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On asymptotic normality of sequential LS-estimate for unstable autoregressive process AR(2)

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ABSTRACT

For estimating parameters in an unstable AR(2) model, the paper proposes a sequential least squares estimate with a special stopping time defined by the trace of the observed Fisher information matrix. It is shown that the sequential LSE is asymptotically normally distributed in the stability region and on its boundary in contrast to the usual LSE, having six different types of asymptotic distributions on the boundary depending on the values of the unknown parameters. The asymptotic behavior of the stopping time is studied.

1. Introduction

Let (x_n) be observations of an autoregressive AR(2) model

$$x_n = \theta_1 x_{n-1} + \theta_2 x_{n-2} + \varepsilon_n, \quad n = 1, 2, \ldots. \tag{1.1}$$

Throughout this paper we assume that (ε_n) is a sequence of independent identically distributed (i.i.d.) random variables with $\mathbf{E}\varepsilon_1 = 0$, $0 < \mathbf{E}\varepsilon_1^2 = \sigma^2 < \infty$, and $x_0 = x_{-1} = 0$. The variance σ^2 is known (or unknown in Section 4). The process (1.1) is assumed to be unstable, that is, both roots of the characteristic polynomial

$$\mathcal{P}(z) = z^2 - \theta_1 z - \theta_2 \tag{1.2}$$

lie on or inside the unit circle. The model (1.1) is a particular case of unstable autoregressive processes AR(p) which have been studied by many authors due to their applications in automatic control, identification and in modeling economic and financial time series (we refer the reader to Anderson [2], Ahtola and Tiao [1], Dickey and Fuller [5], Chan and Wei [4], Greenwood and Shiryaev [8], Greenwood and Wefelmeyer [9], Jeganathan [11,12], Rao [16] for details and further references).

A commonly used estimate of parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ is the least squares estimate (LSE)

$$\boldsymbol{\theta}(n) = (\theta_1(n), \theta_2(n))' = M_n^{-1} \sum_{k=1}^{n} X_{k-1} x_k, \quad M_n = \sum_{k=1}^{n} X_{k-1} X_{k-1}', \tag{1.3}$$

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where $X_k = (x_k, x_{k-1})'$; the prime denotes the transpose; M_n^{-1} denotes the inverse of matrix M_n if $\det M_n > 0$ and $M_n^{-1} = 0$ otherwise.

It is well known that

$$\sqrt{n}(\boldsymbol{\theta}(n) - \boldsymbol{\theta}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, F), \quad \text{as } n \to \infty,$$

for all $\boldsymbol{\theta} \in \Lambda$, where Λ is the stability region of process (1.1), that is,

$$\Lambda = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' : -1 + \theta_2 < \theta_1 < 1 - \theta_2, \ |\theta_2| < 1\}, \tag{1.4}$$

$F = F(\boldsymbol{\theta})$ is a positive definite matrix (see, e.g., [2, Th. 5.5.7]), $\xrightarrow{\mathcal{L}}$ indicates convergence in law. If $\boldsymbol{\theta}$ belongs to the boundary $\partial\Lambda$ of the stability region Λ, the limiting distribution of LSE is no longer normal. Moreover, there is no one universal limiting distribution for all $\boldsymbol{\theta} \in \partial\Lambda$ and the corresponding set of limiting distributions numbers 6 different types depending on the values of roots z_1 and z_2 of the polynomial (1.2). Each limiting distribution of LSE on the boundary coincides with that of the ratio of certain Brownian functionals (we refer the reader to the paper of Chan and Wei [4] for general results on the limiting distributions of the least squares estimates for unstable AR(p) processes and further details).

As is known, a similar situation arises in the case of AR(1) process

$$x_n = \theta x_{n-1} + \varepsilon_n, \tag{1.5}$$

for which the limiting distributions of the least squares estimate are not normal at the end-points $\theta = \pm 1$ of stability interval $(-1, 1)$ (see White [18], Lai and Siegmund [14]). Lai and Siegmund [14] for a first order non-explosive autoregressive process (1.5) proposed to use a sequential sampling scheme and proved that the sequential least squares estimate for θ with the stopping time based on the observed Fisher information is asymptotically normal uniformly in $\theta \in [-1, 1]$ in contrast with the ordinary LSE.

In the paper we develop a sequential sampling scheme for estimating parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ in model (1.1). We will use the sequential least squares estimate defined by the formula

$$\boldsymbol{\theta}(\tau(h)) = M_{\tau(h)}^{-1} \sum_{k=1}^{\tau(h)} X_{k-1} x_k, \tag{1.6}$$

where $\tau = \tau(h)$ is the stopping time for the threshold $h > 0$:

$$\tau = \tau(h) = \inf\left\{n \ge 1 : \sum_{k=1}^{n} (x_{k-1}^2 + x_{k-2}^2) \ge h\sigma^2\right\}, \qquad \inf\{\emptyset\} = +\infty. \tag{1.7}$$

This construction of sequential estimate is similar to that proposed in the paper of Lai and Siegmund for AR(1) which is defined as

$$\hat{\theta}_\tau = \left(\sum_{k=1}^{\tau} x_{k-1}^2\right)^{-1} \sum_{k=1}^{\tau} x_{k-1} x_k, \qquad \tau = \inf\left\{n \ge 1 : \sum_{k=1}^{n} x_{k-1}^2 \ge h\sigma^2\right\}. \tag{1.8}$$

It should be noted, however, that the first factor in (1.6) is a random matrix and not a random variable, as in (1.8), and this makes additional difficulties.

For AR(1) the stopping time (1.8) turns the denominator in the estimate (1.8) practically into a constant $h\sigma^2$ and this allows one to use the central limit theorem for martingales. In the case of AR(2) the stopping time (1.7) enables one to control the inverse matrix $M_{\tau(h)}^{-1}$ in (1.6) only partially since it remains random. Nevertheless, we will see that such a change of time also enables one to improve the properties of the estimate (1.3).

In our paper (2006) we proved the following result.

Theorem 1.1. *For any compact set $K \subset \Lambda_1$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} \left|\mathbf{P}_{\boldsymbol{\theta}}\left(M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \le \boldsymbol{t}\right) - \Phi_2(\boldsymbol{t}/\sigma)\right| = 0,$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2)$, Φ is the standard normal distribution function,

$$\Lambda_1 = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' : \ -1 + \theta_2 < \theta_1 < 1 - \theta_2, \ -1 \le \theta_2 < 1\}, \qquad \boldsymbol{t} = (t_1, t_2)'.$$

This theorem implies, in particular, that estimate (1.6) is asymptotically normal not only inside the stability region (1.4) but also on the part of its boundary $\{\boldsymbol{\theta} = (\theta_1, -1)' : -2 < \theta_1 < 2\}$ in contrast to the LSE (1.3).

The goal of this paper is to prove the asymptotic normality of the estimate (1.6) and (1.7) in the whole region $[\Lambda]$ including its boundary $\partial\Lambda$.

Our main result (Theorem 3.1) claims that, as $h \to \infty$,

$$M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2 I), \tag{1.9}$$

for any $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ inside the stability region Λ (1.4) and on its boundary $\partial\Lambda$, where I is the identity matrix. Thus the sequential estimate (1.6) and (1.7) has a unique normal asymptotic distribution in the closure $[\Lambda]$ of the stability region (1.4). It will be observed that the normalizing factor $M_{\tau(h)}^{1/2}$ and the limiting distribution in (1.9) remain the same in the whole region $[\Lambda]$ in contrast to the case of the LSE (1.3), which has seven different limiting distributions in $[\Lambda]$. The convergence of the sequential estimate (1.6) and (1.7) to the normal distribution in (1.9) is not uniform in $\boldsymbol{\theta}$ for $\boldsymbol{\theta} \in [\Lambda]$. It can be explained by the fact that in the case, when the polynomial (1.2) has one root inside and the other on the unit circle, the rates of information provided by sample values x_n about the unknown parameters θ_1 and θ_2 may differ greatly.

Theorem 3.1 permits setting up tests of hypotheses about $\boldsymbol{\theta}$ and forming asymptotic confidence regions for $\boldsymbol{\theta}$ on the basis of standard normal distribution for the sequential estimate (1.6). It will be noted that solving the above problems on the basis of limiting distributions for LSE (1.3) is complicated by the fact that one needs some knowledge about the location of unknown parameters (see [4] for details) to determine both the normalizing factor for the estimate deviation and the appropriate limiting distribution.

In the paper we study the estimation problem for the unstable AR(2) process and show that one can reduce the number of limiting distributions in the least squares method to one by making use of a special stopping time. It is difficult to conjecture the result of applying the proposed stopping rule for the general AR(p) case but one can hope that it may also be useful in overcoming the drastic differences of behavior of the AR(p) process at different 'unstable' points on the boundary of the stability region.

The remainder of this paper is arranged as follows. Section 2 gives the asymptotic distribution of the stopping time (1.7) (Theorem 2.1) and some properties of the observed Fisher information matrix. In Section 3 the asymptotic normality of sequential estimate (1.6) for unstable AR(2) model is established (Theorem 3.1). Section 4 proposes the sequential estimation scheme for the case of unknown variance σ^2 in model (1.1). Section 5 contains some technical results.

2. Properties of the stopping time $\tau(h)$ and the observed Fisher information matrix M_n

In this section the attention is mainly focused on the case when the unknown parameter $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ belongs to the boundary $\partial\Lambda$ of the stability region (1.4). The boundary $\partial\Lambda$ includes three sides:

$$\begin{aligned} &\Gamma_1 = \{\boldsymbol{\theta} : -\theta_1 + \theta_2 = 1, -2 < \theta_1 < 0\}, \qquad \Gamma_2 = \{\boldsymbol{\theta} : \theta_1 + \theta_2 = 1, 0 < \theta_1 < 2\}, \\ &\Gamma_3 = \{\boldsymbol{\theta} : -2 < \theta_1 < 2, \theta_2 = -1\} \end{aligned} \tag{2.1}$$

and three apexes $(0, 1)$, $(-2, -1)$, $(2, -1)$. Denote

$$A = \begin{pmatrix} \theta_1 & \theta_2 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$W^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} \varepsilon_i, \qquad W_1^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} (-1)^i \varepsilon_i, \quad 0 \le t \le 1, \tag{2.2}$$

and introduce the following functionals

$$\begin{aligned} &\mathcal{J}_1(x; t) = \int_0^t x^2(s)\mathrm{d}s, \qquad \mathcal{J}_2(x; t) = \int_0^t \left(\int_0^s x(u)\mathrm{d}u\right)^2 \mathrm{d}s, \\ &\mathcal{J}_3(x; y; t) = \int_0^t (x^2(s) + y^2(s))\mathrm{d}s, \qquad \mathcal{J}_4(x; t) = \left(\int_0^t x(s)\mathrm{d}s\right)^2. \end{aligned} \tag{2.3}$$

Theorem 2.1. *Let $\tau(h)$ be defined by (1.7), a and b be real roots of the polynomial (1.2), $-1 \le a < b \le 1$. Then, for each $\boldsymbol{\theta} \in \Lambda$,*

$$\mathbf{P}_{\boldsymbol{\theta}} - \lim_{h\to\infty} \tau(h)/h = 1/\mathrm{tr}\, F, \quad F - AFA' = B. \tag{2.4}$$

Moreover, for each $\boldsymbol{\theta} \in \partial\Lambda$, as $h \to \infty$,

$$\frac{\tau(h)}{\psi(\boldsymbol{\theta}, h)} \xrightarrow{\mathcal{L}} \begin{cases} \nu_1(W_1) = \inf\{t \ge 0 : \mathcal{J}_1(W_1; t) \ge 1\} & \text{if } \boldsymbol{\theta} \in \Gamma_1, \\ \nu_2(W) = \inf\{t \ge 0 : \mathcal{J}_1(W; t) \ge 1\} & \text{if } \boldsymbol{\theta} \in \Gamma_2, \\ \nu_3(W, W_1) = \inf\{t \ge 0 : \mathcal{J}_3(W; W_1; t) \ge 1\} & \text{if } \boldsymbol{\theta} \in \Gamma_3 \cup \{(0, 1)\}, \\ \nu_4(W) = \inf\{t \ge 0 : \mathcal{J}_2(W; t) \ge 1\} & \text{if } \boldsymbol{\theta} = (2, -1), \\ \nu_5(W_1) = \inf\{t \ge 0 : \mathcal{J}_2(W_1; t) \ge 1\} & \text{if } \boldsymbol{\theta} = (-2, -1), \end{cases} \tag{2.5}$$

where $\inf\{\emptyset\} = \infty$, Λ *is defined in* (1.4),

$$\psi(\boldsymbol{\theta}, h) = \begin{cases} (1+b)\sqrt{h/2} & \text{if } \boldsymbol{\theta} \in \Gamma_1, \\ (1-a)\sqrt{h/2} & \text{if } \boldsymbol{\theta} \in \Gamma_2, \\ \sqrt{2h}\sin\varphi & \text{if } \boldsymbol{\theta} = (2\cos\varphi, -1)' \in \Gamma_3, \\ \sqrt{2h} & \text{if } \boldsymbol{\theta} = (0, 1), \\ (h/2)^{1/4} & \text{if } \boldsymbol{\theta} \in \{(-2, -1), (2, -1)\}, \end{cases} \tag{2.6}$$

$W(t)$, $W_1(t)$ *are independent standard Brownian motions.*

Proof. Assertion (2.4) easily follows from Lemma 3.12 in [6].

