eta \). Note that \( \alpha \) is independent of \( \rho \) in both cases.
Next we show that there is \( \varepsilon \in E \setminus \{ u \in E : \| u \| \leq \eta \} \) such that \( J_{e,\rho}(\varepsilon) \leq 0 \). Let \( \bar{u}_{\varepsilon} \equiv \bar{u}_{\varepsilon} \) be the solution of the problem

\[
-\text{div}(|Du|^{p-2} Du) = \lambda_0 f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]

obtained in Theorem A with \( \varepsilon_0 = 1/\lambda_0 \). Then \( \max \bar{u}_{\varepsilon_0} \) is near \( \rho_2 \) if \( \varepsilon_0 \) is sufficiently small. Moreover, for a fixed \( \varepsilon_0 > 0 \), it follows from Lemma 5.1 that \( J_{\varepsilon_0}(\bar{u}_{\varepsilon_0}) < 0 \). Since \( \varepsilon_0 \int_{\Omega} |D\bar{u}_{\varepsilon_0}|^p \to 0 \) as \( \varepsilon_0 \to 0 \), then \( \varepsilon_0 \int_{\Omega} \rho |D\bar{u}_{\varepsilon_0}|^2 \to 0 \) for any \( \rho \in [0, 1/2) \) (note that \( p > 2 \)). This implies that for every \( \rho \in (0, 1/2) \), there is \( \varepsilon_0 > 0 \) such that \( J_{\varepsilon_0,\rho}(\bar{u}_{\varepsilon_0}) < 0 \).

Now define

\[
U_{\varepsilon}(x) = \begin{cases} \bar{u}_{\varepsilon_0}(\varepsilon_0/\varepsilon)^{1/p} x, & \text{if } x \in (\varepsilon_0/\varepsilon)^{1/p} \Omega, \\ 0, & \text{if } x \in \Omega \setminus (\varepsilon_0/\varepsilon)^{1/p} \Omega. \end{cases}
\]

Note that \( 0 \in \Omega \) and \( (\varepsilon/\varepsilon_0)^{1/p} \Omega \subset \Omega \) if \( \varepsilon < \varepsilon_0 \). For \( 0 < \varepsilon < \varepsilon_0 \), we have \( \varepsilon \in E \). Then it follows that

\[
J_{\varepsilon,\rho}(U_{\varepsilon}) = \frac{\varepsilon}{2} \int_{(\varepsilon_0/\varepsilon)^{1/p} \Omega} \rho |D\bar{u}_{\varepsilon_0}(\varepsilon_0/\varepsilon)^{1/p} x)|^2 dx + \frac{\varepsilon}{p} \int_{(\varepsilon_0/\varepsilon)^{1/p} \Omega} |D\bar{u}_{\varepsilon_0}(\varepsilon_0/\varepsilon)^{1/p} x)|^p dx 
\]

\[
- \int_{(\varepsilon_0/\varepsilon)^{1/p} \Omega} F(\bar{u}_{\varepsilon_0}(\varepsilon_0/\varepsilon)^{1/p} x) dx 
\]

\[
= (\varepsilon/\varepsilon_0)^{N/p} \left[ (\varepsilon_0/\varepsilon)(\varepsilon_0/\varepsilon)^{2/p} \left( \varepsilon_0/2 \right) \int_{\Omega} \rho |D\bar{u}_{\varepsilon_0}|^2 + J_{\varepsilon_0}(\bar{u}_{\varepsilon_0}) \right] 
\]

\[
< (\varepsilon/\varepsilon_0)^{N/p} J_{\varepsilon_0,\rho}(\bar{u}_{\varepsilon_0}) < 0.
\]

This implies that \( U_{\varepsilon} \) satisfies our requirement. In fact, we have

\[
\varepsilon \| U_{\varepsilon} \|^p = (\varepsilon/\varepsilon_0)^{N/p} \varepsilon_0 \| \bar{u}_{\varepsilon_0} \|^p,
\]

and so,

\[
\| U_{\varepsilon} \| = (\varepsilon/\varepsilon_0)^{(N-p)/p} \| \bar{u}_{\varepsilon_0} \|.
\]

Therefore, for \( p < N \), we have \( \| U_{\varepsilon} \| > \eta \) if we choose \( 0 < \theta < (1/\varepsilon_0)^{N-p/(N-p)} \). For \( p \geq N \), we easily see that \( \| U_{\varepsilon} \| > \eta \) for \( 0 < \theta < 1/(2pC^p) \).
Now we apply the mountain pass lemma \cite{37} to the functional $J_{e, r}$ and obtain a critical point $u_{e, r}$ with critical value $J_{e, r}(u_{e, r}) = \inf_{f \in \bar{P}} \max_{u \in B(0, 1)} J_{e, r}(u)$, where $\bar{P} = \{ g \in C([0, 1]; E) : g(0) = 0, g(1) = U_e \}$. Since $f$ is bounded, by regularity of the $p$-Laplacian, (see \cite{17}), we have that $u_{e, r} \in C^{1, \gamma}(\Omega)$. It is clear that for $0 < e < e_0$ ($e_0$ is independent of $r$) and any $r \in [0, 1/2)$, $u_{e, r}$ is a nontrivial solution of the problem

$$-\varepsilon \text{div}((\rho + |Du|^{p-2}) Du) = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad (5.3)$$

Moreover, the maximum principle implies $0 < \varepsilon u_{e, r} \leq \rho_2$. Since

$$\varepsilon \rho \int_{\Omega} |Du_{e, r}|^2 + \varepsilon \int_{\Omega} |Du_{e, r}|^p = \int_{\Omega} f(u_{e, r}) u_{e, r},$$

then

$$\|u_{e, r}\|_{W^{1, p}(\Omega)} \leq C,$$

where $C$ depends on $\varepsilon$ but is independent of $\rho$. By arguments similar to those in the proof of Proposition 2.2 of \cite{17}, we have $\|u_{e, r}\|_{C^{1, \gamma}(\Omega)} \leq C$ for $0 < \beta < 1$ where $C$ depends on $\varepsilon$ but is independent of $\rho$. (Such results can be obtained directly from \cite{28, 42, 43}.) This implies that for $0 < e < e_0$,

$$u_{e, r} \to u_e \quad \text{in } C^{1}(\Omega) \quad \text{as } r \to 0.$$

Since $J_{e, r}(u_{e, r}) \geq \alpha > 0$, $\alpha$ is independent of $r$, and

$$J_{e, r}(u_{e, r}) \to J_e(u_e) > \alpha \quad \text{as } r \to 0,$$

we know from Lemma 5.1 that $u_e \neq \bar{u}_e$. Moreover,

$$J_e(u_e) = \inf_{f \in \bar{P}} \max_{u \in B(0, 1)} J_e(u).$$

From the maximum principle we have $\max_{\Omega} u_e > \rho_1$. 

**Remark 5.3.** Taking $\rho = 0$, we see that the conclusion of Theorem 5.2 holds also for $1 < p \leq 2$. We present a bad version of Theorem 5.2 here in order to obtain the asymptotic behaviour of $u_e$ as $e \to 0$ below.
6. PROOF OF THEOREM B

To obtain the first claim of Theorem B, we define \( g : [0, 1] \to E \) by \( g(t) = tU_e \). Then we have

\[
J_e(tU_e) = \frac{e}{p} \int_{((e_0/e)^{1/p})}^{t} x \left| Du_e \left( (e_0/e)^{1/p} x \right) \right|^p dx
\]

\[
- \int_{((e_0/e)^{1/p})}^{t} F(tu_e \left( (e_0/e)^{1/p} x \right)) dx
\]

\[
\leq (e/e_0)^{N/p} \left( \frac{e_0}{p} \int_{\Omega} \left| Du_e(x) \right|^p dx \right) \leq C_1 e^{N/p},
\]

and thus \( J_e(u_e) \leq \max_{t \in [0, 1]} J_e(tU_e) \leq C_1 e^{N/p} \). Since \( J_e(u_e) \geq \alpha \), claims (i) and (ii) of Theorem B hold.

