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Webs and Bounded Finitely Additive Measures*

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DEDICATED TO PROFESSOR M. VALDIVIA WHO PIONEERED SOME METHODS IN THIS FIELD AND PROPOSED THE CENTRAL PROBLEM OF THIS PAPER

Let $M = \{\mu_s\colon s \in S\}$ be a family of scalar bounded finitely additive measures defined on a σ-algebra $\mathcal{A}$. The Nikodym–Grothendieck boundedness theorem states that if M is simply bounded in $\mathcal{A}$ then M is uniformly bounded in $\mathcal{A}$. In this paper we prove that if $\mathcal{V} = \{\mathcal{A}_{n_1, n_2, \dots, n_p}\colon p, n_1, n_2 \dots n_p \in \mathbb{N}\}$ is an increasing web in $\mathcal{A}$, then there is a strand $\{\mathcal{A}_{n_1 n_2 \dots n_i}\colon i \in \mathbb{N}\}$ such that if M is simply bounded in one $\mathcal{A}_{n_1 n_2 \dots n_i}$ then M is uniformly bounded in $\mathcal{A}$ (Theorem 3.1). This result is deduced from the fact that if $\mathcal{W} = \{E_{n_1 n_2 \dots n_p}\colon p, n_1, n_2, \dots, n_p \in \mathbb{N}\}$ is a linear increasing web in $l_0^\infty(X, \mathcal{A})$, then there exists a strand $\{E_{n_1 n_2 \dots n_i}\colon i \in \mathbb{N}\}$ such that every $E_{n_1 n_2 \dots n_i}$ is barrelled and dense in $l_0^\infty(X, \mathcal{A})$ (Theorem 2.7). From this strong barrelledness condition previous results of the author jointly with J. C. Ferrando are improved here. These results are related to the classical result of Diestel and Faires in vector measures. © 1997 Academic Press

1. INTRODUCTION

Following Valdivia [17], if $\mathcal{A}$ is a σ-algebra on a set X, then $l_0^\infty(X, \mathcal{A})$ is the normed space generated by the characteristic functions $e(A)$, with $A \in \mathcal{A}$, provided with the norm $\|z\| = \sup\{|z(\omega)|\colon \omega \in X\}$, and its topological dual, $l_0^\infty(X, \mathcal{A})^*$, is the Banach space $ba(\mathcal{A})$ of bounded finitely additive measures with the variation norm. Given $A \in \mathcal{A}$, $l_0^\infty(A, \mathcal{A})$ denotes the subspace of $l_0^\infty(X, \mathcal{A})$ generated by $\{e(B)\colon B \in \mathcal{A},\ B \subset A\}$, and if $u \in$

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$l_0^\infty(X, \mathcal{A})^*$, then $|u|(A)$ is the norm of the restriction of u to $l_0^\infty(A, \mathcal{A})$. We will write $u(A)$ or $u(e(A))$.

If $A \in \mathcal{A}$ and $U \subset l_0^\infty(X, \mathcal{A})$, then we will say that U is a q-neighbourhood of zero in $l_0^\infty(A, \mathcal{A})$ if there exists a finite subset Q_U of $l_0^\infty(X, \mathcal{A})$ such that the closed absolutely convex hull of $U \cup Q_U$ contains a zero neighbourhood of $l_0^\infty(A, \mathcal{A})$. This definition follows from Valdivia [17, Note 1].

We are going to consider the tree with infinitely many branching points, $T_\infty = \bigcup\{\mathbb{N}^k : k = 0, 1, 2, \ldots\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. An increasing web in a set Y (see [3]) is a family $\mathcal{W} = \{E_t : t \in T_\infty\}$ of subsets of Y such that $Y = E_\varnothing = \bigcup\{E_n : n \in \mathbb{N}\}$, $E_n \subset E_{n+1}$, $E_t = \bigcup\{E_{t,n} : n \in \mathbb{N}\}$, and $E_{t,n} \subset E_{t,n+1}$, when $\varnothing \neq t \in T_\infty$ and $n \in \mathbb{N}$. If Y is a vector space and every E_t is a linear subspace of Y, we will then say that $\mathcal{W}$ is a linear increasing web.