For $\boldsymbol{\theta} \in \partial\Lambda$ we decompose the original process (1.1) into two processes $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ using the transformation

$$QX_k = (u_k, v_k)', \tag{2.7}$$

where Q is a non-degenerate constant matrix of size 2×2 which will be chosen later depending on the values of $\boldsymbol{\theta}$. The limiting relation (2.5) for $\boldsymbol{\theta} \in \cup_{i=1}^{3} \Gamma_i$ has been proved in [7], Theorem 2.2. It remains to consider the apexes $(2, -1)$, $(-2, -1)$, $(0, 1)$. Denote

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & -b \\ 1 & -a \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2.8}$$

For $\boldsymbol{\theta} = (2, -1)$, making use of Q_1 in (2.7) one obtains

$$v_k = \sum_{j=1}^{k} \varepsilon_j, \qquad u_k = \sum_{j=1}^{k} (x_j - x_{j-1}) = \sum_{j=1}^{k} v_j = \sum_{j=1}^{k} \sum_{i=1}^{j} \varepsilon_i,$$

$$\sum_{k=1}^{n} \|X_{k-1}\|^2 = \sum_{k=1}^{n} u_{k-1}^2 + \sum_{k=1}^{n} u_{k-2}^2 = 2\sum_{k=1}^{n} u_{k-1}^2 - u_{n-1}^2. \tag{2.9}$$

By the definition of $\tau(h)$ in (1.7), one gets

$$\begin{aligned} \mathbf{P}_{\boldsymbol{\theta}}\left\{\tau(h) \leq th^{1/4}\right\} &= \mathbf{P}_{\boldsymbol{\theta}}\left\{\sum_{k=1}^{[th^{1/4}]} \|X_{k-1}\|^2 \geq h\sigma^2\right\} \\ &= \mathbf{P}_{\boldsymbol{\theta}}\left\{\frac{2}{h\sigma^2}\sum_{k=1}^{[th^{1/4}]} u_{k-1}^2 - \frac{1}{h\sigma^2} u_{[th^{1/4}]-1}^2 \geq 1\right\}. \end{aligned} \tag{2.10}$$

Further we show (by the argument similar to that in the proof of Lemma 2.3 in Section 5) that the sum

$$S_n(t) = \frac{1}{n^4\sigma^2}\sum_{k=1}^{[nt]} u_{k-1}^2 = \mathcal{J}_2(W^{(n)}; t) + g^{(n)}(t),$$

where $g^{(n)}(t)$ is a random process such that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|g^{(n)}(t)| > \delta) = 0.$$

Now we check that

$$\lim_{n\to\infty} u_n^2/n^4 = 0 \quad \mathbf{P}_{\boldsymbol{\theta}}\text{-a.s.} \tag{2.11}$$

By the Cauchy inequality and the law of iterated logarithms we have

$$u_n^2/n^4 \leq n^{-3}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\varepsilon_i\right)^2, \qquad \sum_{k=1}^{n}\frac{1}{k^3}\left(\sum_{j=1}^{k}\varepsilon_i\right)^2 < \infty \quad \mathbf{P}_{\boldsymbol{\theta}}\text{-a.s.}$$

These inequalities, by virtue of the Kronecker Lemma, imply (2.11).

From here and (2.10) and (2.11), we obtain

$$\mathbf{P}_{\boldsymbol{\theta}}\left(\tau(h)/\psi(\boldsymbol{\theta}, h) \leq t\right) = \mathbf{P}_{\boldsymbol{\theta}}(\nu_{\boldsymbol{\theta}}^{(n)} \leq t) + \beta_{\boldsymbol{\theta}}(h),$$

where $\nu_{\boldsymbol{\theta}}^{(n)} = \inf\{t \geq 0 : \mathcal{J}_2(W^{(n)}; t) \geq 1\}$, $\lim_{h\to\infty} \beta_{\boldsymbol{\theta}}(h) = 0$.

$W^{(n)}(t)$ is given in (2.2). This, by the functional Donsker theorem (see [3]), leads to (2.5) for $\boldsymbol{\theta} = (2, -1)$.

The case of the apexes $(0, 1)$, $(-2, -1)$ can be considered similarly with the use of Theorem 5.14 given in Section 5. Hence Theorem 2.1. □

Now we will establish some properties of the observed Fisher information matrix M_n. Introduce the following subsets of the closed region $[\Lambda]$:

$$\Lambda_d = [\Lambda] \setminus \bigcup_{i=1}^{2} B_i, \quad \Lambda_d = \Lambda_{d,1} + \Lambda_{d,2}; \tag{2.12}$$
$$\Lambda_{d,1} = \Lambda_d \cap V_d, \qquad \Lambda_{d,2} = \Lambda_d \setminus \Lambda_{d,1};$$
$$V_d = \left\{ \boldsymbol{\theta} : -2 + \frac{d}{\sqrt{2}} \le \theta_1 \le 0, \ \frac{-\theta_1^2}{4} + \frac{d^2}{8} < \theta_2 \le 1 + \theta_1 \right\}$$
$$\cup \left\{ \boldsymbol{\theta} : 0 \le \theta_1 \le 2 - \frac{d}{\sqrt{2}}, \ \frac{-\theta_1^2}{4} + \frac{d^2}{8} \le \theta_2 \le 1 - \theta_1 \right\};$$

B_i are open balls of radius $d > 0$ centered at the apexes $(-2, -1)$, $(2, -1)$.

In view of Theorem 1.1, it suffices to study the properties of M_n only for the parametric subset $\Lambda_{d,1}$ and the apexes $(-2, -1)$, $(2, -1)$. In the case of $\Lambda_{d,1}$, one can use the transformation (2.7) with the matrix Q_2 and $-1 \le a < b \le 1$. Substituting (2.7) and Q_2 in M_n (1.3) yields

$$M_n = Q^{-1} S_n (Q')^{-1} = Q^{-1} R_n^{-1} J_n R_n^{-1} (Q')^{-1}, \tag{2.13}$$

where $J_n = R_n S_n R_n$, $R_n = \mathrm{diag}\left((u,u)_n^{-1/2}, \quad (v,v)_n^{-1/2}\right)$,

$$J_n = \begin{pmatrix} 1 & \xi_n \\ \xi_n & 1 \end{pmatrix}, \qquad S_n = \begin{pmatrix} (u,u)_n & (u,v)_n \\ (u,v)_n & (v,v)_n \end{pmatrix}; \tag{2.14}$$

$$\xi_n = (u,u)_n^{-1/2} (v,v)_n^{-1/2} (u,v)_n, \quad (u,v)_n = \sum_{k=1}^{n} u_{k-1} v_{k-1}. \tag{2.15}$$

Proposition 2.2. *For any $d > 0$ and $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(\| J_{\tau(h)} - T(\theta_1, \theta_2) \| > \delta \right) = 0; \tag{2.16}$$

$$T(\theta_1, \theta_2) = \begin{pmatrix} 1 & r(a,b) \\ r(a,b) & 1 \end{pmatrix}, \quad r(a,b) = \frac{\sqrt{1-a^2}\sqrt{1-b^2}}{1-ab}. \tag{2.17}$$

The proof of Proposition 2.2 is given in Section 5.

Further we consider the asymptotic behavior of the matrix J_n in the extreme cases when the process x_k is "most" unstable, that is, $\boldsymbol{\theta}$ coincides with one of the apexes $(-2, -1)$, $(2, -1)$ of the parametric region $[\Lambda]$.

For $\boldsymbol{\theta} = (2, -1)$ we take the matrix Q_1 from (2.8). This yields

$$u_k = \sum_{j=0}^{k} \sum_{i=0}^{j} \varepsilon_i, \qquad v_k = \sum_{j=0}^{k} \varepsilon_j, \quad k \ge 1, \ u_0 = v_0 = \varepsilon_0 = 0. \tag{2.18}$$

For $\boldsymbol{\theta} = (-2, -1)$ we take the matrix Q_3 from (2.8). This implies

$$u_k = (-1)^k \sum_{j=1}^{k} \sum_{i=1}^{j} (-1)^i \varepsilon_i, \qquad v_k = \sum_{j=1}^{k} (-1)^j \varepsilon_j.$$

Lemma 2.3. *Let ξ_n be given by (2.15) and $\boldsymbol{\theta} \in \{(-2, -1), (2, -1)\}$. Then*

$$\xi_n \overset{\mathcal{L}}{\Longrightarrow} \begin{cases} \varphi(W) & \text{if } \boldsymbol{\theta} = (2, -1), \\ \varphi(W_1) & \text{if } \boldsymbol{\theta} = (-2, -1), \end{cases} \qquad \text{as } n \to \infty; \tag{2.19}$$

$$\varphi(W) = 2^{-1} \mathcal{J}_2^{-1/2}(W; 1) \mathcal{J}_1^{-1/2}(W; 1) \mathcal{J}_4(W; 1). \tag{2.20}$$

The proof of Lemma 2.3 is given in Section 5.

3. Asymptotic normality

It is known that the sequential least squares estimate (1.6) and (1.7) is asymptotically normal just like the ordinary LSE for any value of $\boldsymbol{\theta}$ in the stability region Λ. Moreover, according to Theorem 1.1, this convergence of sequential LSE to normal law is uniform in $\boldsymbol{\theta}$ belonging to any compact set in Λ supplemented with the part of its boundary corresponding to complex roots of the polynomial (1.2). In this section, we will show that in contrast with the ordinary LSE (cf. Chen and Wei [4]), the sequential LSE is asymptotically normal also on the boundary $\partial\Lambda$ of the stability region Λ.

Theorem 3.1. *Let $\tau(h)$, $\boldsymbol{\theta}(\tau(h))$ and $M_{\tau(h)}$ be as in (1.3), (1.6) and (1.7). Then*

$$\lim_{h\to\infty}\sup_{\mathbf{t}\in R^2}\left|\mathbf{P}_{\boldsymbol{\theta}}\left(M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h))-\boldsymbol{\theta})\le\mathbf{t}\right)-\Phi_2(\mathbf{t}/\sigma)\right|=0, \tag{3.1}$$

for all $\boldsymbol{\theta}\in[\Lambda]$, where $\Phi_2(\mathbf{t})=\Phi(t_1)\Phi(t_2)$, $\mathbf{t}=(t_1,t_2)'$, Φ is the standard normal distribution function; $[\Lambda]$ is the closure of the stability region (1.4).

Proof. By Theorem 1.1, we show (3.1) for $\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1),(-2,-1),(2,-1)\}$. If $\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}$, the minimal and the maximal roots a and b of the polynomial (1.2) satisfy the inequalities $-1\le a<b\le 1$. Using the transformation (2.7) with Q_2 we decompose the original process (1.1) into two processes (u_k) and (v_k) which obey the equations

$$u_k=au_{k-1}+\varepsilon_k,\qquad v_k=bv_{k-1}+\varepsilon_k,\qquad u_0=v_0=0. \tag{3.2}$$

Since the matrix Q_2 is non-degenerate, (2.13) implies

$$M_n^{1/2}=Q_2^{-1}R_n^{-1}J_n^{1/2}. \tag{3.3}$$

Substituting this matrix in the standardized deviation of the sequential estimate (1.6), one gets

$$M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h))-\boldsymbol{\theta})=J_{\tau(h)}^{-1/2}R_{\tau(h)}\sum_{k=1}^{\tau(h)}Q_2X_{k-1}\varepsilon_k=J_{\tau(h)}^{-1/2}\mathbf{Z}_{\tau(h)}, \tag{3.4}$$

where $\mathbf{Z}_n=\left((u,u)_n^{-1/2}\sum_{k=1}^n u_{k-1}\varepsilon_k,(v,v)_n^{-1/2}\sum_{k=1}^n v_{k-1}\varepsilon_k\right)'$. Further we note that Proposition 2.2 implies that, for any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}}\mathbf{P}_{\boldsymbol{\theta}}\left(\|J_{\tau(h)}^{-1/2}-I\|>\delta\right)=0. \tag{3.5}$$

Therefore in order to prove (3.1) for $\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}$ it suffices to establish the following result.

