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Dynamic behavior of an eco-epidemic system with impulsive birth ☆

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A R T I C L E I N F O

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A B S T R A C T

In the paper, we investigate an eco-epidemic system with impulsive birth. The conditions for the stability of infection-free periodic solution are given by applying Floquet theory of linear periodic impulsive equation. And we give the conditions of persistence by constructing a consequence of some abstract monotone iterative schemes. By using the method of coincidence degree, a set of sufficient conditions are derived for the existence of at least one strictly positive periodic solution. Finally, numerical simulation shows that there exists a stable positive periodic solution with a maximum value no larger than a given level. Thus, we can use the stability of the positive periodic solution and its period to control insect pests at acceptably low levels.

1. Introduction

Mathematical biology, namely predator–prey systems and models for transmissible diseases are major fields of study. While a given species spreads a disease, susceptible become infectious by contact with infectious individuals. However, in the natural world, species do not exist alone. The species also competes with the other species for food and space, or is preyed on by other species. So diseases in ecological species cannot be ignored. Models that deal with disease in ecological systems are known as eco-epidemiological systems. This model has been extensively studied by several researchers. For instance, Hadeler and Freedman [1] were the first who described a predator–prey model where the prey was infected by a parasite, and the prey in turn infected the predator with the parasite. They dealt with two different, though closely related problems, persistence of a parasite in a given prey–predator system and parasite mediated coexistence of prey and predator. Venturino [2] proposed a two-dimensional prey–predator model and studied how the presence of the disease among the prey affected the behavior of the model. They concluded that under suitable assumptions the disease acts as a control for the system. Chattopadhyay and Arino [3] proposed and analyzed three species eco-epidemiological system consisting of sound prey, infected prey and the predator population. They showed under certain parametric conditions, the strictly positive interior equilibrium entered into Hopf type bifurcation. And they concluded that there was a threshold level of infection below which all three species studied would persist and above which the disease would be epidemic. Chattopadhyay and Bairagi [4] proposed and analyzed a three-dimensional eco-epidemiological model consisting of susceptible fish population, infected fish population and their predator the Pelican population. They assumed that the predator population only preyed infected fish population. They studied the local stability, global stability and persistence of the system around the positive interior equilibrium. They observed that if the level of the search rate of predator was low, the system around the positive interior equilibrium was stable. But the instability seted in with the increase of search rate level of predator. The model of Chattopadhyay and Arino [5] modifies the eco-epidemiological model proposed by Chattopadhyay and Bairagi in [4]. They

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assumed that Pelican population feeded on both susceptible and infected fish population. Feeding on infected fish enhanced the death rate of Pelican and was considered to contribute negative growth where as feeding on susceptible fish enhanced their growth rate and was considered to contribute positive growth. The main objective of this work is to find out the conditions for which considered system will become eventually disease free. [6] modified the model of Chattopadhyay and Arino [5] by considering standard incidence instead of horizontal incidence. The conditions under which the population reaches the origin either by following the axis or in a spiral pattern were determined.

Keeping the above observations in mind, we consider a prey–predator model with the disease in the prey:

Assumption 1. In the absence of infection, the prey population grows according to a logistic fashion with carrying capacity $\frac{1}{k_1}$, with an intrinsic birth rate constant r_1 such that

$$\dot{N}(t) = N[r_1 - k_1 N].$$

And in the absence of the prey population, the predator also grows according to a logistic fashion with carrying capacity $\frac{1}{k_2}$, with an intrinsic birth rate constant r_2 such that

$$\dot{P}(t) = P[r_2 - k_2 P].$$

Assumption 2. In the presence of infection we assume that the total prey population N is divided into two classes, namely, susceptible population, denoted by S, and infected population, denoted by I. Therefore, at any time t the total number of prey population is

$$N(t) = S(t) + I(t).$$

Assumption 3. We assume that only susceptible prey population S, is capable of reproducing with logistic law and the infective population, I, does not reproduce. However, the infective population, I, still contributes with S to population growth towards the carrying capacity.

Assumption 4. The mode of disease transmission follows the simple law of mass action.

Assumption 5. The disease is only spread among the prey population and the disease is not genetically inherited. The infected population does not recover or become immune.

Assumption 6. We assume that the predator population only preys on the infective population. Since infected preys are weakened and become easier to catch.

From the above assumptions, we can write down the eco-epidemiological model as:

$$\begin{cases} \dot{S}(t) = S\big[r_1 - k_1(S+I)\big] - \lambda SI, \\ \dot{I}(t) = \lambda SI - cIP - \delta I, \\ \dot{P}(t) = P(r_2 - k_2 P) + eIP. \end{cases} \tag{1.1}$$

where λ is the rate of transmission, δ is the death rate of the infection, the predator rate is cIP and the feeding efficiency in turning predation into new predator is $\frac{e}{c}$ and $c > e$.

The model (1.1) is considered the continuous birth. Whereas, many animals give birth only during a fixed period of a year. For example, fox rabies is a highly virulent disease of the fox, and the fox gives birth once a year only in spring. For having more accurate describing to the phenomenon, we need consider to use the impulsive birth instead of the continuous birth. On the other hand, the research on theory and applications of impulsive differential equations have been many nice works [7,8,11–13].

$$\begin{cases} \left.\begin{array}{l} \dot{S}(t) = S\big[-d_1 - k_1(S+I)\big] - \lambda SI \\ \dot{I}(t) = \lambda SI - cIP - \delta I \\ \dot{P}(t) = P(-d_2 - k_2 P) + eIP \end{array}\right\} \quad t \neq t_k,\ k \in Z, \\ \left.\begin{array}{l} S(t_k^+) = (1 + b_{1k})S(t_k^-) \\ I(t_k^+) = I(t_k^-) \\ P(t_k^+) = (1 + b_{2k})P(t_k^-) \end{array}\right\} \quad t = t_k,\ k \in Z. \end{cases} \tag{1.2}$$

where b_{1k}, b_{2k} represent growth of birth pulse for the susceptible prey $S(t)$ and the predator $P(t)$ at t_k, respectively. And we also assume that

(i) there exists a positive integer q, such that $t_{k+q} = t_k + T$, $b_{i(k+q)} = b_{ik} > 0$ $(i = 1, 2)$;
(ii) $t_k \neq 0$ for $k = 1, 2, \ldots,$ and $[0, T] \cap \{t_k\} = \{t_1, t_2, \ldots, t_q\}$.

The purpose of this paper is investigating the dynamics behavior of an eco-epidemic system with impulsive birth. The paper is organized as follows: In Section 2, some notations are introduced and some results needed in later sections are stated. In Section 3, we give the conditions for the existence of periodic infection-free solution. In Section 4, we also give the conditions for persistence by constructing a consequence of some abstract monotone iterative schemes introduced. In Section 5, the conditions for local stability of periodic infection-free solution are given by applying Floquet theory of linear periodic impulsive equation. In Section 6, using the method of coincidence degree, some sufficient conditions for the existence of at least one strictly positive periodic solution are derived. In the final discussion section, numerical experiments were performed to observe the dynamics of the system (1.2).