To show (iii) of Theorem B, we first define a class of perturbations \( g_\theta(s) \) (with \( \theta > 0 \) small) of \( f(s) \) such that

\[
\gamma(\theta) + f(s) \leq g_\theta(s) \leq f(s) + \Gamma(\theta) \quad \text{for } s \in [0, \rho_2 + 1) \quad (6.1)
\]

and \( g_\theta \) satisfies \((H_1)-(H_4)\) in Theorem A.5 of the Appendix for any \( \theta > 0 \) sufficiently small, where \( \gamma(\theta), \Gamma(\theta) \in C^0([0, 1/8]) \) satisfy \( \gamma(0) = \Gamma(0) = 0 \) and \( \Gamma(\theta) \geq \gamma(\theta) > 0 \) for \( 0 < \theta \leq 1/8 \).

We denote the three zeros of \( g_\theta \) by \( \rho_0(\theta), \rho_1(\theta), \) and \( \rho_2(\theta) \). Then it is clear that

\[
0 < \rho_0(\theta) < \rho_1(\theta) < \rho_1 < \rho_2 < \rho_2(\theta)
\]

and

\[
\lim_{\theta \to 0} \rho_0(\theta) = 0, \quad \lim_{\theta \to 0} |\rho_i(\theta) - \rho_i| = 0 \quad \text{for } i = 1, 2,
\]

(6.3)

where \( \rho_1, \rho_2 \) are the positive zeros of \( f \). Moreover, for any \( \theta > 0 \), there is a unique \( \mu(\theta) \in (\rho_1(\theta), \rho_2(\theta)) \) such that

\[
\int_{\rho_0(\theta)}^{\mu(\theta)} g_\theta(s) \, ds = 0.
\]
Noted that the zeros of \( g_{\theta} \) depend on \( \theta \); we assume that \( g_{\theta} \) is chosen so that there exists \( C > 0 \) independent of \( \theta \) such that
\[
|g_{\theta}(s)| \leq C |s - \rho_i(\theta)|^{p-1}
\]
for \( s \) near \( \rho_i(\theta) \) and \( i = 0, 1, 2 \). Moreover, we assume that
\[
\lim_{s \to \rho_i(\theta)} g_{\theta}(s)/(s - \rho_i(\theta))^{p-1} = -m_1(\theta) < 0
\]
and \( m_2 \leq m_1(\theta) \leq m_3 \) for all \( \theta \) small, where \( 0 < m_2 \leq m_3 \) are independent of \( \theta \).

For example, if \( f \) is the example given in the Introduction, we may choose the perturbations \( g_{\theta}(s) \) (\( \theta > 0 \) sufficiently small) to be
\[
g_{\theta}(s) = \begin{cases} 
- m \psi_1^{2}(s) \psi_3^{2}(s), & \text{for } p \geq 2, \\
- m |s-\theta|^{p-2} (s-\theta)((a-\theta)-s)((1+\theta)-s), & \text{for } 1 < p < 2,
\end{cases}
\]
where
\[
\psi_1^{1}(s) = |s-\theta|^{p-2} (s-\theta),
\psi_2^{1}(s) = |(a-\theta)-s|^{p-2} ((a-\theta)-s),
\psi_3^{1}(s) = |(1+\theta)-s|^{p-2} ((1+\theta)-s).
\]
(Note that for \( p > 2 \), \( g_{\theta} \in C^1((0, \infty)) \cap C^0([0, \infty)). \))

We will use the following lemma.

**Lemma 6.1.** Let \( p, \Omega \) be as in Theorem B and \( u_\varepsilon \) be the positive solution obtained in Theorem 5.2. Then there exists \( \varepsilon^* \) such that for \( 0 < \varepsilon < \varepsilon^* \), \( u_\varepsilon \) satisfies
\[
\left| \frac{\partial u_\varepsilon}{\partial v}(x) \right| \leq (C/\varepsilon) e^{-\sigma \varepsilon^{-1/p}} \quad \text{for all } x \in \partial \Omega,
\]
where \( C \) and \( \sigma \) are independent of \( \varepsilon \) and \( v(x) = -n_\varepsilon \) denotes the outward normal vector at \( x \in \partial \Omega \).

**Proof.** We show that there exists \( \bar{\rho} > 0 \) independent of \( \varepsilon \) such that
\[
l_\beta = \sup_{\bar{\rho}} u_\varepsilon < \bar{\mu},
\]
where \( \bar{\mu} \in (\rho_1, \rho_2) \) is such that \( \int_0^\beta f(s) \, ds = 0 \).
By the moving plane method near \( \partial \Omega \), as in the Appendix, we can find \( \bar{\sigma} > 0 \) independent of \( \varepsilon \) and \( \rho \) such that for any \( y \in \Omega^\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \} \), there is a fixed-size cone \( K_y \subset \Omega^\delta \) with vertex at \( y \) and \( u_{\varepsilon, \rho}(y) = \min_{x \in K_y} u_{\varepsilon, \rho}(x) \).

From the proof of Theorem 5.2, for \( 0 < \varepsilon < \varepsilon_0 \), \( u_{\varepsilon, \rho} \rightarrow u_{\varepsilon} \) in \( C^1(\Omega) \) as \( \rho \rightarrow 0 \) and \( u_\varepsilon \) has the same properties as \( u_{\varepsilon, \rho} \). Thus, for any \( y \in \Omega^\delta \), there is a fixed-size cone \( K_y \subset \Omega^\delta \) with vertex at \( y \) and \( u_{\varepsilon}(y) = \min_{x \in K_y} u_{\varepsilon}(x) \).

Let \( \delta \in (0, \bar{\mu} - \rho_1) \). We claim that there exists \( 0 < \varepsilon_1 < \varepsilon_0 \) such that for \( 0 < \varepsilon < \varepsilon_1 \), \( u_{\varepsilon}(y) < \rho_1 + \delta \). To see this, fix \( y \in \Omega^\delta \) and choose a positive radially symmetric function \( \phi \in C^\infty_c(\Omega) \) so that \( \text{supp} \phi \subset K_y \) and \( \rho_1 < \max \phi < \rho_1 + \delta \). Then there exists \( x_1 \in K_y \) and \( \bar{r} > 0 \) such that \( \phi \geq \bar{r} > \rho_1 \) in \( B(x_1, \bar{r}) \subset K_y \). By Theorem A, there exists a constant \( e_1 = e_1(\phi) < \varepsilon_0 \) such that, for \( 0 < \varepsilon < e_1 \), \( u_{\varepsilon} \) is the unique solution between \( \phi \) and \( \rho_2 \).

Suppose that for \( 0 < \varepsilon < e_1 \), \( u_{\varepsilon}(y') \geq \rho_1 + \delta \) at some point \( y' \in \Omega^\delta \). Then \( u_{\varepsilon} \geq \rho_1 + \delta \) in \( K_y \) and consequently \( u_{\varepsilon} \geq \phi' \) for some \( \phi' \in C^\infty_c(\Omega) \), which is a translation of \( \phi \). Since \( K_y \) and \( K_y' \) have the same fixed size, then there exists \( x_2 \in K_y ' \) such that \( B(x_2, \bar{r}) \subset K_y ' \) and \( \phi' \geq \bar{r} > \rho_1 \) in \( B(x, \bar{r}) \) (since \( \phi' \) is a translation of \( \phi \)). Remark 4.1 then implies \( \varepsilon_1(\phi) = e_1(\phi') \). This gives \( u_{\varepsilon} \equiv u_{\varepsilon} \), a contradiction. Let \( \bar{\rho} = \bar{\sigma} \). Thus, (6.5) holds.

Now fix a point \( x^* \in \partial \Omega \). Without loss of generality, we may assume that \( x^* = 0 \) and \( \text{v}(x^*) = e_1 = (1, 0, \ldots, 0) \), where \( \text{v}(x) \) is the outward normal vector at \( x \in \partial \Omega \).

