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On island sequences of labelings with a condition at distance two

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ABSTRACT

An $L(2,1)$-*labeling* of a graph G is a function f from the vertex set of G to the set of nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$, and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between the pair of vertices x, y. The *lambda number* of G, denoted $\lambda(G)$, is the minimum range of labels used over all L(2,1)-labelings of G. An L(2,1)-labeling of G which achieves the range $\lambda(G)$ is referred to as a λ-labeling. A *hole* of an L(2,1)-labeling is an unused integer within the range of integers used. The *hole index* of G, denoted $\rho(G)$, is the minimum number of holes taken over all its λ-labelings. An *island* of a given λ-labeling of G with $\rho(G)$ holes is a maximal set of consecutive integers used by the labeling. Georges and Mauro [J.P. Georges, D.W. Mauro, On the structure of graphs with non-surjective L(2,1)-labelings, SIAM J. Discrete Math. 19 (2005) 208–223] inquired about the existence of a connected graph G with $\rho(G) \geq 1$ possessing two λ-labelings with different ordered sequences of island cardinalities. This paper provides an infinite family of such graphs together with their lambda numbers and hole indices. Key to our discussion is the determination of the path covering number of certain 2-*sparse graphs,* that is, graphs containing no pair of adjacent vertices of degree greater than 2.

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1. Introduction

An *L(2,1)-labeling* of a graph G, first studied by Griggs and Yeh in 1992 [10], is a function f from the vertex set of G to the set of nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$, and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between the pair of vertices x, y. These labelings have been used to model the channel assignment problem [12], wherein transmitters in close proximity receive frequencies that are sufficiently far apart to avoid interference. L(2,1)-labelings and their various generalizations have spawned a vast literature, as described in two recent surveys [3,16].

An L(2,1)-labeling of a graph G that uses labels in the set $\{0, 1, \ldots, k\}$ (not necessarily all of them) is called a k-*labeling*. The minimum k so that G has a k-labeling is called the *lambda number* of G and is denoted by $\lambda(G)$. A $\lambda(G)$-labeling is referred to simply as a λ-labeling. A k-labeling is said to have a *hole* h with $0 < h < k$, if the label h is not used. The minimum number of holes over all λ-labelings of a graph G is called the *hole index* of G and is denoted by $\rho(G)$. It is not difficult to see that if a λ-labeling has exactly $\rho(G)$ holes, then any two holes are non-consecutive. Several papers [1,4,5,7–9,13–15] have studied $\rho(G)$ and have investigated its connections with $\lambda(G)$ and $\Delta(G)$, the maximum degree of vertices in G.

An *island* of a given λ-labeling of a graph G with $\rho(G)$ holes is a maximal set of consecutive integers used by the labeling. The *island sequence* of a given λ-labeling of a graph G with $\rho(G)$ holes is the ordered sequence of island cardinalities in non-decreasing order (note that this definition allows for repeated cardinalities). We say that the λ-labeling *induces* and that G

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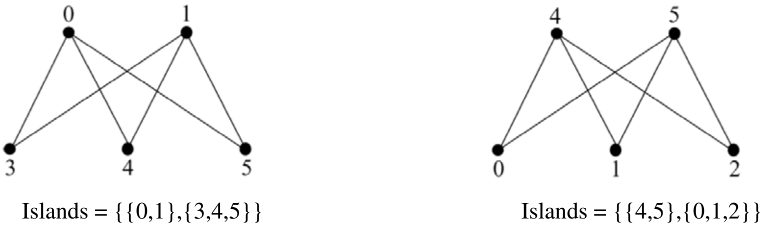

Fig. 1. Two 5-labelings of $K_{2,3}$ with 1 hole and island sequence (2,3).

admits the island sequence. Fig. 1 presents two different λ-labelings of the complete bipartite graph $K_{2,3}$ with $\lambda(K_{2,3}) = 5$, $\rho(K_{2,3}) = 1$ hole, and inducing the same island sequence (2,3).

