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Infinite horizon forward–backward stochastic differential equations☆

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Abstract

A class of systems of infinite horizon forward–backward stochastic differential equations is investigated. Under some monotonicity assumptions, the existence and uniqueness results are established by means of a homotopy method. The global exponential asymptotical stability is also obtained. A comparison theorem is given. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Infinite horizon; Forward–backward stochastic differential equations; Homotopy; Global exponential asymptotical stability; Lyapunov function

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_t\}$ and let $\{B_t\}_{t\geqslant 0}$ be a d-dimensional standard Brownian motion in this space. We will assume that $\{\mathscr{F}_t\}_{t\geqslant 0}$ is the natural filtration of this Brownian motion such that $\mathscr{F}_0$ contains all P-null elements of $\mathscr{F}$. Then $\{\mathscr{F}_t\}$ is right continuous and satisfies the usual hypotheses.

Let $\tau \geqslant 0$ be a given $\mathscr{F}_t$-stopping time with value in $[0, \infty]$. We will denote by $M^2(0, \tau; \mathbb{R}^n)$, the space of all $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted processes (v_t) such that

$$\mathbf{E} \int_0^\tau |v_t|^2 \, \mathrm{d}t < \infty.$$

Obviously $M^2(0, \tau; \mathbb{R}^n)$ is a Hilbert space.

We consider the following classical stochastic differential equations of Itô's type (SDE):

$$\begin{aligned} \mathrm{d}x(t) &= b(t, x(t))\,\mathrm{d}t + \sigma(t, x(t))\,\mathrm{d}B_t, \quad t \geqslant 0, \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

It is well-known that under some suitable conditions on b and σ, the equation has a unique solution $x(t)$. Once $x(t)$ is known, we can consider to solve the following

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backward stochastic differential equation (BSDE) (see Pardoux and Peng, 1990; Duffie and Epstein, 1992a,b):

$$\begin{aligned} -\mathrm{d}y(t) &= f(t,x(t),y(t),z(t))\,\mathrm{d}t - z(t)\,\mathrm{d}B_t, \quad t\in[0,T],\\ y(T) &= \Phi(x(T)). \end{aligned} \tag{1.2}$$

An interesting observation, first revealed in Peng (1991), is that $y(t)$ can be expressed in the form of $y(t)=u(t,x(t))$, where $u(t,x)$ is the solution of the following partial differential equation (PDE):

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 u) + \langle b, Du\rangle + f(t,x,u,Du\sigma) = 0,\\ &u(T,x) = \Phi(x). \end{aligned} \tag{1.3}$$

With this observation a probabilistic interpretation of the solution of the above PDE is introduced in Peng (1991, 1992a,b), Pardoux and Peng (1992), Karoui et al. (1997) etc. This generalizes the well-known Feynman–Kac formula for the nonlinear case.

Note that Eqs. (1.1) and (1.2) are only partially coupled, i.e. they are a special situation of the following fully coupled forward–backward stochastic differential equation (FBSDE):

$$\begin{aligned} \mathrm{d}x(t) &= b(t,x(t),y(t),z(t))\,\mathrm{d}t + \sigma(t,x(t),y(t),z(t))\,\mathrm{d}B_t, \quad x(0)=x_0,\\ -\mathrm{d}y(t) &= f(t,x(t),y(t),z(t))\,\mathrm{d}t - z(t)\,\mathrm{d}B_t, \quad y(T)=\Phi(x(T)). \end{aligned} \tag{1.4}$$

A special form of this FBSDE, called stochastic Hamiltonian equation, is considered in other works (see Bismut, 1978; Kushner, 1972; Bensoussan, 1983a,b; Haussmann, 1976; Peng, 1990). Recently, the existence and uniqueness of this FBSDE received many attentions, see Antonelli (1993) for sufficiently small time duration $[0,T]$; Ma et al. (1994), Ma and Yong (1995) and Duffie et al. (1995) for arbitrarily large duration.

In Hu and Peng (1995) and then in Peng and Wu (1999), a new probabilistic method is applied to solve (1.4) with arbitrarily large time duration $[0,T]$ (see Yong (1997) for a more systematical discussion). A monotonicity condition is needed (see (H1)).

In this paper we are interested in FBSDE of form (1.4) with $T=+\infty$. More specifically, we discuss the problem of existence, uniqueness and the limit behavior of the following FBSDE:

$$\begin{aligned} \mathrm{d}x(t) &= b(t,x(T),y(t),z(t))\,\mathrm{d}t + \sigma(t,x(t),y(t),z(t))\,\mathrm{d}B_t,\\ -\mathrm{d}y(t) &= f(t,x(t),y(t),z(t))\,\mathrm{d}t - z(t)\,\mathrm{d}B_t,\\ x(0) &= x_0, \quad (x(\cdot),y(\cdot),z(\cdot))\in M^2(0,\infty;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}). \end{aligned} \tag{1.5}$$

A special situation is when $\sigma\equiv 0$, $z\equiv 0$, $b=H_y(x,y)$ and $f=H_x(x,y)$, i.e. the following Hamiltonian equation with infinite time horizon:

$$\begin{aligned} &\frac{\mathrm{d}x(t)}{\mathrm{d}t} = H_y(t,x(t),y(t)), \quad x(0)=x_0,\\ &-\frac{\mathrm{d}y(t)}{\mathrm{d}t} = H_x(t,x(t),y(t)), \quad (x(\cdot),y(\cdot))\in L^2(0,\infty;\mathbb{R}^n\times\mathbb{R}^n). \end{aligned} \tag{1.6}$$

This situation arises in problems of singular perturbation of optimal control systems (see Bensoussan, 1988; Bensoussan and Peng, 1986). In those works the following estimate plays an important role:

$$|x(t)| + |y(t)| \leqslant c\mathrm{e}^{-\beta t}.$$

The method introduced in Bensoussan (1988), Bensoussan and Peng (1986) depends essentially on the fact that (1.6) derives an optimal control system. It cannot be applied to a general situation. In this paper we will introduce a very different method to treat the limit behavior of (1.5).

Our method is based on an observation that the scaler product $\langle x(t), y(t)\rangle$ plays a role of Lyapunov function in this problem. We will see that, with this observation, the problem of estimation can be significantly simplified. It also allows us to treat more general situations. For example, we do not have to assume that b, σ and f in (1.5) are deterministic functions.

This paper is organized as follows: in the next section we present the main assumptions in this paper; in Section 3 we give some preliminary results in order to prove the existence and uniqueness theorem in Section 4 and exponential asymptotical stability of FBSDE in Section 6; in Section 5 we give a comparison theorem; finally we discuss a more general situation of FBSDE in Section 7.

For simplicity of notations, in this paper we only consider the case where $n = m$. But using the techniques introduced in Peng and Wu (1999), we can also treat the case $n \neq m$.

2. Setting of the problem

Consider the following infinite horizon FBSDE:

$$\begin{aligned} &\mathrm{d}X(t) = b(t, X(t), Y(t), Z(t))\,\mathrm{d}t + \sigma(t, X(t), Y(t), Z(t))\,\mathrm{d}B_t, \\ &-\mathrm{d}Y(t) = f(t, X(t), Y(t), Z(t))\,\mathrm{d}t - Z(t)\,\mathrm{d}B_t, \\ &X(0) = x_0, \quad (X(\cdot), Y(\cdot), Z(\cdot)) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}), \end{aligned} \tag{2.1}$$

where $x_0 \in \mathbb{R}^n$,

$$b: \Omega \times [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n,$$

$$\sigma: \Omega \times [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d},$$

$$f: \Omega \times [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n.$$

Let us introduce some notations

$$u = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad A(t,u) = \begin{pmatrix} -f \\ b \\ \sigma \end{pmatrix}(t,u).$$

We use the usual inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$ in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$. All the equalities and inequalities mentioned in this paper are in the sense of $\mathrm{d}t \times \mathrm{d}P$ almost surely on $[0,\infty) \times \Omega$.