To construct a family of supersolutions to Eq. (5.1), we consider the following ordinary differential equation with an arbitrary small \( \theta > 0 \)

\[
e(\nu|w|^p-2w')' + g_\theta(w) = 0 \text{ in } (0, \gamma), \quad w'(0) = 0, \quad w(\gamma) = \rho_\theta(\theta), \quad (6.6)
\]

where \( 0 < \gamma < \bar{\theta} \) is independent of \( \varepsilon \) and \( \theta \). From Theorem A.5 there exists \( \bar{\varepsilon}(\theta) > 0 \) such that for \( 0 < \varepsilon < \bar{\varepsilon}(\theta) \), (6.6) has a solution \( w_{\theta, \varepsilon} \) with the following properties:

1. \( |w_{\theta, \varepsilon} - \rho_\theta(\theta)| \leq Ce^{-\sigma \varepsilon^{1/\theta}} \) in any closed subinterval of \( (0, \gamma] \), where \( C > 0, \sigma > 0 \) are independent of \( \varepsilon \) and \( \theta \),

2. \( w_{\theta, \varepsilon}(0) \rightarrow \mu(\theta) \) as \( \varepsilon \rightarrow 0 \).

(From the proof of Theorem A.5, \( \sigma \) in (1) depends on \( m_1(\theta) \), but since \( m_1 \leq m_1(\theta) \leq m_2 \), we can choose \( \sigma \) in (1) that does not depend on \( \theta \), but \( \sigma \) depends on \( m_2 \).)

We claim that we can choose \( \bar{\varepsilon} > 0 \) independent of \( \theta \) such that for \( 0 < \varepsilon < \bar{\varepsilon} \), properties (1) and (2) of \( w_{\theta, \varepsilon} \) still hold. In fact, we easily see that \( w_{\theta, \varepsilon} - \rho_\theta(\theta) \) satisfies the problem (A.19) in Corollary A.7. Then if we
choose $\bar{\varepsilon}$ as in Corollary A.7, we have from Corollary A.7 that if $0 < \varepsilon < \bar{\varepsilon}$, then (1) and (2) hold. Choosing $\varepsilon^* = \min\{\varepsilon_1, \bar{\varepsilon}\}$, we have (6.5), and properties (1) and (2) hold for $0 < \varepsilon < \varepsilon^*$ and any small $\theta > 0$.

We now define $w_{\theta, \varepsilon}(x) = w_{\theta, \varepsilon}(x_1 + t)$ for $0 \leq t \leq \gamma$. Clearly $w_{\theta, \varepsilon, t}$ satisfies

$$\varepsilon \Delta u + g(u) = 0 \quad \text{in} \quad \Omega' = \Omega \cap \{x \in \mathbb{R}^N : x \cdot e_1 = x_1 > -t\}$$

and there exists $t_0 > 0$ (depending on $\theta$ and $\varepsilon$) such that $w_{\theta, \varepsilon, t_0} > u_\varepsilon$ in $\Omega'_{t_0}$ (since $\mu(\theta) \to \bar{\mu}$ as $\theta \to 0$), we can choose $\theta$ so small that $\mu(\theta) > l_\varepsilon$). By the sweeping out result as in Proposition 2.5, we have

$$w_{\theta, \varepsilon, t} \geq u_\varepsilon \quad \text{in} \quad \Omega'_{t}.$$  \hspace{1cm} (6.7)

In fact, by the convexity of $\Omega$, $\Omega' = \Omega \cap \{x \in \mathbb{R}^N : -t < x_1 < 0\}$ and for any $t_0 < t_1 < \gamma$, $w_{\theta, \varepsilon, t_1}(x)|_{x_1 = -t_1} = w_{\theta, \varepsilon}(0) = \mu(\theta) + o(1) > u_\varepsilon(x)|_{x_1 = -t_1}$ for $\varepsilon$ sufficiently small, $w_{\theta, \varepsilon, t_0} > u_\varepsilon$. Thus, arguments similar to those in the proof of Proposition 2.5 imply that

$$w_{\theta, \varepsilon, t}(x) \geq u_\varepsilon(x) \quad \text{for} \quad x \in \Omega'_{t_1} \quad \text{and} \quad t_1 \in [t_0, \gamma].$$

This implies that (6.7) holds.

By compactness of $\partial \Omega$ and the properties of $w_{\theta, \varepsilon, t}$, we obtain from (6.7) that

$$u_\varepsilon \leq \rho_\varepsilon(\theta) + Ce^{-\sigma I_1/\varepsilon} \quad \text{in} \quad \Omega'_{\gamma/2}. \hspace{1cm} (6.8)$$

Since $C$ and $\sigma$ are independent of $\theta$, letting $\theta \to 0$, we have

$$u_\varepsilon \leq Ce^{-\sigma I_1/\varepsilon} \quad \text{in} \quad \Omega'_{\gamma/2}. \hspace{1cm} (6.9)$$

(Note that $\rho_\varepsilon(\theta) \to 0$ as $\theta \to 0$.) On the other hand, it follows from (6.7) that

$$w_{\theta, \varepsilon, t} \geq u_\varepsilon \quad \text{in} \quad \Omega'_{t}, \hspace{1cm} (6.10)$$

where $w_{\theta, \varepsilon} = w_\varepsilon$ is the solution given in Corollary A.7. (In fact, we can show that $w_{\theta, \varepsilon} \to w_\varepsilon$ in $C^1([0, \gamma])$ as $\theta \to 0$). Since $x^* \in \partial \Omega'$, it follows from (6.10) that

$$\frac{\partial u_\varepsilon}{\partial y}(x^*) \leq \frac{\partial w_{\theta, \varepsilon, t}}{\partial y}(0) = w_{0, \varepsilon}(\gamma). \hspace{1cm} (6.11)$$
Notice that \( w_0, e \) satisfies problem (A.19) of the Appendix. It follows from Corollary A.7 that
\[
w_0'(\gamma) \leq (C/e) e^{-\epsilon^{-1/p}},
\]
where \( C \) and \( \sigma \) are independent of \( e \). This completes the proof.

**Remark 6.2.** The proof of Lemma 6.1 also implies that the maximum of \( u_e \) cannot be attained near \( \partial \Omega \). In fact, the proof of Lemma 6.1 shows that \( \max_{\Omega} u_e < \rho_1 \). A similar argument to that in the proof of Lemma 7.2 implies that if \( x_0 \in \Omega \) is a local maximum point of \( u_e \), then \( u_e(x_0) > \rho_1 \).

**Proof of part (iii) of Theorem B.** Pohozaev’s identity for \( u_e \) (see [16]),
\[
(1 - N/p) \epsilon \int_{\Omega} |Du_e| p \, dx + N \int_{\Gamma} F(u_e) \, dx = \frac{\epsilon(1 - 1/p)}{p} \int_{\Omega} |Du_e| p (x \cdot n) \, ds,
\]
and (i) and (ii) of Theorem B and Lemma 6.1 imply that
\[
\epsilon \int_{\Omega} |Du_e| p \, dx = NJ\epsilon(u_e) + \epsilon(1 - 1/p) \int_{\Omega} |Du_e| p (x \cdot v) \, ds,
\]
and
\[
< Ce^{N/p} + \frac{\epsilon^N}{p} < Ce^{N/p}.
\]

Next, for each \( 0 < \sigma < \mu - \rho_1 \), we define \( \Omega_{\epsilon, \rho_1 + \sigma} = \{ x \in \Omega : u_e(x) > \rho_1 + \sigma \} \) and we shall prove that \( |\Omega_{\epsilon, \rho_1 + \sigma}| < Ce^{N/p} \) for some \( C > 0 \) independent of \( \epsilon \). Choose \( k > 0 \) independent of \( \epsilon \) such that \( f(u_e) > k \) in \( \Omega_{\epsilon, \rho_1 + \sigma/2} \); this is possible by assumptions (F_1)–(F_3) and Proposition 4.2, since we know from Proposition 4.2 that \( \max_{\Omega} u_e < \rho_2 - \sigma_1 \) with \( \sigma_1 > 0 \) independent of \( \epsilon \). Then we have
\[
|\Omega_{\epsilon, \rho_1 + \sigma}| < \frac{2}{\sigma} \int_{\Omega_{\epsilon, \rho_1 + \sigma}} (u_e - \rho_1 - \sigma/2) \, dx < \frac{2}{\sigma} \int_{\Omega_{\epsilon, \rho_1 + \sigma/2}} (u_e - \rho_1 - \sigma/2) \, dx
\]
\[
< \frac{2k}{\sigma} \int_{\Omega_{\epsilon, \rho_1 + \sigma/2}} f(u_e)(u_e - \rho_1 - \sigma/2) \, dx
\]
\[
= \frac{2k}{\sigma} \int_{\Omega_{\epsilon, \rho_1 + \sigma/2}} - (u_e - \rho_1 - \sigma/2) \, div(|Du_e|^{p-2} Du_e) \, dx
\]
\[
< \frac{2k}{\sigma} \int_{\Omega_{\epsilon, \rho_1 + \sigma/2}} |Du_e| p \, dx < \frac{2k}{\sigma} \int_{\Omega} |Du_e| p \, dx < \frac{2k}{\sigma} e^{N/p}.
\]
Noting that there is a constant $K > 0$ such that $u^p \leq -KF(u)$ for $0 \leq u \leq \rho_1 + \sigma$, we have