Proposition 3.2. *Let $\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}$. Then, for each constant vector $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)'\in R^2$ with $\|\boldsymbol{\lambda}\|=1$, the random variable*

$$Y_h=\boldsymbol{\lambda}'\mathbf{Z}_{\tau(h)}/\sigma \tag{3.6}$$

is asymptotically normal with mean 0 and unit variance, as $h\to\infty$, that is,

$$\lim_{h\to\infty}\sup_{t\in R}|\mathbf{P}_{\boldsymbol{\theta}}(Y_h\le t)-\Phi(t)|=0.$$

The main difficulty in the analysis of Y_h is that the stopping time (1.7) enables one to control the sums $(u,u)_{\tau(h)}$, $(v,v)_{\tau(h)}$ in the denominators of (3.6) only partially because one of them or both are random variables even in the asymptotic as $h\to\infty$.

The proof of Proposition 3.2 is given in Section 5. The key idea of the proof is to replace Y_h by a more tractable random variable $\tilde{Y}_h$ equivalent to Y_h in distribution by making use of the Skorohod coupling theorem and then apply the Central Limit Theorem for martingales. Section 5 contains also the proof of Theorem 3.1 for the case of $\boldsymbol{\theta}\in\{(-2,-1),(2,-1)\}$. This case is considered separately because the matrix J_n in (3.3) converges, according to Lemma 2.3, only in distribution. □

4. Case of unknown variance

In this section, we extend the sequential estimation scheme to model (1.1) with unknown variance. It is shown that the sequential least squares estimate modified to embrace this case remains asymptotically normal uniformly in $\boldsymbol{\theta}$ for any compact set in the region $\Lambda_1=\Lambda\cup\Gamma_3$ (Theorem 4.1) and it is asymptotically normal in the closure of the stability region $[\Lambda]$ (Theorem 4.2).

Suppose that the variance σ^2 in (1.1) is unknown. A commonly used estimate for σ^2 in autoregressive processes on the basis of observations $(x_1,\ldots,x_n)$ is defined as

$$\hat{\sigma}_n^2=n^{-1}\sum_{k=1}^n(x_k-\boldsymbol{\theta}'(n)X_{k-1})^2, \tag{4.1}$$

where $\boldsymbol{\theta}(n)$ is the least squares estimate of $\boldsymbol{\theta}$ defined in (1.3). Now we must modify the stopping time (1.7). At first sight, to this end one should replace σ^2 in (1.7) by $\hat{\sigma}_n^2$. However, we will use a different modification similar to that proposed by Lai and Siegmund for AR(1) model, which turns out to be more convenient in the theoretical studies. Define the sequential estimate as

$$\boldsymbol{\theta}(\hat{\tau}(h)) = M^{-1}_{\hat{\tau}(h)} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1} x_k, \tag{4.2}$$

$$\hat{\tau}(h) = \inf\left\{ n \geq 3 : \sum_{k=1}^{n} (x^2_{k-1} + x^2_{k-2}) \geq h s^2_n \right\}, \tag{4.3}$$

where $s^2_n = \hat{\sigma}^2_n \vee \delta_n$, δ_n is a sequence of positive numbers with $\delta_n \to 0$.

The main results of this section are stated in the following theorems.

Theorem 4.1. *For any compact set $K \subset \Lambda_1$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} \left| \mathbf{P}_{\boldsymbol{\theta}} \left(M^{1/2}_{\hat{\tau}(h)} (\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta}) / \hat{\sigma}_{\hat{\tau}(h)} \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}) \right| = 0, \tag{4.4}$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2)$, Φ is the standard normal distribution function,

$$\Lambda_1 = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' : -1 + \theta_2 < \theta_1 < 1 - \theta_2, \ -1 \leq \theta_2 < 1\}, \quad \boldsymbol{t} = (t_1, t_2)'.$$

Theorem 4.2. *For any $\boldsymbol{\theta} \in [\Lambda]$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{t}\in R^2} \left| \mathbf{P}_{\boldsymbol{\theta}} \left(M^{1/2}_{\hat{\tau}(h)} (\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta}) / \hat{\sigma}_{\hat{\tau}(h)} \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}) \right| = 0.$$

We omit the proofs of Theorems 4.1 and 4.2 which are similar to those of Theorems 1.1 and 3.1 though they become more laborious.

5. Auxiliary propositions

This section contains the proofs of some results used in this paper.

5.1. Proof of Proposition 2.2

First we prove three lemmas.

Lemma 5.1. *For each $m = 1, 2, \ldots$ and for any $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}}(\tau(h) < m) = 0, \qquad \lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}}(1/\tau(h) > \delta) = 0. \tag{5.1}$$

Proof. These equations follow from the definition of $\tau(h)$ in (1.7). □

Lemma 5.2. *For any $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)} \varepsilon^2_k - \sigma^2 \right| > \delta \right) = 0.$$

Proof. One has $\mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)} \varepsilon^2_k - \sigma^2 \right| > \delta \right)$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}(\tau(h) < m) + \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{n} \sum_{k=1}^{n} \varepsilon^2_k - \sigma^2 \right| > \delta \text{ for some } n \geq m \right).$$

Lemma 5.1 and the strong law of large numbers yields the result. □

Lemma 5.3. *Let $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ be the processes defined in (3.2). Then, for each $d > 0$ and any $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(|\tau(h)(u, u)^{-1}_{\tau(h)} - (1 - a^2)/\sigma^2| > \delta \right) = 0, \tag{5.2}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(|\tau(h)(v, v)^{-1}_{\tau(h)} - (1 - b^2)/\sigma^2| > \delta \right) = 0.$$

Proof. Since these relations are similar, we verify only (5.2). First we show that, for each $d > 0$ and any $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\substack{\boldsymbol{\theta}\in\Lambda_{d,1}, \\ -\theta_1+\theta_2<1}} \mathbf{P}_{\boldsymbol{\theta}} \left(|\tau(h)(u, u)^{-1}_{\tau(h)} - (1 - a^2)/\sigma^2| > \delta \right) = 0. \tag{5.3}$$

Squaring both sides of the first equation in (3.2) and summing give

$$(1-a^2)\sum_{j=1}^{\tau(h)} u_{j-1}^2 = u_0^2 - u_{\tau(h)}^2 + 2a\sum_{j=1}^{\tau(h)} u_{j-1}\varepsilon_j + \sum_{j=1}^{\tau(h)} \varepsilon_j^2.$$

This, in view of (2.16) and the estimate $\sum_{k=1}^{n-1}\varepsilon_k^2 \le 4\sum_{k=1}^{n} u_{k-1}^2$, yields

$$|\tau(h)(u,u)_{\tau(h)}^{-1} - (1-a^2)/\sigma^2| \le \frac{u_{\tau(h)}^2}{\sigma^2 (u,u)_{\tau(h)}} + \frac{2\left|\sum_{k=1}^{\tau(h)} u_{k-1}\varepsilon_k\right|}{\sigma^2 (u,u)_{\tau(h)}} + \frac{\left|\sum_{k=1}^{\tau(h)}(\varepsilon_k^2-\sigma^2)\right|}{(\sigma^2/4)\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2}. \tag{5.4}$$

By Lemma 5.2, we have to show that, for each $d>0$ and any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\theta\in\Lambda_{d,1}\cap\{-\theta_1+\theta_2<1\}} \mathbf{P}_\theta\left(u_{\tau(h)}^2 (u,u)_{\tau(h)}^{-1} > \delta\right) = 0, \tag{5.5}$$

$$\lim_{h\to\infty}\sup_{\theta\in\Lambda_{d,1}\cap\{-\theta_1+\theta_2<1\}} \mathbf{P}_\theta\left(\left|\sum_{k=1}^{\tau(h)} u_{k-1}\varepsilon_k\right| (u,u)_{\tau(h)}^{-1} > \delta\right) = 0. \tag{5.6}$$

One has

$$\mathbf{P}_\theta\left(u_\tau^2 (u,u)_\tau^{-1} > \delta\right) \le \mathbf{P}_\theta(\tau < m) + \mathbf{P}_\theta\left(u_n^2 (u,u)_n^{-1} > \delta \text{ for some } n \ge m\right). \tag{5.7}$$

It is known (see, [14]) that

$$\lim_{m\to\infty}\sup_{|a|\le 1} \mathbf{P}_\theta(u_n^2 (u,u)_n^{-1} > \delta \text{ for some } n\ge m) = 0. \tag{5.8}$$

Applying this and Lemma 5.1 in (5.7) yields (5.5). To prove (5.6) we use the representation

$$\left|\sum_{k=1}^{\tau(h)} u_{k-1}\varepsilon_k\right| \Big/ (u,u)_{\tau(h)} = \zeta_{\tau(h)} \max\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}, \left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/4} \frac{1}{\sqrt[4]{\tau(h)}}\right),$$

where $\zeta_n = |\sum_{k=1}^{n} u_{k-1}\varepsilon_k| / \max\left(n, (u,u)_n^{3/4}\right)$. By Lemmas 5.1 and 5.2 and applying the uniform law of large numbers for martingales (see [14]) one comes to (5.6). Combining (5.4)–(5.5) and Lemma 5.2 one gets (5.3). It remains to show that, for each $d>0$ and $\delta>0$,

$$\lim_{h\to\infty}\sup_{\theta\in\Lambda_{d,1}\cap\{-\theta_1+\theta_2<1\}} \mathbf{P}_\theta(|\tau(h)/(u,u)_{\tau(h)} - (1-a^2)/\sigma^2| > \delta) = 0. \tag{5.9}$$

If $\theta_1+\theta_2=1$, then $a=-1$ and the process u_k in (3.2) satisfies the limiting relation (see, e.g., [15])

$$\liminf_{n\to\infty}\sum_{k=1}^{n} u_k^2/(n^2/\log\log n) = \frac{\sigma^2}{4} \quad \text{a.s.} \tag{5.10}$$

By making use of the inequality

$$\mathbf{P}_\theta\left(\frac{\tau(h)}{(u,u)_{\tau(h)}} > \delta\right) \le P_\theta(\tau < m) + \mathbf{P}_\theta\left(\frac{n}{(u,u)_n} > \delta \text{ for some } n\ge m\right)$$

and (5.10), we come to (5.9). This completes the proof of Lemma 5.3. □

Now we can prove Proposition 2.2. We have to show that, for each $d>0$ and any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\theta\in\Lambda_{d,1}} \mathbf{P}_\theta\left(|\xi_{\tau(h)} - r(a,b)| > \delta\right) = 0. \tag{5.11}$$

Denote

$$\eta_h^{(l)} = A_h \sum_{k=l}^{\tau(h)} u_{k-l} v_{k-l}, \quad l = 1,\dots,\tau(h),\ A_h = (u,u)_{\tau(h)}^{-1/2}(v,v)_{\tau(h)}^{-1/2}. \tag{5.12}$$

From Eq. (3.2), one gets for $l = 1, \dots, \tau(h) - 1$,

$$\sum_{k=l}^{\tau(h)} u_{k-l} v_{k-l} = \sum_{k=l+1}^{\tau(h)} (a u_{k-l-1} + \varepsilon_{k-l})(b v_{k-l-1} + \varepsilon_{k-l})$$
$$= ab \sum_{k=l+1}^{\tau(h)} u_{k-l-1} v_{k-l-1} + a \sum_{k=l+1}^{\tau(h)} u_{k-l-1}\varepsilon_{k-l} + b \sum_{k=l+1}^{\tau(h)} v_{k-l-1}\varepsilon_{k-l} + \sum_{k=l+1}^{\tau(h)} \varepsilon_{k-l}^2.$$

Substituting this in (5.12) yields

$$\eta_h^{(l)} = ab\eta_h^{(l+1)} + z_{\tau(h)-l}, \quad 1 \le l < \tau(h);$$
$$z_{\tau-l} = A_h \left(a \sum_{k=l+1}^{\tau} u_{k-l-1}\varepsilon_{k-l} + b \sum_{k=l+1}^{\tau} v_{k-l-1}\varepsilon_{k-l} + \sum_{k=l+1}^{\tau} \varepsilon_{k-l}^2 \right). \tag{5.13}$$

Putting $\zeta_m = \eta_h^{(\tau(h)-m)}$ one comes to the equation

$$\zeta_m = ab\zeta_{m-1} + z_m, \quad 1 \le m < \tau(h), \qquad \zeta_0 = 0.$$

Solving this equation one finds

$$\xi_{\tau(h)} = \zeta_{\tau(h)-1} = \sum_{j=0}^{\tau(h)-2} (ab)^j z_{\tau(h)-1-j}.$$