Definition 1. A triple of processes $(X, Y, Z) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$ is called an adapted solution of FBSDE (2.1), if (2.1) is satisfied.

The following monotonicity condition is our main assumption:
(H1) there exists a constant $\mu > 0$, such that

$$\langle A(t,u) - A(t,\bar{u}), u - \bar{u}\rangle \leqslant -\mu|u-\bar{u}|^2,$$

$$\forall u = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \quad \forall t \in \mathbb{R}^+.$$

We also assume that
(H2) for each $u \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, A(\cdot,u)$ is an $\mathscr{F}_t$-adapted vector process defined on $[0,\infty)$ with $A(\cdot,0) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$;
(H3) $A(t,u)$ is Lipschitz with respect to u: there exists a constant $l > 0$, such that

$$|A(t,u) - A(t,\bar{u})| \leqslant l|u-\bar{u}|, \quad \forall u,\bar{u} \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \ \forall t \in \mathbb{R}^+.$$

In fact, the assumptions (H2) and (H3) guarantee that

$$A(\cdot,u(\cdot)) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}) \quad \text{for } \forall u(\cdot) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}).$$

3. Preliminaries

In order to prove the existence and uniqueness theorem for FBSDE (2.1), we need the following lemmas. It involves a priori estimates of solutions of the following family of infinite horizon forward–backward stochastic differential equations parametrized by $\alpha \in [0,1]$:

$$\begin{aligned} \mathrm{d}X^\alpha(t) &= [\alpha b(t,u^\alpha(t)) - \mu(1-\alpha)Y^\alpha(t) + \phi(t)]\,\mathrm{d}t \\ &\quad + [\alpha\sigma(t,u^\alpha(t)) - \mu(1-\alpha)Z^\alpha(t) + \psi(t)]\,\mathrm{d}B_t, \\ -\mathrm{d}Y^\alpha(t) &= [\alpha f(t,u^\alpha(t)) + \mu(1-\alpha)X^\alpha(t) + \gamma(t)]\,\mathrm{d}t - Z^\alpha(t)\,\mathrm{d}B_t, \\ X^\alpha(0) &= x_0, \quad (X^\alpha(\cdot), Y^\alpha(\cdot), X^\alpha(\cdot)) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}), \end{aligned} \tag{3.1}$$

where ϕ, ψ and γ are given processes in $M^2(0,\infty)$ with values in $\mathbb{R}^n, \mathbb{R}^{n\times d}$ and $\mathbb{R}^n$, respectively.

Observe that when $\alpha = 1$, $\phi \equiv 0$, $\psi \equiv 0$ and $\gamma \equiv 0$, (3.1) becomes (2.1); when $\alpha = 0$, (3.1) is written in the following simple form:

$$\begin{aligned} \mathrm{d}X^0(t) &= (-\mu Y^0(t) + \phi(t))\,\mathrm{d}t + (-\mu Z^0(t) + \psi(t))\,\mathrm{d}B_t, \\ -\mathrm{d}Y^0(t) &= (\mu X^0(t) + \gamma(t))\,\mathrm{d}t - Z^0(t)\,\mathrm{d}B_t, \\ X^0(0) &= x_0, \quad (X^0, Y^0, Z^0) \in M^2(0,\infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}). \end{aligned} \tag{3.2}$$

We have the following lemma:

Lemma 2. *For each* $x_0 \in \mathbb{R}^n$, $\phi, \gamma, \psi \in M^2(0,\infty)$, *FBSDE* (3.2) *has a unique solution in* $M^2(0,\infty;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$.

In order to prove Lemma 2, we need to introduce a kind of backward stochastic differential equations in infinite horizon case. Consider the following infinite horizon backward stochastic differential equations

$$-\,\mathrm{d}Y_t = (G(t,Y_t,Z_t) + \varphi_t)\,\mathrm{d}t - Z_t\,\mathrm{d}B_t, \quad (Y,Z) \in M^2(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d}), \tag{3.3}$$

where

$$G(t,Y,Z) : \Omega \times [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n.$$

To obtain the existence and uniqueness result for (3.3), we need to introduce, for some given constant $K \in \mathbb{R}$, the space $M^{2,K}_{\mathscr{F}_t}(0,\infty;\mathbb{R}^n)$ of all $\mathscr{F}_t$-adapted processes defined on $[0,\infty)$ with

$$\mathbf{E}\int_0^\infty |v_t|^2 \mathrm{e}^{Kt}\,\mathrm{d}t < \infty.$$

It is seen that $M^{2,K}_{\mathscr{F}_t}(0,\infty;\mathbb{R}^n)$ is a Hilbert space. We assume that

$$(\mathrm{H4})\begin{cases} \text{(i) for } \forall (Y,Z) \in \mathbb{R}^{n+n\times d}, G(\cdot,Y,Z) \text{ is a } \mathscr{F}_t\text{-adapted process defined} \\ \text{on } [0,\infty) \text{ with } G(t,0,0) \equiv 0,\ \forall t \in \mathbb{R}^+; \\ \text{(ii) } G(t,Y,Z) \text{ is Lipschitz with respect to } (Y,Z)\text{: } \exists C_0 > 0 \text{ and } C > 0 \text{ s.t.} \\ |G(t,Y_1,Z_1) - G(t,Y_2,Z_2)| \leqslant C_0|Y_1 - Y_2| + C|Z_1 - Z_2|, \\ \qquad\qquad \forall Y_1,Y_2 \in \mathbb{R}^n,\ Z_1,Z_2 \in \mathbb{R}^{n\times d},\ t \in \mathbb{R}^+; \\ \text{(iii) } G \text{ satisfies `weak monotonicity' condition: } \exists \rho > 0, \text{ s.t.} \\ \langle G(t,Y_1,Z) - G(t,Y_2,Z), Y_1 - Y_2\rangle \leqslant -\rho|Y_1 - Y_2|^2, \\ \qquad\qquad \forall Y_1,Y_2 \in \mathbb{R}^n,\ Z \in \mathbb{R}^{n\times d},\ t \in \mathbb{R}^+. \end{cases}$$

Notice that the above constant K may be positive as well as negative. We also assume

$$K + 2\rho - 2C^2 - \delta > 0. \tag{3.4}$$

We are looking for a pair of processes $(Y_t,Z_t) \in M^{2,K}_{\mathscr{F}_t}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$ satisfying (3.3) where $(\varphi_t) \in M^{2,K}_{\mathscr{F}_t}(0,\infty;\mathbb{R}^n)$ is given. Then we have the following a priori estimate.

Lemma 3. *Let* (Y^1,Z^1) *and* $(Y^2,Z^2) \in M^{2,K}_{\mathscr{F}_t}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$ *be solutions of* (3.3) *correspondent to* $\varphi = \varphi^1$ *and* $\varphi = \varphi^2$, *respectively. Then we have*

$$\mathbf{E}\int_0^\infty \left[(K + 2\rho - 2C^2 - \delta)|Y_t^1 - Y_t^2|^2 + \frac{1}{2}|Z_t^1 - Z_t^2|^2\right] \mathrm{e}^{Kt}\,\mathrm{d}t$$
$$\leqslant \frac{1}{\delta}\mathbf{E}\int_0^\infty |\varphi_t^1 - \varphi_t^2|^2 \mathrm{e}^{Kt}\,\mathrm{d}t, \tag{3.5}$$

where $\delta > 0$ *is defined in* (3.4).