\[
\int_{\Omega} u^p \, dx = \int_{\Omega \setminus B_{\rho_1 + \sigma}} u^p \, dx + \int_{B_{\rho_1 + \sigma}} u^p \, dx
\leq -K \int_{\Omega \setminus B_{\rho_1 + \sigma}} F(u) \, dx + \int_{B_{\rho_1 + \sigma}} u^p \, dx
\leq KJ(x) + K \int_{B_{\rho_1 + \sigma}} F(u) \, dx + \int_{B_{\rho_1 + \sigma}} u^p \, dx
\leq KC_{1}e^{N/p} + (\rho_2^p + KF(\rho_2)) |\Omega \setminus B_{\rho_1 + \sigma}|
\leq CE^{N/p},
\]
and our proof of part (iii) of Theorem B is completed.

Remark 6.3. We easily see that $u_\varepsilon < \bar{u}$ in $\Omega$ as $\varepsilon \to 0$. Thus, $u_\varepsilon$ is a small solution.

7. PROOF OF THEOREM C

Let $x_0$ be a point at which $u_\varepsilon$ attains a local maximum. We know that \{\{x_0\}\} is bounded away from $\partial \Omega$. Let \(U_\varepsilon(x) = u_\varepsilon(e^{1/p}x + x_0)\). By a compactness argument, as in the proof of Proposition 4.2, we see that $U_\varepsilon(x) \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where $U$ with $0 < U < \rho_2$ is a positive solution of the equation $D_p U + f(U) = 0$ in $\mathbb{R}^N$. Moreover, it follows from $\int_{\mathbb{R}^N} u_\varepsilon^p \, dx \leq C \varepsilon^{N/p}$ and the change of variable that $\int_{\mathbb{R}^N} U^p \, dx \leq C$. We know from the proof of (iii) of Theorem B that

\[
\frac{\varepsilon}{p} \int_{\Omega} |Du_\varepsilon|^p \, dx \leq C \varepsilon^{N/p}.
\]
Thus, by the change of variable, we obtain $\int_{\mathbb{R}^N} |DU|^p \, dx \leq C$. Therefore, $U \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$.

Now we show that $f(U) \in L^1(\mathbb{R}^N)$. In fact, it follows from the assumptions in Theorem C that there exists a $\sigma^*$ with $\rho_1 + \sigma^* < \bar{\mu}$ such that $\Omega_{\rho_1 + \sigma^*}$ is a connected convex set. Then the continuity of $u_\varepsilon$ implies that we can choose a convex domain $\Omega_\varepsilon \subset \subset \Omega_{\rho_1 + \sigma^*}$ with boundary suitably smooth such that $\max_{\Omega_{\rho_1 + \sigma^*} \setminus \Omega_\varepsilon} u_\varepsilon < \bar{\mu}$. We claim that

\[
u_\varepsilon \leq C e^{-\sigma_0^{-1/p}} \quad \text{in} \quad \Omega \setminus \Omega_{\rho_1 + \sigma^*},
\] (7.1)
where $C$ and $\sigma$ are independent of $\varepsilon$. In fact, for a fixed $x^* \in \partial \Omega$, without loss of generality, we may assume that $x^* = 0$ and $\nu(x^*) = e_1 = (1, 0, \ldots, 0)$, where $\nu(x)$ is the outward normal vector at $x \in \partial \Omega$. We choose $\gamma > 0$ such that

$$\Omega_1 := \{x \in \Omega, x \cdot e_1 = x_1 > 0\} \subset \{x \in \mathbb{R}^N : x \cdot e_1 = x_1, 0 < x_1 < \gamma\} \quad (7.2)$$

and $\partial \Omega \cap \{x_1 = \gamma\} \neq \emptyset$. Define

$$\Omega' = \Omega \cap \{x \in \mathbb{R}^N : x \cdot e_1 = x_1 > \gamma - t\}$$

and

$$w_{\theta, t, \gamma}(x) = w_{\theta, t}(x_1 + t - \gamma) \quad \text{for} \quad 0 \leq t \leq \gamma$$

where $w_{\theta, t}$ is the solution of (6.6). Clearly $w_{\theta, t, \gamma}$ satisfies

$$\varepsilon A_p u + g_\theta(u) = 0 \quad \text{in} \quad \Omega'$$

and there exists a small $\varepsilon_0 > 0$ (depending on $\theta$ and $\varepsilon$) such that $w_{\theta, t, \gamma} > u_\varepsilon$ in $\Omega_0$. Arguments similar to those in the proof of Lemma 6.1 imply that

$$w_{\theta, t, \gamma} \geq u_\varepsilon \quad \text{in} \quad \Omega' \setminus \Omega_0.$$ \quad (7.3)

Therefore, the compactness of $\partial \Omega$, and the properties of $w_{\theta, t, \gamma}$ imply that

$$u_\varepsilon \leq p_0(\theta) + C e^{-\varepsilon_0^{-1}} \quad \text{in} \quad \Omega \setminus \Omega_{\varepsilon, \varepsilon_0^*}, \quad (7.4)$$

where $\sigma$ and $C$ are independent of $\varepsilon$ and $\theta$. Letting $\theta \to 0$, we obtain our claim (7.1).

Now, with the help of (7.1), we show that

$$\int_B u_\varepsilon^{p-1} \, dx \leq C e^{N/p} \quad (7.5)$$

for $C$ independent of $\varepsilon$. In fact, from the proof of (iii) of Theorem B we have

$$|\Omega_{\varepsilon, \varepsilon_0^*}| \leq C e^{N/p}.$$ 

On the other hand, by (7.1) we also have

$$\int_B u_\varepsilon^{p-1} \, dx = \int_{B \setminus \Omega_{\varepsilon, \varepsilon_0^*}} u_\varepsilon^{p-1} \, dx + \int_{\Omega_{\varepsilon, \varepsilon_0^*}} u_\varepsilon^{p-1} \, dx \leq C e^{N/p}. \quad (7.6)$$
This implies that $\int_{\mathbb{R}^p} U^{p-1} = dx \leq C$. Since $|f(s)| \leq M s^{p-1}$ for $s \in [0, \rho]$, we have

$$\int_{\mathbb{R}^p} |f(U)| dx \leq M \int_{\mathbb{R}^p} U^{p-1} dx \leq C. \quad (7.7)$$

Theorem D implies that $U$ is radially symmetric (since the fact that $f'(s) < 0$ for $s$ near $0$ in $(F_2)$ implies that there exists $\delta > 0$ such that $f$ is nonincreasing for $0 < s < \delta$). Since $\max U \in (\rho_1, \rho_2 - \sigma)$, it follows from [20] that $U(x) \equiv w(r)$ is the unique positive (radial) solution of (1.2). \[\square\]

We now give an estimate for $J_{\epsilon}(u_{\epsilon})$.

**Lemma 7.1.** We have

$$0 < J_{\epsilon}(u_{\epsilon}) \leq e^{N/\epsilon}[I(U) + o(1)], \quad (7.8)$$

where $I(U) = \frac{1}{\epsilon} \int_{\mathbb{R}^p} |DU|^p - \int_{\mathbb{R}^p} F(U)$, $F(u) = \int_0^u f(s) ds$, and $U$, with $\max U \in (\rho_1, \rho_2 - \sigma)$, is the unique positive (radial) solution of Eq. (1.2) in $W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$.