Introducing the sums

$$S_m = \sum_{l=0}^{m} (ab)^l, \quad m \ge 0, \tag{5.14}$$

one can rewrite this formula as follows

$$\xi_{\tau(h)} = S_{\tau(h)-2} z_1 + \sum_{j=0}^{\tau(h)-3} S_j (z_{\tau(h)-1-j} - z_{\tau(h)-2-j}). \tag{5.15}$$

By making use of (5.13) one can easily verify that

$$z_{\tau(h)-1-j} - z_{\tau(h)-2-j} = A_h (a u_{\tau(h)-2-j}\varepsilon_{\tau(h)-j-1} + b v_{\tau(h)-2-j}\varepsilon_{\tau(h)-j-1} + \varepsilon_{\tau(h)-j-1}^2).$$

Substituting this in (5.15) one obtains

$$\xi_{\tau(h)} = \xi_h^{(1)} + \xi_h^{(2)} + \xi_h^{(3)}; \tag{5.16}$$

$$\xi_h^{(1)} = A_h \sum_{k=1}^{\tau(h)-1} S_{\tau(h)-1-k}\varepsilon_k^2, \qquad \xi_h^{(2)} = aA_h \sum_{k=2}^{\tau(h)-1} S_{\tau(h)-1-k} u_{k-1}\varepsilon_k,$$
$$\xi_h^{(3)} = bA_h \sum_{k=2}^{\tau(h)-1} S_{\tau(h)-1-k} v_{k-1}\varepsilon_k. \tag{5.17}$$

To show (5.11) we have to check that, for each $d > 0$ and $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}} \mathbf{P}_\theta(|\xi_h^{(1)} - r(a,b)| > \delta) = 0,$$
$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}} \mathbf{P}_\theta(|\xi_h^{(i)}| > \delta) = 0, \quad i = 2, 3. \tag{5.18}$$

First we will verify the equalities for some subsets of $\Lambda_{d,1}$: for any $q \in]0, 1[$

$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|\le q\}} \mathbf{P}_\theta(|\xi_h^{(1)} - r(a,b)| > \delta) = 0,$$
$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|\le q\}} \mathbf{P}_\theta(|\xi_h^{(i)}| > \delta) = 0, \quad i = 2, 3. \tag{5.19}$$

Denoting $\lim_{n\to\infty} S_n = (1 - ab)^{-1} = S^*$ we rewrite $\xi_h^{(1)}$ as

$$\xi_h^{(1)} = A_h S^* \sum_{k=1}^{\tau(h)-1} \varepsilon_k^2 + W_h, \qquad W_h = A_h \sum_{k=1}^{\tau(h)-1} (S_{\tau(h)-1-k} - S^*)\varepsilon_k^2. \tag{5.20}$$

By Lemmas 5.1 and 5.3 one gets

$$A_h S^* \sum_{k=1}^{\tau-1} \varepsilon_k^2 = \sqrt{1-a^2}\sqrt{1-b^2}(1-ab)^{-1} + \alpha_h, \tag{5.21}$$

where α_h satisfies, for $d > 0$, $0 < q < 1$, and $\delta > 0$, the limiting relation

$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|\le q\}} \mathbf{P}_\theta(|\alpha_h| > \delta) = 0. \tag{5.22}$$

For $|W_h|$, on the set $(\tau(h) > N+1)$, one has the following estimate

$$|W_h| \le \max_{n\ge N} |S_n - S^*| \left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{\frac{1}{2}} \left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{\frac{1}{2}} \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)-1} \varepsilon_k^2$$
$$+ \max_{n\ge 1} |S_n - S^*| \left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{\frac{1}{2}} \left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{\frac{1}{2}} \left(\frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)-1} \varepsilon_k^2 - \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)-N-1} \varepsilon_k^2\right).$$

From here, in view of the inequalities,

$$\max_{n\ge N} |S_n - S^*| \le q^{N+1}/(1-q), \qquad \max_{n\ge 1} |S_n - S^*| \le q/(1-q),$$

by applying Lemmas 5.1–5.3, we obtain

$$\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|\le q\}} \mathbf{P}_\theta(|W_h| > \delta) = 0.$$

This and (5.20)–(5.22) imply (5.19). Similarly one shows (5.19) for $i = 2, 3$.

Thus we have verified all limiting relationships (5.19) which give the asymptotic convergence of random variables $\xi_h^{(i)}$ on the parametric set $\Lambda_{d,1}$ with the additional condition $|ab| \le q$. It remains to show that $\xi_h^{(i)}$ converges on the set $\Lambda_{d,1}$. It will be observed that, by the definition of parametric set $\Lambda_{d,1}$ in (2.12), there exists a number $q^* \in (0, 1)$ such that for all $q^* \le q < 1$ the corresponding set $\Lambda_{d,1} \cap \{\theta : |ab| \le q\}$ contains all points of $\Lambda_{d,1}$ except for those lying in some vicinity of the apex $(0, 1)$. On the other hand, function $r(a, b)$ in (5.18) vanishes when $|ab|$ approaches 1. Therefore, for a given $\delta > 0$, there exists a number $\tilde{q} \ge q^*$ such that, for every $\theta \in \Lambda_{d,1} \cap \{\theta : |ab| \ge \tilde{q}\}$, the inequality $\sqrt{1-a^2}\sqrt{1-b^2} < \delta/3$ holds and therefore

$$r(a, b) < \delta/3. \tag{5.23}$$

Consider $\xi_h^{(1)}$. Since $S_n \le 1$ for negative ab, then, in view of Lemmas 5.1 and 5.3, $|\xi_h^{(1)}|$ can be estimated as

$$|\xi_h^{(1)}| \le \left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{\frac{1}{2}} \left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{\frac{1}{2}} \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)-1} \varepsilon_k^2 = \sqrt{(1-a^2)(1-b^2)} + \alpha_h,$$

where $\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}} \mathbf{P}_\theta(|\alpha_h| > \delta/3) = 0$. From here and (5.23) one has

$$|\xi_h^{(1)} - r(a,b)| \le 2\delta/3 + \alpha_h.$$

Therefore, for any $\Delta > 0$, there exists a number h_0 such that for all $h \ge h_0$

$$\sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|>\tilde{q}\}} \mathbf{P}_\theta(|\xi_h^{(1)} - r(a,b)| > \delta) \le \Delta. \tag{5.24}$$

By (5.20), for any $\Delta > 0$, there exists a number h_1 such that for all $h \ge h_1$

$$\sup_{\theta\in\Lambda_{d,1}\cap\{\theta:|ab|\le\tilde{q}\}} \mathbf{P}_\theta(|\xi_h^{(1)} - r(a,b)| > \delta) \le \Delta. \tag{5.25}$$

Combining (5.24) and (5.25) we come to (5.18).

Further we estimate $|\xi_h^{(2)}|$ for $\theta \in \Lambda_{d,1} \cap \{\theta : |ab| > \tilde{q}\}$ as

$$|\xi_h^{(2)}| \le \left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2} \left(\frac{1}{\tau(h)} \sum_{k=2}^{\tau(h)-1} \varepsilon_k^2\right)^{1/2} = \sqrt{1-b^2} + \alpha_h^{(1)},$$

where $\alpha_h^{(1)}$ satisfies $\lim_{h\to\infty} \sup_{\theta\in\Lambda_{d,1}} \mathbf{P}_\theta(|\alpha_h^{(1)}| > \delta/3) = 0$.

This enables us, as in the case of $\xi_h^{(1)}$, to show (5.18) for $i = 2$. The case of $\xi_h^{(3)}$ can be studied by a similar way. Hence Proposition 2.2. $\square$

5.2. Proof of Lemma 2.3

Consider the case when $\boldsymbol{\theta} = (2, -1)$. Denote

$$f^{(n)}(t) = \frac{1}{n}\sum_{j=0}^{[nt-1]} W^{(n)}_{\frac{j}{n}}, \qquad I_t(f) = \int_0^t f(s)\mathrm{d}s, \tag{5.26}$$

where $W^{(n)}_t$ is given in (2.2). Then the nominator in (2.15) becomes

$$\sum_{k=1}^n u_{k-1}v_{k-1} = n^3 \sum_{k=1}^n f^{(n)}\left(\frac{k}{n}\right) W^{(n)}_{\frac{k-1}{n}}\frac{1}{n}. \tag{5.27}$$

It will be observed that

$$f^{(n)}(t) = I_{\frac{[nt-1]}{n}}(W^{(n)}) + r^{(1)}_n(t), \qquad |r^{(1)}_n(t)| \le \omega(W^{(n)}; [0,1]; 1/n), \tag{5.28}$$

where $\omega(f; E; \delta)$ denotes the oscillation of a function $f : E \to R$ of radius $\delta > 0$, that is, $\omega(f; E; \delta) = \sup_{|x-y|\le\delta, x,y\in E} |f(x) - f(y)|$. By (5.26) and (5.28)

$$r^{(1)}_n(t) \le \max_{1\le i\le n} |\varepsilon_i|/\sqrt{n} \to 0 \quad \text{a.s.} \tag{5.29}$$

Substituting (5.28) in (5.27) yields

$$\sum_{k=1}^n u_{k-1}v_{k-1} = n^3 \sum_{k=1}^n I_{\frac{k-1}{n}}(W^{(n)}) W^{(n)}_{\frac{k-1}{n}}\frac{1}{n} + n^3 r^{(2)}_n, \tag{5.30}$$

where $r^{(2)}_n = \sum_{k=1}^n r^{(1)}_n\left(\frac{k}{n}\right) W^{(n)}_{\frac{k-1}{n}}\frac{1}{n}$. Note that in view of (5.29),

$$|r^{(2)}_n| \le \max_{0\le t\le 1}|W^{(n)}_t| \cdot \max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}} = \max_{1\le k\le n}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^k \varepsilon_i\right| \max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}}.$$

We show that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}\left(|r^{(2)}_n| > \delta\right) = 0. \tag{5.31}$$

By applying the Kolmogorov inequality one gets, for any $\delta > 0$ and $\varDelta > 0$,

$$\mathbf{P}_{\boldsymbol{\theta}}\left(|r^{(2)}_n| > \delta\right) \le \mathbf{P}_{\boldsymbol{\theta}}\left(\max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}} > \varDelta\right) + \frac{\varDelta^2\sigma^2}{\delta^2}.$$

This implies (5.31). Now we rewrite (5.30) as

$$\sum_{k=1}^n u_{k-1}v_{k-1} = n^3\int_0^1 I_t(W^{(n)})W^{(n)}_t\mathrm{d}t + n^3 r^{(3)}_n + n^3 r^{(4)}_n; \tag{5.32}$$

$$|r^{(3)}_n| \le \omega\left(I_{\frac{[tn-1]}{n}}(W^{(n)})W^{(n)}_{\frac{[tn-1]}{n}}; [0,1]; \frac{1}{n}\right) \le 2\max_{1\le k\le n}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^k\varepsilon_i\right| \max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}}$$

$$r^{(4)}_n = A_n + B_n, \qquad A_n = \int_0^1 \left(I_{\frac{[tn-1]}{n}}(W^{(n)}) - I_t(W^{(n)})\right) W^{(n)}_{\frac{[tn-1]}{n}}\mathrm{d}t,$$

$$B_n = \int_0^1 I_t(W^{(n)})\left(W^{(n)}_{\frac{[tn-1]}{n}} - W^{(n)}_t\right)\mathrm{d}t.$$

For A_n and B_n one has the estimates

$$|A_n| \le n^{-1}\max_{0\le t\le 1}|W^{(n)}_t|^2 = n^{-2}\max_{1\le k\le n}\left|\sum_{i=1}^n\varepsilon_i\right|^2;$$

$$|B_n| \le \max_{0\le t\le 1}|I_t(W^{(n)})| \cdot \max_{1\le i\le n}|\varepsilon_i|/\sqrt{n} \le \max_{0\le t\le 1}|W^{(n)}_t| \cdot \max_{1\le i\le n}|\varepsilon_i|/\sqrt{n}.$$

From here and (5.32), it follows that, for any $\delta > 0$,

$$\lim_{n\to\infty}\mathbf{P}_{\boldsymbol{\theta}}(|r^{(i)}_n| > \delta) = 0, \quad i = 3, 4. \tag{5.33}$$

Consider now the sums in the denominator of (2.15). One can show that

$$(u,u)_n = n^4 \int_0^1 I_t^2(W^{(n)})\mathrm{d}t + n^4 r_n^{(5)}, \qquad (v,v)_n = n^2 \int_0^1 \left(W_t^{(n)}\right)^2 \mathrm{d}t + n^2 r_n^{(6)},$$

where $r_n^{(5)}$ and $r_n^{(6)}$ are such that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|r_n^{(i)}| > \delta) = 0, \quad i = 5, 6. \tag{5.34}$$

