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On the Ulam stability of mixed type mappings on restricted domains

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Abstract

In 1941 D.H. Hyers solved the well-known Ulam stability problem for linear mappings. In 1951 D.G. Bourgin was the second author to treat the Ulam problem for additive mappings. In 1982–1998 we established the Hyers–Ulam stability for the Ulam problem of linear and nonlinear mappings. In 1983 F. Skof was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S.M. Jung investigated the Hyers–Ulam stability of additive and quadratic mappings on restricted domains. In this paper we improve the bounds and thus the results obtained by S.M. Jung, in 1998. Besides we establish the Ulam stability of mixed type mappings on restricted domains. Finally, we apply our recent results to the asymptotic behavior of functional equations of different types.

Keywords: Ulam stability problem; Hyers–Ulam stability; Approximately even (or odd) mapping; Restricted domain

1. Introduction

In 1940 and in 1968 Ulam [23] proposed the *general Ulam stability problem*:

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> "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941 Hyers [13] solved this problem for linear mappings. In 1951 Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1978 Rassias [21] employed Hyers' ideas to new linear mappings. In 1980 and in 1987, Fenyö [7,8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Gajda and Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: Aczél [1], Borelli and Forti [2, 9], Cholewa [4], Czerwik [5], Drljević [6], and Kannappan [15]. In 1982–1998 we [16–20] solved the above Ulam problem for different mappings. In 1999 Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1983 Skof [22] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 Jung [14] investigated the Hyers–Ulam stability for additive and quadratic mappings on restricted domains. In this paper we improve the bounds and thus the stability results obtained by Jung, in 1998. Besides we establish the Ulam stability for more general equations of two types on a restricted domain. Finally we apply our recent results to the asymptotic behavior of functional equations of different types.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Definition 1. A mapping $f : X \to Y$ is called *additive* (respectively, *quadratic*) if f satisfies the equation

$$f(x_1 + x_2) = f(x_1) + f(x_2) \tag{1}$$

(respectively, $f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)$) for all $x_1, x_2 \in X$.

Theorem 1. *Let $\delta \geqslant 0$ be fixed. If a mapping $f : X \to Y$ satisfies the quadratic inequality*

$$\big\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2) \big\| \leqslant \delta \tag{2}$$

for all $x_1, x_2 \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that $\| f(x) - Q(x)\| \leqslant \delta/2$ for all $x \in X$.

Definition 2. A mapping $f : X \to Y$ is called *approximately odd* (respectively, *even*) if f satisfies

$$\big\| f(x) + f(-x) \big\| \leqslant \theta \tag{3}$$

(respectively, $\| f(x) - f(-x)\| \leqslant \theta$) for some fixed $\theta \geqslant 0$ and for all $x \in X$.

Definition 3. A mapping $M : X \to Y$ is called *additive* (respectively, *quadratic*) in X if M satisfies the functional equation of two types

$$M\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} M(x_i) = \sum_{1 \leqslant i < j \leqslant 3} M(x_i + x_j) \tag{4}$$

for all $x_i \in X$ $(i = 1, 2, 3)$. We note that all the real mappings $M : R \to R$ of the two types: $M(x) = ax$ or $M(x) = \beta x^2$ satisfy (4) for all $x \in R$ and all arbitrary but fixed $a, \beta \in R$.

We note that the mapping $M : X \to Y$ may be called *mixed type* as it is either additive or quadratic. The same terminology occurs to the mappings M satisfying the following Eq. (25).

2. Stability of the quadratic equation (1) on a restricted domain

Theorem 2. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If a mapping $f : X \to Y$ satisfies the quadratic inequality* (2) *for all $x_1, x_2 \in X$, with $\|x_1\| + \|x_2\| \geqslant d$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leqslant \frac{5}{2}\delta \tag{5}$$

for all $x \in X$.

Proof. Assume $\|x_1\| + \|x_2\| < d$. If $x_1 = x_2 = 0$, then we choose a $t \in X$ with $\|t\| = d$. Otherwise, let

$$t = \left(1 + \frac{d}{\|x_1\|}\right)x_1, \quad \text{if } \|x_1\| \geqslant \|x_2\|;$$
$$t = \left(1 + \frac{d}{\|x_2\|}\right)x_2, \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