2. Definitions and preliminary knowledge

Let $J \subset R$. Denote by $PC(J, R)$ the set of function $f : J \to R$ which are continuous for $t \in J$, $t \neq t_k$, are continuous from the left for $t \in J$ and have discontinuities of the first kind at the points $t_k \in J$. Denote by $PC'(J, R)$ the set of functions $f : J \to R$ with a derivative $\frac{df}{dt} \in PC(J, R)$. Throughout this work we deal with the Banach spaces of T-periodic functions

$$PC_T = \{ f \in PC([0, T], R) \mid f(0) = f(T) \} \quad \big(\text{where } \|f\|_{PC_T} = \sup\{ |f(t)| \colon t \in [0, T] \}\big),$$
$$PC'_T = \{ f \in PC'([0, T], R) \mid f(0) = f(T) \} \quad \big(\text{where } \|f\|_{PC'_T} = \max\{ \|f\|_{PC_T}, \|\dot f\|_{PC'_T} \}\big).$$

Lemma 1. *Let the function $\omega \in PC'([0, \infty), R)$ satisfies the inequalities*

$$\begin{cases} \dot\omega(t) \leqslant f(t)\omega(t) + g(t), & t \neq t_k,\ t > 0, \\ \omega(t_k^+) \leqslant f_k \omega(t_k) + g_k, & t = t_k > 0, \\ \omega(0^+) \leqslant \omega_0, \end{cases} \tag{2.1}$$

where $f(t), g(t) \in PC([0, \infty), R)$ and $f_k > 0$, g_k and ω_0 are constants. Then for $t > 0$

$$\omega(t) \leqslant \omega(0) \prod_{0<t_k<t} f_k \exp\left(\int_0^t f(s)\, ds \right) + \int_0^t \prod_{s \leqslant t_k < t} f_k \exp\left(\int_0^t f(r)\, dr \right) g(s)\, ds + \sum_{0<t_k<t} \prod_{t_k \leqslant t_j < t} f_j \exp\left(\int_{t_k}^t f(s)\, ds \right) g_k.$$

Analogously, we have

$$\omega(t) \geqslant \omega(0) \prod_{0<t_k<t} f_k \exp\left(\int_0^t f(s)\, ds \right) + \int_0^t \prod_{s \leqslant t_k < t} f_k \exp\left(\int_0^t f(r)\, dr \right) g(s)\, ds + \sum_{0<t_k<t} \prod_{t_k \leqslant t_j < t} f_j \exp\left(\int_{t_k}^t f(s)\, ds \right) g_k$$

for all $t > 0$ if all the inequalities of (2.1) are inverse.

Definition 1. (See [10].) The set A is said to be quasiequicontinuous in $[0, T]$ if for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $x \in A$, $k \in Z_+$, $t_1, t_2 \in (t_{k-1}, t_k] \cap [0, T]$, and $|t_1 - t_2| < \delta$, then

$$|x(t_1) - x(t_2)| < \varepsilon.$$

Lemma 2. *(See [10].) The set $A \subset PC_T$ is relatively compact if and only if:*

(1) *A is bounded, that is, $\|f\|_{PC_T} = \sup\{|f(t)| \colon t \in J\} \leqslant M$ for each $x \in A$ and some $M > 0$;*
(2) *A is quasiequicontinuous in J.*

3. Existence of the periodic infection-free solution

We first demonstrate the existence of an infection-free solution, in which infection individuals are entirely absent from the population permanently, i.e. $I(t) = 0$, $t \geqslant 0$. Under this condition, the susceptible prey and the predator must satisfy:

$$\begin{cases} \left.\begin{aligned} \dot S(t) &= S(-d_1 - k_1 S) \\ \dot P(t) &= P(-d_2 - k_2 P) \end{aligned}\right\} & t \neq t_k,\ k \in Z, \\ \left.\begin{aligned} S(t_k^+) &= (1 + b_{1k}) S(t_k^-) \\ P(t_k^+) &= (1 + b_{2k}) P(t_k^-) \end{aligned}\right\} & t = t_k,\ k \in Z. \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that*

$$d_i < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{ik}), \quad i=1,2.$$

Then (3.1) *exists a positive periodic solution.*

Proof. First, we consider the logistic equation with impulsive birth

$$\begin{cases} \dot S(t) = S(t)\big[-d_1 - k_1 S(t)\big], & t \neq t_k,\ k \in Z_+, \\ S(t_k^+) = (1+b_{1k})S(t_k^-), & t = t_k,\ k \in Z_+. \end{cases} \tag{3.2}$$

For (3.2) we carry out the change of variable $x = \frac{1}{S}$ and obtain the linear non-homogeneous impulsive equation

$$\begin{cases} \dot x(t) = d_1 x(t) + k_1, & t \neq t_k,\ k \in Z_+, \\ x(t_k^+) = \dfrac{1}{1+b_{1k}} x(t_k^-), & t = t_k,\ k \in Z_+. \end{cases} \tag{3.3}$$

Let

$$W(t,s) = \prod_{s \leqslant t_k < t} \frac{1}{1+b_{1k}} \exp\big[d_1(t-s)\big] \tag{3.4}$$

be the Cauchy matrix for the respective homogeneous equation. Then

$$x(t) = W(t,0)x(0) + \int_0^t k_1 W(t,s)\,ds$$

of a solution of (3.3). This solution is T-periodic if $x(0) = x(T)$, or if

$$\big(1 - W(0,T)\big)x(0) = \int_T^0 k_1 W(T,s)\,ds. \tag{3.5}$$

Since the multiplier $W(T,0)$ of the homogeneous equation

$$\begin{cases} \dot x(t) = d_1 x(t), & t \neq t_k,\ k \in Z_+, \\ x(t_k^+) = \dfrac{1}{1+b_{1k}} x(t_k^-), & t = t_k,\ k \in Z_+, \end{cases}$$

is less than 1 because we have $d_1 < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k})$, and

$$\int_0^T k_1 W(T,s)\,ds > 0,$$

Eq. (3.5) has a unique solution $x(0)$. To the initial value $x(0)$, so we obtained there corresponds the unique T-periodic solution of (3.3) which in view of (3.4) is positive for each $t \in R$. Denote this solution by $x(t)$. Then the function $\theta_{[-d_1,k_1]} = S(t) = \frac{1}{x(t)}$ is the unique T-periodic solution of (3.2) which is also positive.

Similarly, we obtain

$$\begin{cases} \dot P(t) = P(t)\big[-d_2 - k_2 P(t)\big], & t \neq t_k,\ k \in Z_+, \\ P(t_k^+) = (1+b_{2k})P(t_k^-), & t = t_k,\ k \in Z_+, \end{cases}$$

has a unique positive T-periodic solution $\theta_{[-d_2,k_2]} = P(t)$ if and only if $d_2 < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k})$. This completes the proof. □

Given $S_0 \in R$, we denote by $\Phi_{[-d_1,k_1]}(t, t_0^+, S_0)$ the unique solution of Cauchy problem

$$\begin{cases} \dot S(t) = -d_1 S(t) - k_1 S^2(t), & t \geqslant t_0\ (\neq t_k),\ k \in Z_+, \\ S(t_k^+) = (1+b_{1k})S, & t = t_k,\ k \in Z_+, \\ S(t_0^+) = S_0, \end{cases}$$

which is positive and globally defined for $t \geqslant t_0$. Similarly, given $P_0 \in R$, we denote by $\Phi_{[-d_2,k_2]}(t, t_0^+, P_0)$ the unique solution of Cauchy problem

$$\begin{cases} \dot{P}(t) = -d_2 P(t) - k_2 P^2(t), & t \geqslant t_0\ (\neq t_k),\ k \in Z_+, \\ P(t_k^+) = (1 + b_{2k})P, & t = t_k,\ k \in Z_+, \\ P(t_0^+) = P_0, \end{cases}$$

which is positive and globally defined for $t \geqslant t_0$.

The following result gives the global attractive character of $\theta_{[-d_i,k_i]}$, $i = 1, 2$.