Substituting (5.30) and (5.34) in (2.15) yields

$$\xi_n = \varphi(W^{(n)}) + r_n, \tag{5.35}$$

where r_n, in view of (5.31), (5.33) and (5.34), satisfies, for any $\delta > 0$, the relation

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|r_n| > \delta) = 0.$$

One can check that the functional $\varphi(x)$ given by (2.20) is continuous everywhere in $C[0, 1]$ except for the point $x(t) \equiv 0$. Since the Wiener measure of the set $D = \{x \equiv 0\}$ equals zero we can apply the Donsker theorem to this functional in (5.35). This leads to (2.19). It remains to verify that $0 \le \varphi(W) \le 1$. It is obvious that the function $\varphi(W)$ in (2.20) can be viewed as the inner product of the functions

$$x(t) = \mathcal{J}_2^{-1/2}(W;1) \int_0^t W(s)\mathrm{d}s, \qquad y(t) = \mathcal{J}_1^{-1/2}(W;1)\, W(t).$$

The equality $\varphi(W) = 1$ is possible iff the functions $x(t)$ and $y(t)$ are linearly dependent, that is, $x(t) = Cy(t),\ 0 \le t \le 1$, for some constant C. However this does not hold with probability one, because $x(t)$ is absolutely continuous and $y(t)$ is non-differentiable almost everywhere. Hence the case $\boldsymbol{\theta} = (2, -1)$. Similarly one can show (2.19) for $\boldsymbol{\theta} = (-2, -1)$. Hence Lemma 2.3. □

5.3. *Additional properties of the sums $(u,u)_{\tau(h)}$ and $(v,v)_{\tau(h)}$*

In addition to Lemma 5.3 we will need the following results.

Lemma 5.4. *For each $d > 0$ and $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap(\boldsymbol{\theta}:\theta_1\le 0)} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| h\sigma^2 (u,u)_{\tau(h)}^{-1} - \frac{2(1+ab)}{(1-ab)(1-b^2)} \right| > \delta \right) = 0, \tag{5.36}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|(v,v)_{\tau(h)}^{-1} - (1-b^2)/\tau(h)| > \delta) = 0, \tag{5.37}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap(\boldsymbol{\theta}:\theta_1 > 0)} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| h\sigma^2 (v,v)_{\tau(h)}^{-1} - \frac{2(1+ab)}{(1-ab)(1-a^2)} \right| > \delta \right) = 0, \tag{5.38}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|(u,u)_{\tau(h)}^{-1} - (1-a^2)/\tau(h)| > \delta) = 0. \tag{5.39}$$

Proof of Lemma 5.4. Consider first (5.37) and (5.39). By Lemma 5.3

$$\tau(h)(u,u)_{\tau(h)}^{-1} = 1 - a^2 + \alpha_1(h), \qquad \tau(h)(v,v)_{\tau(h)}^{-1} = 1 - b^2 + \alpha_2(h),$$

where $\alpha_1(h)$ and $\alpha_2(h)$ satisfy, for any $\delta > 0$, the relations

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} P_{\boldsymbol{\theta}}(|\alpha_i(h)| > \delta) = 0, \quad i = 1, 2. \tag{5.40}$$

Therefore

$$(u,u)_{\tau(h)}^{-1} - (1-a^2)/\tau(h) = \alpha_1(h)/\tau(h), \qquad (v,v)_{\tau(h)}^{-1} - (1-b^2)/\tau(h) = \alpha_2(h)/\tau(h).$$

These equalities and (5.40) imply (5.37) and (5.39). Denote

$$t_h = h\sigma^2\, (u,u)_{\tau(h)}^{-1} - 2(1+ab)/((1-ab)(1-b^2)).$$

By the definition of stopping time $\tau(h)$ in (1.7), one has

$$h\sigma^2 = \sum_{k=1}^{\tau(h)-1} \|X_{k-1}\|^2 + \alpha_h \|X_{\tau(h)-1}\|^2 = {\sum_{k=1}^{\tau(h)}}' \|X_{k-1}\|^2,$$

where the prime at the sum sign means that the last addend is taken with the correction factor α_h providing the validity of the left-hand side equality, $0 < \alpha_h \le 1$. This equality implies

$$h\sigma^2(u,u)^{-1}_{\tau(h)} = (u,u)^{-1}_{\tau(h)} \mathrm{tr} \sum_{k=1}^{\tau(h)}{}' X_{k-1}X'_{k-1}$$
$$= \mathrm{tr}\, Q^{-1} \begin{pmatrix} 1 & (u,v)_{\tau(h)}/(u,u)_{\tau(h)} \\ (u,v)_{\tau(h)}/(u,u)_{\tau(h)} & (v,v)_{\tau(h)}/(u,u)_{\tau(h)} \end{pmatrix} (Q^{-1})'. \quad (5.41)$$

By Lemma 5.3

$$\frac{(v,v)_\tau}{(u,u)_\tau} = \frac{\tau}{(u,u)_\tau}\frac{(v,v)_\tau}{\tau} = \left[\frac{1-a^2}{\sigma^2} + \alpha_1(h)\right](v,v)_{\tau(h)}/\tau(h). \quad (5.42)$$

Since, on the set $\Lambda_{d,1} \cap (\boldsymbol{\theta} : \theta_1 \le 0)$, parameter b is bounded away from the end-points of the interval $(-1, 1)$, then, for any $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap(\boldsymbol{\theta}:\theta_1\le 0)} \mathbf{P}_{\boldsymbol{\theta}}\left(|(\tau(h))^{-1}(v,v)_{\tau(h)} - \sigma^2(1-b^2)^{-1}| > \delta\right) = 0.$$

This and (5.42) yield $(v,v)_\tau/(u,u)_\tau = (1-a^2)/(1-b^2) + \alpha_3(h)$, where $\alpha_3(h)$ satisfies the following relation for any $\delta > 0$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|\alpha_3(h)| > \delta) = 0. \quad (5.43)$$

In view of (5.42) and Lemma 5.3 we rewrite the cross-term in (5.41) as

$$\frac{(u,v)_{\tau(h)}}{(u,u)_{\tau(h)}} = \left(\frac{(v,v)_{\tau(h)}}{(u,u)_{\tau(h)}}\right)^{1/2} \frac{(u,v)_{\tau(h)}}{(u,u)^{1/2}_{\tau(h)}(v,v)^{1/2}_{\tau(h)}} = \frac{1-a^2}{1-ab} + \alpha_4(h), \quad (5.44)$$

where $\alpha_4(h)$, in virtue of Proposition 2.2, has the property (5.43). Hence

$$\frac{h\sigma^2}{(u,u)_{\tau(h)}} = \mathrm{tr}\, Q^{-1} \begin{pmatrix} 1 & (1-a^2)/(1-ab) \\ (1-a^2)/(1-ab) & (1-a^2)/(1-b^2) \end{pmatrix} (Q^{-1})' + r_h;$$
$$r_h = \mathrm{tr}\, Q^{-1} \begin{pmatrix} 0 & \alpha_4(h) \\ \alpha_4(h) & \alpha_3(h) \end{pmatrix} (Q^{-1})'.$$

One can easily verify that

$$\mathrm{tr}\, Q^{-1} \begin{pmatrix} 1 & (1-a^2)/(1-ab) \\ (1-a^2)/(1-ab) & (1-a^2)/(1-b^2) \end{pmatrix} (Q^{-1})' = \frac{2(1+ab)}{(1-ab)(1-b^2)}.$$

From here and (5.37), we come to the assertion of Lemma 5.4. □

5.4. The Skorohod coupling theorem. Proof of Proposition 3.2

By Theorem 2.1, on the boundary $\partial\Lambda$ of the stability region (1.4), the stopping time $\tau(h)$ (1.7) converges in distribution to some functional of one or two Brownian motions. In order to prove Proposition 3.2 we need to strengthen this convergence by applying the following result.

Theorem 5.5 (*Extended Skorohod Coupling; See Theorem 4.30 and Corollary 6.12 in [13]*). *Let $f, f_1, f_2, \ldots$ be measurable functions from a Borel space S to a Polish space T, and let $\eta, \eta_1, \eta_2, \ldots$ be random elements in S with $f_n(\eta_n) \xrightarrow{\mathcal{L}} f(\eta)$. Then there exists a probability space with some random elements $\tilde\eta \overset{\mathcal{L}}{=} \eta$ and $\tilde\eta_n \overset{\mathcal{L}}{=} \eta_n$, $n \in \mathbb{N}$, with $f_n(\tilde\eta_n) \to f(\tilde\eta)$ a.s.*

Let $W = (W(t))_{t\ge 0}$ and $W_1 = (W_1(t))_{t\ge 0}$ be independent Brownian motions and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ be a sequence of i.i.d. random variables with $\mathbf{E}\varepsilon_1 = 0$ and $\mathbf{E}\varepsilon_1^2 = \sigma^2$, which does not depend on W, W_1. Random elements

$$\eta = (\varepsilon, W, W_1)$$

take on values in the space $S = \mathbf{R}^\infty \times \mathbf{C}(\mathbf{R}_+) \times \mathbf{C}(\mathbf{R}_+)$, where $\mathbf{C}(\mathbf{R}_+)$ is the set of continuous functions on $\mathbf{R}_+ = [0, \infty)$. Define the metric on S as $\rho(\eta', \eta'') = \rho_1(\varepsilon', \varepsilon'') + \rho_2(W', W'') + \rho_3(W'_1, W''_1)$, where

$$\rho_1(\varepsilon', \varepsilon'') = \sum_{k\ge 1} 2^{-k} \frac{|\varepsilon'_k - \varepsilon''_k|}{1 + |\varepsilon'_k - \varepsilon''_k|},$$

$$\rho_i(x, y) = \sum_{k\ge 1} 2^{-k} \frac{\max_{1\le t\le k} |x(t) - y(t)|}{1 + \max_{1\le t\le k} |x(t) - y(t)|}, \quad i = 2, 3.$$

Let $(S, \mathcal{B}(S), \mathbf{P}_\eta)$ be the corresponding Borel space with the distribution $\mathbf{P}_\eta$ induced by η, that is, $\mathbf{P}_\eta = \mathbf{P}_\varepsilon \times \mathbf{P}_W \times \mathbf{P}_{W_1}$. Now we prove Proposition 3.2.

Assume that $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2$. Consider only the case when $\boldsymbol{\theta} \in \Gamma_1$ (the case $\boldsymbol{\theta} \in \Gamma_2$ is similar). For $\boldsymbol{\theta} \in \Gamma_1$ the processes $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ are described by Eq. (3.2) with $a = -1$ and $|b| < 1$. Let us apply the Skorohod Theorem 5.5 to the functional

$$f_n(\eta) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\tau(h)} v_{k-1}^2, \quad n = [h/2],$$

and put $\eta_n \equiv \eta$. By Lemma 5.3 and Theorem 2.1 we have

$$f_n(\eta) \xrightarrow{\mathcal{L}} \nu_1(W_1)\sigma^2/(1-b) = f(\eta).$$

By Theorem 5.5 there exists $\tilde\eta = (\tilde\varepsilon, \tilde W, \tilde W_1)$ such that

$$\begin{aligned} &\tilde\eta = (\tilde\varepsilon, \tilde W, \tilde W_1) \stackrel{\mathcal{L}}{=} \eta = (\varepsilon, W, W_1), \\ &f_n(\tilde\eta) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\tilde\tau(h)} \tilde v_{k-1}^2 \xrightarrow{\text{a.s.}} \frac{\nu_1(\tilde W_1)\sigma^2}{1-b} = f(\tilde\eta), \quad n = [h/2]. \end{aligned} \tag{5.45}$$

It should be noted that all the sequences $(\tilde x_k)$, $(\tilde u_k)$, $(\tilde v_k)$ and the stopping time $\tilde\tau$ are defined by formulae (1.1), (1.7) and (3.2) with a given $\boldsymbol{\theta} \in \Gamma_1$ replacing in them $\varepsilon = (\varepsilon_k)$ by $\tilde\varepsilon = (\tilde\varepsilon_k)$. Besides we define a counterpart $\tilde Y_h$ for Y_h in (3.6) by the formula

$$\tilde Y_h = \frac{\lambda_1}{\sigma\sqrt{(\tilde u, \tilde u)_{\tilde\tau(h)}}} \sum_{k=1}^{\tilde\tau(h)} \tilde u_{k-1}\tilde\varepsilon_k + \frac{\lambda_2}{\sigma\sqrt{(\tilde v, \tilde v)_{\tilde\tau(h)}}} \sum_{k=1}^{\tilde\tau(h)} \tilde v_{k-1}\tilde\varepsilon_k. \tag{5.46}$$