We note that: $\|t\| = \|x_1\| + d > d$, if $\|x_1\| \geqslant \|x_2\|$; $\|t\| = \|x_2\| + d > d$, if $\|x_1\| \leqslant \|x_2\|$. Clearly, we see

$$\|x_1 - t\| + \|x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d,$$
$$\|x_1 - x_2\| + \|2t\| \geqslant \|x_1 - x_2\| + 2d \geqslant d,$$
$$\|x_1 + t\| + \|-x_2 + t\| \geqslant d,$$
$$\|x_1\| + \|t\| \geqslant d, \qquad \|t\| + \|x_2\| \geqslant d, \qquad \|t\| + \|t\| \geqslant d. \tag{6}$$

These inequalities (6) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Besides from (2) with $x_1 = x_2 = 0$ we get that $\|f(0)\| \leqslant \delta/2$. Therefore from (2), (6), and the new *functional identity*

$$\begin{aligned}
&2\big[f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)-f(0)\big]\\
&\quad=\big[f(x_1+x_2)+f(x_1-x_2-2t)-2f(x_1-t)-2f(x_2+t)\big]\\
&\qquad\quad(\text{with } x_1-t \text{ on } x_1 \text{ and } x_2+t \text{ on } x_2)\\
&\qquad-\big[f(x_1-x_2-2t)+f(x_1-x_2+2t)-2f(x_1-x_2)-2f(2t)\big]\\
&\qquad\quad(\text{with } x_1-x_2 \text{ on } x_1 \text{ and } 2t \text{ on } x_2)\\
&\qquad+\big[f(x_1-x_2+2t)+f(x_1+x_2)-2f(x_1+t)-2f(-x_2+t)\big]\\
&\qquad\quad(\text{with } x_1+t \text{ on } x_1 \text{ and } -x_2+t \text{ on } x_2)\\
&\qquad+2\big[f(x_1+t)+f(x_1-t)-2f(x_1)-2f(t)\big]\\
&\qquad\quad(\text{with } x_1 \text{ on } x_1 \text{ and } t \text{ on } x_2)\\
&\qquad+2\big[f(t+x_2)+f(t-x_2)-2f(t)-2f(x_2)\big]\\
&\qquad\quad(\text{with } t \text{ on } x_1 \text{ and } x_2 \text{ on } x_2)\\
&\qquad-2\big[f(2t)+f(0)-2f(t)-2f(t)\big]\quad(\text{with } t \text{ on } x_1 \text{ and } t \text{ on } x_2),
\end{aligned}$$

we get

$$\begin{aligned}
&2\big\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)-f(0)\big\|\\
&\quad\leqslant \delta+\delta+\delta+2\delta+2\delta+2\delta=9\delta,
\end{aligned}$$

or

$$\big\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)\big\| \leqslant \frac{9}{2}\delta+\big\|f(0)\big\| \leqslant 5\delta. \tag{7}$$

Applying now Theorem 1 and the above inequality (7), one gets that there exists a unique quadratic mapping $Q : X \to Y$ that satisfies the quadratic equation (1) and the inequality (5), such that $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$, completing the proof of Theorem 2. □

Obviously our inequalities (5) and (7) are sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to $(7/2)\delta$ and 7δ, respectively.

We note that if we define $S_2 = \{(x_1, x_2) \in X^2\colon\ \|x_i\| < d,\ i = 1, 2\}$ for some $d > 0$, then $\{(x_1, x_2) \in X^2\colon\ \|x_1\| + \|x_2\| \geqslant 2d\} \subset X^2 \backslash S_2$.

Corollary 1. *If we assume that a mapping $f : X \to Y$ satisfies the quadratic inequality* (2) *for some fixed $\delta \geqslant 0$ and for all $(x_1, x_2) \in X^2 \backslash S_2$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (5) *for all $x \in X$.*

Corollary 2. *A mapping $f : X \to Y$ is quadratic if and only if the asymptotic condition*

$$\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)\| \to 0, \quad \text{as } \|x_1\|+\|x_2\| \to \infty$$

holds.

Proof. Following the corresponding techniques of the proof of Jung [14], in 1998, one gets from Theorem 2 and the above asymptotic condition that f is quadratic. The reverse assertion is obvious. □

However, in 1983 Skof [22] proved an asymptotic property of the additive mappings.

3. Stability of Eq. (4) of two types

In 1998 Jung [14] applied the induction principle and proved the following Lemma 1.