Theorem 3.2. *For any $S_0 > 0$, $P_0 > 0$ we have*

$$\lim_{t\to\infty} \left|\Phi_{[-d_1,k_1]}(t, t_0^+, S_0) - \theta_{[-d_1,k_1]}\right| = 0,$$
$$\lim_{t\to\infty} \left|\Phi_{[-d_2,k_2]}(t, t_0^+, S_0) - \theta_{[-d_2,k_2]}\right| = 0$$

provided that $d_i < \frac{1}{T} \ln \prod_{k=1}^{q}(1 + b_{ik})$, $i = 1, 2$, for any $k \in Z_+$.

This is the direct result of Lemma 3.1 of [7].

Corollary 1. *Let any positive real α_i, β_i $(i = 1, 2)$ be such that $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_2$. Then $\theta_{[\alpha_1,\beta_1]} \leqslant \theta_{[\alpha_2,\beta_2]}$.*

4. The monotone scheme

This section is devoted to describe monotone iterative technique which offers a constructive method yielding monotone sequences that converge to solutions of system (1.2). Now we analyze the properties of the following monotone scheme:

$$I_0 := 0, \qquad S_n := \theta_{[-d_1-(k_1+\lambda)I_{n-1},\,k_1]}, \qquad P_n := \theta_{[-d_2+eI_{n-1},\ k_2]}, \qquad I_n := \theta_{[\lambda S_n - cP_n - \delta,0]}, \quad n \geqslant 1. \tag{4.1}$$

Lemma 3. *For each $n \geqslant 1$ the following inequalities hold*:

$$S_2 \leqslant \cdots \leqslant S_{2n} \leqslant S_{2n-1} \leqslant \cdots \leqslant S_1, \qquad I_2 \leqslant \cdots \leqslant I_{2n} \leqslant I_{2n-1} \leqslant \cdots \leqslant I_1, \qquad P_1 \leqslant \cdots \leqslant P_{2n-1} \leqslant P_{2n} \leqslant \cdots \leqslant P_2. \tag{4.2}$$

Moreover, the limits

$$\overline{S} := \lim_{n\to\infty} S_{2n} \leqslant \underline{S} := \lim_{n\to\infty} S_{2n-1}, \qquad \overline{I} := \lim_{n\to\infty} I_{2n} \leqslant \underline{I} := \lim_{n\to\infty} I_{2n-1}, \qquad \underline{P} := \lim_{n\to\infty} P_{2n-1} \leqslant \overline{P} := \lim_{n\to\infty} P_{2n} \tag{4.3}$$

are well defined and $(\overline{S}, \underline{S}, \overline{I}, \underline{I}, \overline{P}, \underline{P})$ is a component-wise nonnegative T-periodic solution of

$$\begin{cases} \dot{\overline{S}}(t) = \overline{S}(t)\left[-d_1 - (k_1 + \lambda)\underline{I}\right] - k_1\overline{S}^2, \\ \overline{S}(t_k^+) = (1 + d_{1k})\overline{S}(t_k^-), \end{cases} \qquad \begin{cases} \dot{\underline{S}}(t) = \underline{S}(t)\left[-d_1 - (k_1 + \lambda)\overline{I}\right] - k_1\underline{S}^2, \\ \underline{S}(t_k^+) = (1 + d_{1k})\underline{S}(t_k^-), \end{cases}$$

$$\begin{cases} \dot{\overline{I}}(t) = \overline{I}(t)[\lambda\overline{S} - c\overline{P} - \delta], \\ \overline{I}(t_k^+) = \overline{P}(t_k^-), \end{cases} \qquad \begin{cases} \dot{\underline{I}}(t) = \underline{I}(t)[\lambda\underline{S} - c\underline{p} - \delta], \\ \underline{I}(t_k^+) = \underline{I}(t_k^-), \end{cases}$$

$$\begin{cases} \dot{\overline{P}}(t) = \overline{P}(t)[-d_2 + e\underline{I}] - k_2\overline{P}^2, \\ \overline{P}(t_k^+) = (1 + d_{2k})\overline{P}(t_k^-), \end{cases} \qquad \begin{cases} \dot{\underline{P}}(t) = \underline{P}(t)[-d_2 + e\underline{I}] - k_2\underline{P}^2, \\ \underline{P}(t_k^+) = (1 + d_{2k})\underline{P}(t_k^-). \end{cases} \tag{4.4}$$

Proof. We shall argue by induction using Corollary 3.2. By definition of the θ's we have $S_n \geqslant 0$, $I_n \geqslant 0$, $P_n \geqslant 0$ for all $n \geqslant 1$. Thus, it follows from Corollary 3.2 and monotonicity of function that $\theta_{[-d_1,k_1]} \geqslant \theta_{[-d_1-(k_1+\lambda)I_1,k_1]}$ that is, $S_1 \geqslant S_2$. Similarly, we have $P_1 \leqslant P_2$ and $I_1 \geqslant I_2$. Now, we suppose that (4.2) is satisfied for $n \in \{1, \dots, k\}$, we can prove that

$$S_{2k} \leqslant S_{2k+2} \leqslant S_{2k+1} \leqslant S_{2k-1}, \qquad I_{2k} \leqslant I_{2k+2} \leqslant I_{2k+1} \leqslant I_{2k-1}, \qquad P_{2k-1} \leqslant P_{2k+1} \leqslant P_{2k+2} \leqslant P_{2k}.$$

Since we are assuming that $I_{2k-2} \leqslant I_{2k}$, it follow form (4.1) that $S_{2k-1} \geqslant S_{2k+1}$ and $P_{2k-1} \leqslant P_{2k+1}$. On the other hand, we find from $I_{2k+1} \leqslant I_{2k-1}$ that $S_{2k+1} \geqslant S_{2k}$ and $P_{2k+1} \leqslant P_{2k}$. So, $I_{2k+1} \geqslant I_{2k}$. Similarly, we have $S_{2k+2} \leqslant S_{2k+1}$, $P_{2k+2} \leqslant P_{2k+1}$. This completes the proof of (4.2).

The fact that the limits in (4.3) are pointwise well defined follows from the monotonicity of the scheme. On the other hand, it follows from the definition of the θ's that

$$\begin{cases} \dot S_{2n}(t) = S_{2n-1}\left[-d_1-(k_1+\lambda)I_{2n-2}-k_1S_{2n-1}\right], \\ S_{2n-1}\left(t_k^+\right) = (1+b_{1k})S_{2n-1}\left(t_k^-\right), \end{cases} \qquad \begin{cases} \dot S_{2n}(t) = S_{2n}\left[-d_1-(k_1+\lambda)I_{2n-1}-k_1S_{2n}\right], \\ S_{2n}\left(t_k^+\right) = (1+b_{1k})S_{2n}\left(t_k^-\right), \end{cases}$$

$$\begin{cases} \dot I_{2n-1}(t) = I_{2n-1}(t)[\lambda S_{2n-1}-cP_{2n-1}-\delta], \\ I_{2n-1}\left(t_k^+\right) = I_{2n-1}\left(t_k^-\right), \end{cases} \qquad \begin{cases} \dot I_{2n}(t) = I_{2n}(t)[\lambda S_{2n}-cP_{2n}-\delta], \\ I_{2n}\left(t_k^+\right) = I_{2n}\left(t_k^-\right), \end{cases}$$

$$\begin{cases} \dot P_{2n-1}(t) = P_{2n-1}(t)[-d_2+eI_{2n-2}]-k_2P_{2n-1}^2, \\ P_{2n-1}\left(t_k^+\right) = (1+d_{2k})P_{2n-1}\left(t_k^-\right), \end{cases} \qquad \begin{cases} \dot P_{2n}(t) = P_{2n}(t)[-d_2+eI_{2n-1}]-k_2P_{2n}^2, \\ P_{2n}\left(t_k^+\right) = (1+d_{2k})P_{2n}\left(t_k^-\right). \end{cases} \tag{4.5}$$

Thus, since the S_k's, the I_k's, the P_k's are uniformly bounded, we find from (4.5) that each of the sequences S_{2n-1}, S_{2n}, I_{2n-1}, I_{2n}, P_{2n-1}, P_{2n} is quasiequicontinuous. So, it follows from (4.2) and Lemma 2.2 that the limits in (4.3) are uniform; that is, the convergence occurs in PC_T. Now, passing to the limits as $n \to \infty$ in (4.5), we find that the convergence actually occurs in PC_T and so (4.4) holds.