By the construction the distribution of the random variable $\tilde Y_h$ coincides with that of Y_h and therefore, for our purposes, it suffices to study its asymptotic distribution as $h \to \infty$. In view of (5.45) and Lemma 5.4, we rewrite $\tilde Y_h$ as

$$\begin{aligned} &\tilde Y_h = \frac{1}{\sigma^2\sqrt{h}} \sum_{k=1}^{\tilde\tau(h)} \tilde g_{k-1}\tilde\varepsilon_k + r_1(h); \\ &\tilde g_{k-1} = \frac{\lambda_1\sqrt{2}}{1+b}\tilde u_{k-1} + \lambda_2\sqrt{\frac{1-b}{\nu_1(\tilde W_1)}}(2h)^{1/4}\tilde v_{k-1}; \\ &r_1(h) = \frac{1}{\sqrt{h}} t_h^1 \sum_{k=1}^{\tilde\tau(h)} \tilde u_{k-1}\tilde\varepsilon_k + \left(\frac{1}{h}\right)^{1/4} t_h^2 \sum_{k=1}^{\tilde\tau(h)} \tilde v_{k-1}\tilde\varepsilon_k; \\ &t_h^1 = \frac{\lambda_1}{\sigma^2}\left(\left(\frac{h\sigma^2}{(\tilde u, \tilde u)_{\tilde\tau(h)}}\right)^{\frac12} - \frac{\sqrt{2}}{1+b}\right), \qquad t_h^2 = \frac{2^{\frac14}\lambda_2}{\sigma^2}\left(\left(\frac{\sigma^2\sqrt{(h/2)}}{(\tilde v, \tilde v)_{\tilde\tau(h)}}\right)^{\frac12} - \sqrt{\frac{(1-b)}{\nu_1(\tilde W_1)}}\right). \end{aligned} \tag{5.47}$$

Let us show that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}}(|r_1(h)| > \delta) = 0, \tag{5.48}$$

where $P'_{\boldsymbol{\theta}}$ is the distribution of the process $(\tilde x_k)$. We rewrite $r_1(h)$ as

$$r_1(h) = \frac{t_h^1}{\sqrt{h}} \sum_{k=1}^{\tilde\tau(h)-1} \tilde u_{k-1}\tilde\varepsilon_k + \frac{t_h^1}{\sqrt{h}} \tilde u_{\tilde\tau(h)-1}\tilde\varepsilon_{\tilde\tau(h)} + \left(\frac{1}{h}\right)^{1/4} t_h^2 \sum_{k=1}^{\tilde\tau(h)} \tilde v_{k-1}\tilde\varepsilon_k.$$

For any $\delta > 0$ and any $C > 0$, we have the estimate

$$\begin{aligned} \mathbf{P}'_{\boldsymbol{\theta}}(|r_1(h)| > \delta) \le{}& \mathbf{P}'_{\boldsymbol{\theta}}\left(\frac{1}{\sqrt{h}}\left|\sum_{k=1}^{\tilde\tau(h)-1} \tilde u_{k-1}\tilde\varepsilon_k\right| > C\right) + \mathbf{P}'_{\boldsymbol{\theta}}\left(|t_h^1|C > \frac{\delta}{3}\right) + \mathbf{P}'_{\boldsymbol{\theta}}\left(\frac{|\tilde u_{\tilde\tau(h)-1}\tilde\varepsilon_{\tilde\tau(h)}|}{\sqrt{h}} > \sqrt{\frac{\delta}{3}}\right) \\ &+ \mathbf{P}'_{\boldsymbol{\theta}}\left(|t_h^1| > \sqrt{\delta/3}\right) + \mathbf{P}'_{\boldsymbol{\theta}}\left(\frac{1}{h^{1/4}}\left|\sum_{k=1}^{\tilde\tau(h)} \tilde v_{k-1}\tilde\varepsilon_k\right| > C\right) + \mathbf{P}'_{\boldsymbol{\theta}}\left(|t_h^2|C > \frac{\delta}{3}\right). \end{aligned} \tag{5.49}$$

Now we will study each term in the right-hand side of (5.49).

Lemma 5.6. *For each $\boldsymbol{\theta} \in \Gamma_1$,*

$$\lim_{C\to\infty} \sup_{h>0} \mathbf{P}'_{\boldsymbol{\theta}} \left(\frac{1}{\sqrt{h}} \left| \sum_{k=1}^{\tilde{\tau}(h)-1} \tilde{u}_{k-1}\tilde{\varepsilon}_k \right| > C \right) = 0. \tag{5.50}$$

Lemma 5.7. *For each $\delta > 0$,*

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}} \left(h^{-1/2} |\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}| > \delta \right) = 0. \tag{5.51}$$

Lemma 5.8. *For any $0 < C < \infty$ and $a > 0$,*

$$\mathbf{P}'_{\boldsymbol{\theta}} \left(\frac{1}{h^{1/4}} \left| \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \right| \geq C \right) \leq \frac{a}{C} + \mathbf{P}'_{\boldsymbol{\theta}} \left(\frac{1}{\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}^2_{k-1} \geq a \right). \tag{5.52}$$

Lemma 5.9. *For any $a > 0$ and $\Delta > 0$,*

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}} \left(\frac{1}{\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}^2_{k-1} \geq a \right) \leq \mathbf{P}'_{\boldsymbol{\theta}}(\nu_1(\tilde{W}_1) \geq a'), \tag{5.53}$$

where $a' = a\sqrt{2}(1+b)^{-1} \left(\sigma^2(1-b^2)^{-1} + \Delta\right)^{-1}$.

Proof of Lemma 5.6. By the definition of $\tau(h)$ in (1.7), one has

$$\mathbf{E}'_{\boldsymbol{\theta}} \left(\frac{1}{\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)-1} \tilde{u}_{k-1}\tilde{\varepsilon}_k \right)^2 \leq \frac{\sigma^2}{h} \|Q\|^2 \mathbf{E}'_{\boldsymbol{\theta}} \left(\sum_{k=1}^{\tilde{\tau}(h)-1} \|\tilde{X}_{k-1}\|^2 \right) \leq \sigma^4 \|Q\|^2,$$

where $\mathbf{E}'_{\boldsymbol{\theta}}$ is the expectation with respect to $\mathbf{P}'_{\boldsymbol{\theta}}$. This implies (5.50). □

Proof of Lemma 5.7. One has

$$\begin{aligned} \mathbf{P}'_{\boldsymbol{\theta}} \left(|\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}|/\sqrt{h} > \delta \right) &\leq \mathbf{P}'_{\boldsymbol{\theta}} \left(\|Q\| \left(\sqrt{h}\right)^{-1} \|\tilde{X}_{\tilde{\tau}(h)-1}\| \cdot |\tilde{\varepsilon}_{\tilde{\tau}(h)}| > \delta \right) \\ &\leq \mathbf{P}'_{\boldsymbol{\theta}} \left(h^{-1} \|Q\|^2 \|\tilde{X}_{\tilde{\tau}(h)-1}\|^2 C^2 > \delta^2 \right) + C^{-2}\sigma^2. \end{aligned}$$

It remains to show that

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}}(h^{-1} \|\tilde{X}_{\tilde{\tau}(h)-1}\|^2 > \tilde{\delta}) = 0, \tag{5.54}$$

where $\tilde{\delta} = \delta^2 \|Q\|^{-2} C^{-2}$. We have

$$\begin{aligned} \mathbf{P}'_{\boldsymbol{\theta}}(h^{-1} \|\tilde{X}_{\tilde{\tau}(h)-1}\|^2 > \tilde{\delta}) &\leq \mathbf{P}'_{\boldsymbol{\theta}} \left(\|\tilde{X}_{\tau(h)-1}\|^2 \geq \frac{\tilde{\delta}}{\sigma^2} \sum_{k=1}^{\tilde{\tau}(h)-1} \|\tilde{X}_{k-1}\|^2 \right) \\ &\leq \mathbf{P}'_{\boldsymbol{\theta}}(\tilde{\tau}(h) \leq m) + \mathbf{P}'_{\boldsymbol{\theta}} \left(\|\tilde{X}_n\|^2 \geq \frac{\tilde{\delta}}{\sigma^2} \sum_{k=1}^{n} \|\tilde{X}_{k-1}\|^2 \text{ for some } n \geq m \right). \end{aligned}$$

By virtue of the relation (3.3) in [7] and Lemma 5.1, we come to (5.54). □

Further we need the following Lenglart inequality.

Lemma 5.10 (*See,* [17] *Ch. VII, 3, Th. 4*)**.** *Let $(\xi_n, \mathcal{F}_n)$ be non-negative adapted sequence of random variables and $(A_n, \mathcal{F}_n)$ be predictable increasing sequence which dominates (ξ_n) in the sense that, for any stopping time σ with respect to $(\mathcal{F}_n)$, one has $\mathbf{E}\xi_\sigma \leq \mathbf{E}A_\sigma$. Then, for any $\varepsilon > 0$ and $a > 0$,*

$$\mathbf{P}(\sup_{1\leq j\leq \sigma} \xi_j \geq \varepsilon) \leq \varepsilon^{-1}\mathbf{E}(A_\sigma \wedge a) + \mathbf{P}(A_\sigma \geq a).$$

Proof of Lemma 5.8. Denote

$$\xi_n = h^{-1/2}\left(\sum_{k=1}^{n}\tilde{v}_{k-1}\tilde{\varepsilon}_k\right)^2, \quad n \geq 1,\ \xi_0 = 0;$$
$$A_n = h^{-1/2}\sum_{k=1}^{n}\tilde{v}_{k-1}^2, \quad n \geq 1,\ A_0 = 0. \tag{5.55}$$

Let us introduce the filtration $(\mathcal{F}_n)_{n\geq 0}$ with

$$\mathcal{F}_0 = \sigma\{\nu(\tilde{W}_1)\}, \qquad \mathcal{F}_n = \sigma\{\nu(\tilde{W}_1), \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n\}. \tag{5.56}$$

Note that for each stopping time σ with respect to this filtration, one has

$$\mathbf{E}'_{\boldsymbol{\theta}}\xi_\sigma \leq \mathbf{E}'_{\boldsymbol{\theta}}A_\sigma.$$

Therefore the processes (5.55) satisfy the conditions of Lemma 5.10. Applying this lemma with $\sigma = \tilde{\tau}(h)$ yields (5.52):

$$\begin{aligned}\mathbf{P}'_{\boldsymbol{\theta}}(\xi_{\tilde{\tau}(h)} \geq C) &\leq C^{-1}\mathbf{E}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \wedge a) + \mathbf{P}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \geq a)\\ &\leq aC^{-1} + \mathbf{P}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \geq a). \quad \Box\end{aligned}$$

Proof of Lemma 5.9. For any $\Delta > 0$, one has

$$\begin{aligned}\mathbf{P}'_{\boldsymbol{\theta}}\left(h^{-1/2}(\tilde{v}, \tilde{v})_{\tilde{\tau}(h)} \geq a\right) \leq \mathbf{P}'_{\boldsymbol{\theta}}&\left(\frac{\tilde{\tau}(h)}{(1+b)\sqrt{h/2}}\frac{(1+b)}{\sqrt{2}}(\sigma^2(1-b^2)^{-1} + \Delta) \geq a\right)\\ &+ \mathbf{P}'_{\boldsymbol{\theta}}\left(|\tilde{\tau}(h)^{-1}(\tilde{v}, \tilde{v})_{\tilde{\tau}(h)} - \sigma^2/(1-b^2)| \geq \Delta\right).\end{aligned}$$

From here, by Theorem 2.1 and Lemma 5.3, we come to (5.53). $\Box$

Now we are ready to show (5.48). Limiting in (5.49) $h \to \infty$ and taking into account Lemmas 5.4 and 5.6–5.9 and (5.45) we obtain

$$\limsup_{h\to\infty}\mathbf{P}'_{\boldsymbol{\theta}}(|r_1(h)| > \delta) \leq \sup_{h>0}\mathbf{P}'_{\boldsymbol{\theta}}\left(\frac{1}{\sqrt{h}}\left|\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}^2\tilde{\varepsilon}_k\right| > C\right) + a/C + \mathbf{P}'_{\boldsymbol{\theta}}(\nu_1(\tilde{W}_1) \geq a').$$

In view of Lemma 5.6, limiting $C \to \infty$ and then $a \to \infty$, we come to (5.48).

