On the maximal distance between triangular embeddings of a complete graph

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Abstract

The distance \(d(f, f')\) between two triangular embeddings \(f\) and \(f'\) of a complete graph is the minimal number \(t\) such that we can replace \(t\) faces in \(f\) by \(t\) new faces to obtain a triangular embedding isomorphic to \(f'\). We consider the problem of determining the maximum value of \(d(f, f')\) as \(f\) and \(f'\) range over all triangular embeddings of a complete graph. The following theorem is proved: for every integer \(s \geq 9\), if \(4s + 1\) is prime and \(2\) is a primitive root modulo \((4s + 1)\), then there are nonorientable triangular embeddings \(f\) and \(f'\) of \(K_{12s + 4}\) such that \(d(f, f') \geq (1/2)(4s + 1)(12s + 4) - O(s)\), where \((4s + 1)(12s + 4)\) is the number of faces in a triangular embedding of \(K_{12s + 4}\). Some number-theoretical arguments are advanced that there may be an infinite number of odd integers \(s\) satisfying the hypothesis of the theorem.

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1. Introduction

A triangular embedding of a complete graph in a surface is a cellular embedding of the graph such that all faces are 3-gonal.

Constructing triangular embeddings of complete graphs was a major step in proving the Map Color Theorem [8]. Now it is known [1,3,6,7] that the size of the set of all nonisomorphic triangular embeddings of a complete graph \(K_n\) grows dramatically as \(n\) increases. It is of great mathematical interest to investigate the structure of the set. The first step here is to investigate differences between two triangular embeddings.
Theorem 1. For every integer $d(f, f')$ the distance between two embeddings from the set (see [2]). We will consider the face set of a triangular embedding of a graph as the set of unordered triples $[x, y, z]$, where a triple $[x, y, z]$ denotes the face incident with the vertices $x$, $y$, and $z$ of the embedded graph. Let $f$ and $f'$ be two triangular embeddings of a complete graph $K$. For a bijection $\phi : V(K) \rightarrow V(K)$ denote by $M(f, f'|\phi)$ the set of faces $[x, y, z]$ of $f$ such that $[\phi(x), \phi(y), \phi(z)]$ is not a face of $f'$. The distance $d(f, f')$ between the embeddings $f$ and $f'$ is the minimal value of $|M(f, f'|\phi)|$ as $\phi$ ranges over all bijections between the vertices of $K$. Note that if $f$ and $f'$ are isomorphic, then $d(f, f') = 0$. Roughly speaking, the distance $d(f, f')$ is the minimal number $t$ such that we can replace $t$ faces in $f$ by $t$ new faces to obtain a triangular embedding isomorphic to $f'$.

The minimal nonzero distance between two triangular embeddings $f$ and $f'$ of a complete graph was investigated in [2] where it was shown that 4 is the minimal nonzero distance in the case when $f$ and $f'$ are both nonorientable, and that 6 is the minimal distance in each of the cases when $f$ and $f'$ are orientable, and when $f$ is orientable and $f'$ is nonorientable.

In the present paper, we consider the problem of determination of the maximum value of $d(f, f')$ as $f$ and $f'$ range over all triangular embeddings of a complete graph $K_n$. Every triangular embedding of $K_n$ contains a disc consisting of $n - 1$ faces incident with the same vertex, that is, every two triangular embeddings of $K_n$ have at least $n - 1$ faces in common (up to relabeling of the vertices of the graph), hence, $d(f, f') \leq n(n - 1)/3 - (n - 1)$, where $n(n - 1)/3$ is the number of faces in a triangular embedding of $K_n$.

Conjecture 1. There is a constant $0 < c < 1$ such that for every $n$ such that $K_n$ has triangular embeddings, for every two triangular embeddings $f$ and $f'$ of $K_n$

$$d(f, f') \leq c \cdot \frac{n(n - 1)}{3},$$

where $n(n - 1)/3$ is the number of faces in a triangular embedding of $K_n$.

In other words, the conjecture claims that every two triangular embeddings of a complete graph $K_n$ have many, namely at least $(1 - c) \cdot \frac{n(n - 1)}{3}$, faces in common (up to relabeling of vertices of the complete graph). Either answer to the conjecture, positive or negative, will give us an insight into how different two triangular embeddings of a complete graph can be.