Now, given $(S_0, I_0, P_0) \in R^3$, we denote the unique positive solution of the Cauchy problem

$$\begin{cases} \dot S(t) = S\left[-d_1-k_1(S+I)\right]-\lambda SI, \\ \dot I(t) = \lambda SI - cIP - \delta I, \\ \dot P(t) = P(-d_2-k_2P)+eIP, \\ S\left(t_k^+\right) = (1+b_{1k})S\left(t_k^-\right), \\ I\left(t_k^+\right) = I\left(t_k^-\right), \\ P\left(t_k^+\right) = (1+b_{2k})P\left(t_k^-\right), \\ S\left(t_0^+\right) = S_0, \qquad I\left(t_0^+\right) = I_0, \qquad P\left(t_0^+\right) = P_0 \end{cases} \tag{4.6}$$

by $(S(t;t_0,S_0,I_0,P_0), I(t;t_0,S_0,I_0,P_0), P(t;t_0,S_0,I_0,P_0))$. □

Theorem 4.1. *For any $t_0 \in R$ and $(S_0, I_0, P_0) \in R_+^3$ the following estimates hold true*:

$$\begin{aligned} &\lim_{t\to\infty}\sup\left[S(t;t_0,S_0,I_0,P_0)-\underline{S}(t)\right] \leqslant 0 \leqslant \lim_{t\to\infty}\inf\left[S(t;t_0,S_0,I_0,P_0)-\overline{S}(t)\right], \\ &\lim_{t\to\infty}\sup\left[I(t;t_0,S_0,I_0,P_0)-\underline{I}(t)\right] \leqslant 0 \leqslant \lim_{t\to\infty}\inf\left[I(t;t_0,S_0,I_0,P_0)-\overline{I}(t)\right], \\ &\lim_{t\to\infty}\sup\left[P(t;t_0,S_0,I_0,P_0)-\overline{P}(t)\right] \leqslant 0 \leqslant \lim_{t\to\infty}\inf\left[P(t;t_0,S_0,I_0,P_0)-\underline{P}(t)\right]. \end{aligned} \tag{4.7}$$

Proof. To simplicity the notation we set

$$\big(S(t), I(t)\big), P(t) := \big(S(t;t_0,S_0,I_0,P_0), I(t;t_0,S_0,I_0,P_0), P(t;t_0,S_0,I_0,P_0)\big).$$

Consider the iterative scheme

$$i_0 := 0, \qquad s_n := \Phi_{[-d_1-(k_1+\lambda)I_{n-1},k_1]}, \qquad p_n := \Phi_{[-d_2+eI_{n-1},k_2]}, \qquad i_n := \Phi_{[\lambda S_n - cP_n - \delta, 0]}, \quad n \geqslant 1. \tag{4.8}$$

Arguing as in the proof Lemma 4.1, but this time applying Lemma 3.2 of [7], it can be easily seen that for any $n \geqslant 1$ we have

$$\begin{aligned} &s_2 \leqslant \cdots \leqslant s_{2n} \leqslant S \leqslant s_{2n-1} \leqslant \cdots \leqslant S_1, \qquad i_2 \leqslant \cdots \leqslant i_{2n} \leqslant I \leqslant i_{2n-1} \leqslant \cdots \leqslant i_1, \\ &p_1 \leqslant \cdots \leqslant p_{2n-1} \leqslant P \leqslant p_{2n} \leqslant \cdots \leqslant p_2. \end{aligned} \tag{4.9}$$

In particular, $(S(t), I(t), P(t))$ is well defined for any $t \geqslant t_0$. Finally, an induction argument together with Lemma 3.2 and Theorem 3.1 of [7] show that

$$\lim_{t\to\infty}\left|s_n(t)-S_n(t)\right| = \lim_{t\to\infty}\left|i_n(t)-I_n(t)\right| = \lim_{t\to\infty}\left|p_n(t)-P_n(t)\right| = 0, \quad n \geqslant 1.$$

This identity together with the uniformly in the convergence of (4.3) complete the proof. □

5. Local stability of the periodic infection-free solution

In the section, we characterize whether the periodic infection-free solution is local stable. To prove the result we shall use the following characterization of the linear stability of the semi-trivial state by using Floquet theory of linear homogeneous periodic impulsive equation (see [10]).

Theorem 5.1. *If*

$$d_i < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{ik}), \quad i=1,2,$$

and

$$\frac{\lambda k_2}{c}\left[-d_1+\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k})\right] < k_1\left[-d_2+\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k})\right]+\frac{k_1k_2\delta}{c},$$

then the periodic infection-free solution of system (1.2) *is local stable, otherwise it is unstable.*

Proof. The stability of a T-periodic solution $(\widetilde{S},0,\widetilde{P})$ of (1.2) may be determined by considering the behavior of small-amplitude perturbations of the solution.

Define $(S(t),I(t),P(t))=(\widetilde{S}+x(t),y(t),\widetilde{P}+z(t))$, there may be written in matrix form as:

$$\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix}=\Phi(t)\begin{pmatrix} x(0)\\ y(0)\\ z(0)\end{pmatrix},$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt}=\begin{pmatrix} -d_1-2k_1\widetilde{S} & -(k_1+\lambda)\widetilde{S} & 0\\ 0 & \lambda\widetilde{S}-c\widetilde{P}-\delta & 0\\ 0 & c\widetilde{P} & -d_2-2k_2\widetilde{P}\end{pmatrix}\Phi(t),$$

with $\Phi(0)=I$, the identity matrix. The resetting impulsive condition of (1.2) becomes

$$\begin{pmatrix} x(t_k^+)\\ y(t_k^+)\\ z(t_k^+)\end{pmatrix}\begin{pmatrix} 1+b_{1k} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1+b_{2k}\end{pmatrix}\begin{pmatrix} x(t_k^-)\\ y(t_k^-)\\ z(t_k^-)r\end{pmatrix}.$$

Hence, the stability of the solution is determined by the eigenvalues of

$$M=\begin{pmatrix} 1+b_{1k} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1+b_{2k}\end{pmatrix}\Phi(T)$$

if all eigenvalues have absolute value less than 1, the solution is stable, which

$$\mu_1=\prod_{k=1}^{q}\exp\int_0^T(-d_1-2k_1\widetilde{S})\,ds=\prod_{k=1}^{q}\exp\left[\int_0^T(-d_1-k_1\widetilde{S})\,ds-\int_0^T k_1\widetilde{S}\,ds\right]=\exp\left[d_1T-\ln\prod_{k=1}^{q}(1+b_{1k})\right],$$

$$\mu_2=\exp\left\{\frac{\lambda}{k_1}\left[\prod_{k=1}^{q}(1+b_{1k})-d_1T\right]-\frac{c}{k_2}\left[\prod_{k=1}^{q}(1+b_{2k})-d_2T\right]-\delta T\right\},$$

$$\mu_3=\exp\left[d_2T-\ln\prod_{k=1}^{q}(1+b_{2k})\right].$$

$\mu_1<1$ if $d_1<\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k})$ and $\mu_3<1$ if $d_2<\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k})$, and $\mu_2<1$ if $\frac{\lambda k_2}{c}[-d_1+\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k})]<k_1[-d_2+\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k})]+\frac{k_1k_2\delta}{c}$. So, $(\widetilde{S},0,\widetilde{P})$ is locally stable. The proof is completed. □