Proof. We apply Itô's formula to $|Y_t^1 - Y_t^2|^2 e^{Kt}$

$$
\begin{aligned}
&|Y_0^1 - Y_0^2|^2 + \mathbf{E}\int_0^\infty e^{Kt}[K|Y_t^1 - Y_t^2|^2 + |Z_t^1 - Z_t^2|^2]\,dt \\
&\quad = \mathbf{E}\int_0^\infty 2e^{Kt}\langle Y_t^1 - Y_t^2, G(t, Y_t^1, Z_t^1) - G(t, Y_t^2, Z_t^2) + \varphi_t^1 - \varphi_t^2\rangle\,dt \\
&\quad \leqslant \mathbf{E}\int_0^\infty 2e^{Kt}[-\rho|Y_t^1 - Y_t^2|^2 + |Y_t^1 - Y_t^2|(C|Z_t^1 - Z_t^2| + |\varphi_t^1 - \varphi_t^2|)]\,dt \\
&\quad \leqslant \mathbf{E}\int_0^\infty e^{Kt}[(-2\rho + 2C^2 + \delta)|Y_t^1 - Y_t^2|^2 + \frac{1}{2}|Z_t^1 - Z_t^2|^2 + \frac{1}{\delta}|\varphi_t^1 - \varphi_t^2|^2]\,dt.
\end{aligned}
$$

From this it follows that (3.5) holds true. □

We now consider the existence and uniqueness theorem for (3.3).

Theorem 4. *Under hypothesis* (H4), *for each* $(\varphi_t) \in M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n)$, (3.3) *has a unique solution* (Y_t, Z_t) *in* $M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$.

Proof. The uniqueness is an immediate consequence of the a priori estimate (3.5). We now prove the existence. For $k = 1, 2, 3, \ldots,$ we set

$$\varphi_t^k \equiv \mathbf{1}_{[0,k]}(t)\varphi_t, \quad t \in [0,\infty).$$

It is seen that the $\{(\varphi_t^K)\}$ converges to (φ_t) in $M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n)$.

We now consider, for each k, the solution $(Y_t^k, Z_t^k) \in M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$ of the following BSDE:

$$-\,dY_t^k = [G(t, Y_t^k, Z_t^k) + \varphi_t^k]\,dt - Z_t^k\,dB_t, \quad (Y^k, Z^k) \in M^2(0,\infty;\mathbb{R}^{n+n\times d}). \tag{3.6}$$

Owing to (H4)-(i), (3.6) can be solved as follows: on $[0,k]$, the solution coincides with the solution of BSDE

$$
\begin{aligned}
&-dY_t^k = [G(t, Y_t^k, Z_t^k) + \varphi_t]\,dt - Z_t^k\,dB_t, \quad t \in [0,k], \\
&Y_k^k = 0
\end{aligned}
$$

on (k,∞), it identically equals zero. It follows from a priori estimate (3.5) that $\{(Y_t^k, Z_t^k)\}$ is a Cauchy sequence in $M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$. It is easy to check that the limit (Y_t, Z_t) in $M_{\mathscr{F}_t}^{2,K}(0,\infty;\mathbb{R}^n \times \mathbb{R}^{n\times d})$ solves (3.3). The proof is complete. □

Proof of Lemma 2. The proof of uniqueness is similar to the one of Theorem 6 below. We only need to find a solution of (3.2). We consider the following linear infinite horizon BSDE:

$$-d\,p(t) = [-\mu p(t) + \phi(t) + \gamma(t)]\,dt + [\psi(t) - (1+\mu)q(t)]\,dB_t, \quad t \in [0,\infty).$$

In Theorem 4, we take $K = C = 0$ and $\rho = \delta = \mu$, it follows that the above equation has a unique solution $(p,q) \in M^2(0,\infty;\mathbb{R}^{n+n\times d})$. Once (p,q) is solved, we can solve the following SDE:

$$\mathrm{d}x(t) = [-\mu x(t) + p(t) + \phi(t)]\,\mathrm{d}t + [\psi(t) - \mu q(t)]\,\mathrm{d}B_t, \quad x(0) = x_0.$$

It has a unique solution $x(\cdot) \in M^2(0,\infty;\mathbb{R}^n)$.

It is easy to check that $(X^0(\cdot), Y^0(\cdot), Z^0(\cdot)) = (x(\cdot), x(\cdot) + p(\cdot), q(\cdot))$ is a solution of (3.2). The proof is complete. □

The following lemma gives a priori estimate for the "existence interval" of (3.1) with respect to $\alpha \in [0,1]$.

Lemma 5. *Under assumptions* (H1)–(H3), *there exists a positive constant* δ_0 *such that if, a priori, for some* $\alpha_0 \in [0,1)$, *for each* $x_0 \in \mathbb{R}^n$, $\phi, \gamma, \psi \in M^2(0,\infty)$, (3.1) *has a unique solution* $(X^{\alpha_0}(\cdot), Y^{\alpha_0}(\cdot), Z^{\alpha_0}(\cdot))$ *in* $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$, *then for each* $\delta \in [0,\delta_0]$, *for each* $x_0 \in \mathbb{R}^n$, $\phi, \gamma, \psi \in M^2(0,\infty)$, (3.1) *also has a unique solution* $(X^{\alpha_0+\delta}(\cdot), Y^{\alpha_0+\delta}(\cdot), Z^{\alpha_0+\delta}(\cdot))$ *in* $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$.

Proof. Since for each $x_0 \in \mathbb{R}^n$, $\phi \in M^2(0,\infty,\mathbb{R}^n)$, $\gamma \in M^2(0,\infty;\mathbb{R}^n)$, $\psi \in M^2(0,\infty;\mathbb{R}^{n\times d})$, there exists a unique solution of (3.1) for $\alpha = \alpha_0$, thus for each triple $u = (x,y,z) \in M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$, there exists a unique triple $U = (X,Y,Z) \in M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$ satisfying the following equations:

$$\begin{aligned}
\mathrm{d}X(t) &= [\alpha_0 b(t,U(t)) - \mu(1-\alpha_0)Y(t) + \delta(b(t,u(t)) + \mu y(t)) + \phi(t)]\,\mathrm{d}t \\
&\quad + [\alpha_0\sigma(t,U(t)) - \mu(1-\alpha_0)Z(t) + \delta(\sigma(t,u(t)) + \mu z(t)) + \psi(t)]\,\mathrm{d}B_t, \\
-\mathrm{d}Y(t) &= [\alpha_0 f(t,U(t)) + \mu(1-\alpha_0)X(t) + \delta(f(t,u(t)) - \mu x(t)) + \gamma(t)]\,\mathrm{d}t \\
&\quad - Z(t)\,\mathrm{d}B_t, \\
X(0) &= x_0.
\end{aligned}$$

We will prove that the mapping defined by

$$U = I_{\alpha_0+\delta}(u): M^2(0,\infty;\mathbb{R}^{n+n+n\times d}) \to M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$$

is contractive. Let $u' = (x', y', z') \in M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$ and $U' = (X', Y', Z') = I_{\alpha_0+\delta}(u')$, and

$$\begin{aligned}
\hat{u} &= (\hat{x}, \hat{y}, \hat{z}) = (x - x', y - y', z - z'), \\
\hat{U} &= (\hat{X}, \hat{Y}, \hat{Z}) = (X - X', Y - Y', Z - Z').
\end{aligned}$$