**Proof.** The main idea in this lemma is similar to that in the proofs of Proposition 3.2 of [9] and Theorem 3.5 of [25]. Note that $\tilde{u}_1$, $u_{\epsilon}$ in Proposition 3.2 of [9] are $0$ here. We have from [20] that the problem

$$\varepsilon A_{\epsilon} u + f(u) = 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

where $B$ is the unit ball in $\mathbb{R}^N$, has exactly two positive radial solutions $\tilde{u}_1$ and $\tilde{u}_2$ for $\varepsilon$ sufficiently small. Moreover, we also know from [20] that $\tilde{u}_1$ and $\tilde{u}_2$ satisfy $(\tilde{u}_1)'(r) < 0$, $(\tilde{u}_2)'(r) < 0$ for $r \in (0, 1)$; $\tilde{u}_1$ is the maximal positive solution (and is close to $\rho_2$) on $B$; $\tilde{\mu} < \max \tilde{u}_2 < \rho_2$ and if we let $\tilde{\Omega}(r) = \tilde{u}_2^*(\varepsilon^{1/\mu})$, then $\tilde{\Omega} \rightarrow U$ in $C^1_{\text{loc}}(0, \infty)$ as $\varepsilon \rightarrow 0$. The arguments in the proofs of Theorems 5.2 and B imply that $\tilde{u}_2$ is a mountain pass solution. (To obtain the symmetry results, in [20], the author used the results in [3] and [4].) Moreover, the proofs of Lemma 6.1 and (iii) of Theorem B imply that for any $0 < \sigma^* < \tilde{\mu} - \rho_1$, there exists $\tilde{C} = C(\sigma^*) > 0$ independent of $\varepsilon$ such that the set $\{r \in [0, 1] : \tilde{u}_2(r) > \rho_1 + \sigma^*\} \subseteq [0, \varepsilon^{1/\tilde{C}}]$. The proof of (7.1) implies that $\tilde{u}_2^*(r) \leq C e^{-a\varepsilon^{-1/p}}$ for $e^{1/\tilde{C}} \leq r \leq 1$, where $C$ and $\sigma$ are independent of $\varepsilon$. Hence if we use the radial functional $\tilde{J}_l$ for the equation and use the minimax on all nondecreasing paths joining $0$ and $\tilde{\Omega}(x)$, where

$$\tilde{\Omega}(x) = \begin{cases} \tilde{u}_2^*(\varepsilon^{1/\mu} x), & \text{if } x \in (\varepsilon/\varepsilon_0)^{1/p} B, \\ 0, & \text{if } x \in B \setminus (\varepsilon/\varepsilon_0)^{1/p} B, \end{cases}$$
for some small $\varepsilon_0 > 0$ with $\varepsilon < \varepsilon_0$, the energy $\hat{e}_s = \hat{J}_s(\hat{u}^2) = e^{N/\rho}(I(U) + o(1))$. 
(This can easily be obtained by the same argument as in the proof of Proposition 3.2 of [9]. Note that $\hat{U}_s$ is a radial function, $U$ decays exponentially as $r \to \infty$, and $\hat{u}^2$ is exponentially small in any closed interval in $(0, 1)$.) Define $p_1(t), \hat{p}(t) = (1 - l(x)) p_1(t)$ to be as in the proof of Proposition 3.2 of [9]. Then, from the proof of Theorem 5.2, we have 

$$J_s(\hat{p}(1)) = \hat{J}_s(\hat{U}_s) < 0.$$ 

Furthermore, also from that proof, we have $\|\hat{p}(1)\| > \eta$, where $\eta$ is as in that proof. Therefore, 

$$J_s(\hat{p}(t)) = \hat{J}_s(p_1(t)).$$ 

Hence, 

$$J_s(u^2) = c_s \leq e^{N/\rho}(I(U) + o(1)).$$

The next lemma is similar to Lemma 4.2 of [30].

**Lemma 7.2.** Let $\phi \in C^1(B_b)$ ($B_b$ is the ball with centre 0 and radius $b$) be a radial function and satisfy $\phi(0) = 0$ and $(|\phi'(r)|^{r-2} \phi'(r))' < 0$ for $0 \leq r \leq b$. Then there exists $\delta_1 > 0$ such that if $\psi \in C^1(B_b)$ satisfies 

(i) $D\psi(0) = 0$ and

(ii) $\|\text{div}(|D\psi|^{r-2} D\psi - |D\phi|^{r-2} D\phi)\|_{L^\infty(B_b)} < \delta_1$, 

then $D\psi \neq 0$ for $x \neq 0$.

**Proof.** By replacing $\psi_x$ and $\phi_x$ in the proof of Lemma 4.2 of [30] by $|D\psi|^{r-2} \psi_x$ and $|D\phi|^{r-2} \phi_x$, the proof of this lemma follows easily from the cited proof.

Applying this lemma and Theorem D, we can show that if $P^1_s, P^2_s$ are two local maximum points of $u^2$, then 

$$e^{-1/\rho} |P^1_s - P^2_s| \to \infty$$

as $\varepsilon \to 0$.

We first claim that if $P_s$ is a local maximum point of $u_s$, then $u_s(P_s) > p_1$. In fact, since $P_s$ is a local maximum point of $u_s$, there is a ball $B_s = B_s(P_s) \subset \Omega$ with $\varepsilon > 0$ sufficiently small (which may depend on $\varepsilon$) such that $\partial u_s / \partial v \leq 0$ on $\partial B_s(P_s)$. Then multiplying both sides of (5.1) by $u_s$,
and integrating on \( B_t \), we obtain \( \int_{B_t} f(u_\varepsilon) \, u_\varepsilon \, dx > 0 \). This implies that there exists \( 0 < \xi_0 \leq \xi \) such that \( f(u_\varepsilon) > 0 \) on \( B_{\xi_0} \). Thus, \( u_\varepsilon(P_t) > \rho_1 \). This also implies that \( P_{t_1}^1 \) and \( P_{t_2}^1 \) are bounded away from \( \partial \Omega \).

Now defining \( U_{t_1}^1(y) = u_\varepsilon(\varepsilon^{1/p}y + P_{t_1}^1) \), we see that \( U_{t_1}^1 \to U \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \) as \( \varepsilon \to 0 \). We also know that for any \( R > 0 \),

\[
\|\text{div}(|Du_{t_1}^1|^{p-2} \, Du_{t_1}^1 - |Du|^{p-2} \, Du)\|_{L^\infty(B_R)} \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

since \( f(U_{t_1}^1) \to f(U) \) in \( B_R^c \) as \( \varepsilon \to 0 \). To obtain (7.9), as in the proof of Lemma 4.2 of [30], we only need prove that \( U_{t_1}^1 \) has exactly one local maximum point in \( B_R \). This fact can be obtained by arguments similar to those in the paragraph immediately after the proof of Lemma 4.2 of [30] and using Lemma 7.2. (Note that \(|U(r)'|^{p-2} U(r)' < 0 \) for \( r \in [0, b] \) and some \( b > 0 \) (since \(|U'|^{p-2} U' \)'(0) < 0 \) (see [20])), so that \( U \) has exactly one local maximum point in \( \mathbb{R}^N \).)

The proof that \( u_\varepsilon \) has only one local maximum point in \( \Omega \) (and thus the proof of Theorem C) is obtained by arguments similar to those in the proof of Theorem 1.1 of [9]. (Note that \( a_1, a_2, \) and \( u_\varepsilon \) in the cited theorem are 0, \( \rho_1 \), and 0, respectively, in our case.)

**APPENDIX A**

**Theorem A.1.** Let \( p > 2, \ \Omega \) be a convex domain with smooth boundary. Let \( u_{\varepsilon, \rho} \in C^2(\bar{\Omega}) \) be a positive solution of the problem

\[
-\varepsilon \text{div}(\rho + |Du|^{p-2} \, Du) = f(u) \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega, \quad (A.1)
\]

where \( 0 < \varepsilon < \varepsilon_0 \) (with \( \varepsilon_0 \) as in the proof of Theorem B), \( \rho \in (0, 1/2) \). Then there exists \( \sigma > 0 \), independent of \( \rho \) and \( \varepsilon \), such that for any \( y \in \Omega^\ast \), there is a fixed size cone \( K_y \subset \Omega \) with vertex at \( y \) and \( u_{\varepsilon, \rho}(y) = \min_{x \in K_y} u_{\varepsilon, \rho}(x) \).