If $\mathcal{W} = \{E_t : t \in T_\infty\}$ is a linear increasing web in $l_0^\infty(X, \mathcal{A})$ and $s \in \mathbb{N}$ then there exists an $E_{n_1 n_2 \ldots n_s}$ which is barrelled and dense in $l_0^\infty(X, \mathcal{A})$ (see Valdivia [17, Theorem 1], Rodríguez-Salinas [13, Theorem 1], and [6, Theorem 1; 10, Theorem 1]). When this result was obtained, Professor Valdivia suggested studying the existence of an infinite branch $\gamma = \{\varnothing, (n_1), (n_1, n_2), \ldots, (n_1, n_2, \ldots, n_k), \ldots\}$ in T_∞ such that every $E_t, t \in \gamma$, is barrelled and dense in $l_0^\infty(X, \mathcal{A})$. The aim of this paper is to answer this question in the positive. This problem is proposed in [8, Chap. 11, problem 11.10] and also in [9] in an equivalent form; here Ferrando and Sanchez Ruiz ask if $l_0^\infty(X, \mathcal{A})$ is baireled [9, Definition 1].

2. THE MAIN RESULT

If $t = (n_1, \ldots, n_i, \ldots, n_q) \in T_\infty$, set $|t| = q$ to be the rank of t, $P_i t = (n_1, \ldots, n_i)$, for $1 \leq i \leq |t|$, and $P_0 t = \varnothing$. If $T \subset T_\infty$ then $P_i T := \{P_i t : t \in T, |t| \geq i\}$.

DEFINITION 2.1. A non-void subset T of $T_\infty - \{\varnothing\}$ is called a υ-web if it verifies:

(1) If $t \in T$ and $1 \leq i \leq |t|$ then cardinal $\{P_i s : s \in T,\ i \leq |s|, P_{i-1} s = P_{i-1} t\} = \infty$.

(2) If $t \in T$, then $\{u \in T : |u| > |t|, P_{|t|-1}(u) = P_{|t|-1}(t)\} = \varnothing$.

(3) For each sequence $\{t_n \in T : n \in \mathbb{N}, |t_n| \geq n\}$, there is a p with $P_p t_p \neq P_p t_{p+1}$.

An element $t = (n_1, n_2, \ldots, n_s) \in T$ determines the branch $\gamma_t = \{\varnothing, (n_1), (n_1, n_2), \ldots, (n_1, n_2, \ldots, n_s)\}$, and the set $\mathcal{B}_T = \bigcup\{\gamma_t : t \in T\}$ will be called the υ-tree determined by the υ-web T. $\mathcal{B}_T$ does not contain infinite branches and $(n_1, n_2, \ldots, n_i) \in T$ implies that $(n_1, n_2, \ldots, n_{i-1}, p) \in T$ for infinitely many values of p.

We have $(n_1, \dots, n_{i-1}, n_i) \in T$ if, and only if, there is no $(n_1, \dots, n_{i-1}, p, q) \in T$.

Examples of υ-webs are $\mathbb{N}^p$, $\bigcup\{\{i\} \times \mathbb{N}^i \colon i \in \mathbb{N}\}$, and the infinite subsets of $\mathbb{N}$, which will be called trivial υ-webs.

Notice that if T is a υ-web and if its subset T^* verifies the preceding condition (1) then T^* is a υ-web.

If $W = \{E_t \colon t \in T_\infty\}$ is an increasing web in Y and T is a υ-web, then $Y = \bigcup\{E_n \colon n \in P_1 T\}$ and if $s \in P_p T - T$ then $E_s = \bigcup\{E_{(s,n)} \colon (s, n) \in P_{p+1} T\}$. Then we have:

PROPOSITION 2.2. $Y = \cup\{E_t \colon t \in T\}$.