In the present paper, using current graphs, we show how to construct two triangular embeddings of a complete graph such that the distance between the embeddings is large. The main result of the paper is the following.

Theorem 1. For every integer $s \geq 9$, if $4s + 1$ is prime and 2 is a primitive root modulo $4s + 1$, then there are nonorientable triangular embeddings $f$ and $f'$ of $K_{12s+4}$ such that

$$d(f, f') \geq \frac{1}{2}(4s + 1)(12s + 4) - O(s),$$

where $(4s + 1)(12s + 4)$ is the number of faces in a triangular embedding of $K_{12s+4}$.

Recall that 2 is a primitive root modulo prime $p$, if $p - 1$ is the least possible integer $h$ such that $2^h \equiv 1 \mod p$. 

Now we advance some number-theoretical arguments that there may be an infinite number of integers \( s \) satisfying the hypothesis of the theorem. Artin’s conjecture [5, Chapter 4] states that given an integer \( m \) such that \( m \) is not the square of another integer and \( m \neq -1 \), there are infinitely many prime integers \( p \) such that \( m \) is a primitive root modulo \( p \). As for the number 2, there is a stronger version of Artin’s conjecture: the prime numbers \( p \) such that 2 is a primitive root modulo \( p \) have positive density in the set of all prime numbers. These conjectures are still very much open.

By the law of quadratic reciprocity [5, Chapter 5], 2 can be a primitive root modulo \( p \) only if the prime \( p \) is of the form \( 8k + 3 \) or \( 8k + 5 \). Since 8 and 5 are relatively prime, then, by Dirichlet’s theorem [5, Chapter 16], the arithmetic progression \( 8k + 5, k = 1, 2, 3, \ldots \) contains an infinite number of prime integers.

Hence, it might be suggested that the arithmetic progression \( 4s + 1, s = 9, 10, 11, \ldots \) contains infinitely many prime integers \( p \) of the form \( 8k + 5 \) such that 2 is a primitive root modulo \( p \).

Using a computer, the author obtained that the intervals \([0, 1000], [1001, 2000], [2001, 3000], [3001, 4000], [4001, 5000]\) contain, correspondingly, 193, 149, 154, 141, 150 values of \( k \), such that \( 8k + 5 \) is prime and 2 is a primitive root modulo \( 8k + 5 \).

The main idea of the proof of Theorem 1 is as follows. Two faces of an embedding are adjacent if they share a common edge. By a link for vertices \( x \) and \( y \) of an embedding we mean any pair \([x, v, w], [y, v, w]\) of adjacent triangular faces of the embedding. To prove the theorem, we use current graphs (cascades, for short) to construct triangular embeddings \( f \) and \( f' \) of the same complete graph such that in \( f \) some pairs of vertices have a large number of links and in \( f' \) every pair of vertices has a small number of links. Note that in a triangular embedding every link is uniquely determined by the common edge of the adjacent faces, and every edge is a common edge of some two adjacent faces. Hence, the total number of links in a triangular embedding equals the number of edges.

The paper is organized as follows. In Section 2 the proof of Theorem 1 is given. The proof uses two constructions of cascades given in Section 4. To facilitate construction checking in Section 4, some material about cascades is given in Section 3.

2. Proof of Theorem 1

In Section 4, for every integer \( s \geq 9 \) such that \( 4s + 1 \) is prime and 2 is a primitive root modulo \( 4s + 1 \), we construct nonorientable triangular embeddings \( f \) and \( f' \) of \( K_{12s+4} \) such that: in \( f \) there are \( 3(12s + 4) \) different pairs of vertices such that 12s + 4 of them have at least \( 2(2s - 1) \) links, and the other 2(12s + 4) pairs have at least \( s \) links; in \( f' \) every pair of vertices has at most eight links.

Without loss of generality, we can assume that \( f \) and \( f' \) have the same vertex set and that the vertices of the embeddings have been relabeled so that now the number of faces \([x, y, z]\) of \( f \) that are not faces of \( f' \) is \( d(f, f') \).

Now we show that (1) holds for these \( f \) and \( f' \).

For all two vertices \( x \) and \( y \), denote by \( W(x, y) \) (resp. \( W'(x, y) \)) the set of all links between \( x \) and \( y \) in \( f \) (resp. \( f' \)) and denote by \( R(x, y) \) the set of all faces such that the faces enter into links from \( W(x, y) \) and are not faces of \( f' \).