Georges and Mauro [7] provided examples of non-connected graphs admitting at least two different island sequences, and they left it as an open problem to determine whether there exist any connected graphs with the same property. We solve this problem by providing an infinite family of connected graphs that each admit at least two different island sequences.

In Section 2, we study *2-sparse graphs*, our term for graphs that contain no two adjacent vertices of degree greater than 2, and we answer the question posed by Georges and Mauro [7] by showing that most complements of 2-sparse trees are connected graphs admitting at least two different island sequences. In Section 3, we extend some of the results in Section 2 and prove that if G is a connected 2-sparse graph different from a cycle, then $\lambda(G^c) = p+m-2$, and $\rho(G^c) = p+m-n-1$ whenever $p+m-n \geq 2$, where G^c denotes the complement of G, and n, m, p are respectively the number of vertices, the number of edges, and the number of degree 1 vertices of G. The paper is concluded in Section 4.

2. A family of graphs with distinct island sequences

In this section, we use the concept of a path covering and a specific family of trees to answer the open question of Georges and Mauro [7] concerning whether there exist any connected graphs that admit two or more distinct island sequences.

A *path covering* of a graph G is a set of vertex disjoint paths of G containing all the vertices of G. The *path covering number* of G, denoted by $c(G)$, is the minimum number of paths in a path covering of G. A path covering with exactly $c(G)$ paths is called a *minimum path covering*.

We say that a graph is *2-sparse* if it contains no pair of adjacent vertices of degree greater than 2. A vertex in a graph is *heavy* if it has degree greater than 2, otherwise we say this vertex is *light*. Thus, a graph is 2-sparse if it does not contain any pairs of adjacent heavy vertices.

We now introduce some notation and lemmas that will be used to determine the path covering number $c(T)$ when T is a 2-sparse tree different from a path. We note that determining the path covering number of arbitrary graphs is NP-hard, but there are polynomial-time algorithms to determine the path covering number of trees (e.g., [2,6,11]). Surprisingly, however, not many exact values for the path covering number of special families of trees are known.

If e is an edge of a graph G then $G-e$ is the graph obtained by deleting e from G but not its end vertices. If H is a subgraph of G, the graph $G-H$ is the graph obtained by deleting the vertices of H from G and any edge incident to a vertex in H. (Note that if e is an edge in G and H is the subgraph of G containing e only, then the graphs $G-e$ and $G-H$ are different as the ends of e are in $G-e$ but not in $G-H$.) If f is an edge not in G but its ends are in G, then $G+f$ is the graph obtained by adding f to G. If two graphs G and H are disjoint, the graph $G+H$ is defined as the graph with vertex and edge sets given respectively by the union of the vertex and edge sets of G and H.

The following two lemmas provide connections between path coverings and heavy or light vertices, and they will be used in what follows.

Lemma 2.1. *If v is a heavy vertex in a 2-sparse graph G, then v is an internal vertex in every minimum path covering of G.*

Proof. Assume for contradiction that the heavy vertex v is the end of a path P in a minimum path covering of the 2-sparse graph G. Since v is heavy and G is 2-sparse, there are at least two light vertices incident to v, which are not in P. Therefore these vertices must be ends of paths in the same path covering. Select one of these paths, different from P, and call it Q. Let f be the edge incident to v and to one of the ends of Q. Since P and Q are different, $(P+Q)+f$ is a path. Replacing paths P and Q with $(P+Q)+f$, we obtain a path covering of G with a smaller number of paths, contradicting the minimality of the original path covering. Hence, heavy vertices in 2-sparse graphs cannot be endpoints of any path within a minimum path covering. ■

Lemma 2.2. *If e is an edge incident to two light vertices of a tree T, then e is contained in every minimum path covering of T.*

Proof. Let u and w be the two light vertices incident to e. Assume for contradiction that there is a minimum path covering of T not containing e. Since u and w are light, there exists a path P ending at u and a path Q ending at w in this minimum path covering such that neither P nor Q contains e. Since T does not contain cycles, P and Q are different and therefore $(P+Q)+e$ is a path. Replace paths P and Q with $(P+Q)+e$ to obtain a path covering of T with smaller number of paths, contradicting the minimality of the initial path covering. ■