Proof. If $x \in \cup\{E_n \colon n \in P_1 T\} - \cup\{E_t \colon t \in T\}$, then there exists $t_1 \in T$, $t_1 \neq P_1 t_1 = \alpha_1$ with $x \in E_{\alpha_1}$.

If we have obtained in T the elements $t_i \neq P_i t_i = (\alpha_1, \alpha_2, \dots, \alpha_i)$, for $1 \leq i \leq n-1$ such that $x \in E_{\alpha_1, \alpha_2, \dots, \alpha_i}$, then $x \in \bigcup\{E_{\alpha_1, \dots, \alpha_{n-1}, q} \colon (\alpha_1, \dots, \alpha_{n-1}, q) \in P_n T\} - \cup\{E_t \colon t \in T\}$. Therefore there exists in T an element $t_n \neq P_n t_n = (\alpha_1, \dots, \alpha_n)$ such that $x \in E_{\alpha_1 \dots \alpha_n}$. Clearly the sequence $\{t_n\}$ contradicts the preceding condition (3). ∎

If the υ-web $T = T_1 \bigcup T_2$ is trivial and T_1 does not contain any υ-web then T_2 contains a υ-web. The next proposition extends this property.

PROPOSITION 2.3. *If T_1 is a subset of the υ-web T, and T_1 does not contain any υ-web, then $T - T_1$ contains a υ-web.*

Proof. We may assume that T is not trivial. Clearly there is $a \in \mathbb{N}$ such that for $a_1 \in J_1 = \{m \in P_1 T \colon m > a\}$ there does not exist any υ-web U_{a_1} with $\{a_1\} \times U_{a_1}$ contained in T_1 (if this were false T_1 would contain a υ-web), and then two cases can occur:

(1) If there exists $(a_1, m) \in T$ then there is a trivial υ-web T_{a_1} with $\{a_1\} \times T_{a_1} = I_{a_1} \subset T - T_1$. The set I_{a_1} is infinite and we define $J_{a_1} = \varnothing$.

(2) There is no natural number m such that $(a_1, m) \in T$. Then, as in the beginning of the proof, there exists a natural number b such that for $a_2 > b$ there does not exist any υ-web $U_{a_1 a_2}$ such that $\{(a_1, a_2)\} \times U_{a_1 a_2}$ is contained in T_1. Now the set $J_{a_1} = \{(a_1, a_2) \in P_2 T,\ a_2 > b\}$ is infinite and we write $I_{a_1} = \varnothing$.

We will finish the first step of this induction by writing $I_2 = \cup\{I_{a_1} \colon a_1 \in J_1\}$ and $J_2 = \cup\{J_{a_1} \colon a_1 \in J_1\}$.

If $(a_1, a_2) \in J_2$ and there exists $(a_1, a_2, m) \in T$ then there is a trivial υ-web $T_{a_1 a_2}$ with $\{(a_1, a_2)\} \times T_{a_1 a_2} \subset T - T_1$. Then the set $I_{a_1, a_2} = \{(a_1, a_2)\} \times T_{a_1 a_2}$ is infinite and we define $J_{a_1 a_2} = \varnothing$. If $(a_1, a_2, m) \notin T$ for each $m \in \mathbb{N}$, then there exists $c \in \mathbb{N}$ such that for $a_3 > c$ there does not exist any υ-web $U_{a_1a_2a_3}$ such that $\{(a_1, a_2, a_3)\} \times U_{a_1a_2a_3} \subset T_1$. In this case we write $I_{a_1a_2} = \emptyset$ and $J_{a_1a_2} = \{(a_1, a_2, a_3) \in P_3T, a_3 > c\}$. Now $J_{a_1a_2}$ is infinite and we define $I_3 = \cup\{I_{a_1a_2}\colon (a_1, a_2) \in J_2\}$ and $J_3 = \cup\{J_{a_1a_2}\colon (a_1, a_2) \in J_2\}$.