Given a vertex \( x \) and a face \( \theta = [x, y, z] \) of \( f \), denote by \( x_\theta \) the vertex different from \( x \) such that there is a face \([x_\theta, y, z]\) of \( f \). Every face \( \theta = [x, y, z] \) of \( f \) that is not a face of \( f' \) enters into exactly three different \( R(v, w) \), namely, \( R(x, x_\theta), R(y, y_\theta), \) and \( R(z, z_\theta) \). We obtain that the
number \( d(f, f') \) of faces of \( f \) not entering into \( f' \) is

\[
\frac{1}{3} \sum_{(x, y)} |R(x, y)|,
\]

where the sum runs over all unordered pairs \( x, y \) of vertices of the complete graph.

Suppose \( |W(x, y)| > |W'(x, y)| \) for some vertices \( x \) and \( y \). Then at least \( h = |W(x, y)| - |W'(x, y)| \) links between \( x \) and \( y \) in \( f \) are not links in \( f' \), that is, at least \( h \) faces entering into the \( h \) links are not faces of \( f' \). Hence, we have

\[
|R(x, y)| \geq |W(x, y)| - |W'(x, y)|
\]

for these \( x \) and \( y \).

Denote by \( T \) the set of \( 3(12s + 4) \) pairs of vertices such that \( 12s + 4 \) of the pairs have in \( f \) at least \( 2(2s - 1) \) links, and the other \( 2(12s + 4) \) pairs have in \( f \) at least \( s \) links. Since \( |W'(x, y)| \leq 8 \) for all \( x \) and \( y \), we have for \( s \geq 9 \),

\[
d(f, f') = \frac{1}{3} \sum_{(x, y)} |R(x, y)| \geq \frac{1}{3} \sum_{(x, y) \in T} |R(x, y)|
\]

\[
\geq \frac{1}{3} \sum_{(x, y) \in T} \{|W(x, y)| - |W'(x, y)|\}
\]

\[
\geq \frac{1}{3} \{(12s + 4)(2(2s - 1) - 8) + 2(12s + 4)(s - 8)\}
\]

\[
= \frac{1}{2}(4s + 1)(12s + 4) - \frac{55}{6}(12s + 4).
\]

The proof is complete.  

3. Cascades

In this section we briefly review the theory of cascades in the form used in the paper. The reader is referred to [4,8] for more detailed development of the material sketched herein. We assume that the reader is familiar with current graphs, derived graphs and their embeddings generated by current graphs. At the end of the section we give a description of the set of links in the derived embedding.

Permutations are written in cyclic form: \((\delta_1, \delta_2, \ldots, \delta_m)\).

Let \( G \) be a connected graph whose edges have all been given plus and minus directions. Hence, each edge \( e \) gives rise to two reverse arcs \( e^+ \) and \( e^- \) of \( G \). A rotation of a vertex of \( G \) is a cyclic permutation of all arcs directed from the vertex.

Let each edge of \( G \) be assigned the type 0 or 1. Given the edge types, let \( \lambda \) be a function from the arc set of \( G \) into a group \( \Phi \) such that \( \lambda(e^-) = (\lambda(e^+))^{-1} \) for every edge \( e \) of type 0, and \( \lambda(e^-) = \lambda(e^+) \) for every edge \( e \) of type 1. The values of \( \lambda \) are called currents and \( \Phi \) is called the current group. If an edge \( e \) of type 0 is a loop and \( \lambda(e^-) = \lambda(e^+) \) (that is, \( \lambda(e^+) \) is of order 2 in \( \Phi \)), then the arcs \( e^+ \) and \( e^- \) are identified and this arc is called an end arc.

The graph \( G \) with given vertex rotations and edge types determines a cellular embedding \( G \to S \) of \( G \) in a surface \( S \) (see [4, p. 113]). If this embedding has exactly one face, then the pair \( (G \to S, \lambda) \) (with the given vertex rotation and edge types) is called a cascade.

We restrict ourselves in this paper to cascades with abelian current group only. The group operation is written additively.

