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Asymptotic Expansion for the Initial Value Problem of the Sunflower Equation

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In this paper, we consider the initial value problem (IVP) for the sunflower equation,

$$\varepsilon x''(t)+ax'(t)+b\sin x(t-\varepsilon)=0,$$

with $x(t)=\phi(t)$, $t\in[-\varepsilon,0]$; $x'(0)=y_0$. We prove that the solution of the above-mentioned IVP has an asymptotic expansion uniformly valid on $[-\varepsilon,\infty)$, for ε sufficiently small.

1. INTRODUCTION

It is pointed out in [2] that the stem's upper part performs a rotating movement in some plants (particularly the sunflower). This phenomenon is related to the nonhomogeneous distribution of the plant's growth factors along the stem, which tends to incline towards the least-developed side by the action of gravity. We assume that the mathematical model describing the motion of the sunflower is the equation

$$\varepsilon x''(t)+ax'(t)+b\sin x(t-\varepsilon)=0, \qquad \varepsilon>0, \tag{1.1}$$

according to the theory described in [4], and already used in [1, 2, 5]. In (1.1) the function $x(t)$ is the angle of the plant with the vertical, the time lag ε is the geotropic reaction, and a and b are positive parameters which can be obtained experimentally.

In [2, 5], numerical and analytical studies have proved the existence of periodic solutions for a range of values of each of the parameters a, b, and ε. In [3], the equation was studied when an external periodic perturbation is applied.

In this paper we study the behaviour of solutions of (1.1) as $\varepsilon\to 0$. More precisely, we construct asymptotic expansions for the solutions of (1.1), uniformly valid for every $t\geqslant-\varepsilon$ and ε sufficiently small.

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2. The Main Result

Equation (1.1) is equivalent to the following system:

$$\begin{aligned} x'(t) &= y(t). \\ \varepsilon y'(t) &= -ay(t) - b \sin x(t-\varepsilon). \end{aligned} \tag{2.1}$$

We consider (2.1) for $t > 0$ subject to the initial conditions

$$\begin{aligned} x(t) &= \phi(t), \qquad t \in [-\varepsilon, 0], \\ y(0) &= y_0, \end{aligned} \tag{2.2}$$

where ϕ is a continuous function.

Because our objective consists in studying the behaviour of the solutions of (2.1) as $\varepsilon \to 0$, we may suppose without loss of generality that $\varepsilon \leqslant \varepsilon_0 = a^2\pi/b(a+1)$. Under this condition, in [5] it was proved that all solutions of (2.1) are bounded.

The main result of this paper is the following:

THEOREM 1. *The initial value problem* (2.1)–(2.2) *has a unique bounded solution* $(x(t, \phi, y_0, \varepsilon), y(t, \phi, y_0, \varepsilon))$ *defined for every* $t > -\varepsilon$ *and* $0 < \varepsilon \leqslant \varepsilon_0$, *which is such that*:

(a) *if* $\phi(0) \neq (2k+1)\pi$, *for all* $k \in Z$, *then for each integer* $N \geqslant 0$ *and* $t \geqslant -\varepsilon$, *the following asymptotic approximations hold*:

$$x(t, \phi, y_0, \varepsilon) = \sum_{n=0}^{N} \varepsilon^n (x_n(t) + x_n^*(t/\varepsilon)) + \varepsilon^{N+1} R_1(t, \varepsilon), \tag{2.3}$$

$$y(t, \phi, y_0, \varepsilon) = \sum_{n=0}^{N} \varepsilon^n (y_n(t) + y_n^*(t/\varepsilon)) + \varepsilon^{N+1} R_2(t, \varepsilon). \tag{2.4}$$

(b) *if* $\phi(0) = (2k+1)\pi$ *for some* $k \in Z$, *then* (2.3) *and* (2.4) *hold just for* $N = 0$.