Lemma 1. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1+x_2) - f(x_1+x_3) - f(x_2+x_3) + \sum_{i=1}^{3} f(x_i) \right\| \leqslant \delta \tag{8}$$

for some fixed $\delta \geqslant 0$ and for all $x_i \in X$ $(i = 1, 2, 3)$. It then holds that

$$\left\| f(x) - \frac{2^n+1}{2^{2n+1}} f(2^n x) + \frac{2^n-1}{2^{2n+1}} f(-2^n x) \right\| \leqslant 3\delta \sum_{i=1}^{n} 2^{-i} \left(= 3\delta(1-2^{-n})\right), \tag{9}$$

for all $x \in X$ and $n \in N = \{1, 2, \ldots\}$.

In this paper, *without the induction principle*, we prove the above-mentioned Lemma 1.

Proof. Let us denote

$$a_i = \frac{2^i+1}{2^{2i+1}}, \qquad A_i(x) = 3f(2^{i-1}x) + f(-2^{i-1}x) - f(2^i x),$$

$$b_i = -\frac{2^i-1}{2^{2i+1}}, \qquad B_i(x) = 3f(-2^{i-1}x) + f(2^{i-1}x) - f(-2^i x),$$

$$T_i(x) = a_i f(2^i x) + b_i f(-2^i x), \qquad S_n(x) = T_0(x) - T_n(x),$$

such that $T_0(x) = f(x)$, for all $x \in X$, $i \in N_n = \{1, 2, \ldots, n\}$, and $n \in N$.

We note that

$$a_{i-1} = 3a_i + b_i, \qquad b_{i-1} = a_i + 3b_i$$

hold for any $i \in N_n = \{1, 2, \ldots, n\}$, $n \in N$.

From these identities we get that

$$\begin{aligned} T_{i-1}(x) - T_i(x) &= a_{i-1} f(2^{i-1}x) + b_{i-1} f(-2^{i-1}x) - T_i(x) \\ &= (3a_i + b_i) f(2^{i-1}x) + (a_i + 3b_i) f(-2^{i-1}x) - a_i f(2^i x) - b_i f(-2^i x) \\ &= a_i\big[3f(2^{i-1}x) + f(-2^{i-1}x) - f(2^i x)\big] \\ &\quad + b_i\big[3f(-2^{i-1}x) + f(2^{i-1}x) - f(-2^i x)\big], \end{aligned}$$

or the formula

$$T_{i-1}(x) - T_i(x) = a_i A_i(x) + b_i B_i(x) \tag{10}$$

holds for any $i \in N_n = \{1, 2, \ldots, n\}$, $n \in N$.

We note that

$$S_n(x) = T_0(x) - T_n(x) = \sum_{i=1}^{n} \big[T_{i-1}(x) - T_i(x)\big].$$

Therefore from this formula and (10) one obtains the new formula

$$S_n(x) = \sum_{i=1}^{n} \big[a_i A_i(x) + b_i B_i(x)\big]. \tag{11}$$

Replacing $x_i = 0$ $(i = 1, 2, 3)$ in (8) one gets

$$\|f(0)\| \leqslant \delta. \tag{12}$$

Setting $x_1 = x$, $x_2 = x$, $x_3 = -x$ in (8) we find from (12) that

$$\|3f(x) + f(-x) - f(2x) - 2f(0)\| \leqslant \delta$$

or

$$\|3f(x) + f(-x) - f(2x)\| \leqslant 3\delta \tag{13}$$

holds for all $x \in X$.

Substituting $-x$ for x in (13), one obtains

$$\|3f(-x) + f(x) - f(-2x)\| \leqslant 3\delta. \tag{14}$$

Placing $2^{i-1}x$ on x in (13) and (14) we get

$$\|A_i(x)\| \leqslant 3\delta \quad \text{and} \quad \|B_i(x)\| \leqslant 3\delta \tag{15}$$

for all $i \in N_n$, $n \in N$.

Thus from the formula (11), the inequalities (15), and the triangle inequality we prove

$$\begin{aligned}\|S_n(x)\| &\leqslant \sum_{i=1}^{n}\bigl[|a_i|\,\|A_i(x)\| + |b_i|\,\|B_i(x)\|\bigr]\\ &\leqslant 3\delta\sum_{i=1}^{n}\left[\frac{2^i+1}{2^{2i+1}}+\frac{2^i-1}{2^{2i+1}}\right] = 3\delta\sum_{i=1}^{n}2^{-i} = 3\delta(1-2^{-n}),\end{aligned}\tag{16}$$

for all $x \in X$ and $n \in N$, completing the proof of this Lemma 1. □

In 1998 Jung [14] applied Lemma 1 *on approximately even mappings* f and proved the following Theorem 3.