If \((a_1, a_2, \ldots, a_m)\) is the rotation of a vertex of a cascade \( (G \to S, \lambda) \), where \( \lambda(a_i) = \delta_i \) for \( i = 1, 2, \ldots, m \), then the cyclic sequence \((\delta_1, \delta_2, \ldots, \delta_m)\) is called the current rotation of the
vertex and the element $\delta_1 + \delta_2 + \cdots + \delta_m$ is the excess of the vertex. If the excess of a vertex equals zero, then we say that the vertex satisfies KCL (Kirchhoff’s Current Law).

In Section 4 we will consider a cascade $\langle G \rightarrow S, \lambda \rangle$ with current group $\Phi$, satisfying the following construction principles (A1)–(A4):

(A1) Each vertex is tri-valent or one-valent.
(A2) A nonzero current from $\Phi$ is assigned to each arc. If $|\Phi|$ is even, then every element of $\Phi$ of order $|\Phi|/2$ is the current on an end arc. For every pair $\{\delta, -\delta\}$ of inverse elements of $\Phi$, there is exactly one edge of the cascade such that the arcs of the edge carry currents from $\{\delta, -\delta\}$.
(A3) The excess of every one-valent vertex has order 3 or $|\Phi|$.
(A4) Each tri-valent vertex satisfies KCL.

The cascade generates the derived cellular embedding of the derived graph $K_\langle \Phi \rangle$. The vertex set of the derived graph is the set of all elements of $\Phi$. The edge set of the derived graph consists of the edges $[x, x + \delta]$ for all $x \in \Phi$ and all nonzero elements $\delta$ of $\Phi$. The cascade determines vertex rotations and edge types of the derived graph, thereby generating the derived embedding. Given the vertex rotations and edge types of $G$, we can construct an oriented boundary walk (called circuit) of the unique face of the embedding $G \rightarrow S$. Then the edge types of the derived graph are given by (B)

(B) Let the arcs of an edge of $G$ carry currents from $\{\delta, -\delta\}$. If the circuit traverses the edge in opposite directions (resp. twice in the same direction), then all edges $[x, x + \delta], x \in \Phi$ of the derived graph are of type 0 (resp. 1).

The face set of the derived embedding is determined as follows. There is a mapping from the face set onto the vertex set of the cascade. Given a vertex of the cascade, the faces mapping onto the vertex are the faces induced by the vertex, and they are determined by Theorem 4.4.1 [4], which extends to the nonorientable case as well. By the theorem, for cascades with current group $\Phi$ satisfying (A1)–(A4), the face set of the derived embedding is as follows:

(C) A one-valent vertex with excess $\delta$ of order $|\Phi|$ induces one face: this face is $|\Phi|$-gonal, is incident with all vertices of the derived graph, and the cyclic sequence of vertices obtained by listing the incident vertices when traversing the boundary cycle of the face in some chosen direction is $(0, \delta, 2\delta, \ldots, (|\Phi| - 1)\delta)$. A one-valent vertex with excess $\delta$ of order 3 induces $|\Phi|/3$ triangular faces $[x, x + \delta, x + 2\delta], x = 0, \delta, 2\delta, \ldots, (|\Phi|/3 - 1)\delta$. A tri-valent vertex with current rotation $(\beta, \gamma, \delta)$ induces $|\Phi|$ triangular faces $[x, x + \beta, x + \beta + \gamma], x \in \Phi$.

A cascade $\langle G \rightarrow S, \lambda \rangle$ can be represented as a figure of $G$ where the rotations of vertices are indicated. The black vertices denote a clockwise rotation and the white vertices a counterclockwise rotation. Each pair of reverse arcs with different (inverse) currents is represented by one of the arcs. Each pair of reverse arcs with the same current is depicted as a broken arc. The end arc, as is customary, is depicted as a straight line without an arrow, with a vertex at one and without a vertex at the other end.

We need to know how to determine the number of links between pairs of vertices in the embedding generated by a cascade. The cascade is associated with the embedded dual voltage graph such that the cascade and the embedded dual voltage graph generate the same derived embedding of the same derived graph (see [4]). Every link of the derived embedding is uniquely determined by the common edge of the adjacent triangular faces, we will say that the edge determines the link. Given an edge $e$ of the cascade, the dual edge $e^*$ of the embedded dual voltage graph lifts to the edges of the derived embedding which are in the fiber over the edge $e^*$ (see [4]) and either each of the edges determines a link or none of the edges determines a link.
If each of the edges determines a link, the links are called the *links* of the derived embedding *induced* by the edge $e$ of the cascade. Clearly, the link set of the derived embedding consists of the links induced by edges of the cascade. For the cascades under consideration, every link of the derived embedding is induced either by an edge joining two tri-valent vertices or by an edge joining a tri-valent vertex and a one-valent vertex with excess of order 3.

