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On Some Mixed Monotonicity Problems*

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1. The well-known Gronwall–Bellman inequality has been employed extensively in a variety of problems in the study of ordinary differential equations. This inequality has been generalized and extended in various contexts, in particular to vector forms by Opial [5] and others. While considering these inequalities, the central problem is always to estimate a function satisfying a differential inequality by the maximal and minimal solutions of a related differential system. It was proved by Kamke [4] that in case of non-uniqueness of solutions, extremal solutions do exist under certain monotonicity conditions. Burton and Whyburn [2], and more recently Ziebur [7], have proved the existence of these extremal solutions even under the more general mixed monotonicity conditions.

In this paper we show that in all these problems, the existence of extremal solutions is a consequence of a lattice fixed-point theorem—a technique employed by Hanson and Waltman [3] in the context of another problem. The lattice fixed-point theorem proves not only the existence of a solution but also the existence of extremal solutions at once, and is thus ideally suited for applications to problems of this kind.

In Section 2, we give Theorem (2.2) which is a generalization of that used in [3] and in Sections 3 and 4, we will apply this theorem to the problems considered in [2, 4, 5, 7]. We believe that this gives a unified approach to several problems and furthermore has the advantages of brevity and elegance.

2. Let X be any nonempty set and L be a lattice with a partial order α. Suppose that G and H are given functions $X \times L \to L$ and $\mathscr{F} \subset L^X$. Consider the functional equation

$$z(x) = H(x, G(x, z)), \tag{2.1}$$

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where $x \in X$, $z \in \mathscr{F}$. $\mathscr{F}$ is partially ordered by the usual "pointwise" order; that is, $z_1 \alpha z_2$ iff $z_1(x) \alpha z_2(x)$ for all $x \in X$. We can now state the following theorem.

THEOREM 2.2. *If*

(i) *$\mathscr{F}$ is a complete lattice,*

(ii) *$z_1 \alpha z_2$ implies $H(x, G(x, z_1)) \alpha H(x, G(x, z_2))$, then* (2.1) *has a largest solution and a least solution in $\mathscr{F}$.*

Since the operator $T : \mathscr{F} \to \mathscr{F}$ defined by $(Tz)(x) = H(x, G(x, z))$ is isotone by virtue of hypothesis (ii), the fixed points of T form a nonempty complete lattice by a well-known theorem of Tarski [7, p. 286]. Hence, there exist a largest and a smallest fixed point of T, which proves the Theorem.

3. We shall now take up a problem considered in [2]. Consider a system

$$y' = f(x, y), \qquad y(x_0) = \eta, \tag{3.1}$$

where $y, f \in R^n$ and the function f is defined and continuous on the hyper-rectangle

$$P : | x - x_0 | \leqslant a, \qquad \| y - \eta \| \leqslant b, \qquad a > 0, \qquad b > 0.$$

Let, for $(x, y) \in P$, $| f_i(x, y)| \leqslant M_i$, $i = 1, 2,..., n$ and $M = (M_1, M_2, ..., M_n)$. Define $c = \min[a, (b/2\| M \|)]$. Suppose that the subscripts p and q range over the integers 1 to k and $k + 1$ to n respectively, where $0 \leqslant k \leqslant n$. Assume further that $f(x, y)$ satisfies the following mixed monotonicity conditions:

$$\left.\begin{array}{l} \text{(i)} \;\; f_p(x, y) \text{ is nondecreasing in } y_p \text{ and nonincreasing in } y_q. \\ \text{(ii)} \;\; f_q(x, y) \text{ is nonincreasing in } y_p \text{ and nondecreasing in } y_q. \end{array}\right. \tag{3.2}$$

We note in passing that the conditions (3.2) are slightly more general than the conditions (π_k) in [2], in that we are replacing strict increasing (decreasing) by nondecreasing (nonincreasing).

To apply Theorem (2.2) to this problem, let L_i denote the lattice of real numbers equipped with order α_i, where

$$\alpha_i = \begin{cases} \leqslant, & 1 \leqslant i \leqslant k \\ \geqslant, & k + 1 \leqslant i \leqslant n; \end{cases}$$

and $L = (L, \alpha)$ be the product lattice of L_i, $i = 1$ to n. Let

$$X = [x_0, x_0 + c) \qquad \text{and} \qquad G(x, z) = f(x, z)$$

where $z : X \to L$. Observe that $f : X \times L \to L$ and the conditions (3.2) can now be stated briefly as: $z_1 \alpha z_2$ implies $f(x, z_1) \alpha f(x, z_2)$; and Theorem 3 of [2] can be stated as follows:

THEOREM 3.3. *If $z_1 \alpha z_2$ implies $f(x, z_1) \alpha f(x, z_2)$, then the system* (3.1) *has a unique maximal and a unique minimal solutions in $[x_0, x_0 + c)$.*

Proof. Let us take

$$H(x, G(x, z)) = \eta + \int_{x_0}^{x} f(s, z(s))\, ds. \tag{3.4}$$

Using the triangle inequalities on (3.4) and the boundedness of f, we can find an $\Omega \in L$ such that $-\Omega \alpha H \alpha \Omega$. We now take $\mathscr{F}$ to be the class of all functions $g : X \to L$ which are bounded by $-\Omega$ and Ω and such that each component of g is Lebesgue-integrable on X.