We continue the induction in an obvious way. If a J_i were empty then the inductive process would be finite. Finally we are going to prove that $I = I_2 \cup I_3 \cup \cdots$ is a υ-web, obviously contained in $T - T_1$. I is non-empty, because if $I = \emptyset$ and $\alpha_1 \in J_1$ we may determine a sequence $(\alpha_1, \alpha_2) \in J_2$, $(\alpha_1, \alpha_2, \alpha_3) \in J_3, \dots$. Therefore there exists a sequence $\{t_n \in T,\ n \in \mathbb{N}\}$ such that $P_n t_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$, contradicting condition (3). I verifies condition (1) of the υ-web definition because given $(a_1, \dots, a_i, \dots, a_n) \in I$ we have that the sets J_1, $J_{a_1,\dots,a_i, 1 \le i \le n-2}$, and $I_{a_1a_2\dots a_{n-1}}$ are infinite. ∎

Remark 2.4. If the υ-web T is the union $T_1 \cup T_2 \cup \cdots \cup T_p$ then a T_i must contain a υ-web. The next lemma follows from this remark and from [17, Proposition 5].

LEMMA 2.5. *Let us suppose that $A \in \mathcal{A}$, T is a υ-web and that for each $(n_1, n_2, \dots, n_p) \in T$, $U_{n_1n_2\dots n_p}$ is a closed absolutely convex subset of $l_0^\infty(X, \mathcal{A})$ which is not a q-neighbourhood of zero in $l_0^\infty(A, \mathcal{A})$. Let α be a positive number. If $x_1, x_2, \dots,$ and x_n are n vectors of $l_0^\infty(X, \mathcal{A})$ and $(n_1^i, n_2^i, \dots, n_{p(i)}^i) \in T$, for $1 \le i \le k$, then there are in A k pairwise disjoint subsets $A_i \in \mathcal{A}$, and k bounded measures $u_i \in (U_{n_1^i, n_2^i, \dots, n_{p(i)}^i})^0$, $1 \le i \le k$, such that*

$$|u_i(A_i)| \ge \alpha, \qquad \sum \{|u_i(x_s)|\colon 1 \le s \le n\} \le 1.$$

Moreover, there is a υ-web $T^* \subset T$, containing the elements $(n_1^i, n_2^i, \dots, n_{p(i)}^i)$, $1 \le i \le k$, and such that if $(n_1, n_2, \dots, n_p) \in T^*$ then $U_{n_1n_2\dots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(A - \cup\{A_i, 1 \le i \le k\}, \mathcal{A})$.

Proof. By [17, Proposition 4] there is a partition of the set A in $p = p(1) + p(2) + \cdots + p(k) + 2$ subsets, $B_1, B_2, \dots, B_p \in \mathcal{A}$, and p linear forms, $\lambda_1, \lambda_2, \dots, \lambda_p \in (U_{n_1^1, n_2^1, \dots, n_{p(1)}^1})^0$ such that

$$|\lambda_i(B_i)| \ge \alpha, \qquad \sum \{|\lambda_i(x_l)|\colon 1 \le l \le n\} \le 1.$$

Proposition 3 of [17] enables us to obtain a $B_{h(i)}$ such that $U_{n_1^i, n_2^i, \dots, n_{p(i)}^i}$ is not a q-neighbourhood of zero in $l_0^\infty(B_{h(i)}, \mathcal{A})$, $1 \le i \le k$.