Since $\hat{U} \in M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$, there exists a sequence $0 \leqslant T_1 < T_2 < \cdots < T_i < \cdots$, $T_i \to \infty$ $(i \to \infty)$, such that $\lim_{i\to\infty} \mathbf{E}\langle \hat{X}(T_i), \hat{Y}(T_i)\rangle = 0$. Applying Itô's formula to $\langle \hat{X}, \hat{Y}\rangle$ on $[0,T_i]$ and by virtue of (H1) and (H2), we have, since $\hat{X}(0) = 0$

$$\begin{aligned}
&\mathbf{E}\langle \hat{X}(T_i), \hat{Y}(T_i)\rangle - \mathbf{E}\langle \hat{X}(0), \hat{Y}(0)\rangle \\
&\quad = \alpha_0 \mathbf{E}\int_0^{T_i} \langle (A(s,U) - A(s,U')), \hat{U}\rangle\,\mathrm{d}s
\end{aligned}$$

$$-(1-\alpha_0)\mu\mathbf{E}\int_0^{T_i}(\langle\hat{X},\hat{X}\rangle+\langle\hat{Y},\hat{Y}\rangle+\langle\hat{Z},\hat{Z}\rangle)\,\mathrm{d}s$$

$$+\delta\mathbf{E}\int_0^{T_i}(\mu\langle\hat{X},\hat{x}\rangle+\mu\langle\hat{Y},\hat{y}\rangle+\mu\langle\hat{Z},\hat{z}\rangle+\langle\hat{X},-\bar{f}\rangle+\langle\hat{Y},\bar{b}\rangle+\langle\hat{Z},\bar{\sigma}\rangle)\,\mathrm{d}s$$

$$\leqslant\mathbf{E}\int_0^{T_i}[-\mu|\hat{U}|^2+\delta\mu(|\hat{X}||\hat{x}|+|\hat{Y}||\hat{y}|+|\hat{Z}||\hat{z}|)+\delta l|\hat{u}|(|\hat{X}|+|\hat{Y}|+|\hat{Z}|)]\,\mathrm{d}s,$$

where

$$\bar{f}=f(s,u)-f(s,u'),\quad \bar{b}=b(s,u)-b(s,u'),\quad \bar{\sigma}=\sigma(s,u)-\sigma(s,u').$$

Let $i\to\infty$, then we can get

$$\mu\mathbf{E}\int_0^{\infty}|\hat{U}|^2\,\mathrm{d}s\leqslant\delta\mathbf{E}\int_0^{\infty}\frac{\mu+3l}{2}(|\hat{U}|^2+|\hat{u}|^2)\,\mathrm{d}s.$$

We now choose $\delta_0=\mu/2(\mu+3l)$, it is clear that, for each fixed $\delta\in[0,\delta_0]$, the mapping $I_{\alpha_0+\delta}$ is contractive in the following sense:

$$\mathbf{E}\int_0^{\infty}|\hat{U}|^2\,\mathrm{d}s\leqslant\frac{1}{3}\mathbf{E}\int_0^{\infty}|\hat{u}|^2\,\mathrm{d}s.$$

It follows immediately that this mapping has a unique fixed point $U^{\alpha_0+\delta}=(X^{\alpha_0+\delta},Y^{\alpha_0+\delta},Z^{\alpha_0+\delta})$ which is the solution of (3.1) for $\alpha=\alpha_0+\delta$. The proof is complete. □

4. Existence and uniqueness

Now we can obtain one of the main results in this paper — the existence and uniqueness theorem for solutions of FBSDE (2.1).

Theorem 6. *Under assumptions* (H1)–(H3), *for each* $x_0\in\mathbb{R}^n$ (2.1) *has a unique solution in* $M^2(0,\infty;\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d})$.

Proof. (*Uniqueness*): Let $U=(X,Y,Z)$ and $U'=(X',Y',Z')$ be two solutions of (2.1). We set

$$\hat{U}=U-U'=(X-X',Y-Y',Z-Z')=(\hat{X},\hat{Y},\hat{Z}).$$

Similar to Lemma 5, there exist $T_i\to\infty$ $(i\to\infty)$ such that

$$\lim_{i\to\infty}\mathbf{E}\langle\hat{X}(T_i),\hat{Y}(T_i)\rangle=0.$$

Applying Itô's formula to $\langle\hat{X}(\cdot),\hat{Y}(\cdot)\rangle$,

$$\mathbf{E}\langle\hat{X}(T_i),\hat{Y}(T_i)\rangle-\mathbf{E}\langle\hat{X}(0),\hat{Y}(0)\rangle=\mathbf{E}\int_0^{T_i}\langle A(t,U)-A(t,U'),\hat{U}\rangle\,\mathrm{d}t.$$

Let $i\to\infty$, this with the monotonicities of A implies

$$\mu\mathbf{E}\int_0^{\infty}(|\hat{X}(t)|^2+|\hat{Y}(t)|^2+|\hat{Z}(t)|^2)\,\mathrm{d}t\leqslant 0.$$

Thus $\hat{X}\equiv0$, $\hat{Y}\equiv0$, $\hat{Z}\equiv0$, the uniqueness is proved.

(*Existence*): By Lemma 2, for any $x_0 \in \mathbb{R}^n, \phi, \gamma, \psi \in M^2(0,\infty)$, (3.1) has a unique solution in $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$ as $\alpha = 0$.

It follows from Lemma 5 that there exists a positive constant $\delta_0 = \delta_0(l,\mu)$ such that for any $\delta \in [0,\delta_0]$ and any $\phi, \gamma, \psi \in M^2(0,\infty)$, (3.1) has a unique solution in $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$ for $\alpha = 0 + \delta$. Since δ_0 depends only on (l,μ), we can repeat this process for N times with $1 \leqslant N\delta_0 < 1 + \delta_0$. In particular, for $\alpha = 1$ with $\phi \equiv 0$, $\gamma \equiv 0$ and $\psi \equiv 0$, (3.1) has a unique solution in $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$. The proof is complete. □

5. Comparison theorem

In this section we give an important property of FBSDE (2.1) — comparison theorem.

Let $(X_1(\cdot), Y_1(\cdot), Z_1(\cdot))$ and $(X_2(\cdot), Y_2(\cdot), Z_2(\cdot))$ be, respectively, the solutions of (2.1) corresponding to $X_1(0) = x_1 \in \mathbb{R}^n$ and $X_2(0) = x_2 \in \mathbb{R}^n$. Set

$$\begin{aligned}\hat{U} &= U_1 - U_2 = (X_1, Y_1, Z_1) - (X_2, Y_2, Z_2)\\ &= (X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2) = (\hat{X}, \hat{Y}, \hat{Z}).\end{aligned}$$

Lemma 7. *We make assumptions* (H1)–(H3). *Then* $\langle \hat{X}(t), \hat{Y}(t)\rangle \geqslant 0$, $\forall t \in \mathbb{R}^+$. *Moreover,*

$$(\hat{X}(t), \hat{Y}(t), \hat{Z}(t))\mathbf{1}_{[\tau,\infty)}(t) \equiv 0,$$

where τ *is a* $\mathscr{F}_t$*-stopping time defined by*

$$\tau \doteq \inf\{t \geqslant 0; \langle \hat{X}(t), \hat{Y}(t)\rangle = 0\}.$$

Proof. We chose T_i $(i = 1, 2, \ldots,)$, such that $\mathbf{E}\langle \hat{X}(T_i), \hat{Y}(T_i)\rangle \to 0$. Clearly, for a given time $t \in \mathbb{R}^+$

$$\lim_{i\to\infty} \mathbf{E}^{\mathscr{F}_t}\langle \hat{X}(T_i), \hat{Y}(T_i)\rangle = 0.$$