Definition 2. It is said that (1.2) is persistence if there exist positive numbers $0<\rho<\varrho$ such that any component-wise positive solution of (1.2) satisfies

$$\rho\leqslant\liminf_{t\to\infty}S(t)\leqslant\limsup_{t\to\infty}S(t)\leqslant\varrho,\qquad \rho\leqslant\liminf_{t\to\infty}I(t)\leqslant\limsup_{t\to\infty}I(t)\leqslant\varrho,\qquad \rho\leqslant\liminf_{t\to\infty}P(t)\leqslant\limsup_{t\to\infty}P(t)\leqslant\varrho.$$

The next result gives some sufficient conditions for persistence.

Theorem 5.2. *If*

$$d_i < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{ik}), \quad i=1,2,$$

$$\frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k}) - d_1 > \frac{1}{T}\int_0^T (k_1+\lambda)\theta_{[\lambda\theta_{[-d_1,k_1]}-c\theta_{[-d_2,k_2]}-\delta,0]}, \tag{5.1}$$

then (1.2) *is persistence.*

Proof. Let S_n, I_n, P_n, $n \geqslant 1$, denote the sequences given by the scheme (4.1). Since $d_i < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{ik})$, $i=1,2$, we have $S_1 > 0$, $P_1 > 0$, and the sequence I_n is positive if $I(0) > 0$. Moreover, condition (5.1) implies that $S_2 > 0$. Now, Theorem 4.1 completes the proof. □

6. Existence of the positive *T*-periodic solution

In this section, we study the existence of strictly positive periodic solution of (1.2). For the reader's convenience, we shall first summarize below a few concepts and results from Mawhin [9] that will be used in this section.

Let X, Z be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$. It follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 4. *(See [9].) Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) *$Lx \neq \lambda Nx$, for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{dom} L$.*
(ii) *For each $x \in \operatorname{Ker} L \cap \partial\Omega$, $QNx \neq 0$, and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Theorem 6.1. *If*

$$d_i < \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{ik}), \quad i=1,2,$$

and

$$\frac{\lambda k_2}{c}\left[-d_1 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k})\right] > k_1\left[-d_2 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k})\right] + \frac{k_1k_2\delta}{c},$$

then the system (1.2) *has at least one T-periodic positive solution.*

Proof. Making the changes of variable

$$S(t) = \exp\bigl(x_1(t)\bigr), \qquad I(t) = \exp\bigl(x_2(t)\bigr), \qquad P(t) = \exp\bigl(y(t)\bigr),$$

then system (1.2) is reformulated as

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{x}_1(t) = -d_1 - k_1\bigl(e^{x_1(t)} + e^{x_2(t)}\bigr) - \lambda e^{x_2(t)} \triangleq F(t) \\ \dot{x}_2(t) = \lambda e^{x_2(t)} - ce^{y(t)} - \delta \triangleq G(t) \\ \dot{y}(t) = -d_2 - k_2 e^{y(t)} + ee^{x_2(t)} \triangleq H(t) \end{array}\right\} \quad t \neq t_k,\ k \in Z, \\ \left.\begin{array}{l} x_1(t_k^+) - x_1(t_k^-) = \ln(1+b_{1k}) \\ x_2(t_k^+) - x_2(t_k^-) = 0 \\ y(t_k^+) - y(t_k^-) = \ln(1+b_{2k}) \end{array}\right\} \quad t = t_k,\ k \in Z. \end{array}\right. \tag{6.1}$$

Let $\operatorname{Dom} L = PC'_T \times PC'_T \times PC'_T$ and

$$L : \operatorname{Dom} L \to Z, \quad \text{with } \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \to \left(\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix}, \left\{\begin{pmatrix} \Delta x_1(t_k) \\ \Delta x_2(t_k) \\ \Delta y(t_k) \end{pmatrix}\right\}_{k=1}^{q}\right),$$

and let $N : PC'_T \times PC'_T \times PC'_T \to Z$ with

$$N\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \left(\begin{pmatrix} F(t) \\ G(t) \\ H(t) \end{pmatrix}, \left\{ \begin{pmatrix} \ln(1+b_{1k}) \\ 0 \\ \ln(1+b_{2k}) \end{pmatrix} \right\}_{k=1}^{q} \right).$$

Obviously,

$$\operatorname{Ker} L = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in R^3,\ t \in [0,T], \right\}$$

and

$$\operatorname{Im} L = \left\{ Z = \left(\begin{pmatrix} f \\ g \\ h \end{pmatrix}, \left\{ \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \right\}_{k=1}^{q} \right) \in Z\colon \begin{pmatrix} \int_0^T f(t)\,dt + \sum_{k=1}^q a_k = 0 \\ \int_0^T g(t)\,dt + \sum_{k=1}^q b_k = 0 \\ \int_0^T h(t)\,dt + \sum_{k=1}^q c_k = 0 \end{pmatrix} \right\},$$

and $\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L$.

So, $\operatorname{Im} L$ is closed in Z, L is a Fredholm mapping of index zero. Define

$$P\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \frac{1}{T}\begin{pmatrix} \int_0^T x_1 + \sum_{k=1}^q a_k \\ \int_0^T x_2 + \sum_{k=1}^q b_k \\ \int_0^T y + \sum_{k=1}^q c_k \end{pmatrix},$$

$$QZ = Q\left(\begin{pmatrix} f \\ g \\ h \end{pmatrix}, \left\{ \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \right\}_{k=1}^{q} \right) = \left(\frac{1}{T}\begin{pmatrix} \int_0^T x_1(t)\,dt + \sum_{k=1}^q a_k \\ \int_0^T x_2(t)\,dt + \sum_{k=1}^q b_k \\ \int_0^T y(t)\,dt + \sum_{k=1}^q c_k \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}_{k=1}^{q} \right).$$

It is easy to show that P and Q are continuous projectors satisfying

$$\operatorname{Im} P = \operatorname{Ker} Q, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q).$$

Furthermore, through an easy computation, we can find that the inverse (L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ of L_P has the form

$$K_P Z = \begin{pmatrix} \int_0^T f + \sum_{k>t_k} a_k - \frac{1}{T}\int_0^T \int_0^t f(s)\,ds\,dt - \sum_{k=1}^q a_k + \frac{1}{T}\sum_{k=1}^q a_k t_k \\ \int_0^T g + \sum_{k>t_k} b_k - \frac{1}{T}\int_0^T \int_0^t g(s)\,ds\,dt - \sum_{k=1}^q b_k + \frac{1}{T}\sum_{k=1}^q b_k t_k \\ \int_0^T h + \sum_{k>t_k} c_k - \frac{1}{T}\int_0^T \int_0^t h(s)\,ds\,dt - \sum_{k=1}^q c_k + \frac{1}{T}\sum_{k=1}^q c_k t_k \end{pmatrix}.$$