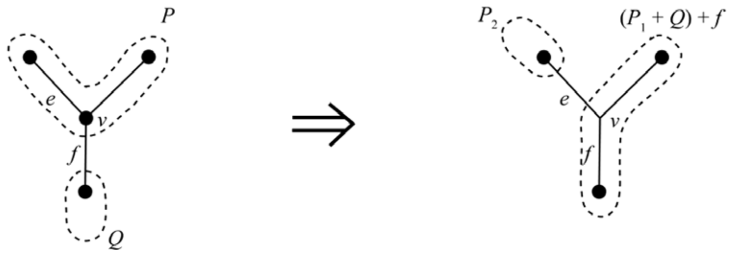

Fig. 2. The swapping construction.

A *vine* S of a tree T is defined as a maximal path in T such that one endpoint is a leaf of T and each vertex in S is a light vertex in T. Clearly, if T has at least 2 vertices, then it contains a vine. Moreover, if T is not a path, then there is a unique heavy vertex of T adjacent to one of the ends of S. We call such a vertex the *center* of vine S. The following lemma illuminates a special property of vines of 2-sparse trees.

Lemma 2.3. *Let S be a vine in a 2-sparse tree T. Then S is a subgraph of every minimum path covering of T.*

Proof. Let $S = v_0v_1 \dots v_k$ where v_0 is the leaf of T in S, and v_i and v_{i+1} are adjacent for $i = 0, 1, \dots, k-1$. Given a minimum path covering of T, let P be the path containing v_0 in this path covering. Let j be the largest integer so that $v_0, v_1, \dots, v_j$ are in P. If $j < k$ then v_j and v_{j+1} are two light vertices and by Lemma 2.2, the edge between v_j and v_{j+1} must also be in P, a contradiction. So, $j = k$ and therefore S is a subgraph of the given minimum path covering. ■

We now introduce a construction that will be used below in Theorem 2.4. Let P and Q be two different paths in a given path covering of a graph G such that P contains an edge e incident to an internal vertex v of P, and one end of Q is adjacent to v through an edge f. Note that $P - e$ has two connected components, namely the paths P_1 and P_2 where v is an end of P_1. Since P and Q are different paths, $(P_1 + Q) + f$ is a path. We can replace paths P and Q with paths $(P_1 + Q) + f$ and P_2 to obtain another path covering of G with the same number of paths. For convenience, we will say that this new path covering was obtained from the original one by *swapping e with f*. Fig. 2 illustrates this swapping construction.

Theorem 2.4. *Let T be a 2-sparse tree with p leaves. If T is not a single vertex, then $c(T) = p - 1$.*

Proof. Since T has more than one vertex, $p \geq 2$. The proof will proceed by induction on p. If $p = 2$, then T is a path and $c(T) = 1 = p - 1$. Suppose $p > 2$ and that the result holds for all 2-sparse trees with k leaves where $2 \leq k < p$. Let S be a vine in T with center v. Thus $T - S$ is a 2-sparse tree with $p - 1 \geq 2$ leaves and by the induction hypothesis, $c(T - S) = (p - 1) - 1 = p - 2$. If we add S to any minimum path covering of $T - S$ with $p - 2$ paths, we obtain a path covering of T with $p - 1$ paths and therefore $c(T) \leq p - 1$. Assume for contradiction that $c(T) \leq p - 2$ and consider an arbitrary minimum path covering of T. From Lemma 2.3, S is a subgraph of some path P in this minimum path covering. If P does not contain v, the center of S, then $P = S$ and by deleting S from the current path covering we would obtain a path covering of $T - S$ with $c(T) - 1 \leq p - 3$ paths, contradicting $c(T - S) = p - 2$. Thus, P must contain v and by Lemma 2.1, v is internal in P. Let e be the edge in P incident to v and to one of the ends of S. Let f be another edge adjacent to v, not contained in P (f must exist because v is heavy). The other end of f is light and consequently it must be the end of another path Q different from P in the minimum path covering of T. By swapping e with f, we obtain another path covering of T with $c(T)$ paths, one of which is exactly S. Thus, by deleting S, we obtain a path covering of $T - S$ with $c(T) - 1 \leq p - 3$ paths, which again contradicts $c(T - S) = p - 2$. We conclude that $c(T) = p - 1$. ■