We then apply Itô's formula to $\langle \hat{X}(\cdot), \hat{Y}(\cdot)\rangle$:

$$\begin{aligned}\mathbf{E}^{\mathscr{F}_t}\langle \hat{X}(T_i), \hat{Y}(T_i)\rangle - \langle \hat{X}(t), \hat{Y}(t)\rangle &= \mathbf{E}^{\mathscr{F}_t}\int_t^{T_i} \langle A(s,U_1) - A(s,U_2), \hat{U}\rangle\,\mathrm{d}s\\ &\leqslant -\mu\mathbf{E}^{\mathscr{F}_t}\int_t^{T_i} |\hat{U}(s)|^2\,\mathrm{d}s, \quad \forall t \geqslant 0.\end{aligned}$$

Let $i \to \infty$, then we have

$$\langle \hat{X}(t), \hat{Y}(t)\rangle \geqslant \mu\mathbf{E}^{\mathscr{F}_t}\int_t^{\infty} |\hat{U}(s)|^2\,\mathrm{d}s \geqslant 0, \quad \forall t \geqslant 0.$$

It then follows that, for the above given stopping time τ

$$\langle \hat{X}(T), \hat{Y}(T)\rangle - \langle \hat{X}(\tau \wedge T), \hat{Y}(\tau \wedge T)\rangle \geqslant 0, \quad \forall T \geqslant 0.$$

Since

$$\mathbf{E}(\langle \hat{X}(T), \hat{Y}(T)\rangle - \langle \hat{X}(\tau \wedge T), \hat{Y}(\tau \wedge T)\rangle) = \mathbf{E}\int_{\tau\wedge T}^{T} \langle A(t,U_1) - A(t,U_2), \hat{U}\rangle \,\mathrm{d}t$$

$$\leqslant -\mu \mathbf{E}\int_{\tau\wedge T}^{T} |\hat{U}(t)|^2 \,\mathrm{d}t,$$

we have then

$$\hat{X}(t)\mathbf{1}_{[\tau\wedge T,T]}(t) = \hat{Y}(t)\mathbf{1}_{[\tau\wedge T,T]}(t) = 0, \quad \forall T \geqslant 0.$$

Thus

$$\hat{X}(t)\mathbf{1}_{[\tau,\infty)}(t) \equiv \hat{Y}(t)\mathbf{1}_{[\tau,\infty)}(t) \equiv 0, \quad \forall t \geqslant 0.$$

Following the existence and uniqueness theorem, $\hat{Z}(t)\mathbf{1}_{[\tau,\infty)}(t) \equiv 0$, $\forall t \geqslant 0$. The proof is complete. □

We now consider a special situation of (2.1), namely $n=1$. Thus $X(\cdot)$ and $Y(\cdot)$ are all R-valued processes. We can assert the following comparison theorem.

Theorem 8. *We set* $n=1$. *Assuming that* (2.1) *satisfies assumptions* (H1)–(H3), *we have that if* $\hat{X}(0) > 0$ *then* $\hat{Y}(0) > 0$, *furthermore*, $X_1(t) \geqslant X_2(t)$ *and* $Y_1(t) \geqslant Y_2(t)$, $\forall t \geqslant 0$.

Proof. We set

$$\tau_x \doteq \inf\{t \geqslant 0;\ \hat{X}(t) = 0\},$$

$$\tau_y \doteq \inf\{t \geqslant 0;\ \hat{Y}(t) = 0\}.$$

Clearly,

$$\tau \leqslant \tau_x, \quad \tau \leqslant \tau_y.$$

From the above lemma,

$$\hat{X}(t) = 0, \quad \forall t \geqslant \tau_x,$$

$$\hat{Y}(t) = 0, \quad \forall t \geqslant \tau_y.$$

It follows that $\hat{X}(t) \geqslant 0$ and $\hat{Y}(t) \geqslant 0$, $\forall t \geqslant 0$. The proof is complete. □

6. Exponential asymptotical stability

In this section we will show asymptotic behavior of the adapted solutions of (2.1). Let $\tau \geqslant 0$ be a fixed time. We note $(X^{\tau,x}(t), Y^{\tau,x}(t), Z^{\tau,x}(t))$ stands for the solution of (2.1) with initial value conditions $X(\tau) = x$, i.e., it satisfies the following equations:

$$\begin{aligned} &\mathrm{d}X(t) = b(t, X(t), Y(t), Z(t))\,\mathrm{d}t + \sigma(t, X(t), Y(t), Z(t))\,\mathrm{d}B_t,\\ &-\mathrm{d}Y(t) = f(t, X(t), Y(t), Z(t))\,\mathrm{d}t - Z(t)\,\mathrm{d}B_t, \\ &X(\tau) = x,\ (X(\cdot), Y(\cdot), Z(\cdot)) \in M^2(\tau, \infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}), \end{aligned} \tag{6.1}$$

where $x \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$.

Similar to Theorem 6, we easily prove the existence and uniqueness theorem for adapted solutions of (6.1) under the same assumptions.

Theorem 9. *Under assumptions* (H1)–(H3), *for each* $x \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$ (6.1) *has a unique solution in* $M^2(\tau, \infty; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$.

We further assume that

(H5) $A(\cdot, 0) \equiv 0$ if and only if $u = 0$.

Actually by Theorem 9, $(X(\cdot) \equiv 0, Y(\cdot) \equiv 0, Z(\cdot) \equiv 0)$ is the unique trivial solution of (6.1) for $x = 0$. We first give some properties of the solution $(X^{\tau,x}(t), Y^{\tau,x}(t), Z^{\tau,x}(t))$ of (6.1).

Lemma 10. *Assume* (H1)–(H3) *and* (H5). *Then we have*

(i) $Y^{\tau,0}(\tau) = 0$,

(ii) $Y^{\tau,x}(\tau)$ *satisfies Lipschitz condition with respect to* x: *there exists a constant* $\beta > 0$ *independent of* τ, *such that* $\forall x_1, x_2 \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$,

$$|Y^{\tau,x_1}(\tau) - Y^{\tau,x_2}(\tau)| \leqslant \beta |x_1 - x_2|,$$

(iii) $Y^{\tau,x}(\tau)$ *satisfies monotonicity condition on* x: *there exists a constant* $\alpha > 0$ *independent of* τ, *such that* $\forall x_1, x_2 \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$,

$$\langle Y^{\tau,x_1}(\tau) - Y^{\tau,x_2}(\tau), x_1 - x_2 \rangle \geqslant \alpha |x_1 - x_2|^2.$$

Proof. Let $(X_1(t), Y_1(t), Z_1(t))$ and $(X_2(t), Y_2(t), Z_2(t))$ be, respectively, the solutions of (6.1) correspondent to $X_1(\tau) = x_1 \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$ and $X_2(\tau) = x_2 \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$. We set

$$\hat{X}(t) = X_1(t) - X_2(t), \ \hat{Y}(t) = Y_1(t) - Y_2(t), \ \hat{Z}(t) = Z_1(t) - Z_2(t),$$

$$\hat{b} = b(t, X_1(t), Y_1(t), Z_1(t)) - b(t, X_2(t), Y_2(t), Z_2(t)),$$

$$\hat{f} = f(t, X_1(t), Y_1(t), Z_1(t)) - f(t, X_2(t), Y_2(t), Z_2(t)),$$

$$\hat{\sigma} = \sigma(t, X_1(t), Y_1(t), Z_1(t)) - \sigma(t, X_2(t), Y_2(t), Z_2(t)).$$

(i) It is an immediate consequence of the existence and uniqueness for solutions of (6.1).