The remainders R_1, R_2 are continuous uniformly bounded functions on $[0, \infty) \times [0, \varepsilon_0]$. The functions x_n, x_n^*, y_n, and y_n^* for $n = 0, 1, 2, \ldots$ are solutions of the following equations:

$$x_0'(t) = -(b/a) \sin x_0(t), \qquad x_0(0) = \phi(0), \tag{2.5}$$

$$y_0(t) = x_0'(t), \qquad x_0^*(\tau) = 0, \tag{2.6}$$

$$\dot{y}_0^*(\tau) = -ay_0^*(\tau), \qquad y^*(0) = y_0 - y_0(0), \tag{2.7}$$

$$\dot{x}_n^*(\tau) = y_{n-1}^*(\tau), \qquad x_n^*(\tau) \to 0 \text{ as } \tau \to +\infty, \tag{2.8}$$

$$x_n'(t) = -(b/a)x_n(t)\cos x_0(t) - (1/a)\,y_{n-1}'(t) - (b/a)F_n,$$

$$x_n(0) = -x_n^*(0), \tag{2.9}$$

$$y_n(t) = x_n'(t), \tag{2.10}$$

$$\dot{y}_n^*(\tau) = \begin{cases} -ay_n^*(\tau), & \tau \in [0, 1], \\ -ay_n^*(\tau) - b(x_n^*(\tau-1)\cos\phi(0) + F_n^*), & \tau > 1, \end{cases} \tag{2.11}$$

$$y_n^*(0) = -y_n(0),$$

$$n = 1, 2, 3, ...; \quad t \geqslant 0; \quad \tau \geqslant 0, \quad (\ )' = d/dt, \quad (\dot{\ }) = d/d\tau.$$

The functions F_n and F_n^* are given by

$$F_n = U_n \cos x_0(t) + V_n \sin x_0(t), \qquad n = 1, 2, ..., \tag{2.12}$$

$$F_1^* = 0,$$

$$F_n^* = \sum_{1 < |\gamma| < n} (x_1^*(\tau-1))^{\gamma_1} \times \cdots \times (x_{n-1}^*(\tau-1))^{\gamma_{n-1}} Q_\gamma(\tau), \quad n = 2, 3, ..., \tag{2.13}$$

where U_n and V_n are of the form

$$\sum a_{\alpha\beta}(x_0^{(\beta_0)}(t))^{\alpha_0} \times \cdots \times (x_{n-1}^{(\beta_{n-1})}(t))^{\alpha_{n-1}} \qquad (|\alpha| < n, |\beta| < n, \beta_0 > 1),$$

$$\alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}), \qquad \beta = (\beta_0, \beta_1, ..., \beta_{n-1}), \qquad \gamma = (\gamma_1, ..., \gamma_{n-1}),$$

with α_i β_i, γ_i as nonnegative integers, $|\alpha| = \sum_{i=0}^{n-1} \alpha_i$, $|\beta| = \sum_{i=0}^{n-1} \beta_i$, $|\gamma| = \sum_{i=1}^{n-1} \gamma_i$, Q_γ as polynomials in τ of order equal or smaller than $(n-1)$, and $x_i^{(\beta_i)}$ as the β_i derivative of x_i with respect to t.

3. Algorithm for the Construction of the Asymptotic Expansions

We seek the unique solution of the IVP (2.1)–(2.2) in the form

$$x(t, \varepsilon) + x^*(\tau, \varepsilon), \qquad y(t, \varepsilon) + y^*(\tau, \varepsilon), \tag{3.1}$$

where $\tau = t/\varepsilon$ and

$$\begin{aligned} x(t, \varepsilon) &\sim \sum_{n=0}^{\infty} \varepsilon^n x_n(t), & y(t, \varepsilon) &\sim \sum_{n=0}^{\infty} \varepsilon^n y_n(t), \\ x^*(\tau, \varepsilon) &\sim \sum_{n=0}^{\infty} \varepsilon^n x_n^*(\tau), & y^*(\tau, \varepsilon) &\sim \sum_{n=0}^{\infty} \varepsilon^n y_n^*(\tau). \end{aligned} \tag{3.2}$$

Substituting (3.1) into (2.1), we get

$$\varepsilon x'(t, \varepsilon) + \dot{x}^*(\tau, \varepsilon) = \varepsilon y(t, \varepsilon) + \varepsilon y^*(\tau, \varepsilon), \tag{3.3}$$

$$\varepsilon y'(t, \varepsilon) + \dot{y}^*(\tau, \varepsilon) = (-ay(t, \varepsilon) - bF) + (-ay^*(\tau, \varepsilon) - bF^*), \tag{3.4}$$

where

$$\begin{aligned} F &= \sin x(t-\varepsilon, \varepsilon), \\ F^* &= \sin(x(\varepsilon(\tau-1), \varepsilon) + x^*(\tau-1, \varepsilon)) - \sin x(\varepsilon(\tau-1), \varepsilon). \end{aligned} \tag{3.5}$$

In order to simplify the computations, first we show that $x_0^*(\tau) = 0$ for all $\tau > 0$. Substituting (3.2) into (3.3) and setting $\varepsilon = 0$, we get that $\dot{x}_0^*(\tau) = 0$. Integrating this equation under the condition $x_0^*(\tau) \to 0$ as $\tau \to +\infty$, it follows our claim.