It is now easy to show that the hypotheses of Theorem (2.2) are satisfied. The function H is clearly isotone. To show that $\mathscr{F}$ is a complete lattice, let $\mathscr{L}_i(x)$ be the class of Lebesgue-integrable functions $X \to L_i$. This is a Dedekind-complete lattice [1, p. 361] and hence so is the cartesian product $\prod_{i=1}^{n} \mathscr{L}_i(x)$. $\mathscr{F}$, being the lattice interval $[-\Omega, \Omega]$ in the product, is a complete lattice. Theorem (2.2) now guarantees a solution for (3.1) and in fact, a largest solution and a smallest solution. These are precisely the solutions referred to as $k \max(n - k) \min$ and $k \min(n - k) \max$ solutions respectively in [2].

If in conditions (3.2), we interchange the words "nonincreasing" and "nondecreasing," we can still prove the existence of the extremal solutions under these new conditions. All we do in the foregoing proof is to replace L by its dual lattice. The altered conditions correspond to $(\pi_k{}^*)$ in [2].

Finally, we observe that, for $k = n$ and $k = 0$, conditions (3.2) reduce to: $f(x, y)$ is nondecreasing in y. Further, the $k \max(n - k) \min$ and $k \min(n - k) \max$ solutions reduce respectively to the maximal and minimal solutions. Thus the problem considered in [4] becomes a special case.

4. We shall now write a variation of Theorem (3.3) and point out some special cases.

THEOREM 4.1. *Assume the hypotheses of Theorem* (3.3) *hold.*

(i) *If there exists a $\phi \in \mathscr{F}$ such that*

$$\phi(x) \,\alpha\, \eta + \int_{x_0}^{x} f(s, \phi(s))\, ds,$$

then there exists a unique largest solution $U(x)$ *of* (3.1) *such that*

$$\phi(x) \propto U(x), \qquad x \in [x_0, x_0 + c).$$

(ii) *If there exists a* $\psi \in \mathscr{F}$ *such that*

$$\eta + \int_{x_0}^{x} f(s, \psi(s))\, ds \propto \psi(x),$$

then there exists a unique smallest solution $u(x)$ *of* (3.1) *such that*

$$u(x) \propto \psi(x), \qquad x \in [x_0, x_0 + c).$$

Proof. The proof of Theorem (3.3) still goes thru if, instead of the lattice interval $[-\Omega, \Omega]$, we consider the intervals $[\phi, \Omega]$ and $[-\Omega, \psi]$.

For $k = n$ and $k = 0$, the Theorem (4.1) clearly reduces to that of Opial [5].

We shall also show that, as a special case of Theorem (4.1), we can easily obtain the following result of Ziebur [7]. If

$$P\phi(t) = \eta + \int_0^t f(s, \phi(s))\, ds$$

then the operator equation

$$x(t) = P^2 x(t) \tag{4.2}$$

has extremal solutions, where the vector-valued function $f(t, u)$ is continuous and either nondecreasing or nonincreasing in u. The equation (4.2) is equivalent to a $2n$-vector system

$$\begin{aligned} x'(t) &= f(t, y(t)), \\ y'(t) &= f(t, x(t)), \end{aligned} \qquad (x(0), y(0)) = (\eta, \eta), \tag{4.3}$$

where $y(t) = Px(t)$.

Hence, (4.3) can be considered as system (3.1) with $2n$ replacing n and n replacing k. Now, if $f(t, u)$ is increasing in u then clearly (3.1) reduces to Kamke's case. If, on the other hand, $f(x, u)$ is decreasing in u, then we may consider f in the first equation of (4.3) as nondecreasing in components of x and nonincreasing in components of y; and similarly in the second equation, f may be considered nonincreasing in x and nondecreasing in y. It is clear that, the conditions (3.2) are satisfied and Theorem 2 of [7] is a special case of Theorem (4.1).

An alternative, and a simpler, viewpoint is to observe that if an operator T is isotone or antiisotone, T^2 is isotone. Hence Theorem (3.3) now applies directly with P^2 replacing T.

REFERENCES

1. G. BIRKHOFF, "Lattice Theory," American Mathematical Society, Rhode Island, 1967.
2. L. P. BURTON AND W. M. WHYBURN, Minimax solutions of ordinary differential systems, *Proc. Amer. Math. Soc.* **3** (1952), 794–803.
3. D. L. HANSON AND P. WALTMAN, A note on a functional equation, *J. Math. Anal. Appl.* **10** (1965), 330–333.
4. E. KAMKE, Zur Theorie der Systems gewöhnlicher Differential-gleichungen, *Acta. Math.* **58** (1932), 57–85.
5. Z. OPIAL, Sur un système d'inégalités intégrales, *Ann. Polon. Math.* **3** (1957), 200–209.
6. A. TARSKI, A lattice-theoretical fixpoint theorem and its applications, *Pacific J. Math.* **5** (1955), 285–310.
7. A. D. ZIEBUR, On the Gronwall–Bellman lemma, *J. Math. Anal. Appl.* **22** (1968), 92–95.