Thus

$$QN\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \left(\begin{pmatrix} \frac{1}{T}\int_0^T F(t)\,dt + \frac{1}{T}\sum_{k=1}^q \ln(1+b_{1k}) \\ \frac{1}{T}\int_0^T G(t)\,dt \\ \frac{1}{T}\int_0^T H(t)\,dt + \frac{1}{T}\sum_{k=1}^q \ln(1+b_{2k}) \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}_{k=1}^{q} \right),$$

$$K_P(I-Q)N\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} \int_0^t F(s)\,ds + \sum_{t>t_k} \ln(1+b_{1k}) \\ \int_0^t G(s)\,ds \\ \int_0^t H(s)\,ds + \sum_{t>t_k} \ln(1+b_{2k}) \end{pmatrix} - \begin{pmatrix} \frac{1}{T}\int_0^T \int_0^t F(s)\,ds\,dt + \sum_{k=1}^q \ln(1+b_{1k}) \\ \frac{1}{T}\int_0^T \int_0^t G(s)\,ds\,dt \\ \frac{1}{T}\int_0^T \int_0^t H(s)\,ds\,dt + \sum_{k=1}^q \ln(1+b_{2k}) \end{pmatrix}$$
$$- \begin{pmatrix} (\frac{t}{T} - \frac{1}{2})\{\int_0^T F(s)\,ds + \sum_{k=1}^q \ln(1+b_{1k})\} \\ (\frac{t}{T} - \frac{1}{2})\int_0^T G(s)\,ds \\ (\frac{t}{T} - \frac{1}{2})\int_0^T H(s)\,ds + \sum_{k=1}^q \ln(1+b_{2k}) \end{pmatrix}.$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Using Lemma 2.2 and Arela–Ascoli theorem, it is not difficult to show that $K_P(I-Q)N(\bar{\Omega})$ is relatively compact for any open bounded set $\Omega \subset X$. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of Lemma 6.1. Corresponding to the operator equation $LU = \beta NU$, $\beta \in (0,1)$, $U = (u_1, u_2, v)^\top$, we have

$$\begin{cases} \left.\begin{array}{l} u_1'(t) = \beta\big[-d_1 - k_1\big(e^{u_1(t)} + e^{u_2(t)}\big) - \lambda e^{u_2(t)}\big] \\ u_2'(t) = \beta\big[\lambda e^{u_1(t)} - ce^{v(t)} - \delta\big] \\ v'(t) = \beta\big[-d_2 - k_2 e^{v(t)} + ee^{u_2(t)}\big] \end{array}\right\} & t \neq t_k,\ k \in Z, \\ \left.\begin{array}{l} \Delta u_1(t) = \beta \ln(1 + b_{1k}) \\ \Delta u_2(t) = 0 \\ \Delta v(t) = \beta \ln(1 + b_{2k}) \end{array}\right\} & t = t_k,\ k \in Z. \end{cases} \tag{6.2}$$

Suppose that $U = (u_1, u_2, v)^\top$ is a T-periodic solution of system (6.2) for some $\beta \in (0, 1)$. In what follows M_i denotes a fixed constant independent of β.

Integrating (6.2) over the interval $[0, T]$, we obtain

$$\int_0^T \big[d_1 + k_1\big(e^{u_1(t)} + e^{u_2(t)}\big) + \lambda e^{u_2(t)}\big]\,dt = \sum_{k=1}^{q} \ln(1 + b_{1k}), \tag{6.3}$$

$$\int_0^T \big[\lambda e^{u_1(t)} - ce^{v(t)} - \delta\big]\,dt = 0, \tag{6.4}$$

$$\int_0^T \big[d_2 + k_2 e^{v(t)} - ee^{u_2(t)}\big]\,dt = \sum_{k=1}^{q} \ln(1 + b_{2k}). \tag{6.5}$$

It follows from (6.3), (6.4) and (6.6) that $\|e^{u_1}\|_{PC_T} + \|e^{u_2}\|_{PC_T} \leqslant M_1$ and $\|e^{v}\|_{PC_T} \leqslant M_2$. Combing the bound with (6.2) one obtains the bound $\|\dot u_1\|_{PC'_T} + \|\dot u_2\|_{PC'_T} + \|\dot v\|_{PC'_T} \leqslant M_3$.

Since $U = (u_1, u_2, v) \in PC'_T \times PC'_T \times PC'_T$, there exist $\xi_1,\ \eta_1,\ \zeta_1$ in $[0, T]$ such that

$$u_1(\xi_1) = \min_{t\in[0,T]} u_1(t), \qquad u_2(\eta_1) = \min_{t\in[0,T]} u_2(t), \qquad v(\zeta_1) = \min_{t\in[0,T]} v(t). \tag{6.6}$$

From (6.3) and (6.6), we see that $u_1(\xi_1) \leqslant M_4$, $u_2(\eta_1) \leqslant M_5$, $v(\zeta_1) \leqslant M_6$.

So, we have $u_1(t) \leqslant u_1(\xi_1) + \|\dot u_1\|_{PC'_T} \leqslant M_3 + M_4 \triangleq H_1$, $u_2(t) \leqslant u_2(\eta_1) + \|\dot u_2\|_{PC'_T} \leqslant M_5 + M_3 \triangleq H_2$, $v(t) \leqslant v(\zeta_1) + \|\dot v\|_{PC'_T} \leqslant M_3 + M_6 \triangleq H_3$.

On the other hand, there exist $\xi_2,\ \eta_2,\ \zeta_2$ in $[0, T]$, such that

$$u_1(\xi_2) = \max_{t\in[0,T]} u_1(t), \qquad u_2(\eta_2) = \max_{t\in[0,T]} u_2(t), \qquad v(\zeta_2) = \max_{t\in[0,T]} v(t).$$

In the above formula, if $\xi_2,\ \eta_2,\ \zeta_2$ are not impulsive points, we have $u_1(\xi_2^+) = u_1(\xi_2)$, $u_2(\eta_2^+) = u_2(\eta_2)$, $v(\zeta_2^+) = v(\zeta_2)$; if some of $\xi_2,\ \eta_2,\ \zeta_2$ are impulsive points, with loss of generality, let $\xi_2 = t_k$, we have $u_1(\xi_2^+) = u_1(t_k^+)$.

From (6.4) that

$$\frac{1}{T}\int_0^T e^{v(t)} = \frac{\lambda}{c}\frac{1}{T}\int_0^T e^{u_1(t)} - \frac{\delta}{c}. \tag{6.7}$$

Substitute (6.7) into (6.5) that

$$\frac{\lambda k_2}{c}\frac{1}{T}\int_0^T e^{u_1(t)} - \frac{1}{T}\int_0^T ee^{u_2(t)} - \frac{k_2\delta}{c}\,dt = -d_2 + \frac{1}{T}\sum_{k=1}^{q}\ln(1 + b_{2k}). \tag{6.8}$$

Combing (6.3) and (6.8), we have

$$e\left[-d_1 + \frac{1}{T}\ln\prod_{k=1}^{q}(1 + b_{1k})\right] + (k_1 + \lambda)\left[-d_2 + \frac{1}{T}\ln\prod_{k=1}^{q}(1 + b_{2k})\right] + \frac{k_2\delta}{c}(k_1 + \lambda) = \left[ek_1 + (k_1 + \lambda)\frac{\lambda k_2}{c}\right]\frac{1}{T}\int_0^T e^{u_1(t)}\,dt.$$

According to the conditions of Theorem 6.1, the first term in the inequality is positive. Therefore, we can find a positive constant M_7 such that $u_1(\xi_2) \geqslant M_7$.