Now, taking into account that $x_0^*(\tau) = 0$ and substituting the series for $x(t, \varepsilon)$ and $x^*(\tau, \varepsilon)$ into (3, 5), a straightforward computation shows that F and F^* can be formally represented by

$$\begin{aligned} F &= \sin x_0(t) + \sum_{n=1}^{\infty} \varepsilon^n (x_n(t) \cos x_0(t) + F_n), \\ F^* &= \sum_{n=1}^{\infty} \varepsilon^n (x_n^*(\tau-1) \cos \phi(0) + F_n^*), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} F_1 &= -x_0'(t) \cos x_0(t), \\ F_2 &= (x_0''(t)/2 - x_1'(t)) \cos x_0(t) - (x_1(t) - x_0'(t))^2 \sin x_0(t)/2, \\ F_1^* &= 0, \\ F_2^* &= -\sin \phi(0)[((\tau-1)x_0'(0) + x_1(0))x_1^*(\tau-1) + (x_1^*(\tau-1))^2]; \end{aligned}$$

in general F_n and F_n^* are given by (2.12) and (2.13), respectively.

Next, let us substitute the series (3.2) and (3.6) into (3.3)–(3.4). Considering the terms dependent on t and τ separately and equating coefficients of like powers of ε, we obtain the sequence of differential equations (2.5)–(2.11) for the determination of x_n, y_n, x_n^*, y_n^*. To solve the above-mentioned equations, we must determine the initial conditions. It is known that

$$\begin{aligned} x(t, \varepsilon) + x_\cdot^*(t/\varepsilon, \varepsilon) &= \phi(t), \qquad t \in [-\varepsilon, 0], \\ y(0, \varepsilon) + y^*(0, \varepsilon) &= y_0. \end{aligned} \tag{3.7}$$

Since ϕ is independent of ε and $x_0^*(\tau)=0$, (3.7) leads to

$$\begin{aligned} x_0(t) &= \phi(t), && t \in [-\varepsilon, 0],\\ x_n(t) &= -x_n^*(t/\varepsilon), && t \in [-\varepsilon, 0],\\ y_0^*(0) &= y_0 - y_0(0), &&\\ y_n^*(0) &= y_n(0). && \end{aligned} \tag{3.8}$$

Finally, requiring that each term $x_n^*(\tau)$ tends to zero as $\tau \to +\infty$, it follows that (3.8) together with Eqs. (2.5)–(2.11) permit us to carry out the computations of the coefficients of (3.2).

4. Properties of the Coefficients of the Asymptotic Expansions

The next result is the key to the proof of Theorem 1.

LEMMA 2. (a) *There exist polynomials $P_n(\tau)$ and $Q_n(\tau)$ of order n, such that for every $\tau \geqslant 0$, the following estimates hold:*

$$|x_{n+1}^*(\tau)| \leqslant P_n(\tau) \exp(-a\tau), \tag{4.1}$$

$$|y_n^*(\tau)| \leqslant Q_n(\tau) \exp(-a\tau), \tag{4.2}$$

$n = 0, 1, 2, \ldots$.

(b) *If $\phi(0) \neq (2k+1)\pi$ for all $k \in Z$, then for any $n = 0, 1, 2, \ldots$ x_n and y_n are bounded, indefinitely differentiable and all their derivatives are bounded on $[0, \infty)$.*

(c) *If $\phi(0) = (2k+1)\pi$ for some $k \in Z$, then only x_0 and y_0 are bounded on $[0, \infty)$.*

Proof. (a) By induction. From (2.7) and (2.8), we get

$$\begin{aligned} y_0^*(\tau) &= y_0^*(0) \exp(-a\tau), && \tau \geqslant 0,\\ x_1^*(\tau) &= -\int_\tau^\infty y_0^*(s)\, ds = -(y_0^*(0)/a) \exp(-a\tau), && \tau \geqslant 0. \end{aligned} \tag{4.3}$$

Hence, (4.1) and (4.2) are true for $n=0$.