Theorem 3. *Assume an approximately even mapping* $f : X \to Y$ *satisfies the quadratic inequality* (8). *Then there exists a unique quadratic mapping* $Q : X \to Y$ *which satisfies the quadratic equation* (4) *and the inequality*

$$\|f(x) - Q(x)\| \leqslant 3\delta \tag{17}$$

for a fixed $\delta \geqslant 0$ *and for all* $x \in X$.

Note that the right-hand side of (17) contains no θ term. In 1998 Jung [14] applied Lemma 1 *on approximately odd mappings* f and proved also the following Theorem 4.

Theorem 4. *Assume an approximately odd mapping* $f : X \to Y$ *satisfies the additive inequality* (8). *Then there exists a unique additive mapping* $A : X \to Y$ *which satisfies the additive equation* (4) *and the inequality*

$$\|f(x) - A(x)\| \leqslant 3\delta \tag{18}$$

for a fixed $\delta \geqslant 0$ *and for all* $x \in X$.

4. Stability of Eq. (4) on a restricted domain

In this section, we establish the Hyers–Ulam stability on a more general restricted domain.

Theorem 5. *Let* $d > 0$ *and* $\delta \geqslant 0$ *be fixed. If an approximately even mapping* $f : X \to Y$ *satisfies the quadratic inequality* (8) *for all* $x_i \in X$ $(i = 1, 2, 3)$ *with* $\sum_{i=1}^{3}\|x_i\| \geqslant d$, *then there exists a unique quadratic mapping* $Q : X \to Y$, *such that*

$$\|f(x) - Q(x)\| \leqslant 15\delta \tag{19}$$

for all $x \in X$.

Proof. Assume $\sum_{i=1}^{3}\|x_i\| < d$. If $x_i = 0$ $(i = 1, 2, 3)$, then we choose a $t \in X$ with $\|t\| \geqslant 2d$. Otherwise, choose a $t \in X$ with $\|t\| \geqslant d$; clearly

$$\begin{aligned} &\|x_1 - t\| + \|x_2\| + \|x_3 + t\| \geqslant 2\|t\| - \sum_{i=1}^{3}\|x_i\| \geqslant d,\\ &\|x_1\| + \|x_2\| + \|-t\| \geqslant d, \qquad \|x_2\| + \|x_3\| + \|t\| \geqslant d,\\ &\|x_2\| + \|-t\| + \|t\| = 2\|t\| + \|x_2\| \geqslant d. \end{aligned} \tag{20}$$

Besides from (8) with $x_i = 0$ $(i = 1, 2, 3)$ we get that $\|f(0)\| \leqslant \delta$.

Therefore from (8), (20), and the new *functional identity*

$$\begin{aligned} &f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i) + f(0)\\ &\quad = \big[f(x_1 + x_2 + x_3) - f(x_1 + x_2 - t) - f(x_1 + x_3) - f(x_2 + x_3 + t)\\ &\qquad + f(x_1 - t) + f(x_2) + f(x_3 + t)\big]\\ &\qquad\quad (\text{with } x_1 - t \text{ on } x_1,\ x_2 \text{ on } x_2,\ \text{and } x_3 + t \text{ on } x_3)\\ &\qquad + \big[f(x_1 + x_2 - t) - f(x_1 + x_2) - f(x_1 - t) - f(x_2 - t) + f(x_1)\\ &\qquad\quad + f(x_2) + f(-t)\big]\\ &\qquad\quad (\text{with } x_1 \text{ on } x_1,\ x_2 \text{ on } x_2,\ \text{and } -t \text{ on } x_3)\\ &\qquad + \big[f(x_2 + x_3 + t) - f(x_2 + x_3) - f(x_2 + t) - f(x_3 + t) + f(x_2)\\ &\qquad\quad + f(x_3) + f(t)\big]\\ &\qquad\quad (\text{with } x_2 \text{ on } x_1,\ x_3 \text{ on } x_2,\ \text{and } t \text{ on } x_3)\\ &\qquad - \big[f(x_2) - f(x_2 - t) - f(x_2 + t) - f(0) + f(x_2) + f(-t) + f(t)\big]\\ &\qquad\quad (\text{with } x_2 \text{ on } x_1,\ -t \text{ on } x_2,\ \text{and } t \text{ on } x_3), \end{aligned}$$

we get

$$\begin{aligned} &\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3)\right.\\ &\quad \left. + \sum_{i=1}^{3} f(x_i) + f(0)\right\| \leqslant \delta + \delta + \delta + \delta = 4\delta, \end{aligned}$$

or

$$\begin{aligned} &\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i)\right\|\\ &\quad \leqslant 4\delta + \|f(0)\| \leqslant 5\delta. \end{aligned} \tag{21}$$

Applying the Theorem 3 and the inequality (21), we prove that there exists a unique quadratic mapping $Q : X \to Y$ that satisfies the quadratic equation (4) and the inequality (19), completing the proof of the Theorem 5. □

Obviously, our inequalities (19) and (21) are also sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to 21δ and 7δ, respectively.