Theorem A.1 follows from Theorem A.2 and an idea similar to that in the proof of Lemma A in the Appendix of [25].

We first show that the moving plane method can be used near \( \partial \Omega \). We first introduce some notation as in [14]. Let \( \gamma \) be a unit vector in \( \mathbb{R}^N \) and \( T_1 \) be the hyperplane \( \{ \gamma \cdot x = \lambda \} \). For \( \lambda \) large, \( T_1 \) is disjoint from \( \tilde{\Omega} \). Let the plane move continuously toward \( \Omega \), preserving the same normal, that is, decrease \( \lambda \), until \( T_1 \) begins to intersect \( \tilde{\Omega} \). From that moment, at every stage the plane \( T_1 \) cuts off from \( \Omega \) an open cap \( \Sigma(\lambda) \). Let \( \Sigma'(\lambda) \) denote the reflection of \( \Sigma(\lambda) \) in the plane \( T_1 \). For convenience, let \( \gamma \) be the unit vector \((1, 0, ..., 0)\) and assume \( \max_{x \in \tilde{\Omega}} X_1 = \lambda_0 \).
Theorem A.2. Let \( p > 2, \Omega \) be a convex domain with smooth boundary \( \partial \Omega \). Let \( u_{\rho, \epsilon} \in C^1(\overline{\Omega}) \) be a positive solution of (A.1). Then there exist \( \delta > 0 \) independent of \( \rho \) and \( \epsilon \) such that for \( \lambda_0 - \delta \leq \lambda < \lambda_0 \),

\[
\frac{\partial u_{\rho, \epsilon}}{\partial x_1} < 0 \quad \text{on } \Sigma(\lambda). \tag{A.2}
\]

The lemma below shows that the moving plane procedure can be started.

Lemma A.3. Let \( \epsilon > 0, \rho \in (0, 1/2) \) be fixed, and \( x_0 \in \partial \Omega \) with \( \nu_1(x_0) > 0 \). For some \( \alpha > 0 \) assume \( u_{\rho, \epsilon} \) is a \( C^1 \)-function in \( \Omega_\alpha \), where \( \Omega_\alpha = \Omega \cap \{ |x - x_0| < \alpha \} \), \( u_{\rho, \epsilon} > 0 \) in \( \Omega_\alpha \), and \( u_{\rho, \epsilon} = 0 \) on \( \partial \Omega \cap \{ |x - x_0| < \alpha \} \). Then there exists \( \delta > 0 \) (depending on \( \epsilon \) and \( \rho \)) such that in \( \Omega \cap \{ |x - x_0| < \delta \} \), \( (u_{\rho, \epsilon})_x < 0 \).

Proof. Since \( u_{\rho, \epsilon} > 0 \) in \( \Omega_\alpha \), necessarily, \( (u_{\rho, \epsilon})_x \leq 0 \) on \( S := \partial \Omega \cap \{ |x - x_0| < \alpha \} \), and hence \( (u_{\rho, \epsilon})_x \equiv (u_{\rho, \epsilon})_x \leq 0 \) on \( S \), for by decreasing \( \alpha \) if necessary, we may assume \( \nu_1 > 0 \).

If the lemma were false there would be a sequence of points \( x_i \to x_0 \), with \( (u_{\rho, \epsilon})_x (x_i) \geq 0 \). For \( j \) large, the interval in the \( x_1 \) direction going from \( x_i \) to \( \partial \Omega \) hits \( S \) at a point where \( (u_{\rho, \epsilon})_x \leq 0 \). Consequently, we have \( (u_{\rho, \epsilon})_x (x_0) = 0 \). Letting \( g(s) = f(s) + Ms^{p-1} \) with \( M > 0 \) sufficiently large, from the assumptions of \( f, g \) is strictly increasing on \( (0, \rho_1) \). Therefore,

\[
-\epsilon \text{div}((\rho + |Du_{\rho, \epsilon}|^{p-2}) Du_{\rho, \epsilon}) + Mu_{\rho, \epsilon}^{p-1} \geq 0 \quad \text{in } \Omega.
\]

By a modified version of the strong maximum principle in [34] and [44] we get \( (u_{\rho, \epsilon})_x < 0 \) on \( \partial \Omega \) and so \( (u_{\rho, \epsilon})_x (x_0) < 0 \), a contradiction.

The next lemma implies that the moving plane procedure can be continued.

Lemma A.4. If for some \( \lambda \) satisfying \( \lambda^* < \lambda < \lambda_0 \),

\[
(u_{\rho, \epsilon})_x \leq 0 \quad \text{and } u_{\rho, \epsilon}(x) \leq u_{\rho, \epsilon}(x') \quad \text{but } u_{\rho, \epsilon}(x) \neq u_{\rho, \epsilon}(x') \text{ in } \Sigma(\lambda),
\]

then \( u_{\rho, \epsilon}(x) < u_{\rho, \epsilon}(x') \) in \( \Sigma(\lambda) \) and \( (u_{\rho, \epsilon})_x < 0 \) on \( \Omega \cap T \).

Proof. Let \( v_{\rho, \epsilon}(x) = u_{\rho, \epsilon}(x) \) and \( w_{\rho, \epsilon}(x) = v_{\rho, \epsilon}(x) - u_{\rho, \epsilon}(x) \). Then \( w_{\rho, \epsilon}(x) \geq 0 \) in \( \Sigma(\lambda) \). We also have that \( w_{\rho, \epsilon} \) satisfies the equation

\[
-\epsilon \sum_{ij} \frac{\partial}{\partial x_i} \left[ (\rho \delta_{ij} + a_{ij}(x)) \frac{\partial w_{\rho, \epsilon}}{\partial x_j} \right] + M(p-1) \xi_{\rho, \epsilon}^{p-1}(x) w_{\rho, \epsilon} \geq 0 \quad \text{in } \Sigma(\lambda), \tag{A.3}
\]
where \( \xi_{e, \rho}(x) \in (u_{e, \rho}(x), v_{e, \rho}(x)) \) and \( a_{ij, e, \rho}(x) \) is as in [23]. Since \( \rho > 0 \), we have that the operator in (A.3) is uniformly elliptic. Since \( w_{e, \rho} = 0 \) on \( T_i \cap \Omega \) it follows from the maximum principle that \( w_{e, \rho} > 0 \in \Sigma(\lambda) \) and \( (w_{e, \rho})_i > 0 \) on \( T_i \). But on \( T_i \), \( (w_{e, \rho})_i = -2(u_{e, \rho})_i \), and the lemma is proved.

Now, by Lemmas A.3 and A.4, we give the proof of Theorem A.2 using an idea similar to that of [14]. In fact, if

\[
\lambda^{**} = \inf \{ \lambda \leq \lambda_0; (u_{e, \rho})_1 < 0, u_{e, \rho}(x) < u_{e, \rho}(x^*) \text{ for } x \in \Sigma(\lambda) \},
\]

it follows from the arguments of [14] that at least one of the following occurs:

(i) \( \Sigma'(\lambda^{**}) \) becomes internally tangent to \( \partial \Omega \) at some point \( P \) not on \( T^{**} \),

(ii) \( T^{**} \) is orthogonal to \( \partial \Omega \) at some point \( Q \in T^{**} \cap \partial \Omega \).

Note that \( \lambda^{**} \) is independent of \( \varepsilon \) and \( \rho \). The proof of Theorem A.2 now follows from the compactness of \( \bar{\Omega} \).

In the proof of the following theorem we use \( C \) and \( \sigma \) to denote positive constants which are independent of \( \varepsilon \) but may change line to line.