On the other hand, from (2.11), we obtain

$$y_1^*(\tau) = \begin{cases} y_1^*(0) \exp(-a\tau), & \tau \in [0, 1],\\ y_1^*(1) \exp(-a(\tau-1)) - b \cos \phi(0) \int_1^\tau x_1^*(s-1) & \\ \quad \times \exp(-a(\tau-s))\, ds, & \tau > 1. \end{cases} \tag{4.4}$$

By virtue of (4.3) and (4.4), it follows that (4.2) holds for $n=1$. The inequality (4.1) is an immediate consequence of the validity of (4.2) for $n=1$ and the fact that

$$x_2^*(\tau) = -\int_\tau^\infty y_1^*(s)\,ds.$$

Suppose now that (4.1) and (4.2) are true for $n-1$. From (2.11) we obtain

$$y_n^*(\tau) = \begin{cases} y_n^*(0)\exp(-a\tau), & \tau \in [0, 1], \\ y_n^*(1)\exp(-a(\tau-1)) - b\int_1^\tau \exp(-a(\tau-s)) & \\ \quad \times [x_n^*(s-1)\cos\phi(0) + F_n^*]\,ds, & \tau > 1. \end{cases} \tag{4.5}$$

Recalling the dependence of F_n^* of $x_1^*, ..., x_{n-1}^*$, the induction hypothesis, and (4.5), we certainly get (4.2) for n. The validity of (4.1) for n follows from (4.2) and (2.8).

(b) It is known that x_0 is a solution of (2.5). Hence, x_0 is indefinitely differentiable. Since $\phi(0) \neq (2k+1)\pi$ for any $k \in Z$, it follows that $x_0'(t) \to 0$ as $t \to +\infty$. From this, it is easy to get $x_0^{(n)}(t) \to 0$ as $t \to +\infty$, for all $n = 1, 2, ...$. Taking into account that $y_0(t) = x_0'(t)$, we obtain that $y_0^{(n)}(t) \to 0$ as $t \to +\infty$, for all $n = 0, 1, 2, ...$.

Suppose now that our claim is true for $n-1$. By virtue of the dependence of F_n of $x_0, x_1, ..., x_{n-1}$ (see (2.12)), it follows the boundedness of F_n on $[0, \infty)$.

Integrating Eq. (2.9), we have

$$x_n(t) = -x_n^*(0)\exp(-b/a)\int_0^t \cos x_0(s)\,ds$$
$$-(1/a)\int_0^t \exp(-b/a)\int_s^t \cos x_0(u)\,du (y_{n-1}'(s) + bF_n)\,ds. \tag{4.6}$$

On the other hand, since $x_0(t) \to 2k\pi$ as $t \to +\infty$, for some $k \in Z$, there exists a positive number T such that

$$\cos x_0(t) \geq \tfrac{1}{2}, \qquad t > T. \tag{4.7}$$

Since F_n and y_{n-1}' are bounded, (4.6) and (4.7) imply the boundedness of x_n. The remainder of our assertion certainly follows from (2.9) and (2.10).

Finally, part (c) is obtained, observing that in this case $x_0(t)=(2k+1)\pi$, $y_0(t)=0$ for all $t\geqslant 0$, because when $\phi(0)=(2k+1)\pi$ Eq. (2.9) leads to

$$x'_n(t)=(b/a)x_n(t)-(y'_{n-1}(t)+bF_n)/2, \qquad n=1,2,...,$$

which has unbounded solutions on $[0,\infty)$. Q.E.D.

5. Proof of Theorem 1

Assume that $\phi(0)\neq(2k+1)\pi$ for any $k\in Z$, and let us denote by x_ε, y_ε the unique solution of the IVP (2.1)–(2.2).

Let us set

$$\begin{aligned} u(t)&=x_\varepsilon(t)-X_N(t,\varepsilon)\\ v(t)&=y_\varepsilon(t)-Y_N(t,\varepsilon), \end{aligned} \tag{5.1}$$

where

$$X_N(t,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n(x_n(t)+x_n^*(t/\varepsilon)),$$

$$Y_N(t,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n(y_n(t)+y_n^*(t/\varepsilon)).$$

It is clear that our problem is equivalent to proving that

$$u(t)=O(\varepsilon^{N+1}), \qquad v(t)=O(\varepsilon^{N+1}),$$

uniformly on $[-\varepsilon,\infty)$.