So we have

$$\tilde{Y}_h = \frac{1}{\sigma^2\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k + r_1(h), \tag{5.57}$$

where $r_1(h)$ satisfies (5.48) and

$$\tilde{g}_{k-1} = (\lambda_1\sqrt{2}/(1+b))\tilde{u}_{k-1} + \left(\lambda_2\sqrt{1-b}\Big/\left(\sigma\sqrt{\nu_1(\tilde{W}_1)}\right)\right)(2h)^{1/4}\tilde{v}_{k-1}.$$

For a given $h > 0$, we define the random variable

$$\tau_0(h) = \inf\left\{n \geq 1 : \sum_{k=1}^{n}\tilde{g}_{k-1}^2 \geq h\sigma^2\right\}, \qquad \inf\{\emptyset\} = \infty, \tag{5.58}$$

which is a stopping time with respect to the filtration $(\mathcal{F}_n)$ in (5.56), and rewrite $\tilde{Y}_h$ from (5.57) as

$$\tilde{Y}_h = \frac{1}{\sigma^2\sqrt{h}}\sum_{k=1}^{\tau_0(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k + r_1(h) + r_2(h);$$
$$r_2(h) = \frac{1}{\sigma^2\sqrt{h}}\left(\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k - \sum_{k=1}^{\tau_0(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k\right). \tag{5.59}$$

Now we observe that the first term in the right-hand side of (5.59) is a martingale with respect to the filtration $(\mathcal{F}_n)$ in (5.56) stopped at the time (5.58). According to the Theorem 2.1 from [14], it is asymptotically normal with mean 0 and unit variance as $h \to \infty$. Therefore to end the proof of Theorem 3.1 for $\boldsymbol{\theta} \in \Gamma_1$ it remains to prove that, for any $\delta > 0$,

$$\lim_{h\to\infty}\mathbf{P}'_{\boldsymbol{\theta}}(|r_2(h)| > \delta) = 0. \tag{5.60}$$

First we will establish the following results.

Lemma 5.11. *For each $\boldsymbol{\theta} \in \Gamma_1$ and any $\delta > 0$,*

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}} \left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta \right) = 0. \tag{5.61}$$

Lemma 5.12. *For each $\boldsymbol{\theta} \in \Gamma_1$ and any $\delta > 0$, $\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}}(U_h > \delta) = 0$, where $U_h = \frac{1}{h} \sum_{k=\tilde{\tau}(h)\wedge\tau_0(h)+1}^{\tilde{\tau}(h)\vee\tau_0(h)} \tilde{g}^2_{k-1}$.*

Proof of Lemma 5.11. One has the inclusions, for any $\Delta > 0$,

$$\left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta \right) \subseteq \left(\|J_{\tau_0(h)-1} - I\| > \Delta \right) \cup A, \tag{5.62}$$

where I is 2×2 identity matrix,

$$A = \left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta, \; \|J_{\tau_0(h)-1} - I\| \le \Delta \right).$$

By the relation below (5.57) one gets

$$\tilde{g}^2_n = (z_1 \tilde{u}_n + z_2 \tilde{v}_n)^2 \le 2z_1^2 \tilde{u}^2_n + 2z_2^2 \tilde{v}^2_n, \tag{5.63}$$

$$z_1 = \lambda_1 \sqrt{2}/(1+b), \qquad z_2 = \lambda_2 \sqrt{1-b}(2h)^{1/4} \Big/ \left(\sigma \sqrt{\nu_1(\tilde{W}_1)} \right);$$

$$\begin{aligned} \sum_{k=1}^{n} \tilde{g}^2_{k-1} &= \sum_{k=1}^{n} \left(Z'(\tilde{u}_{k-1}, \tilde{v}_{k-1})' \right)^2 = Z' R_n^{-1} J_n R_n^{-1} Z \\ &= Z' R_n^{-1} (J_n - I) R_n^{-1} Z + Z' R_n^{-2} Z, \quad Z = (z_1, z_2)'; \end{aligned}$$

$$Z' R_n^{-2} Z = z_1^2 \, (u, u)_n + z_2^2 \, (v, v)_n. \tag{5.64}$$

From here it follows that

$$\begin{aligned} A &\subset \left(\tilde{g}^2_{\tau_0-1} > \delta Z' \tilde{R}^{-2}_{\tau_0-1} Z (1 - \|\tilde{J}_{\tau_0-1} - I\|), \|\tilde{J}_{\tau_0-1} - I\| \le \Delta \right) \\ &\subset (\tau_0 \le m) \cup \left(2z_1^2 \tilde{u}^2_n > \delta 2^{-1} (1-\Delta) Z' \tilde{R}^{-2}_n Z \text{ for some } n \ge m \right) \cup \left(2z_2^2 \tilde{v}^2_n > \delta 2^{-1} (1-\Delta) Z' \tilde{R}^{-2}_n Z \text{ for some } n \ge m \right) \\ &\subset (\tau_0 \le m) \cup \left(2\tilde{u}^2_n > \delta 2^{-1} (1-\Delta) \, (\tilde{u}, \tilde{u})_n \text{ for some } n \ge m \right) \cup \left(2\tilde{v}^2_n > \delta 2^{-1} (1-\Delta) \, (\tilde{v}, \tilde{u})_n \text{ for some } n \ge m \right). \end{aligned} \tag{5.65}$$

Combining inclusions (5.62) and (5.65) yields

$$\begin{aligned} \left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta \right) &\subset \left(\|\tilde{J}_{\tau_0(h)-1} - I\| > \Delta \right) \cup (\tau_0(h) \le m) \cup \left(\tilde{u}^2_n > \delta' (\tilde{u}, \tilde{u})_n \text{ for some } n \ge m \right) \\ &\quad \cup \left(\tilde{v}^2_n > \delta' (\tilde{v}, \tilde{v})_n \text{ for some } n \ge m \right), \quad \delta' = \delta \, 4^{-1} (1-\Delta). \end{aligned}$$

This implies

$$\begin{aligned} \mathbf{P}'_{\boldsymbol{\theta}} \left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta \right) &\le \mathbf{P}'_{\boldsymbol{\theta}} (\|\tilde{J}_{\tau_0(h)-1} - I\| > \Delta) + \mathbf{P}'_{\boldsymbol{\theta}} (\tau_0(h) \le m) + \mathbf{P}'_{\boldsymbol{\theta}} \left(\tilde{u}^2_n > \delta' (\tilde{u}, \tilde{u})_n \text{ for some } n \ge m \right) \\ &\quad + \mathbf{P}'_{\boldsymbol{\theta}} \left(\tilde{v}^2_n > \delta' (\tilde{v}, \tilde{v})_n \text{ for some } n \ge m \right). \end{aligned} \tag{5.66}$$

By the same argument as in the proofs of Lemma 5.1 and Proposition 2.2, one can show that for every $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2$ and for each $m = 1, 2, \ldots$ and any $\Delta > 0$, respectively,

$$\lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}} (\tau_0(h) \le m) = 0, \qquad \lim_{h\to\infty} \mathbf{P}'_{\boldsymbol{\theta}} (\|\tilde{J}_{\tau_0(h)-1} - I\| > \Delta) = 0.$$

The last two terms in (5.66) also converge to zero by the well-known property of AR(1)-processes with parameter in the interval $[-1, 1]$ (see, [14]). This completes the proof of Lemma 5.11. $\square$

Proof of Lemma 5.12. Note that $U_h = h^{-1}|(\tilde{g},\tilde{g})_{\tilde{\tau}(h)} - (\tilde{g},\tilde{g})_{\tau_0(h)}|$.

In view of (5.63) this quantity can be estimated as

$$U_h = h^{-1}\left|Z'\tilde{R}^{-1}_{\tilde{\tau}(h)}(\tilde{J}_n - I)\tilde{R}^{-1}_{\tilde{\tau}(h)}Z + Z'\tilde{R}^{-2}_{\tilde{\tau}(h)}Z - (\tilde{g},\tilde{g})_{\tau_0(h)}\right|$$
$$\le h^{-1}\|\tilde{J}_n - I\|Z'\tilde{R}^{-2}_{\tilde{\tau}(h)}Z + |h^{-1}Z'\tilde{R}^{-2}_{\tilde{\tau}(h)}Z - 1| + \tilde{g}^2_{\tau_0(h)-1}/(\tilde{g},\tilde{g})_{\tau_0(h)-1}. \quad (5.67)$$

Now we show that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|h^{-1}Z'\tilde{R}^{-2}_{\tilde{\tau}(h)}Z - 1| > \delta) = 0. \quad (5.68)$$

Using (5.64) and taking into account that for $\boldsymbol{\theta} \in \Gamma_1,\ z_1 = \lambda_1\sqrt{2}/(1+b),\ z_2 = \lambda_2(2h)^{1/4}\sqrt{1-b}/(\sigma\sqrt{\nu_1(\tilde{W}_1)})$ we obtain

$$h^{-1}Z'\tilde{R}^{-2}_{\tilde{\tau}(h)}Z - 1 = z_1^2\,h^{-1}(\tilde{u},\tilde{u})_{\tilde{\tau}(h)} + z_2^2\,h^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} - 1$$
$$= \lambda_1^2\left(\frac{2}{(1+b)^2h}(\tilde{u},\tilde{u})_{\tilde{\tau}(h)} - 1\right) + \lambda_2^2\left(\frac{(1-b)(2h)^{1/2}}{\sigma^2\nu_1(\tilde{W}_1)}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} - 1\right).$$

This, in view of (5.36) gives (5.68). Now by applying Proposition 2.2 and Lemma 5.11 to (5.67) we come to the desired result. Hence Lemma 5.12. □

The case $\boldsymbol{\theta} = (0, 1)$. Then $a = -1$, $b = 1$ and Eqs. (3.2) yield

$$u_k = (-1)^k\sum_{j=1}^{k}(-1)^j\varepsilon_j, \qquad v_k = \sum_{j=1}^{k}\varepsilon_j.$$

By Corollary 5.15 one has

$$\sigma^{-2}\left(n^{-2}(u,u)_n,\ n^{-2}(v,v)_n\right) \xrightarrow{\mathcal{L}} (\mathcal{J}_1(W_1;1),\ \mathcal{J}_1(W;1)). \quad (5.69)$$

Introduce a sequence of functionals

$$f_n(\eta) = \left((\sigma n)^{-2}(u,u)_n,\ (\sigma n)^{-2}(v,v)_n,\ \tau(h)/\sqrt{2h}\right), \quad n = [h].$$

From the definition of $\tau(h)$, Theorem 2.1 and (5.69) it follows that

$$f_n(\eta) \xrightarrow{\mathcal{L}} (\mathcal{J}_1(W_1;1), \mathcal{J}_1(W;1), \nu_3(W,W_1)) = f(\eta).$$

By Theorem 5.5 there exists $\tilde{\eta}$ such that $\tilde{\eta} \overset{\mathcal{L}}{=} \eta$ and

$$f_n(\tilde{\eta}) \xrightarrow{\text{a.s.}} f(\tilde{\eta}) = \left(\mathcal{J}_1(\tilde{W}_1;1), \mathcal{J}_1(\tilde{W};1), \nu_3(\tilde{W},\tilde{W}_1)\right).$$

On the basis of this $\tilde{\eta}$ we define as before $(\tilde{x}_k)$, $(\tilde{u}_k)$, $(\tilde{v}_k)$ and $\tilde{Y}_h$. It should be noted that the ratio $t_n = (\tilde{u},\tilde{u})_n/(\tilde{v},\tilde{v})_n$ satisfies the limiting relation

$$\lim_{h\to\infty} t_{\tilde{\tau}(h)} = \kappa, \quad \kappa = \mathcal{J}_1(\tilde{W}_1;1)/\mathcal{J}_1(\tilde{W};1). \quad (5.70)$$