We note that if we define $S_3 = \{(x_1, x_2, x_3) \in X^3\colon\ \|x_i\| < d,\ i = 1, 2, 3\}$ for some fixed $d > 0$, then $\{(x_1, x_2, x_3) \in X^3\colon\ \sum_{i=1}^{3} \|x_i\| \geqslant 3d\} \subset X^3 \backslash S_3$.

Corollary 3. *If we assume that an approximately even mapping $f : X \to Y$ satisfies the inequality* (8) *for some fixed $\delta \geqslant 0$ and for all $(x_1, x_2, x_3) \in X^3 \backslash S_3$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (19) *for all $x \in X$.*

Corollary 4. *An approximately even mapping $f : X \to Y$ is quadratic if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + \sum_{i=1}^{3} f(x_i)\right\| \to 0,$$
$$\text{as } \sum_{i=1}^{3} \|x_i\| \to \infty,$$

holds.

Similarly, we prove the following Theorem 6.

Theorem 6. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If an approximately odd mapping $f : X \to Y$ satisfies the additive inequality* (8) *for all $x_i \in X$ $(i = 1, 2, 3)$ with $\sum_{i=1}^{3} \|x_i\| \geqslant d$, then there exists a unique additive mapping $A : X \to Y$, such that*

$$\left\| f(x) - A(x)\right\| \leqslant 15\delta \tag{22}$$

for all $x \in X$.

Obviously, our inequalities (21) and (22) are also sharper than the corresponding inequalities of Jung [14], where the right-hand sides were equal to 7δ and 21δ, respectively.

Corollary 5. *If we assume that an approximately odd mapping $f : X \to Y$ satisfies the inequality* (8) *for some fixed $\delta \geqslant 0$ and for all $(x_1, x_2, x_3) \in X^3 \backslash S_3$, then there exists a unique additive mapping $A : X \to Y$ satisfying* (22) *for all $x \in X$.*

Corollary 6. *An approximately odd mapping $f: X \to Y$ is additive if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{3} x_i\right) - f(x_1+x_2) - f(x_1+x_3) - f(x_2+x_3) + \sum_{i=1}^{3} f(x_i) \right\| \to 0,$$
$$\text{as } \sum_{i=1}^{3} \|x_i\| \to \infty,$$

holds.

Remark 1. From (3) for approximately even mappings, the quadratic inequality (8) (with $x_1 = x$, $x_2 = x$, $x_3 = -x$), and the triangle inequality, one obtains that

$$\begin{aligned} 4\left\| f(x) - 2^{-2} f(2x)\right\| &\leqslant \left\|3f(x) + f(-x) - f(2x) - 2f(0)\right\| \\ &\quad + \left\|-\left[f(-x) - f(x)\right]\right\| + \left\|2f(0)\right\| \\ &\leqslant \delta + \theta + 2\delta = 3\delta + \theta, \end{aligned}$$

or

$$\left\| f(x) - 2^{-2} f(2x)\right\| \leqslant \left(\delta + \frac{\theta}{3}\right)\left(1 - 2^{-2}\right).$$

According to our works [19,20] on quadratic mappings, one proves that

$$\left\| f(x) - 2^{-2n} f(2^n x)\right\| \leqslant \left(\delta + \frac{\theta}{3}\right)\left(1 - 2^{-2n}\right),$$

holds for all $n \in N$, and all $x \in X$, which yields there is a unique quadratic mapping $Q: X \to Y$, such that $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ and

$$\left\| f(x) - Q(x)\right\| \leqslant \delta + \frac{\theta}{3}. \tag{23}$$

But this inequality is also sharper than the corresponding inequality of Jung [14], where the right-hand side was equal to $\delta + \theta/2$.