By [17, Proposition 3], given a $U_{n_1n_2\dots n_q}$ there exists a B_h such that $U_{n_1n_2\dots n_q}$ is not a q-neighbourhood of zero in $l_0^\infty(B_h, \mathcal{A})$. This observation and the preceding remark enables us to obtain the following two consequences:

(1) There exists a $B_{h(0)}$ and a υ-web $T_0 \subset T$ such that if $(n_1, n_2, \dots, n_p) \in T_0$ then $n_1 > \max\{n_1^1, n_1^2, \dots, n_1^k\}$ and $U_{n_1n_2\dots n_p}$ is not a

q-neighbourhood of zero in $l_0^\infty(B_{h(0)}, \mathcal{A})$. In fact, let $T_h = \{(n_1, n_2, \ldots, n_q) \in T\colon U_{n_1 n_2 \ldots n_q}$ is not a q-neighbourhood of zero in $l_0^\infty(B_h, \mathcal{A})\}$. By [17, Propostion 3], $T = T_1 \cup T_2 \cdots \cup T_p$; the remark implies that there is a $T_{h(0)}$, $1 \le h(0) \le p$, that contains a v-web T_*. The v-web $T_0 = \{(n_1, n_2, \ldots, n_q) \in T_*\colon n_1 > \max((n_1^1, n_1^2, \ldots, n_1^k)\}$ fulfills the properties we are looking for.

(2) For each i and m, such that $1 \le i \le k$ and $2 \le m \le p(i)$, there exists a $B_{h(i,m)}$ and a v-web $T_{i,m}$ such that every

$$\left(n_1^i, n_2^i, \ldots, n_{m-1}^i, n_m, \ldots, n_{m+q}\right) \in \left\{\left(n_1^i, n_2^i, \ldots, n_{m-1}^i\right)\right\} \times T_{i,m}$$

belongs to T, verifies that $n_m^i < n_m$ and $U_{n_1^i, n_2^i, \ldots, n_{m-1}^i, n_m, \ldots, n_{m+q}}$ is not a q-neighbourhood of zero in $l_0^\infty(B_{h(i,m)}, \mathcal{A})$. The proof is like in the above case, changing T for $T^m = \{(n_m, \ldots, n_{m+q})\colon (n_1^i, n_2^i, \ldots, n_{m-1}^i, n_m, \ldots, n_{m+q}) \in T\}$ and T_h for $T_h^m = \{(n_m, \ldots, n_{m+q}) \in T^m\colon U_{n_1^i, n_2^i, \ldots, n_{m-1}^i, n_m, \ldots, n_{m+q}}$ is not a q-neighbourhood of zero in $l_0^\infty(B_h, \mathcal{A})\}$. Now $T^m = T_1^m \cup T_2^m \cdots \cup T_p^m$.

The quantity of sets $B_{h(i)}, B_{h(0)}$, and $B_{h(i,m)}$ obtained is less than or equal to $k + 1 + (p(1) - 1) + \cdots + (p(k) - 1) = p - 1$. Hence there must be a B_h which has not been used. Let us define $A_1 = B_h$ and $u_1 = \lambda_h$. Then we have that $u_1 \in (U_{n_1^1, n_2^1, \ldots, n_{p(1)}^1})^0$ and

$$|u_1(A_1)| > \alpha, \qquad \sum \left\{|u_1(x_i)|\colon 1 \le i \le n\right\} \le 1.$$

Moreover, the union of $\{(n_1^i n_2^i \ldots n_{p(i)}^i), 1 \le i \le k\}$, the v-web T_0, and the cartesian products $\{(n_1^i, n_2^i, \ldots, n_{m-1}^i)\} \times T_{i,m}$, $1 \le i \le k$, $2 \le m \le p(i)$ is a v-web T_1^* such that if $(n_1, n_2, \ldots, n_p) \in T_1^*$ then $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(A - A_1, \mathcal{A})$, since $A - A_1$ must contain some B_k such that $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(B_k, \mathcal{A})$.