**Theorem A.5.** Assume that \( p > 2 \) and \( h \in C^1((0, \infty)) \cap C^0([0, \infty)) \) satisfies

\[
(\text{H}_1) \quad h(0) > 0, \quad \lim_{t \to 0^+} h(s)/s^{p-1} > 0.
\]

\[
(\text{H}_2) \quad \text{There are exactly three numbers } 0 < z_1 < z_2 < z_3 \text{ such that } h(z_i) = 0 \text{ for } i = 1, 2, 3, \text{ there exists } \delta > 0 \text{ and } M > 0 \text{ sufficiently large such that } h'(s) < 0 \text{ for } s \in (z_3 - \delta, z_1) \text{ and } h(s) \leq M(z_1 - s)^{p-1} \text{ for } s \in [0, z_1].
\]

\[
(\text{H}_3) \quad \text{There exists } \delta > 0 \text{ such that } h'(s) < 0 \text{ for } s \in (z_1 - \delta, z_1 + \delta) \setminus \{z_1\}, \quad \lim_{s \to z_1^-} h(s)/(s - z_1)^{p-1} = -m_1 < 0, \quad \text{and } h(s) \leq M(z_1 - s)^{p-1} \text{ for } s \in [0, z_1].
\]

\[
(\text{H}_4) \quad \int_{z_1}^{z_2} h(s) \, ds > 0.
\]

Then for a given \( \gamma > 0 \), there exists \( \bar{\varepsilon} > 0 \) such that for \( 0 < \varepsilon < \bar{\varepsilon} \), the ordinary differential equation

\[
\varepsilon |w'|^{p-2} w' + h(w) = 0 \quad \text{in } (0, \gamma), \quad w'(0) = 0, \quad w(\gamma) = z_1 \tag{A.4}
\]

possesses a positive solution \( w_\varepsilon(x) \) with the following properties:

(i) \( w_\varepsilon(0) \to \mu \text{ as } \varepsilon \to 0, \quad w_\varepsilon(0) \in (\mu, z_3), \quad w_\varepsilon'(x) < 0 \text{ for } x \in (0, \gamma), \)

where \( \mu \in (z_2, z_3) \) is the unique point such that \( \int_{z_1}^{z_2} h(s) \, ds = 0. \)
(ii) \(|w_e - z_1| < Ce^{-\sigma x^{1/p}}\) in any closed interval in \((0, \gamma]\), where \(C > 0, \sigma > 0\) are independent of \(e\). Moreover,
\[|w'_e(\gamma)| \leq (C/\varepsilon) e^{-\sigma x^{1/p}}.\]  
(A.5)

**Proof.** Let \(\tilde{h}(s) = h(s + z_1)\). It follows from the assumptions on \(h\) that \(z_2 - z_1, z_3 - z_1\) are the only two positive zeros of \(\tilde{h}\). Moreover,
\[\int_0^{\gamma + 1} \tilde{h}(s) ds = 0.\]
Now we consider the initial value problem
\[(|v'_e|^p - 2 v'_e)' + \tilde{h}(v) = 0 \quad \text{in} \ (0, R), \quad v(0) = \alpha, \ v'(0) = 0. \]  
(A.6)

Since \(\tilde{h}\) satisfies the conditions (F1)–(F3) (with \(r_1 = z_2 - z_1, r_2 = z_3 - z_1\)), it is known from [20, 21] that there exists a unique solution \(v_\alpha\) of (A.6) for \(\alpha \in [\mu - z_1, z_3 - z_1]\).

For \(\alpha \in [\mu - z_1, z_3 - z_1]\) define
\[I_\pm = \{\alpha: \exists 0 < R < \infty \text{ such that } v_\alpha(R) = 0\},\]
\[I_0 = \{\alpha: v_\alpha(x) > 0, v'_\alpha(x) < 0 \text{ for every } x > 0 \text{ and } \lim_{x \to \infty} v_\alpha(x) = 0\},\]
\[I_\ell = \{\alpha: \exists R > 0 \text{ such that } v'_\alpha(R) = 0 \text{ and } 0 < v_\alpha(R) < \alpha\}.
\]

From the first integral of the equation in (A.6), we see that
\[I_\pm \cup I_\ell \cup I_0 = [\mu - z_1, z_3 - z_1], \quad I_\ell = (\mu - z_1, z_3 - z_1), \quad I_0 = \{\mu - z_1\}.
\]
This implies that \(v_{\mu - z_1}\) satisfies \(\lim_{x \to \infty} v_{\mu - z_1}(x) = 0\). By arguments similar to [20], we see that
\[\lim_{x \to \infty} \sup v_{\mu - z_1}(x) e^{\left(\frac{m_1 - \varepsilon}{p - 1}\right)x} < \infty \]  
(A.7)
and
\[\lim_{x \to \infty} \frac{v'_{\mu - z_1}(x)}{v_{\mu - z_1}(x)} = -(m_1/(p - 1))^{1/p} \]  
(A.8)
for any \(\eta \in (0, m_1/(p - 1))\). This implies that \(v_{\mu - z_1}\) decays exponentially fast at \(\infty\).

Setting \(y_\alpha(x) = z_1 + v_\alpha(x)\), \(y_\alpha\) is a solution of
\[(|y'|^p - 2 y')' + h(y) = 0 \quad y(0) = z_1 + \alpha, \quad y'(0) = 0. \]  
(A.9)

Moreover, \(y'_{\mu - z_1}(x) < 0 \) for \(x \in (0, \infty)\) with \(y_{\mu - z_1}(0) = \mu\) and \(y_{\mu - z_1} \to z_1\) decays exponentially fast at \(\infty\). On the other hand, for \(\alpha \in (\mu - z_1, z_3 - z_1)\), (A.9) has a solution \(y_\alpha\) for which there exists \(R_\alpha\) such that \(y_\alpha(R_\alpha) = z_1\).
and \( y_s > z_i \) in \([0, R_\ast)\) (since \( L = (\mu - z_1, z_3 - z_i)\)). By the continuous dependence of \( y_s \) and \( R_\ast \) on \( \alpha \), we see that \( \lim_{\alpha \to (u-z_i)^{+}} y_s(0) = \mu \). These imply that there exists \( \bar{R} \) sufficiently large such that for any \( R > \bar{R} \), (A.9) has a solution \( y_\beta(x) \) with \( y_\beta(R) = z_1 \) and \( y_\beta(0) = \mu \) as \( R \to \infty \). Defining \( \varepsilon = (R/\gamma)^{-\phi}, \bar{\varepsilon} = (\bar{R}/\gamma)^{-\phi} \), and \( w_\varepsilon(x) = y_\varepsilon((R/\gamma) x) \) for \( 0 < \varepsilon < \bar{\varepsilon} \), we have that \( w_\varepsilon \) is a positive solution of (A.4) and \( \lim_{\varepsilon \to 0} w_\varepsilon(0) = \mu \). This shows (i).

To prove (ii), we need the following sweeping out result.

**Lemma A.6.** Assume that \( p > 2 \) and \( h \) satisfies (H1)-(H4). Let \([a, b]\) be a closed interval in \( \mathbb{R} \) and \( \{v_i \in \mathcal{C}'(0, \gamma) \cap \mathcal{C}^n([0, \gamma]) \) satisfies \( u > 0 \) in \([0, \gamma]\) and \( u'(x) < 0 \) for \( x \in (0, \gamma) \) with \( \max u < z_3 - z_1 \) and

\[
\epsilon(|v_i'|^{p-2} v_i')' + h(z_i + v_i) \geq 0
\]

for all \( t \in [a, b] \); \( v_i \) satisfies \( v_i'(x) \leq 0 \) for \( r \in (0, \gamma) \). If \( u \in \mathcal{C}'((0, \gamma)) \cap \mathcal{C}^n([0, \gamma]) \) satisfies \( u > 0 \) in \([0, \gamma]\) and \( u'(x) < 0 \) for \( x \in (0, \gamma) \) with \( \max u < z_3 - z_1 \) and

\[
\epsilon(|u'|^{p-2} u')' + h(z_1 + u) \leq 0 \quad \text{in } (0, \gamma)
\]

\[
u \geq v_i \quad \text{at } x = \gamma \text{ for all } t \in [a, b] \text{ and}
\]

\[
u \geq v_u \quad \text{in } [0, \gamma]
\]

then \( u \geq v_b \) in \([0, \gamma]\). 