Theorem 2.4 is important because it adds to the limited library of known path covering numbers. Also, when combined with the following prior results by Georges et al. [9] and Georges and Mauro [7], it implies Corollary 2.7 below.

Result 2.5 (*Georges et al. [9]*). Let G be a graph on n vertices. Then $c(G^c) \geq 2$ if and only if $\lambda(G) = n + c(G^c) - 2$.

Result 2.6 (*Georges and Mauro [7]*). Let G be a graph on n vertices and $\lambda(G) \geq n - 1$. Then $\rho(G) = c(G^c) - 1$.

Corollary 2.7. *Let T be a 2-sparse tree with n vertices and p leaves. If T is not a path, then $\lambda(T^c) = n+p-3$ and $\rho(T^c) = p-2$.*

Proof. Since T is not a path, $p \geq 3$. By Theorem 2.4, $c(T) = p-1 \geq 2$ and therefore Result 2.5 implies $\lambda(T^c) = n+c(T)-2 = n + p - 3 \geq n - 1$. Consequently, Result 2.6 implies $\rho(T^c) = c(T) - 1 = p - 2$. ■

Theorem 2.4 and Corollary 2.7 are instrumental in answering Georges and Mauro's question [7] on the existence of a connected graph admitting at least two different island sequences. Before presenting the main result of this section, we need to extend the classical definition of stars. We define a *generalized star* as a tree with exactly one heavy vertex v, where all vines have the same number of vertices.

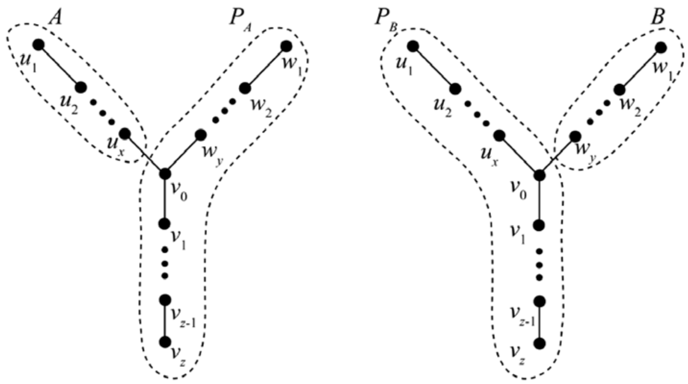

Fig. 3. Two minimum path coverings of a 2-sparse tree T with 3 leaves, T not a generalized star.

Theorem 2.8. *Let T be a 2-sparse tree. If T is neither a path nor a generalized star, then T^c admits at least two different island sequences.*

Proof. Since T is not a path, the number of leaves $p \geq 3$. The proof proceeds by induction on p. For the base case, consider $p = 3$. T must have exactly three vines $A = u_1u_2 \ldots u_x$, $B = w_1w_2 \ldots w_y$, and $C = v_1v_2 \ldots v_z$ with same center v_0 adjacent to u_x, w_y, and v_1. Since T is not a generalized star, we may assume without loss of generality that $x < y$. By Theorem 2.4, $c(T) = 2$, and by Corollary 2.7, $\lambda(T^c) = x + y + z + 1$ and $\rho(T^c) = 1$. So the vine A together with the path P_A induced by the vertices in $\{w_1, w_2, \ldots, w_y\} \cup \{v_0, v_1, \ldots, v_z\}$ form a minimum path covering of T. Similarly, the vine B together with the path P_B induced by the vertices in $\{u_1, u_2, \ldots, u_x\} \cup \{v_0, v_1, \ldots, v_z\}$ form another minimum path covering of T. Fig. 3 shows these two minimum path coverings. These path coverings induce the following λ-labelings f_A and f_B of T^c with exactly $\rho(T^c) = 1$ hole each:

$$f_A(u_i) = i - 1, \quad \text{for } i = 1, 2, \ldots, x, \qquad f_A(w_i) = x + i, \quad \text{for } i = 1, 2, \ldots, y,$$
$$f_B(w_i) = i - 1, \quad \text{for } i = 1, 2, \ldots, y, \qquad f_B(u_i) = y + i, \quad \text{for } i = 1, 2, \ldots, x,$$
$$f_A(v_i) = f_B(v_i) = x + y + i + 1, \quad \text{for } i = 0, 1, \ldots, z.$$

(We refer the reader to [9] for a complete proof of the more general result which states that a minimum path covering of the complement of a graph G induces a $\lambda(G)$-labeling of G with $c(G^c) - 1$ holes.) The island sequence of f_A is $(x, y+z+1)$ as $x < y$. On the other hand, the island sequence of f_B is either $(y, x+z+1)$ or $(x+z+1, y)$, both different from $(x, y+z+1)$. Therefore T^c admits two different island sequences.

For the inductive step, assume that $p > 3$ and that for each 2-sparse tree which is not a generalized star and which has k $(3 \leq k < p)$ leaves, its complement admits at least two different island sequences. Consider an arbitrary 2-sparse tree T with $p > 3$ leaves such that T is not a generalized star. In the remainder of this paragraph, we will select a vine S of T and prove that $T - S$ is not a generalized star by showing that either $T - S$ has more than one heavy vertex or it has two vines with different numbers of vertices. Let S_1 be an arbitrary vine of T. Clearly $T - S_1$ has at least three vines since T has $p > 3$ leaves. If $T - S_1$ is not a generalized star, select $S = S_1$. Let us consider the case where $T - S_1$ is a generalized star with center a and vines with q vertices each. If a is also the center of S_1 in T, then the number of vertices in S_1 is not q, thus by selecting any vine S of $T - S_1$, $T - S$ has two vines with different numbers of vertices. We still need to discuss the case where the center b of S_1 in T is different from a. Let S be a vine of $T - S_1$ not containing b. If a is a heavy vertex in $T - S$, then $T - S$ has two heavy vertices a, b. If a is a light vertex in $T - S$, then $T - S$ has exactly three vines with center b: S_1, the vine S_2 containing a, and the vine S_3 not containing a. The vine S_2 has more than q vertices because it contains a vine of $T - S_1$ and its center a; the vine S_3 has less than q vertices because it has center b and is contained in a vine of $T - S_1$ with center a.

We conclude that $T - S$ is not a generalized star. Moreover, $T - S$ is a 2-sparse tree with $p - 1 \geq 3$ leaves, so by the induction hypothesis for such trees with fewer than p leaves, $H = (T - S)^c$ admits two different island sequences. Let g_1 and g_2 be the two $\lambda(H)$-labelings of H with $\rho(H)$ holes inducing these two island sequences. Extend both labelings g_1 and g_2 to $S = v_1v_2 \ldots v_s$ by assigning label $\lambda(H) + i + 1$ to vertex v_i for each $i = 1, 2, \ldots, s$. It is not difficult to verify that these two new labelings are $(\lambda(H)+s+1)$-labelings of T^c with $c(T-S)+1$ holes. But by Corollary 2.7, $\lambda(H)+s+1 = n+p-3 = \lambda(T^c)$, and by Theorem 2.4, $c(T - S) + 1 = p - 1 = c(T)$. Therefore T^c admits two different island sequences. ■

The connectedness of the family of graphs in Theorem 2.8 is established in the next lemma.