(ii) Applying Itô's formula to $|Y_1(t) - Y_2(t)|^2$, we get

$$\begin{aligned} &|Y_1(\tau) - Y_2(\tau)|^2 \\ &\quad = \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^\infty [2\langle \hat{Y}(s), \hat{f} \rangle - |\hat{Z}(s)|^2] \, ds \\ &\quad \leqslant \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^\infty [2l |\hat{Y}(s)| (|\hat{X}(s)| + |\hat{Y}(s)| + |\hat{Z}(s)|) + |\hat{Z}(s)|^2] \, ds \\ &\quad \leqslant \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^\infty (4l + 1)(|\hat{X}(s)|^2 + |\hat{Y}(s)|^2 + |\hat{Z}(s)|^2) \, ds. \end{aligned} \tag{6.2}$$

Applying Itô's formula to $\langle X_1(t)-X_2(t), Y_1(t)-Y_2(t)\rangle$ and by virtue of assumption (H1), we have

$$-\langle X_1(\tau)-X_2(\tau), Y_1(\tau)-Y_2(\tau)\rangle \leqslant -\mu \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty (|\hat{X}(s)|^2+|\hat{Y}(s)|^2+|\hat{Z}(s)|^2)\,\mathrm{d}s. \tag{6.3}$$

Therefore

$$\begin{aligned}
&\mu \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty (|\hat{X}(s)|^2+|\hat{Y}(s)|^2+|\hat{Z}(s)|^2)\,\mathrm{d}s\\
&\quad\leqslant \langle X_1(\tau)-X_2(\tau), Y_1(\tau)-Y_2(\tau)\rangle\\
&\quad\leqslant |X_1(\tau)-X_2(\tau)|\cdot|Y_1(\tau)-Y_2(\tau)|\\
&\quad\leqslant \frac{4l+1}{2\mu}|X_1(\tau)-X_2(\tau)|^2+\frac{\mu}{2(4l+1)}|Y_1(\tau)-Y_2(\tau)|^2.
\end{aligned}\tag{6.4}$$

From (6.2) and (6.4), we deduce

$$|Y_1(\tau)-Y_2(\tau)|^2 \leqslant \frac{(4l+1)^2}{\mu^2}|X_1(\tau)-X_2(\tau)|^2,$$

thus

$$|Y^{\tau,x_1}(\tau)-Y^{\tau,x_2}(\tau)| \leqslant \beta|x_1-x_2|, \quad \forall x_1,x_2\in L^2(\Omega,\mathscr{F}_\tau,P;\mathbb{R}^n),$$

where $\beta=(4l+1)/\mu>0$.

(iii) Applying Itô's formula to $|X_1(t)-X_2(t)|^2$, we get

$$\begin{aligned}
&-|X_1(\tau)-X_2(\tau)|^2\\
&\quad= \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty [2\langle \hat{X}(s),\hat{b}\rangle+|\hat{\sigma}|^2]\,\mathrm{d}s\\
&\quad\geqslant -\mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty [2l|\hat{X}(s)|(|\hat{X}(s)|+|\hat{Y}(s)|+|\hat{Z}(s)|)\\
&\qquad +l^2(|\hat{X}(s)|+|\hat{Y}(s)|+|\hat{Z}(s)|)^2]\,\mathrm{d}s.
\end{aligned}$$

By virtue of (6.3), for a sufficiently small $\alpha>0$,

$$\begin{aligned}
&\langle X_1(\tau)-X_2(\tau), Y_1(\tau)-Y_2(\tau)\rangle-\alpha|X_1(\tau)-X_2(\tau)|^2\\
&\quad\geqslant \mu \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty (|\hat{X}(s)|^2+|\hat{Y}(s)|^2+|\hat{Z}(s)|^2)\,\mathrm{d}s\\
&\qquad -\alpha \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^\infty [2l|\hat{X}(s)|(|\hat{X}(s)|+|\hat{Y}(s)|+|\hat{Z}(s)|)\\
&\qquad +l^2(|\hat{X}(s)|+|\hat{Y}(s)|+|\hat{Z}(s)|)^2]\,\mathrm{d}s\\
&\quad\geqslant 0.
\end{aligned}$$

Thus (iii) holds true. The proof is complete. □

Now we define a random function:

$$v(\tau,x) \doteq Y^{\tau,x}(\tau): \mathbb{R}^+ \times \mathbb{R}^n \to L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n).$$

Then from the previous lemma we have

Lemma 11. *Assume* (H1)–(H3) *and* (H5). *Then we have*

(i) $v(\tau,0)=0$;

(ii) $v(\tau,x)$ *satisfies Lipschitz condition with respect to* x, *i.e.*,

$$|v(\tau,x_1)-v(\tau,x_2)| \leqslant \beta|x_1-x_2|, \quad \forall x_1,x_2 \in \mathbb{R}^n;$$

(iii) $v(\tau,x)$ *satisfies monotonicity condition on* x, *i.e.*,

$$\langle v(\tau,x_1)-v(\tau,x_2), x_1-x_2\rangle \geqslant \alpha|x_1-x_2|^2, \quad \forall x_1,x_2 \in \mathbb{R}^n.$$

To proceed with our discussion, the following notion is in order.

Definition 12. For a fixed $\tau \in \mathbb{R}^+$ a family of subsets $\{A_i\}_{i=1}^N \subset \mathscr{F}_\tau$ is said to be a partition of $(\Omega, \mathscr{F}_\tau)$ if

$$A_i \in \mathscr{F}_\tau,\ i=1,\ldots,N; \quad A_i \cap A_j = \emptyset, \text{ for } i \neq j; \quad \bigcup_{i=1}^N A_i = \Omega.$$

We now claim

Theorem 13. *For each* $\zeta \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^n)$ we have

$$v(\tau,\zeta) = Y^{\tau,\zeta}(\tau). \tag{6.5}$$

Proof. We first consider the case when ζ is a step function:

$$\zeta = \sum_{i=1}^N \mathbf{1}_{A_i} x_i,$$

where $\{A_i\}_{i=1}^N$ is a partition of $(\Omega, \mathscr{F}_\tau)$, $x_i \in \mathbb{R}^n$, $i=1,2,\ldots,N$. For each i, let

$$(X_t^i, Y_t^i, Z_t^i) \equiv (X^{\tau,x_i}(t), Y^{\tau,x_i}(t), Z^{\tau,x_i}(t)),$$

then (X_t^i, Y_t^i, Z_t^i) is the solution of the following FBSDE:

$$X_t^i = x_i + \int_\tau^t b(s, X_s^i, Y_s^i, Z_s^i)\,\mathrm{d}s + \int_\tau^t \sigma(s, X_s^i, Y_s^i, Z_s^i)\,\mathrm{d}B_s,$$

$$Y_t^i = \int_t^\infty f(s, X_s^i, Y_s^i, Z_s^i)\,\mathrm{d}s - \int_t^\infty Z_s^i\,\mathrm{d}B_s, \quad t \in [\tau,\infty).$$