It is reasonable to ask for which pairs of vertices there is a link induced by a given edge of a cascade. To answer the question we define the index of an edge of a cascade with current group $\Phi$. The index, being an element of $\Phi$, is defined up to inversion.

If an edge joins tri-valent vertices with current rotations $(\gamma, \delta, \beta)$ and $(\eta, \bar{\delta}, \epsilon)$, respectively, where $\bar{\delta} \in \{\delta, -\delta\}$, then the index of the edge is $\gamma + \epsilon$ if $\bar{\delta} = -\delta$, and $\gamma - \eta$ if $\bar{\delta} = \delta$. Since KCL holds at the vertices, we have $\gamma + \epsilon = -(\eta + \beta)$ for $\bar{\delta} = -\delta$, hence the index is well defined.

If an edge joins a tri-valent vertex with current rotation $(\gamma, \delta, \beta)$ and a one-valent vertex with excess $-\delta$, then the index of the edge is $\gamma - \delta$.

Using (C), we can construct links induced by an edge (see Fig. 1).

Considering Fig. 1, we see that the index is defined so that the following holds:

(D) Given a cascade with current group $\Phi$, an edge with index $\Phi$ induces exactly $|\Phi|$ links: one link for every pair $x, x + \Delta$ of vertices, $x \in \Phi$.

Since $x = (x + \Delta) - \Delta$, the set of all pairs $x, x + \Delta$ of vertices, $x \in \Phi$, is the set of all pairs $y, y - \Delta$ of vertices, $y \in \Phi$. This explains why the index can be defined up to inversion.

4. Constructions

In this section two constructions of cascades are given that yield the nonorientable triangular embeddings $f$ and $f'$ of $K_{12s+4}$.

The cascade in Fig. 2 generates the embedding $f$ of $K_{12s+4}$. The reader can easily check that the cascade satisfies the construction principles (A1)–(A4). In this figure, as in Fig. 3 also, the
index of an edge is indicated in a box connected by a line with the edge. Inspecting Fig. 2, the reader can check that the indices on the edges are as follows: the indices 1, 2, and 3 occur $s$, $s$, and $2(2s - 1)$ times, respectively; each of the indices $3s - 2$, $3s - 1$, and $3s + 5$ occurs once. By (D), in $f$ every pair $x, x + 3$ of vertices has $2(2s - 1)$ links, every pair $x, x + 1$ (resp. $x, x + 2$) of vertices has $s$ links, and we see that all these $3(12s + 4)$ pairs of vertices are different.

Now consider the cascade in Fig. 3. The current group is $\mathbb{Z}_{4s+1} \times \mathbb{Z}_3$. The number $4s + 1$ is prime and 2 is a primitive root modulo $4s + 1$. If a current is given as $(\delta, \tau)$, where $\delta$ is an arithmetical expression, then the arithmetic is performed in the ring $\mathbb{Z}_{4s+1}$ (the multiplication is defined in a natural way). The cascade generates an embedding of $K_{12s+3}$ such that one face is $(12s + 3)$-gonal and incident with all vertices of the complete graph, all other faces are triangular (the embedding will be referred to as the basic embedding and the vertices of the embedded graph are called the basic vertices). If we place a new vertex $w$ inside the $(12s + 3)$-gonal face and join the vertex with all vertices on the boundary of the face (as shown in Fig. 4), we obtain the embedding $f'$ of $K_{12s+4}$.
To explain the current assignment of the cascade, we need properties (E1) and (E2) of integers $2^t$ in the case when 2 is a primitive root modulo prime $4s+1$. Recall, that given a prime number $p = 4s+1$, the integer 2 is a primitive root modulo $p$ if $p - 1$ is the least positive integer $h$ such that $2^h - 1 \equiv 0 \mod p$.