Lemma 2.9. *Let T be a 2-sparse tree. If T is neither a path nor a star, then its complement T^c is connected.*

Proof. Since T is not a path, T has at least 3 leaves. Let v_1, v_2, v_3 be different leaves of T and let u, z be any two different vertices (note that $\{v_1, v_2, v_3\}$ and $\{u, z\}$ might intersect). To prove that T^c is connected, we will show that there exists a path connecting u and z in T^c. This statement is trivially verified when u and z are adjacent in T^c. Let us then assume that

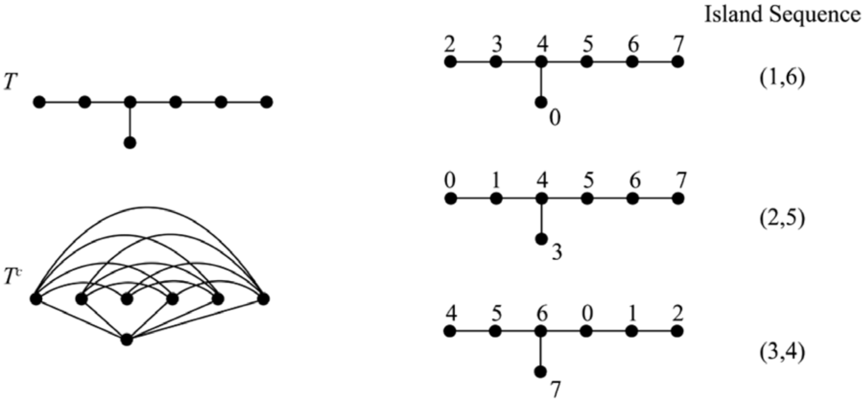

Fig. 4. 2-sparse tree T, its complement T^c,and three λ-labelings of T inducing different island sequences.

u and z are not adjacent in T^c, or equivalently, u and z are adjacent in T. First note that for each $i = 1, 2, 3$, either u or z or both will be adjacent to v_i in T^c because v_1, v_2,v_3 have degree 1 in T. If u and z are both adjacent in T^c to the same v_i, then uv_iz is a path in T^c. If u and z are adjacent in T^c to v_i and v_j, respectively, with $i \neq j$, then uv_iv_jz is a path in T^c (since T is connected and is not a path, v_i and v_j cannot be adjacent in T). The only case left to be examined is when exactly one of u and z is adjacent in T^c to every v_i, and the other is adjacent in T to every v_i. We may assume without loss of generality that u is adjacent in T^c to every v_i, and z is adjacent in T to every v_i. In this case, every v_i is adjacent in T^c to all the other vertices except for z because every v_i has degree 1 in T. Since T is a tree but not a star, there exists a vertex w not contained in $\{u, z, v_1, v_2, v_3\}$ so that w is adjacent to z in T^c. So uv_1wz is a path in T^c and the proof is complete. ■

Theorem 2.8 and Lemma 2.9 together settle the question posed by Georges and Mauro [7]. For clarity, we state this result as the following corollary.

Corollary 2.10. *There exists an infinite family of connected graphs that admit at least two different island sequences.*

In Fig. 4, we provide three copies of the same 2-sparse tree T where the vertex labelings are actually λ-labelings of the complement T^c inducing different island sequences.

3. Some invariants of 2-sparse graphs and their complements

In this section we extend the results in Theorem 2.4 and Corollary 2.7 to more general 2-sparse graphs. We begin with the following preliminary result.

Lemma 3.1. *Let G be a 2-sparse graph. If e is an edge incident to a heavy vertex and e is contained in a cycle in G, then $c(G - e) = c(G)$.*