Remark 2. From (3) for approximately odd mappings, the additive inequality (8) (with $x_1 = x$, $x_2 = x$, $x_3 = -x$), and the triangle inequality, one gets that

$$\begin{aligned} 2\left\| f(x) - 2^{-1} f(2x)\right\| &\leqslant \left\|3f(x) + f(-x) - f(2x) - 2f(0)\right\| \\ &\quad + \left\|-\left[f(-x) + f(x)\right]\right\| + \left\|2f(0)\right\| \\ &\leqslant \delta + \theta + 2\delta = 3\delta + \theta, \end{aligned}$$

or

$$\left\| f(x) - 2^{-1} f(2x)\right\| \leqslant (3\delta + \theta)\left(1 - 2^{-1}\right).$$

According to our works [16–18] on additive mappings, one proves that

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leqslant (3\delta + \theta)(1 - 2^{-n}),$$

holds for all $n \in N$, and all $x \in X$, which yields that there is a unique additive mapping $A : X \to Y$, such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and

$$\left\| f(x) - A(x) \right\| \leqslant 3\delta + \theta. \tag{24}$$

In the following definition we *generalize* the above functional equation (4).

Definition 4. A mapping $M : X \to Y$ is called *additive* (respectively *quadratic*) *in* R^4 if M satisfies the functional equation of two types

$$M\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\leqslant i<j\leqslant 4} M(x_i + x_j) = \sum_{i=1}^{4} M(x_i) + \sum_{1\leqslant i<j<k\leqslant 4} M(x_i + x_j + x_k) \tag{25}$$

for all $x_i \in X$ $(i = 1, 2, 3, 4)$.

5. Stability of Eq. (25)

In this section, we establish the Hyers–Ulam stability for new equations.

Theorem 7. *Assume an approximately even mapping $f : X \to Y$ satisfies the following quadratic inequality*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\leqslant i<j\leqslant 4} f(x_i + x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\leqslant i<j<k\leqslant 4} f(x_i + x_j + x_k) \right\| \leqslant \delta, \tag{26}$$

for some fixed $\delta \geqslant 0$ and $\theta \geqslant 0$ and for all $x_i \in X$ $(i = 1, 2, 3, 4)$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the quadratic equation (25) *and the inequality*

$$\left\| f(x) - Q(x) \right\| \leqslant \delta + \frac{5}{6}\theta \tag{27}$$

for all $x \in X$.

Proof. Replacing $x_i = 0$ $(i = 1, 2, 3, 4)$ in (26), we find $\|f(0)\| \leqslant \delta$. Thus, substituting $x_i = x$ $(i = 1, 2)$ and $x_j = -x$ $(j = 3, 4)$ in (26), one gets

$$\left\|4f(x) + 4f(-x) - f(2x) - f(-2x)\right\| \leqslant 6\delta \tag{28}$$

for all $x \in X$. Therefore from (28), (3) for approximately even mappings, the quadratic inequality (26), and the triangle inequality, we obtain that

$$\begin{aligned} 2\left\|4f(x) - f(2x)\right\| &\leqslant \left\|4f(x) + 4f(-x) - f(2x) - f(-2x)\right\| \\ &\quad + \left\|-4\left[f(-x) - f(x)\right]\right\| + \left\|f(-2x) - f(2x)\right\| \\ &\leqslant 6\delta + 4\theta + \theta = 6\delta + 5\theta, \end{aligned}$$

or

$$\left\|f(x) - 2^{-2}f(2x)\right\| \leqslant \frac{3}{4}\delta + \frac{5}{8}\theta \left(= \left(\delta + \frac{5}{6}\theta\right)(1 - 2^{-2})\right).$$

According to our works [19,20] on quadratic mappings, one proves that

$$\left\|f(x) - 2^{-2n}f(2^n x)\right\| \leqslant \left(\delta + \frac{5}{6}\theta\right)(1 - 2^{-2n}), \tag{29}$$

holds for any $n \in N$, and all $x \in X$. Similarly from (25) we get, by induction on n, that

$$Q(x) = 2^{-2n}Q(2^n x), \tag{30}$$

holds for any $n \in N$, and all $x \in X$.

By (29), for $n \geqslant m > 0$, and $h = 2^m x$, we have

$$\begin{aligned} &\left\|2^{-2n}f(2^n x) - 2^{-2m}f(2^m x)\right\| \\ &\quad = 2^{-2m}\left\|2^{-2(n-m)}f(2^{n-m} \cdot 2^m x) - f(2^m x)\right\| \\ &\quad = 2^{-2m}\left\|2^{-2(n-m)}f(2^{n-m}h) - f(h)\right\| \\ &\quad \leqslant 2^{-2m}\left(\delta + \frac{5}{6}\theta\right)\left(1 - 2^{-2(n-m)}\right) = \left(\delta + \frac{5}{6}\theta\right)(2^{-2m} - 2^{-2n}) \\ &\quad < \left(\delta + \frac{5}{6}\theta\right)2^{-2m} \to 0, \quad \text{as } m \to \infty. \end{aligned} \tag{31}$$