For $0 \leq m < n \leq 2s - 1$, we have $0 < 2(n-m) < 4s = p - 1$, hence, $(2^{n-m} + 1)((2^{n-m} - 1) = 2^{2(n-m)} - 1 \not\equiv 0 \mod p$, whence $2^{n-m} + 1 \not\equiv 0 \mod p$, and since $2^m \not\equiv 0 \mod p$, we obtain $2^m(2^{n-m} + 1) = 2^n + 2^m \not\equiv 0 \mod p$. Since $2^{p-1} = 2^{4s} - 1 = (2^{2s} - 1)(2^{2s} + 1) \equiv 0 \mod p$ and $2^{2s} - 1 \equiv 0 \mod p$, we have $2^{2s} + 1 \equiv 0 \mod p$. Now, for every $0 \leq h \leq 2s - 1$, we have $2^h + 2^{2s} + 2^h = 2^h(2^{2s} + 1) \equiv 0 \mod p$, hence, $2^h$ and $2^h + 2^{2s}$ are inverse elements of $Z_{4s+1}$. So we obtain the following:

(E1) The set $\{1, 2, \ldots, 2^{2s-1}\}$ contains exactly one element from every pair of mutually inverse elements of $Z_{4s+1}$.

Since $4s + 1$ is prime, the mapping $\delta \to 3\delta$ for every $\delta \in Z_{4s+1}$ is an automorphism of $Z_{4s+1}$, so we have the following:

(E2) The set $\{3, 3 \cdot 2, \ldots, 3 \cdot 2^{2s-1}\}$ contains exactly one element from every pair of mutually inverse elements of $Z_{4s+1}$.

Now, taking (E1) into account, the reader can easily check that the cascade in Fig. 3 satisfies the construction principles (A1)–(A4). Here an obvious question to ask is what is special about 2 as a primitive root in $Z_{4s+1}$ versus other primitive roots? Clearly, the property (E1) applies to all primitive roots. But the choice of 2 is very suitable when assigning currents so that KCL holds...
at every tri-valent vertex (to check KCL at a tri-valent vertex in Fig. 3 we check that the sum of some two currents equals the third current).

Inspecting Fig. 3, the reader can check that the indices are as follows: for \( t = 0, 1, \ldots, 2s - 3 \), the index \((3 \cdot 2^t, 0)\) occurs thrice; the index \((3 \cdot 2^{2s-2}, 0)\) occurs twice; each of the indices \((2^{2s-2}, 0)\), \((2^{2s-1}, 0)\), \((2^{2s-1}, 2)\), and \((1, 0)\) occurs once.

By (E2), two pairs \((x, x + (3 \cdot 2^m, 0))\) and \((y, y + (3 \cdot 2^n, 0))\) are different pairs of vertices for all \(0 \leq m < n \leq 2s - 2\) and \(x, y \in Z_{4s+1} \times Z_3\). Hence, in the basic embedding every pair of vertices can have at most seven links: at most three links induced by edges with indices \((3 \cdot 2^m, 0)\) and at most four links induced by edges with indices \((2^{2s-2}, 0)\), \((2^{2s-1}, 0)\), \((2^{2s-1}, 2)\), and \((1, 0)\).

After placing a new vertex \(w\) inside the \((12s + 3)\)-gonal face of the basic embedding (see Fig. 4), every two basic vertices gain at most one new link, and for every vertex \(x\) on the boundary of the face, the vertex \(w\) has a link with the vertex \(x - (1, 0)\). Since every basic vertex appears exactly once on the boundary of the face, the vertex \(w\) has exactly one link with every basic vertex. Hence in the embedding \(f'\) every pair of vertices has at most eight links.

Now it remains to show that the embeddings \(f\) and \(f'\) are nonorientable. To prove that the derived embedding of the derived graph is nonorientable it suffices to show [4, p. 110] that the derived graph has a cycle with an odd number of type 1 edges. The reader can easily verify that in the cascade in Fig. 2 (resp. Fig. 3) each of arcs with currents 1 and 2 (resp. \((1,0)\) and \((2,0)\)) is traversed by the circuit twice in the same direction. Hence, taking (B) into account, we see that all edges \([z, z + 1]\) and \([z, z + 2]\) (resp. \([z, z + (1, 0)]\) and \([z, z + (2, 0)]\) of the derived graph are of type 1. It follows that all three edges of the cycle \([0, 2], [2, 1], [1, 0]\) (resp. \([(0, 0), (2, 0)]\), \([(2, 0), (1, 0)]\), \([(1, 0), (0, 0)]\)) of the derived graph are of type 1, hence the derived embedding is nonorientable.

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