Proof. Let e be an edge in the 2-sparse graph G incident to a heavy vertex v so that e is contained in a cycle in G. Clearly, $c(G - e) \geq c(G)$. Consider an arbitrary minimum path covering of G. We will use this minimum path covering to construct a path covering of $G - e$ with exactly $c(G)$ paths, which would imply that $c(G - e) = c(G)$. By Lemma 2.1, there exists a path P in the minimum path covering so that v is internal in P. If e is not in P, then this path covering of G is obviously a path covering of $G - e$ with $c(G)$ paths. So, we will consider the case when e is in P. Let g be the edge in P that is incident to v and e. Since v is heavy and G is 2-sparse, the $k \geq 1$ vertices $v_1, v_2, \ldots, v_k$ adjacent to v but not incident to e or g are light and must be the ends of paths in the minimum path covering of G. If one of these k paths, call it Q, is different from P, then let f be the edge incident to v and to one of the ends of Q, say v_i for some $1 \leq i \leq k$. By swapping e with f (recall that this construction was defined immediately preceding Theorem 2.4), we obtain a path covering of $G - e$ with $c(G)$ paths. So, it remains to consider the case where each of the k light vertices $v_1, v_2, \ldots, v_k$ are ends of P. Clearly, this is only possible if $k = 1$ or $k = 2$.

If $k = 2$, then v has degree 4, and v_1 and v_2 are the two ends of P. For each $i = 1, 2$, let e_i be the edge incident to v and v_i. Replacing P with the path $(P + e_1 + e_2) - e - g$, we obtain a path covering of $G - e$ with $c(G)$ paths. Fig. 5 illustrates this construction.

If $k = 1$, then v has degree 3 and v_1 is one of the ends of P. To simplify the notation, we will refer to v_1 as w. Let e_3 be the edge incident to v and w. If v and w are in different connected components of $P - e$, then $(P + e_3) - e$ is a path, and by replacing P with $(P + e_3) - e$, we obtain a path covering of $G - e$ with $c(G)$ paths. Fig. 6 illustrates this construction. We must now consider the case where v and w are in the same connected component of $P - e$. Let P_1 be the connected component of $P - e$ containing v and w, and let P_2 be the other component. Recall that e is contained in a cycle C of G. Since v has degree 3, C must necessarily contain either edge e_3 or edge g but not both. It then follows that C and P_1 must have a heavy vertex

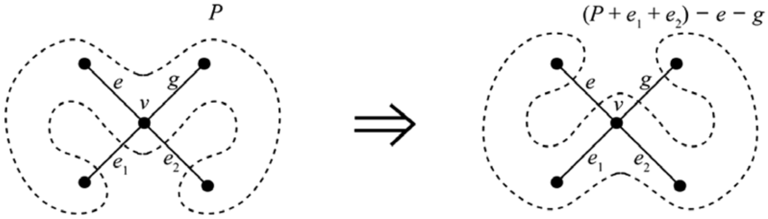

Fig. 5. Construction of a path covering of $G - e$ with $c(G)$ paths for the case $k = 2$.

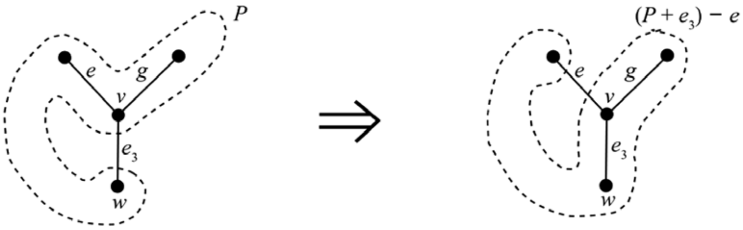

Fig. 6. Construction of a path covering of $G - e$ with $c(G)$ paths for the case $k = 1$ and v, w in different connected components of $P - e$.

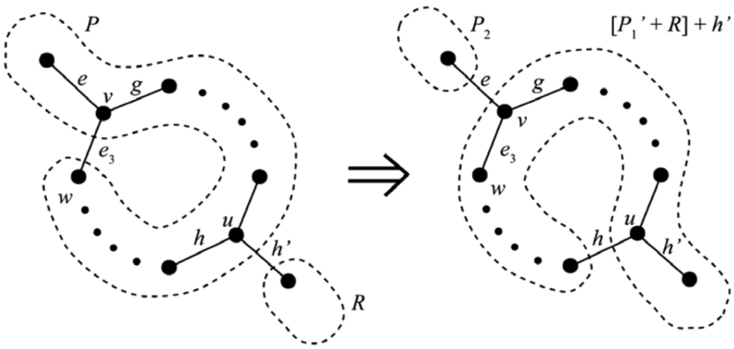

Fig. 7. Construction of a path covering of $G - e$ with $c(G)$ paths for the case $k = 1$ and v, w in the same connected component of $P - e$.