From (31) and the completeness of Y we get that the Cauchy sequence $\{2^{-2n}f(2^n x)\}$ converges. Therefore we [19,20] may apply a direct method to the definition of Q such that $Q(x) = \lim_{n\to\infty} 2^{-2n}f(2^n x)$ holds for all $x \in X$. From the quadratic inequality (26), it follows that

$$\begin{aligned} &\Bigg\|Q\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\leqslant i<j\leqslant 4} Q(x_i + x_j) - \sum_{i=1}^{4} Q(x_i) \\ &\quad - \sum_{1\leqslant i<j<k\leqslant 4} Q(x_i + x_j + x_k)\Bigg\| \leqslant 2^{-2n}\delta \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

for all $x_i \in X$ $(i = 1, 2, 3, 4)$. Thus it is obvious that Q satisfies the quadratic equation (25). Analogously, by (3), we can show that $Q(0) = 0$ (with $x_i = 0$ $(i = 1, 2, 3, 4)$ in (25)) and that Q is even from (3) with $2^n x$ on place of x, $\|Q(x) - Q(-x)\| \leqslant 2^{-2n}\theta \to 0$, as $n \to \infty$, or $Q(-x) = Q(x)$.

According to (29), one gets that the inequality (27) holds. Assume now that there is another quadratic mapping $Q' : X \to Y$ which satisfies the quadratic equation (25), the formula (30) and the inequality (27). Therefore

$$\begin{aligned}\big\|Q(x) - Q'(x)\big\| &= 2^{-2n}\big\|Q(2^n x) - Q'(2^n x)\big\| \\ &\leqslant 2^{-2n}\big[\big\|Q(2^n x) - f(2^n x)\big\| + \big\|f(2^n x) - Q'(2^n x)\big\|\big] \\ &\leqslant 2\left(\delta + \frac{5}{6}\theta\right)2^{-2n} \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

or

$$Q'(x) = Q(x),$$

for all $x_i \in X$, completing the proof of our Theorem 7. □

6. Stability of Eq. (25) on a restricted domain

In this section, we establish the Hyers–Ulam stability for even more general equations of two types on a restricted domain.

Theorem 8. *Let $d > 0$, $\delta \geqslant 0$ and $\theta \geqslant 0$ be fixed. If an approximately even mapping $f : X \to Y$ satisfies the quadratic inequality* (26) *for all $x_i \in X$ $(i = 1, 2, 3)$ with $\sum_{i=1}^{4} \|x_i\| \geqslant d$, then there exists a unique quadratic mapping $Q : X \to Y$, such that*

$$\big\|f(x) - Q(x)\big\| \leqslant 5\left(\delta + \frac{\theta}{6}\right) \tag{32}$$

for all $x \in X$.

Proof. Assume $\sum_{i=1}^{4} \|x_i\| < d$. We choose a $t \in X$ with $\|t\| \geqslant 2d$. Clearly, we see

$$\begin{aligned}&\|x_1 - t\| + \|x_2\| + \|x_3 + t\| + \|x_4\| \geqslant 2\|t\| - \sum_{i=1}^{4} \|x_i\| \geqslant d, \\ &\|x_1\| + \|x_2\| + \|x_4\| + \|-t\| = \|t\| + \big(\|x_1\| + \|x_2\| + \|x_4\|\big) \geqslant d, \\ &\|x_2\| + \|x_3\| + \|x_4\| + \|t\| \geqslant d, \\ &\|x_2\| + \|x_4\| + \|t\| + \|-t\| \geqslant d.\end{aligned} \tag{33}$$

Besides from (26) with $x_i = 0$ $(i = 1, 2, 3, 4)$ we get that $\|f(0)\| \leqslant \delta$. Therefore from (26), (33), and the following new *functional identity*