We multiple $\mathbf{1}_{A_i}$ on both sides of the above equations and then take the summation for all i. From the simple fact $\sum_i \varphi(x_i)\mathbf{1}_{A_i} = \varphi(\sum_i x_i \mathbf{1}_{A_i})$ it follows that

$$\sum_{i=1}^N \mathbf{1}_{A_i} X_t^i = \sum_{i=1}^N \mathbf{1}_{A_i} x_i + \int_\tau^t b\left(s, \sum_{i=1}^N \mathbf{1}_{A_i} X_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} Z_s^i\right) \mathrm{d}s$$

$$+\int_\tau^t \sigma\left(s,\sum_{i=1}^N \mathbf{1}_{A_i}X_s^i,\sum_{i=1}^N \mathbf{1}_{A_i}Y_s^i,\sum_{i=1}^N \mathbf{1}_{A_i}Z_s^i\right)\mathrm{d}B_s$$

and

$$\sum_{i=1}^N \mathbf{1}_{A_i}Y_t^i=\int_t^\infty f\left(s,\sum_{i=1}^N \mathbf{1}_{A_i}X_s^i,\sum_{i=1}^N \mathbf{1}_{A_i}Y_s^i,\sum_{i=1}^N \mathbf{1}_{A_i}Z_s^i\right)\mathrm{d}s-\int_t^\infty \sum_{i=1}^N \mathbf{1}_{A_i}Z_s^i\,\mathrm{d}B_s.$$

It follows from the uniqueness of (6.1) that

$$(X^{\tau,\zeta}(t),Y^{\tau,\zeta}(t),Z^{\tau,\zeta}(t))=\left(\sum_{i=1}^N \mathbf{1}_{A_i}X_t^i,\sum_{i=1}^N \mathbf{1}_{A_i}Y_t^i,\sum_{i=1}^N \mathbf{1}_{A_i}Z_t^i\right).$$

It then follows from the definition $v(\tau,x_i)=Y_\tau^i$ that

$$Y^{\tau,\zeta}(\tau)=\sum_{i=1}^N \mathbf{1}_{A_i}Y_\tau^i=\sum_{i=1}^N \mathbf{1}_{A_i}v(\tau,x_i)=v\left(\tau,\sum_{i=1}^N \mathbf{1}_{A_i}x_i\right)=v(\tau,\zeta).$$

Thus (6.5) holds true for the case where ζ is a step function.

If $\zeta\in L^2(\Omega,\mathscr{F}_\tau,P;\mathbb{R}^n)$ we can choose a sequence of step functions $\{\zeta_i\}$ that converges to ζ in $L^2(\Omega,\mathscr{F}_\tau,P;\mathbb{R}^n)$. From Lemmas 10 and 11 it follows that

$$\mathbf{E}|Y^{\tau,\zeta_i}(\tau)-Y^{\tau,\zeta}(\tau)|^2\leqslant\beta^2\mathbf{E}|\zeta_i-\zeta|^2\to 0$$

and

$$\mathbf{E}|v(\tau,\zeta_i)-v(\tau,\zeta)|^2\leqslant\beta^2\mathbf{E}|\zeta_i-\zeta|^2\to 0,$$

respectively. These with $v(\tau,\zeta_i)=Y^{\tau,\zeta_i}(\tau)$ yield (6.5). The proof is complete. □

According to the existence and uniqueness theorem,

$$Y^{\tau,x}(t)=v(t,X^{\tau,x}(t)),\quad \forall x\in L^2(\Omega,\mathscr{F}_\tau,P;\mathbb{R}^n),\ \forall t\geqslant\tau\geqslant 0.$$

Now we set

$$V(t,x)=\langle x,v(t,x)\rangle=\langle x,Y^{t,x}(t)\rangle.$$

Therefore, we observe that

$$V(t,X^{\tau,x}(t))=\langle X^{\tau,x}(t),v(t,X^{\tau,x}(t))\rangle=\langle X^{\tau,x}(t),Y^{\tau,x}(t)\rangle.$$

Remark 1. In fact $V(t,X^{\tau,x}(t))$ is a Lyapunov function for (2.1).

Then from Lemma 11 we easily get

Lemma 14. *Under assumptions* (H1)–(H3) *and* (H5), *we have*

$$\alpha|x|^2\leqslant V(t,x)\leqslant\beta|x|^2,\quad \forall x\in\mathbb{R}^n,\ \forall t\geqslant\tau\geqslant 0.$$

By a similar procedure, it is not difficult to obtain the following result.

Lemma 15. *Under assumptions* (H1)–(H3) *and* (H5), *we have*

$$\alpha|X^{\tau,x}(t)|^2\leqslant V(t,X^{\tau,x}(t))\leqslant\beta|X^{\tau,x}(t)|^2,\quad \forall x\in L^2(\Omega,\mathscr{F}_\tau,P;\mathbb{R}^n),\ \forall t\geqslant\tau\geqslant 0.$$

Now we can assert the exponential decay properties theorem which is the other main object of this paper.

Theorem 16. *Under assumptions* (H1)–(H3) *and* (H5), *we have that the adapted solutions of FBSDE* (2.1) *are exponential decaying in the following sense: there exist two constants* $r>0$ *and* $R>0$, *such that* $\forall t \geqslant \tau \geqslant 0$

$$\mathbf{E}^{\mathscr{F}_\tau}(|X^{\tau,x}(t)|^2) \leqslant R|x|^2 \mathrm{e}^{-r(t-\tau)}, \qquad \mathbf{E}^{\mathscr{F}_\tau}(|Y^{\tau,x}(t)|^2) \leqslant R|x|^2 \mathrm{e}^{-r(t-\tau)}.$$

Proof. Applying Itô's formula to $V(t,X^{\tau,x}(t))$, we have

$$\begin{aligned}
&\mathbf{E}^{\mathscr{F}_\tau} V(t,X^{\tau,x}(t)) - V(\tau,X^{\tau,x}(\tau)) \\
&\quad = \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^t \langle Y^{\tau,x}(s), b(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle \,\mathrm{d}s \\
&\qquad - \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^t \langle X^{\tau,x}(s), f(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle \,\mathrm{d}s \\
&\qquad + \mathbf{E}^{\mathscr{F}_\tau} \int_\tau^t \langle Z^{\tau,x}(s), \sigma(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle \,\mathrm{d}s \\
&\quad = -\frac{\mu}{\beta}\mathbf{E}^{\mathscr{F}_\tau} \int_\tau^t V(s,X^{\tau,x}(s))\,\mathrm{d}s + \mathbf{E}^{\mathscr{F}_\tau}\int_\tau^t \lambda(s)\,\mathrm{d}s,
\end{aligned}$$

where by virtue of (H1) and Lemma 15,

$$\begin{aligned}
\lambda(s) &= \langle Y^{\tau,x}(s), b(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle \\
&\quad - \langle X^{\tau,x}(s), f(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle \\
&\quad + \langle Z^{\tau,x}(s), \sigma(s,X^{\tau,x}(s),Y^{\tau,x}(s),Z^{\tau,x}(s))\rangle + \frac{\mu}{\beta} V(s,X^{\tau,x}(s)) \\
&\leqslant 0.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
\mathbf{E}^{\mathscr{F}_\tau} V(t,X^{\tau,x}(t)) &= V(\tau,X^{\tau,x}(\tau))\mathrm{e}^{-(\mu/\beta)(t-\tau)} + \int_\tau^t \mathrm{e}^{-(\mu/\beta)(t-s)} \mathbf{E}^{\mathscr{F}_\tau}\lambda(s)\,\mathrm{d}s \\
&\leqslant V(\tau,X^{\tau,x}(\tau))\mathrm{e}^{-(\mu/\beta)(t-\tau)} \\
&\leqslant \beta|x|^2\mathrm{e}^{-(\mu/\beta)(t-\tau)}, \quad \forall t\geqslant\tau\geqslant 0.
\end{aligned}$$

Again from Lemma 15 we obtain

$$\mathbf{E}^{\mathscr{F}_\tau}(|X^{\tau,x}(t)|^2) \leqslant \frac{\beta}{\alpha}|x|^2\mathrm{e}^{-r(t-\tau)}, \quad \forall t\geqslant\tau\geqslant 0,$$

where $r=\mu/\beta>0$. From Lemma 10(ii), we can get the estimate for $Y^{\tau,x}(t)$

$$\mathbf{E}^{\mathscr{F}_\tau}(|Y^{\tau,x}(t)|^2) \leqslant \frac{\beta^3}{\alpha}|x|^2\mathrm{e}^{-r(t-\tau)}, \quad \forall t\geqslant\tau\geqslant 0.$$

Thus the desired result is obtained. □

In fact we may borrow a definition on global exponential asymptotical stability similar to ordinary differential equations.