u in common different from v. Let h be an edge in P_1 incident to u. Clearly, h is different from e_3 and g, otherwise the heavy vertices u and v would be adjacent in a 2-sparse graph. So, $P_1' = (P_1 + e_3) - h$ is a path containing the same vertices as P_1 and ending at u. Let h' be an edge incident to u but not in P_1. The other end of h' is a light vertex, so there is a path R in the minimum path covering ending at this light vertex. Thus, $[P_1' + R] + h'$ is a path. Replacing P and R with P_2 and $[P_1' + R] + h'$ we obtain a path covering of $G - e$ with $c(G)$ paths. (Note that if $P \neq R$, then we are replacing two paths on the minimum path covering with two new paths; if $P = R$, then we are replacing only one path on the minimum path covering with one new path.) Fig. 7 illustrates this final construction for one of the possible two choices for h as an edge in P_1 incident to u. The illustration for the other case is similar. ■

We are now prepared to present our main result in this Section, the determination of the path covering number of non-cycle connected 2-sparse graphs.

Theorem 3.2. *Let G be a connected 2-sparse graph with $m \geq 1$ edges, n vertices, and p vertices of degree 1. If G is not a cycle, then $c(G) = p + m - n$.*

Proof. The proof proceeds by induction on the number of edges m. If $m = 1$, then G is a path with $n = 2$, $p = 2$ and $c(G) = 1 = p + m - n$. Let us assume that $m > 1$ and that the result holds for any connected non-cycle 2-sparse graph with k edges where $1 \leq k < m$. Consider G a connected non-cycle 2-sparse graph with m edges, n vertices, and p vertices of degree 1. If G is a tree, then $m = n - 1$ and p is the number of leaves of G, thus the result follows from Theorem 2.4. Suppose G is not a tree, so it contains a cycle C. Since G is not a cycle, C must contain a heavy vertex v. Let e be an edge in C incident to v. Note that the other end of e must necessarily be a vertex of degree 2. Therefore, $G - e$ is a connected non-cycle 2-sparse graph with $m - 1$ edges, n vertices, $p + 1$ vertices of degree 1, and by the inductive hypothesis $c(G - e) = (p + 1) + (m - 1) - n = p + m - n$. But by Lemma 3.1, $c(G) = c(G - e)$ and the result follows. ■

Our final corollary below determines the lambda number and hole index of complements of certain non-cycle connected 2-sparse graphs.

Corollary 3.3. *Let G be a connected 2-sparse graph with $m \geq 1$ edges, n vertices, and p vertices of degree 1. If G is not a cycle and $p + m - n \geq 2$ then $\lambda(G^c) = p + m - 2$ and $\rho(G^c) = p + m - n - 1$.*

Proof. Suppose that G is not a cycle. By Theorem 3.2, $c(G) = p + m - n \geq 2$ and therefore Result 2.5 implies $\lambda(G^c) = n + c(G) - 2 = n + (p + m - n) - 2 = p + m - 2$. Since $p + m - n \geq 2$, we have that $\lambda(G^c) = p + m - 2 \geq n$ and by Result 2.6, $\rho(G^c) = c(G) - 1 = (p + m - n) - 1$. ■

4. Conclusions

In this paper, we have answered the open question of Georges and Mauro [7] regarding the existence of connected graphs that admit at least two distinct island sequences. We solved this problem by studying complements of 2-sparse trees. It would be interesting to investigate the existence of other families that admit multiple island sequences.

We also determined the path covering number of 2-sparse trees and of more general connected non-cycle 2-sparse graphs. Additionally, we determined the lambda number and hole index for complements of non-path 2-sparse trees and for complements of certain non-cycle 2-sparse graphs. We hope that these results will be extended to include more general trees and graphs.

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