$$
\begin{aligned}
& f\left(\sum_{i=1}^{4} x_i\right) - f(x_1+x_2+x_3) - f(x_1+x_2+x_4) - f(x_1+x_3+x_4) \\
&\quad - f(x_2+x_3+x_4) + f(x_1+x_2) + f(x_1+x_3) + f(x_1+x_4) \\
&\quad + f(x_2+x_3) + f(x_2+x_4) + f(x_3+x_4) - \sum_{i=1}^{4} f(x_i) - f(0) \\
&= f\left(\sum_{i=1}^{4} x_i\right) - f(x_1+x_2+x_3) - f(x_1+x_2+x_4-t) \\
&\quad - f(x_1+x_3+x_4) - f(x_2+x_3+x_4+t) + f(x_1+x_2-t) \\
&\quad + f(x_1+x_3) + f(x_1+x_4-t) + f(x_2+x_3+t) + f(x_2+x_4) \\
&\quad + f(x_3+x_4+t) - f(x_1-t) - f(x_2) - f(x_3+t) - f(x_4) \\
&\qquad (\text{with } x_1 - t \text{ on } x_1,\ x_2 \text{ on } x_2,\ x_3 + t \text{ on } x_3,\ \text{and } x_4 \text{ on } x_4) \\
&\quad + \big[f(x_1+x_2+x_4-t) - f(x_1+x_2+x_4) - f(x_1+x_2-t) \\
&\qquad - f(x_1+x_4-t) - f(x_2+x_4-t) + f(x_1+x_2) + f(x_1+x_4) \\
&\qquad + f(x_1-t) + f(x_2+x_4) + f(x_2-t) + f(x_4-t) - f(x_1) \\
&\qquad - f(x_2) - f(x_4) - f(-t) \big] \\
&\qquad (\text{with } x_1 \text{ on } x_1,\ x_2 \text{ on } x_2,\ x_4 \text{ on } x_3,\ \text{and } -t \text{ on } x_4) \\
&\quad + \big[f(x_2+x_3+x_4+t) - f(x_2+x_3+x_4) - f(x_2+x_3+t) \\
&\qquad - f(x_2+x_4+t) - f(x_3+x_4+t) + f(x_2+x_3) + f(x_2+x_4) \\
&\qquad + f(x_2+t) + f(x_3+x_4) + f(x_3+t) + f(x_4+t) - f(x_2) \\
&\qquad - f(x_3) - f(x_4) - f(t) \big] \\
&\qquad (\text{with } x_2 \text{ on } x_1,\ x_3 \text{ on } x_2,\ x_4 \text{ on } x_3,\ \text{and } t \text{ on } x_4) \\
&\quad - \big[f(x_2+x_4) - f(x_2+x_4+t) - f(x_2+x_4-t) - f(x_2) - f(x_4) \\
&\qquad + f(x_2+x_4) + f(x_2+t) + f(x_2-t) + f(x_4+t) + f(x_4-t) \\
&\qquad + f(0) - f(x_2) - f(x_4) - f(t) - f(-t) \big] \\
&\qquad (\text{with } x_2 \text{ on } x_1,\ x_4 \text{ on } x_2,\ t \text{ on } x_3,\ \text{and } -t \text{ on } x_4),
\end{aligned}
$$

we get

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\leqslant i<j\leqslant 4} f(x_i+x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\leqslant i<j<k\leqslant 4} f(x_i+x_j+x_k)\right\| \leqslant 4\delta + \|f(0)\| \leqslant 5\delta. \tag{34}$$

Applying the Theorem 7 and the inequality (34), we prove that there exists a unique quadratic mapping $Q : X \to Y$ that satisfies the quadratic equation (25) and the inequality (32), completing the proof of the Theorem 8. □

We note that if we define $S_4 = \{(x_1, x_2, x_3, x_4) \in X^4\colon \|x_i\| < d,\ i = 1, 2, 3, 4\}$ for some fixed $d > 0$, then $\{(x_1, x_2, x_3, x_4) \in X^4\colon \sum_{i=1}^{4} \|x_i\| \geqslant 4d\} \subset X^4 \backslash S_4$.

Corollary 7. *If we assume that an approximately even mapping $f : X \to Y$ satisfies the inequality* (26) *for some fixed $\delta \geqslant 0$ and $\theta \geqslant 0$, and for all $(x_1, x_2, x_3, x_4) \in X^4 \backslash S_4$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (32) *for all $x \in X$.*

Corollary 8. *An approximately even mapping $f : X \to Y$ is quadratic and satisfies the quadratic equation* (25) *if and only if the following asymptotic condition*

$$\left\| f\left(\sum_{i=1}^{4} x_i\right) + \sum_{1\leqslant i<j\leqslant 4} f(x_i+x_j) - \sum_{i=1}^{4} f(x_i) - \sum_{1\leqslant i<j<k\leqslant 4} f(x_i+x_j+x_k)\right\| \to 0, \quad \text{as } \sum_{i=1}^{4} \|x_i\| \to \infty,$$

holds.

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