**Proof.** The proof of this lemma is similar to the proof of the generalized sweeping principle of Serrin in [18]. Let \( h(s) = h(z_1 + s) \). By the assumptions on \( h \), there exists \( M > 0 \) such that \( l(s) := h(s) + Ms^{p-1} \) is strictly increasing on \([0, z_3 - z_1]\). Since \( u'(x) < 0, v_i'(x) \leq 0 \) for \( x \in (0, \gamma) \), if \( t \in [a, b] \) is such that \( u \geq v_i \) in \([0, \gamma]\), then we have

\[
-\epsilon(|v_i'|^{p-2} v_i')' + M u^{p-1} \geq l(u) \geq l(v_i) \geq -\epsilon(|v_i'|^{p-2} v_i')' + M v_i^{p-1}.
\]

For any \( x_0 > 0 \) small, we have that

\[
-\epsilon(\mathcal{P}(\xi, u', v_i') (u - v_i')')' + M \mathcal{P}(\xi, u, v_i') (u - v_i) \geq 0 \quad \text{in } [x_0, \gamma], \quad (A.10)
\]

where \( \mathcal{P}(\xi, y, z) = (p-1)(1/\xi |\xi y + (1 - \xi) z|^{p-2} \, d\xi \). The operator in (A.10) is uniformly elliptic. This gives the conclusion of this lemma in \([x_0, \gamma]\). Letting \( x_0 \to 0 \) completes the proof.

Now we give the proof of (ii). We shall show that for any \( x_1 \in (0, \gamma) \),

\[
|w_\varepsilon - z_i| \leq Ce^{-\frac{m_1}{p-1} x_1^p \, e^{-1/R_{\gamma}(1/2)}}.
\]
Define $\bar{w}_e(x) = v_{\mu-z_1}(e^{-1/p}x)$. Then $\bar{w}_e(x) \leq Ce^{-((\mu-z_1)/p-1)x}$ for $x e^{(\mu-z_1)/p-1} \leq 1$. On the other hand, $(w_i - z_1)^{-1} (\mu-z_1) \to 0$ as $\varepsilon \to 0$. Thus, $w_i(x_i/2) - z_1 < \mu - z_1$ for sufficiently small $\varepsilon > 0$, and consequently $w_i(x) - z_1 < \bar{w}_e(x-x_i/2)$ for $x e^{(x_i/2, \gamma]}$ by the sweeping out result in Lemma A.6. In fact, $v_i(x) := w_i(x+t) - z_1$ for $t e^{[0, c]}$ are a class of solutions of

$$e(|v'|^{p-2}v')' + h(z_1 + v) = 0$$

and $v_i(\gamma) = 0$ for $t e^{[0, \gamma]}$ (we assume that $w_i(x) - z_1 \equiv 0$ for $x > \gamma$).

Moreover, $v_i = w_i(x+\gamma) - z_1 \leq \bar{w}_e(x-x_i/2)$ for $x e^{(x_i/2, \gamma]}$. Then, Lemma A.6 implies that $v_i(x) \leq \bar{w}_e(x-x_i/2)$ and thus

$$w_i(x) - z_1 \leq \bar{w}_e(x_i/2) \leq Ce^{-((\mu-z_1)/p-1)(x_i/2)}.$$ 

This also implies that for any closed interval $K \subset (0, \gamma]$ there exists $\sigma > 0$ (depending on $K$) such that for $x e^{K}$,

$$|w_i(x) - z_1| \leq Ce^{-\sigma x}.$$ 

This completes the proof of (ii).

Now we show (A.5). Define $w_i(x) = z_1 + v_i(x)$. Then $v_i$ satisfies the equation

$$e(|v'|^{p-2}v')' + \bar{h}(v_i) = 0 \quad \text{in} \ (0, \gamma).$$ (A.11)

Let $\psi_e = -(|v'_i/v_i|^{p-2}v'_{i}/v_i)$). By a routine calculation, we have

$$\psi_{i}^{\rho/(p-1)} = \frac{\bar{h}(v_i)}{e |v_i|^{p-1} \psi_i^{\rho/(p-1)} + (p-1)}. \quad \text{(A.12)}$$

$$-(p-1)(\psi_i^{\rho/(p-1)})' = \frac{\bar{h}(v_i)}{e |v_i|^{p-1} \psi_i^{\rho/(p-1)} + (p-1)}. \quad \text{(A.13)}$$

Since $\lim_{s \to 0} \bar{h}(s)/s^{p-1} = -m_1 < 0$, $v'_i < 0$ in $(0, \gamma)$, and $v_i \to 0$ in $[x^*/2, x^*]$ as $\varepsilon \to 0$ for any $x^* e^{(\gamma/2, \gamma]}$, there exists $\bar{e} > 0$ such that

$$\frac{\bar{h}(v_i)}{v_i^{p-1}} > -(m_1 + 1) \quad \text{in} \ [x^*/2, x^*]$$

for all $0 < \varepsilon < \bar{e}$. Suppose that $\psi_i^{\rho/(p-1)} \leq (p-1) e/(2(m_1 + 1))$ in $[x^*/2, x^*]$. Then

$$-(p-1)(\psi_i^{\rho/(p-1)})' = -\frac{(m_1 + 1)}{e} \psi_i^{\rho/(p-1)} + (p-1). \quad \text{(A.14)}$$
and thus, 
\[ (\psi_{e^{-1/(p-1)}})'< -\frac{1}{2} \quad \text{in} \quad [x^*/2, x^*]. \] \hspace{1cm} (A.15)

Integrating (A.15) on \((x^*/2, x^*)\), we have 
\[ \psi_{e^{-1/(p-1)}}(x^*) - \psi_{e^{-1/(p-1)}}(x^*/2) < -(x^*/4). \] \hspace{1cm} (A16)

This is a contradiction since under our assumption the left hand side of (A.16) tends to 0 as \(e \to 0\). This also implies that there exists \( \bar{x} \in (x^*/2, x^*) \) such that 
\[ \psi_{e^{-p/(p-1)}}(\bar{x}) > \frac{(p-1)\, e}{2(m_1+1)}. \] \hspace{1cm} (A17)

Then we have 
\[ |v'(\bar{x})|^p < \frac{2(m_1+1)}{(p-1)\, e} \left( v_x(\bar{x}) \right)^p < (C/e)^p \, e^{-\frac{C}{p-1}} \, e^{-1/p} \left( \frac{m_1}{p-1} \right)^{1/p}, \] \hspace{1cm} (A18)

since \( \bar{x} > x^*/2 > \gamma/4 \). As \( x^* \in (\gamma/2, \gamma) \) is arbitrary and \( |v'(\gamma)|^p < |v'_{x}(x)|^p \) when \( x \) is near \( \gamma \) (note that \( h(s) < 0 \) for \( s \in (0, z_2 - z_1) \)) then (A.18) implies (A.5). This completes the proof of Theorem A.5.

Choosing \( z_1 \) in Theorem A.5 to be 0, the following corollary is easily obtained.

**Corollary A.7.** Assume \( p > 2 \) and \( f \) satisfies (F1)–(F4). Then for a given \( \gamma > 0 \), there exists \( \bar{e} > 0 \) such that for \( 0 < e < \bar{e} \), the ordinary differential equation
\[ \varepsilon |w|^{p-2} w' + f(w) = 0 \quad \text{in} \quad (0, \gamma), \quad w'(0) = 0, \quad w(\gamma) = 0 \] \hspace{1cm} (A.19)
possesses a positive solution \( w_\varepsilon(x) \) with the following properties:

(i) \( w_\varepsilon(0) \to \bar{\mu} \) as \( \varepsilon \to 0 \), \( w_\varepsilon(x) \in (\bar{\mu}, \rho_2) \), \( w_\varepsilon'(x) < 0 \) for \( x \in (0, \gamma) \), where \( \bar{\mu} \in (\rho_1, \rho_2) \) is the unique point such that \( \int_0^\gamma f(s) \, ds = 0 \).

(ii) \( |w_\varepsilon| \leq C \, e^{-\varepsilon \sigma} \) in any closed interval in \((0, \gamma]\). Moreover,
\[ |w_\varepsilon'(\gamma)| \leq \frac{(C/\varepsilon)^{\sigma}}{e^{-\varepsilon^{-1/p}}}, \] \hspace{1cm} (A.20)
where \( C \) and \( \sigma \) are independent of \( \varepsilon \).
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