Similarly, we can obtain that

$$\frac{\lambda k_2}{c}\left[-d_1 + \frac{1}{T}\ln\prod_{k=1}^{q}(1 + b_{1k})\right] - k_2\left[-d_2 + \frac{1}{T}\ln\prod_{k=1}^{q}(1 + b_{2k})\right] - \frac{k_1 k_2\delta}{c} = \left(\frac{\lambda k_2}{c} + ek_1\right)\frac{1}{T}\int_0^T e^{u_2(t)},$$

that is $u_2(\eta_2) \leqslant M_8$.

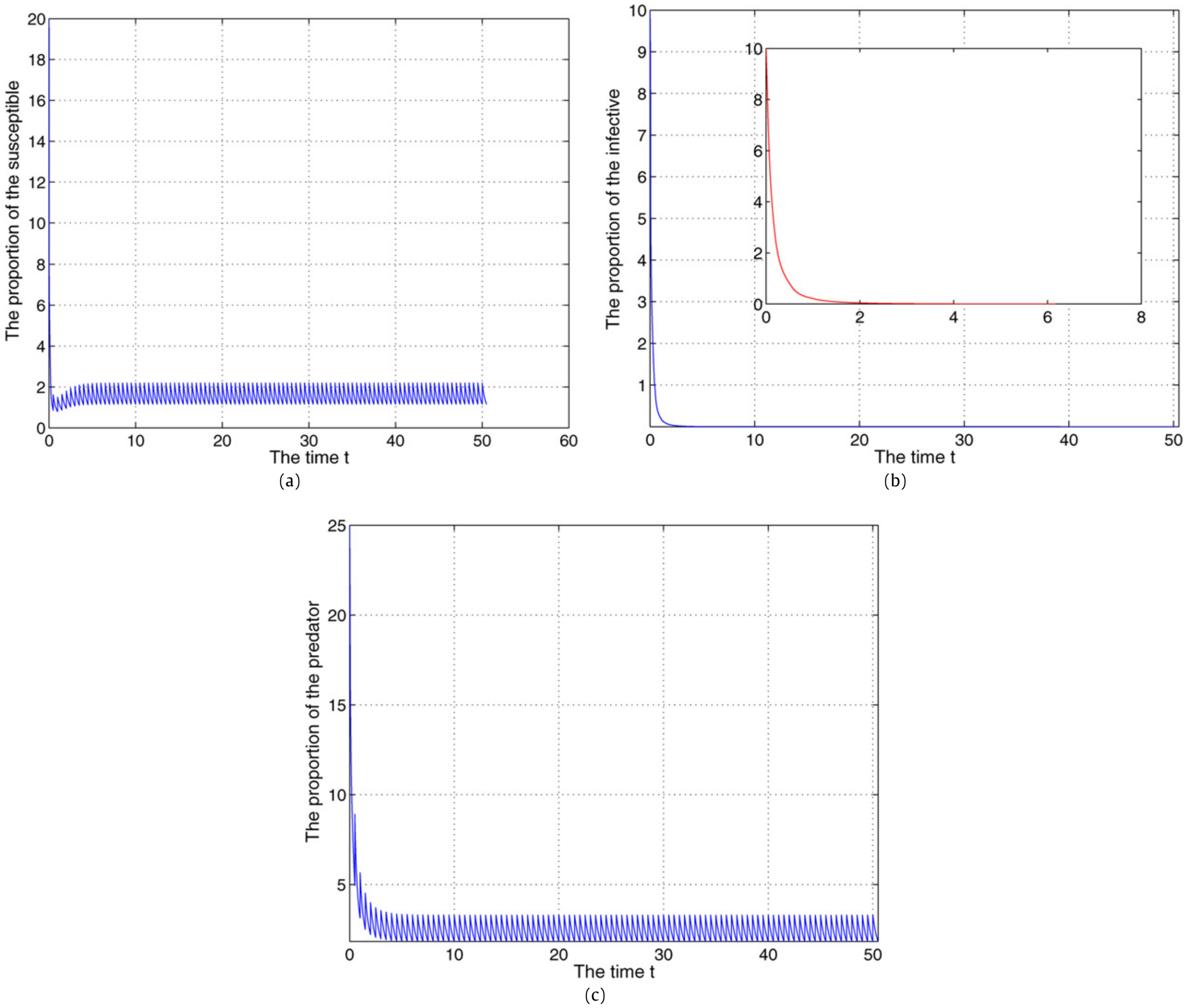

Fig. 1. (a)–(c) show the time series and the orbits of the system (1.2) with the parameters satisfying the conditions of Theorem 5.1.

From (6.7), we have $v(\zeta_2) \leqslant M_9$.

Finally, $u_1(t) \geqslant u_1(\xi_2) + \|\dot{u}_1\|_{PC'_T} \geqslant M_7 - M_3 \triangleq W_1$, $u_2(t) \geqslant u_2(\eta_1) + \|\dot{u}_2\|_{PC'_T} \geqslant M_8 - M_3 \triangleq W_2$, $v(t) \geqslant v(\zeta_2) + \|\dot{v}\|_{PC'_T} \geqslant M_9 - M_3 \triangleq W_3$.

Let

$$\sup_{t\in[0,T]} |u_1(t)| \leqslant \max\{|H_1|, |W_1|\} = B_1,$$
$$\sup_{t\in[0,T]} |u_2(t)| \leqslant \max\{|H_2|, |W_2|\} = B_2,$$
$$\sup_{t\in[0,T]} |v(t)| \leqslant \max\{|H_3|, |W_3|\} = B_3.$$

Clearly

$$\begin{cases} k_1\left(e^{u_1(t)} + e^{u_2(t)}\right) + \lambda e^{u_2(t)} = -d_1 + \dfrac{1}{T}\ln\displaystyle\prod_{k=1}^{q}(1+b_{1k}), \\ \lambda e^{u_2(t)} - ce^{v(t)} - \delta = 0, \\ k_2 e^{v(t)} - ee^{u_2(t)} = -d_2 + \dfrac{1}{T}\ln\displaystyle\prod_{k=1}^{q}(1+b_{2k}). \end{cases} \tag{6.9}$$