Definition 17. The trivial solution of FBSDE (2.1) is called global exponential asymptotically stable, if $\exists\gamma>0$, $\forall\delta>0$, $\exists\Gamma(\delta)>0$, such that as $|x|<\delta$, $\forall t\geqslant\tau\geqslant 0$

$$\mathbf{E}^{\mathscr{F}_\tau}(|X^{\tau,x}(t)|^2)\leqslant\Gamma(\delta)|x|^2\mathrm{e}^{-\gamma(t-\tau)},\qquad \mathbf{E}^{\mathscr{F}_\tau}(|Y^{\tau,x}(t)|^2)\leqslant\Gamma(\delta)|x|^2\mathrm{e}^{-\gamma(t-\tau)}.$$

Then Theorem 16 shows that the trivial solution of (2.1) is global exponential asymptotically stable.

Remark 2. If we replace the monotonicity assumption (H1) with (H1)′
(H1)′ $\langle A(t,u)-A(t,\bar u),u-\bar u\rangle\geqslant\mu|u-\bar u|^2,\ \mu>0,$

$$\forall u=\begin{pmatrix}X\\Y\\Z\end{pmatrix},\quad \bar u=\begin{pmatrix}\bar X\\\bar Y\\\bar Z\end{pmatrix}\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d},\quad \forall t\in\mathbb{R}^+.$$

Then we have similar results.

Remark 3. The case in Remark 2 is equivalent to the following formulation:

$$\begin{aligned}&\mathrm{d}X(t)=b(t,X(t),Y(t),Z(t))\,\mathrm{d}t+\sigma(t,X(t),Y(t),Z(t))\,\mathrm{d}B_t,\\&-\mathrm{d}Y(t)=f(t,X(t),Y(t),Z(t))\,\mathrm{d}t-Z(t)\,\mathrm{d}B_t,\\&Y(0)=y_0,\quad (X(\cdot),Y(\cdot),Z(\cdot))\in M^2(0,\infty;\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d})\end{aligned}\tag{6.6}$$

under the assumptions (H1)–(H3) and (H5).

Remark 4. It is not difficult to check that the above results are true in the case that $\tau\geqslant 0$ is a given $\mathscr{F}_t$-stopping time.

7. Generalization of infinite horizon FBSDE

In this section we consider the following generalized form of FBSDE, i.e., a class of fully coupled system of infinite horizon forward–backward stochastic differential equations

$$\begin{aligned}&\mathrm{d}X(t)=b(t,X(t),Y(t),Z(t))\,\mathrm{d}t+\sigma(t,X(t),Y(t),Z(t))\,\mathrm{d}B_t,\\&-\mathrm{d}Y(t)=f(t,X(t),Y(t),Z(t))\,\mathrm{d}t-Z(t)\,\mathrm{d}B_t,\\&X(0)=h(Y(0)),\quad (X,Y,Z)\in M^2(0,\infty;\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}),\end{aligned}\tag{7.1}$$

where $h(Y){:}\mathbb{R}^n\to\mathbb{R}^n$. We impose the assumptions as follows:
(H6) $h(Y)$ satisfies monotonicity condition, i.e.

$$\langle h(Y)-h(\bar Y),Y-\bar Y\rangle\leqslant 0,\quad \forall Y,\bar Y\in\mathbb{R}^n,$$

(H7) $h(Y)$ satisfies Lipschitz condition with respect to Y.

Analogous to the preceding treatment, we can obtain the same results for (7.1). We need consider the following system:

$$\begin{aligned} dX^\alpha(t) &= [\alpha b(t,u^\alpha(t)) - \mu(1-\alpha)Y^\alpha(t) + \phi(t)]\,dt \\ &\quad +[\alpha\sigma(t,u^\alpha(t)) - \mu(1-\alpha)Z^\alpha(t) + \psi(t)]\,dB_t, \\ -dY^\alpha(t) &= [\alpha f(t,u^\alpha(t)) + \mu(1-\alpha)X^\alpha(t) + \gamma(t)]\,dt - Z^\alpha(t)\,dB_t, \\ X^\alpha(0) &= \alpha h(Y^\alpha(0)) + x_0, \end{aligned} \tag{7.2}$$

where $x_0 \in \mathbb{R}^n, \phi, \psi$ and γ are given processes in $M^2(0,\infty)$ with values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}^n$, respectively. It is obvious that in the case $\alpha = 1$, the fact that (7.2) has a unique solution in $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$ for each $x_0 \in \mathbb{R}^n$, $\phi,\psi,\gamma \in M^2(0,\infty)$ implies that the existence and uniqueness for solutions of (7.1) hold. When $\alpha = 0$, (7.2) is the following linear system:

$$\begin{aligned} dX^0(t) &= (-\mu Y^0(t) + \phi(t))\,dt + (-\mu Z^0(t) + \psi(t))\,dB_t, \\ -dY^0(t) &= (\mu X^0(t) + \gamma(t))\,dt - Z^0(t)\,dB_t, \\ X^0(0) &= x_0, \quad (X^0, Y^0, Z^0) \in M^2(0,\infty;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}). \end{aligned} \tag{7.3}$$

We have following lemma corresponding to Lemma 2:

Lemma 18. *For each* $x_0 \in \mathbb{R}^n$, ϕ, γ, $\psi \in M^2(0,\infty)$, (7.3) *has a unique solution in* $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$.

Corresponding to Lemma 5, we have

Lemma 19. *Under assumptions* (H1)–(H3), (H6) *and* (H7), *there exists a positive constant* δ_0 *such that if, a priori, for some* $\alpha_0 \in [0,1)$, *for each* $x_0 \in \mathbb{R}^n$, ϕ, γ, $\psi \in M^2(0,\infty)$, (7.2) *has a unique solution* $(X^{\alpha_0}(\cdot), Y^{\alpha_0}(\cdot), Z^{\alpha_0}(\cdot))$ *in* $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$, *then for each* $\delta \in [0,\delta_0]$, *for each* $x_0 \in \mathbb{R}^n$, ϕ, γ, $\psi \in M^2(0,\infty)$, (7.2) *also has a unique solution* $(X^{\alpha_0+\delta}(\cdot), Y^{\alpha_0+\delta}(\cdot), Z^{\alpha_0+\delta}(\cdot))$ *in* $M^2(0,\infty;\mathbb{R}^{n+n+n\times d})$.

Corresponding to Theorem 6, we have

Theorem 20. *Under assumptions* (H1)–(H3), (H6) *and* (H7), (7.1) *has a unique solution* $(X,Y,Z) \in M^2(0,\infty;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$.

Remark 5. Via the same procedure we can obtain the corresponding comparison theorem and the global exponential asymptotical stability of solutions for (7.1).

Remark 6. Under the monotonicity assumptions (H1)$'$ and

$$\langle h(Y) - h(\bar{Y}), Y - \bar{Y}\rangle \geqslant 0, \quad \forall Y, \bar{Y} \in \mathbb{R}^n,$$

the similar results hold true.

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