Further by making use of the equality

$$\sum_{k=1}^{\tilde{\tau}(h)}\|\tilde{X}_{k-1}\|^2 = \operatorname{tr}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{X}_{k-1}\tilde{X}'_{k-1} = \operatorname{tr}Q^{-1}\begin{pmatrix}(\tilde{u},\tilde{u})_{\tilde{\tau}(h)} & (\tilde{u},\tilde{v})_{\tilde{\tau}(h)}\\ (\tilde{u},\tilde{v})_{\tilde{\tau}(h)} & (\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\end{pmatrix}(Q^{-1})'$$

one gets

$$\frac{\sum_{k=1}^{\tilde{\tau}(h)}\|\tilde{X}_{k-1}\|^2}{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}} = \operatorname{tr}Q^{-1}\begin{pmatrix}1 & t^{-1/2}_{\tilde{\tau}(h)}\xi_{\tilde{\tau}(h)}\\ t^{-1/2}_{\tilde{\tau}(h)}\xi_{\tilde{\tau}(h)} & t^{-1}_{\tilde{\tau}(h)}\end{pmatrix}(Q^{-1})',$$

where ξ_n is defined in (2.15). Since for any $\delta > 0$

$$\lim_{h\to\infty}\mathbf{P}'_{\boldsymbol{\theta}}\left(|\xi_{\tilde{\tau}(h)}|t^{-1/2}_{\tilde{\tau}(h)} > \delta\right) = 0,$$

from here it follows that

$$\mathbf{P}'_{\boldsymbol{\theta}} - \lim_{h\to\infty}\frac{\sum_{k=1}^{\tilde{\tau}(h)}\|\tilde{X}_{k-1}\|^2}{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}} = \operatorname{tr}Q^{-1}\begin{pmatrix}1 & 0\\ 0 & 1/\kappa\end{pmatrix}(Q^{-1})' = \frac{1}{2}\left(1+\frac{1}{\kappa}\right).$$

On the other hand, by the definition of stopping time $\tau(h)$ in (1.7) and Proposition 2.2 one has $\mathbf{P}'_{\boldsymbol{\theta}}-\lim_{h\to\infty} h^{-1}\sum_{k=1}^{\tilde{\tau}(h)}\|\tilde{X}_{k-1}\|^2 = 1$. As result,

$$\mathbf{P}'_{\boldsymbol{\theta}} - \lim_{h\to\infty} h\left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}\right)^{-1} = 2^{-1}\left(1+\kappa^{-1}\right). \tag{5.71}$$

This and (5.70) give

$$\mathbf{P}'_{\boldsymbol{\theta}} - \lim_{h\to\infty} h\left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\right)^{-1} = 2^{-1}(1+\kappa). \tag{5.72}$$

Now we rewrite (5.47) as

$$\begin{aligned}
&\tilde{Y}_h = \frac{1}{\sigma^2\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k + r_1(h);\\
&\tilde{g}_{k-1} = \lambda_1\sqrt{(1+\kappa^{-1})/2}\,\tilde{u}_{k-1} + \lambda_2\sqrt{(1+\kappa)/2}\,\tilde{v}_{k-1},\\
&r_1(h) = \frac{\lambda_1}{\sigma^2}\left((\tilde{u},\tilde{u})^{-1/2}_{\tilde{\tau}(h)} - \sqrt{(1+\kappa^{-1})/2}\right)\sum_{k=1}^{\tilde{\tau}(h)}\tilde{u}_{k-1}\tilde{\varepsilon}_k + \frac{\lambda_2}{\sigma^2}\left((\tilde{v},\tilde{v})^{-1/2}_{\tilde{\tau}(h)} - \sqrt{(1+\kappa)/2}\right)\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}\tilde{\varepsilon}_k.
\end{aligned} \tag{5.73}$$

Taking into account (5.71) and (5.72), one can show along the lines of the proof of Proposition 3.2 that, for any $\delta > 0$, $\lim_{h\to\infty}\mathbf{P}'_{\boldsymbol{\theta}}(|r_1(h)| > \delta) = 0$.

Further analysis of (5.73) repeats the case of $\boldsymbol{\theta}\in\Gamma_1$ and is omitted.

This completes the proofs of Proposition 3.2 and Theorem 3.1 for $\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}$. □

5.5. *Proof of Theorem 3.1 for the case of multiple roots*

Assume that $\boldsymbol{\theta} = (2,-1)$ (the proof for the case $\boldsymbol{\theta} = (-2,-1)$ is similar). This corresponds to the multiple root of the polynomial (1.2): $a = b = 1$. Since matrix Q in (2.7) is degenerate, we use the matrix Q_1 from (2.8) to transform the original process $(x_k)_{k\ge 0}$ into two components $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$. This leads to the equations: $u_k = x_k,\ v_k = x_k - x_{k-1}$ with the solutions given by the formulas (2.18). Now we introduce a sequence of functionals

$$f_n(\eta) = \left(\xi_n,\,(\sigma n)^{-2}\sum_{k=1}^{n} v_{k-1}^2,\,\tau(h)/(h/2)^4\right),\quad n = [h],$$

where ξ_n is defined in (2.15). By Lemma 2.3 and Theorem 2.1 we have

$$f_n(\eta) \xrightarrow{\mathcal{L}} (\varphi(W),\,\mathcal{J}_1(W;1),\,\nu_4(W)) = f(\eta).$$

By Theorem 5.5 there exists $\tilde{\eta}$ such that $\tilde{\eta} \overset{\mathcal{L}}{=} \eta$ and

$$f_n(\tilde{\eta}) \xrightarrow{\text{a.s.}} f(\tilde{\eta}) = \left(\varphi(\tilde{W}),\,\mathcal{J}_1(\tilde{W};1),\,\nu_4(\tilde{W})\right). \tag{5.74}$$

On the basis of $\tilde{\eta}$ we define $(\tilde{x}_k)$, $(\tilde{u}_k)$, $(\tilde{v}_k)$ and $\tilde{Y}_h$. In view of (5.74) we have

$$\begin{aligned}
&\lim_{h\to\infty} J_{\tilde{\tau}(h)} = T_1 \quad \text{a.s., };\\
&J_n = \begin{pmatrix} 1 & \tilde{\xi}_n\\ \tilde{\xi}_n & 1\end{pmatrix},\qquad T_1 = \begin{pmatrix} 1 & \varphi(\tilde{W})\\ \varphi(\tilde{W}) & 1\end{pmatrix}\\
&\tilde{\xi}_n = (\tilde{u},\tilde{u})_n^{-1/2}(\tilde{v},\tilde{v})^{-1/2}(\tilde{u},\tilde{v})_n.
\end{aligned} \tag{5.75}$$

Besides, we will need the relations

$$\begin{aligned}
&\lim_{h\to\infty}\frac{1}{h}\sum_{k=1}^{\tilde{\tau}(h)}\|\tilde{X}_{k-1}\|^2 = \lim_{h\to\infty}\frac{1}{h}\left(2(\tilde{u},\tilde{u})_{\tilde{\tau}(h)-1} - \tilde{u}^2_{\tilde{\tau}(h)-1}\right) = \sigma^2,\\
&\lim_{h\to\infty}\frac{1}{\sigma^2 h^{1/2}}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} = 2^{-1/2}\mathcal{J}_1(\tilde{W};1)\nu_4^2(\tilde{W}) := \mu,
\end{aligned} \tag{5.76}$$

which directly follow from (5.74).

Consider now the standardized deviation of the sequential estimate (1.6):

$$M^{1/2}_{\tau(h)}\left(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}\right) = M^{-1/2}_{\tau(h)}\sum_{k=1}^{\tau(h)} X_{k-1}\varepsilon_k.$$

Its distribution coincides with that of the vector $\tilde{M}_{\tilde{\tau}(h)}^{-1/2}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{X}_{k-1}\tilde{\varepsilon}_k$ constructed from $(\tilde{x}_k)$, $(\tilde{\varepsilon}_k)$. Representing the matrix

$$\tilde{M}_n=\sum_{k=1}^{n}\tilde{X}_{k-1}\tilde{X}_{k-1}'$$

in the form (2.13) yields

$$\tilde{M}_{\tilde{\tau}(h)}^{-1/2}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{X}_{k-1}\tilde{\varepsilon}_k=\tilde{J}_{\tilde{\tau}(h)}^{-1/2}\tilde{R}_{\tilde{\tau}(h)}\sum_{k=1}^{\tilde{\tau}(h)}Q\tilde{X}_{k-1}\tilde{\varepsilon}_k=\tilde{J}_{\tilde{\tau}(h)}^{-1/2}T_1^{1/2}T_1^{-1/2}\tilde{Z}_{\tilde{\tau}(h)},$$

where $\tilde{Z}_n=\left((\tilde{u},\tilde{u})_n^{-1/2}\sum_{k=1}^{n}\tilde{u}_{k-1}\tilde{\varepsilon}_k,(\tilde{v},\tilde{v})_n^{-1/2}\sum_{k=1}^{n}\tilde{v}_{k-1}\tilde{\varepsilon}_k\right)'$. Taking into account (5.75), it suffices to establish the following result.

Lemma 5.13. *For each constant vector $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)'$ with $\|\boldsymbol{\lambda}\|=1$, the random variable $\tilde{Y}_h=\boldsymbol{\lambda}'T_1^{-1/2}\tilde{Z}_{\tilde{\tau}(h)}/\sigma$ is asymptotically normal with mean 0 and unit variance as $h\to\infty$.*

Proof of Lemma 5.13. Represent $\tilde{Y}_h$ as

$$\begin{aligned}
\tilde{Y}_h&=\frac{1}{\sigma\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k+r_1(h);\\
\tilde{g}_{k-1}&=\boldsymbol{\lambda}'T_1^{-1/2}\left(\sqrt{2}\tilde{u}_{k-1},(h^{1/4}\sqrt{\mu})^{-1}\tilde{v}_{k-1}\right)',\\
r_1(h)&=\frac{\boldsymbol{\lambda}'T_1^{-1/2}}{\sigma}\begin{pmatrix}\left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}^{-1/2}-\sqrt{2/h}\right)\sum_{k=1}^{\tilde{\tau}(h)}\tilde{u}_{k-1}\tilde{\varepsilon}_k\\ \left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}^{-1/2}-(h^{1/4}\sqrt{\mu})^{-1}\right)\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}\tilde{\varepsilon}_k\end{pmatrix}.
\end{aligned}\tag{5.77}$$

By an argument similar to that in the proof of Proposition 3.2, one can verify that $r_1(h)$ satisfies (5.48). Further analysis of $\tilde{Y}_h$ holds true.

Let us check only that $\lim_{h\to\infty}(\sigma^2h)^{-1}(\tilde{g},\tilde{g})_{\tilde{\tau}(h)}=1$. Using the definition of $\tilde{g}$ from (5.77) yields

$$\frac{(\tilde{g},\tilde{g})_{\tilde{\tau}(h)}}{\sigma^2h}=\frac{\boldsymbol{\lambda}'T_1^{\frac{-1}{2}}}{\sigma^2}\begin{pmatrix}(2/h)(\tilde{u},\tilde{u})_{\tilde{\tau}(h)} & h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)}\\ h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)} & (\mu\sqrt{h})^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\end{pmatrix}T_1^{\frac{-1}{2}}\boldsymbol{\lambda}.$$

Now using (5.76) we rewrite the cross term as

$$h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)}=\left((h/2)^{-1}(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}\right)^{1/2}\left((\mu\sqrt{h})^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\right)^{1/2}\tilde{\xi}_{\tilde{\tau}(h)}.$$

From here, (5.75) and (5.76) it follows that

$$\lim_{h\to\infty}h^{-3/4}\sqrt{2/h}(\tilde{u},\tilde{v})_{\tilde{\tau}(h)}=\varphi(\tilde{W}).$$

Hence $\lim_{h\to\infty}(\sigma^2h)^{-1}(\tilde{g},\tilde{g})_{\tilde{\tau}(h)}=\sigma^{-2}\boldsymbol{\lambda}'T_1^{-1/2}T_1T_1^{-1/2}\boldsymbol{\lambda}=\boldsymbol{\lambda}'\boldsymbol{\lambda}=1$. This completes the proof of Theorem 3.1 for $\boldsymbol{\theta}\in\{(-2,-1),(2,-1)\}$. □

Theorem 5.14. *Let $W^{(n)}=(W^{(n)}(t))_{0\le t\le 1}$ and $W_1^{(n)}=(W_1^{(n)}(t))_{0\le t\le 1}$ be defined by (2.2). Then for the random functions*

$$X_n=\left(X_n(s,t)=(W^{(n)}(s),W_1^{(n)}(t)):\ 0\le s\le 1,0\le t\le 1\right)$$

with values in the product of the Skorohod spaces $\mathcal{D}[0,1]\times\mathcal{D}[0,1]$, one has

$$X_n\xrightarrow{\mathcal{L}}(W,W_1),$$

where W and W_1 are independent standard Brownian motions.

This result is a straightforward consequence of Theorem 3.3 in Helland [10]. This functional central limit theorem implies the following result.

Corollary 5.15. *Let $u_k=(-1)^k\sum_{j=1}^{k}(-1)^j\varepsilon_j$, $v_k=\sum_{j=1}^{k}\varepsilon_j$. Then*

$$\left((\sigma n)^{-2}(u,u)_n,\ (\sigma n)^{-2}(v,v)_n\right)\xrightarrow{\mathcal{L}}(\mathcal{J}_1(W_1;1),\ \mathcal{J}_1(W;1)).\tag{5.78}$$

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