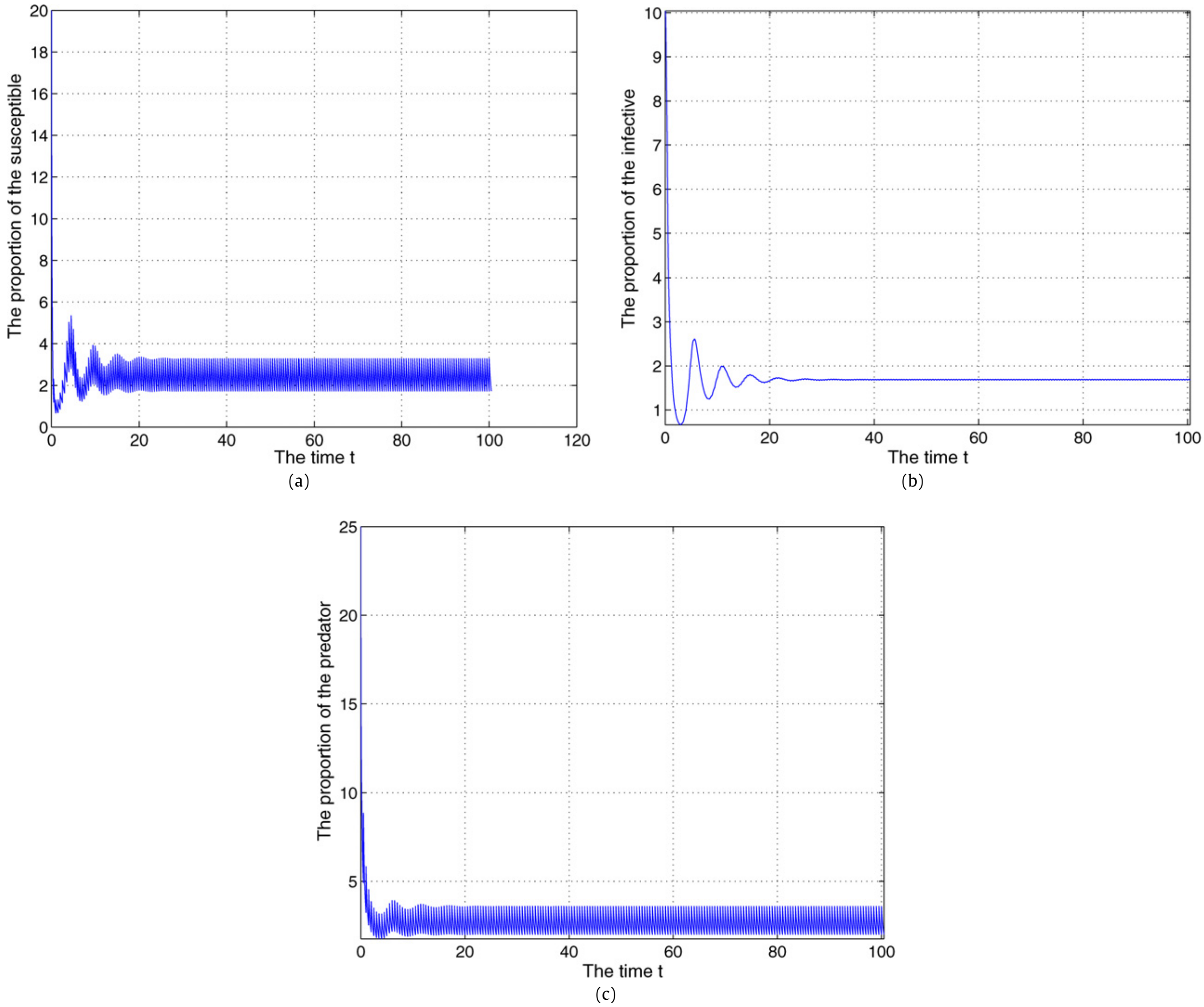

Fig. 2. (a)–(c) show the time series and the orbits of the system (1.2) with the parameters satisfying the conditions of Theorem 6.1.

By the assumption in Theorem 6.1, it is not difficult to show that the system (6.9) has a unique positive periodic solution $(u_1^*, u_2^*, v^*)^\top$.

Set $\Omega = \{U = (u_1, u_2, v) \in PC'_T \times PC'_T \times PC'_T\colon\ \|U\| \leqslant B\}$ where $B = B_1 + B_2 + B_3 + C$ and C is taken sufficiently large such that the unique solution of (6.2) satisfies $\|(u_1^*, u_2^*, v^*)^\top\| < C$.

It is clear that Ω verifies the requirement (i) in Lemma 6.1. When $U \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, U is a constant vector in R^3 with $\|U\| = B$.

Then

$$QNU = \left(\begin{pmatrix} -k_1(e^{u_1(t)} - e^{u_2(t)}) - \lambda e^{u_2(t)} - d_1 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k}) \\ \lambda e^{u_2(t)} - ce^{v(t)} - \delta \\ -k_2 e^{v(t)} + ee^{u_2(t)} - d_2 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k}) \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}_{k=1}^{q}\right).$$

Let $J : \operatorname{Im} Q \to \operatorname{Ker} L$, $(r, 0) \to r$,

$$JQNU = \begin{pmatrix} -k_1(e^{u_1(t)} - e^{u_2(t)}) - \lambda e^{u_2(t)} - d_1 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{1k}) \\ \lambda e^{u_2(t)} - ce^{v(t)} - \delta \\ -k_2 e^{v(t)} + ee^{u_2(t)} - d_2 + \frac{1}{T}\ln\prod_{k=1}^{q}(1+b_{2k}) \end{pmatrix}.$$

And by the assumption in Theorem 6.1, we have

$$\deg\bigl(JQNU, \Omega \cap \operatorname{Ker} L, (0,0,0)\bigr) = \operatorname{sign}\bigl[(-1)k_1k_2\lambda\bigr]e^{u_1^*(t)+u_2^*(t)+v^*(t)} = -1 \neq 0.$$

By now we have proved that Ω verifies all the requirements in Lemma 6.1. Hence, the system (1.2) has at least one T-periodic solution in $\bar{\Omega}$. $\square$

7. Discussion

In the paper, an eco-epidemiological system with the disease in the prey and impulsive birth are investigated. Applying the Floquet theory and small perturbation skills, we proved that the infection-free periodic solution is locally asymptotically stable when the impulsive period is less than the critical value. And using monotone iterative method, the conditions for persistence are given in Section 5. We use the method of coincidence degree to show the existence of at least one T-periodic positive solution.

By considering some hypothetical set of parametric values and the initial values (20, 15, 25), the following interesting dynamic behavior of the system (1.2) was observed.

Let $\lambda = 0.5$, $d_1 = 0.03$, $d_2 = 0.2$, $k_1 = 0.8$, $k_2 = 0.4$, $\delta = 0.15$, $c = 0.6$, $e = 0.4$ and choose b_{1k}, b_{2k} as impulsive birth variables. If we choose $b_{11} = 0.9$, $b_{21} = 0.8$, $T = 0.5$, then $\frac{1}{T}\ln(1 + b_{11}) \approx 1.284 > 0.03$, $\frac{1}{T}\ln(1 + b_{21}) \approx 1.756 > 0.2$, so the conditions of Theorem 5.1 are satisfied. From Fig. 1, we may observe the system (1.2) has an asymptotically stable infection-free periodic solution which the infective population becomes extinct.

Let $\lambda = 0.5$, $d_1 = 0.03$, $d_2 = 0.2$, $k_1 = 0.1$, $k_2 = 0.5$, $\delta = 0.15$, $c = 0.4$, $e = 0.2$ and choose b_{1k}, b_{2k} as impulsive birth variables. If we choose $b_{11} = 0.9$, $b_{21} = 0.8$, $T = 0.5$, then $\frac{1}{T}\ln(1 + b_{11}) \approx 1.284 > 0.03$, $\frac{1}{T}\ln(1 + b_{21}) \approx 1.756 > 0.2$, and $\frac{\lambda k_2}{c}[-d_1 + \frac{1}{T}\ln(1 + b_{11})] \approx 0.784 > k_1[-d_2 + \frac{1}{T}\ln(1 + b_{21})] + \frac{k_1 k_2 \delta}{c} \approx 0.116$, so the conditions of Theorem 6.1 are satisfied. From Fig. 2, we may observe the dynamic behavior of the system (1.2).

Comparing system (1.1) and system (1.2), we can conclude that these systems have the same result. The system (2.1) possesses two equilibria, $E_4 = (S_4, 0, P_4)$ and $E_* = (S_*, I_*, P_*)$. If $\lambda r_1 k_2 < c k_1 r_2 + k_1 k_2 \delta$, E_4 is globally stable; if $\lambda r_1 k_2 > c k_1 r_2 + k_1 k_2 \delta$, E_* is globally stable. The system (1.2) exists a local stable periodic infection-free solution when the conditions of Theorem 5.1 is right. When the stability of the periodic infection-free solution is lost, we can show that the system (1.2) is permanent and there exists at least one T-periodic positive solution if the conditions of Theorem 6.1 is right. Therefore, in order to drive the infected prey to extinction, we can make the impulsive period of birth smaller.

But the uniqueness and global stability of positive T-periodic solution of system (1.2) would be great interest. We leave the problem for the future work.

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