Differentiating (5.1) and substituting into (2.1), we get the following system for the determination of u and v,

$$u'(t)=v(t)+Y_N(t,\varepsilon)-X'_N(t,\varepsilon),$$

$$\varepsilon v'(t)=\begin{cases}-av(t)-aY_N(t,\varepsilon)-\varepsilon Y'_N(t,\varepsilon)\\ \qquad -b\sin\phi(t-\varepsilon), \qquad t\in[0,\varepsilon]\\ -av(t)-b\sin(u(t-\varepsilon)+X_N(t-\varepsilon,\varepsilon))\\ \qquad -aY_N(t,\varepsilon)-\varepsilon Y'_N(t,\varepsilon), \qquad t>\varepsilon,\end{cases} \tag{5.2}$$

subject to the following initial conditions

$$u(t)=0, \qquad t\in[-\varepsilon,0], \qquad v(0)=0. \tag{5.3}$$

The conditions (5.3) are an immediate consequence of the definition of u, v and relation (3.8).

Taking into account the definition of X_N, Y_N and Eqs. (2.5)–(2.11), a straightforward computation shows that

$$Y_N(t, \varepsilon) - X'_N(t, \varepsilon) = \varepsilon^N y_N^*(t/\varepsilon), \tag{5.4}$$

$$\varepsilon Y'_N(t, \varepsilon) + aY_N(t, \varepsilon)$$
$$= \begin{cases} -b\left[\sin x_0(t) + \sum_{n=1}^{N} \varepsilon^n (x_n(t)\cos x_0(t) + F)\right] \\ \quad + \varepsilon^{N+1} y'_N(t), \qquad t \in [0, \varepsilon] \\ -b\left[\sin x_0(t) + \sum_{n=1}^{N} \varepsilon^n ((x_n(t)\cos x_0(t) + F_n^*) \right. \\ \quad \left. + (x_n^*(\tau - 1)\cos\phi(0) + F_n^*)\right] + \varepsilon^{N+1} y'_N(t), \qquad t > \varepsilon. \end{cases} \tag{5.5}$$

Let us set

$$H_N(t, \varepsilon) = \begin{cases} -aY_N(t, \varepsilon - Y'_N(t, \varepsilon) - b\sin\phi(t-\varepsilon), & t \in [0, \varepsilon] \\ -aY_N(t, \varepsilon) - \varepsilon Y'_N(t, \varepsilon) - b\sin X_N(t-\varepsilon, \varepsilon), & t > \varepsilon. \end{cases} \tag{5.6}$$

From the form of the expansion of F, F^* and Lemma 2, we obtain that

$$H_N(t, \varepsilon) = O(\varepsilon^{N+1}). \tag{5.7}$$

By virtue of the definition of H_N, the system (5.2) can be rewritten as

$$\begin{aligned} u'(t) &= v(t) + \varepsilon^N y_N^*(t/\varepsilon) \\ \varepsilon v'(t) &= -av(t) - b[\sin(u(t-\varepsilon) + X_N(t-\varepsilon, \varepsilon)) \\ &\quad - \sin X_N(t-\varepsilon, \varepsilon)] + H_N(t, \varepsilon). \end{aligned} \tag{5.8}$$

In order to complete the proof of Theorem 1, it is sufficient to integrate, step by step, system (5.8), recalling Lemma 2, (5.7), and the fact that

$$|\sin(u(t-\varepsilon) + X_N(t-\varepsilon, \varepsilon)) - \sin X_N(t-\varepsilon, \varepsilon)| < |u(t-\varepsilon)|.$$

Part (b) of Theorem 1 can be proved in the same manner as that of part (a). Q.E.D.

6. Conclusion

According to the proposed algorithm, we should be able to construct the asymptotic expansion of any order of approximation uniformly valid on

$[0, \infty)$, under the condition $\phi(0) \neq (2k+1)\pi$. In the case where $\phi(0) = (2k+1)\pi$, we were able to obtain only the first approximation term. This is related to the instability of the solutions $x_0(t) = (2k+1)\pi$ of Eq. (2.5). Let us observe that if we are interested in the construction of the asymptotic expansions uniformly valid only on intervals of the form $[0, T]$, $T > 0$, the above-mentioned algorithm provides the answer without any restriction on the initial function.

Finally, we point out that, in the context of the studied model, $x(0) = (2k+1)\pi$ corresponds to when the plant forms a right angle with the vertical; e.g., the plant falls to the ground.

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