Repeating the above process with the set $A - A_1$, the v-web T_1^*, and the sets $U_{n_1^i n_2^i \ldots n_{p(i)}^i}$, for $i = 2, 3, \ldots, k$ and 1, we obtain some $A_2 \in \mathcal{A}$, $A_2 \subset A - A_1$, $u_2 \in (U_{n_1^2, n_2^2, \ldots, n_{p(2)}^2})^0$ and a v-web $T_2^* (\subset T_1^*)$, such that:

(1) T_2^* contains $(n_1^i n_2^i \ldots n_{p(i)}^i)$, $1 \le i \le k$,

(2) If $(n_1, n_2, \ldots, n_p) \in T_2^*$, then $U_{n_1, n_2, \ldots, n_p}$ is not a q-neighbourhodd of zero in $l_0^\infty(A - (A_1 \cup A_2), \mathcal{A})$,

(3) $|u_2(A_2)| > \alpha$, $\sum \{|u_2(x_i)|\colon 1 \le i \le n\} \le 1$.

If we continue in the same way we obtain the sets $A_1, A_2, \ldots, A_k$, the linear forms $u_1, u_2, \ldots, u_k$ and the v-web $T_k^* = T^* \subset T$ satisfying the lemma. ▮

PROPOSITION 2.6. Let $W = \{E_{n_1 n_1 \ldots n_p}\colon p, n_1, n_2, \ldots, n_p \in \mathbb{N}\}$ be a linear increasing web in $l_0^\infty(X, \mathcal{A})$ and let T be a v-web. Then there exists some $(n_1, n_2, \ldots, n_p) \in T$ such that $E_{n_1 n_2 \ldots n_p}$ is barrelled.

Proof. Let us suppose that there exists a υ-web T such that for every $(n_1, n_2, \ldots, n_p) \in T$, $E_{n_1 n_2 \ldots n_p}$ is not barrelled and let $U_{n_1 n_2 \ldots n_p}$ be the closure in $l_0^\infty(X, \mathcal{A})$ of a barrel $W_{n_1 n_2 \ldots n_p}$ of $E_{n_1 n_2 \ldots n_p}$ which is not a zero neighbourhood in $E_{n_1 n_2 \ldots n_p}$. By [17, Proposition 7], $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(X, \mathcal{A})$, and then, by recurrence we will obtain:

- A υ-web $\{i^* \in T: i \in \mathbb{N}\}$.
- A sequence $\{I_j = \{1, 2, \ldots, j, \ldots, r(j)\} \subset \mathbb{N},\ j \in \mathbb{N}\}$, with $j \le r(j) \le r(j+1)$, $j \in \mathbb{N}$.
- A family $\{A_{ij}: i \in I_j, j \in \mathbb{N}\}$ of pairwise disjoint sets belonging to $\mathcal{A}$.
- A family of measures $\{u_{ij}: i \in I_j,\ j \in \mathbb{N}\}$, $u_{ij} \in U_{i^*}^0$, such that for $i \in I_j$ and $j \in \mathbb{N}$

$$|u_{ij}(A_{ij})| > j, \qquad \sum \left\{|u_{ij}(A_{lk})|: l \in I_k, 1 \le k \le j-1\right\} \le 1.$$

Indeed, let us start by taking 1^* as the first element $(n_1, n_2, \ldots, n_2) \in T$ with respect to the lexicographic order. The preceding lemma enables us to determine $A_{11} \in \mathcal{A}$, $u_{11} \in U_{1^*}^0$ and a υ-web T_1, contained in T, such that $|u_{11}(A_{11})| > 1$, $1^* \in T_1$ and if $(n_1, n_2, \ldots, n_p) \in T_1$, then $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(X - A_{11}, \mathcal{A})$. For the sake of simplicity we will write $A_1 = X - A_{11}$, and this step of the induction concludes by writing $I_1 = \{1\}$.

Let us now assume that after applying the above reasoning h times we have obtained

- The finite sequence $\{1^*, 2^*, \ldots, s^*\} \subset T$, the family $\{I_j = \{1, \ldots, j, \ldots, r(j)\} \subset \mathbb{N},\ j \le h\}$, with $j \le r(j) < r(j+1)$ and $r(h) = s$, and the υ-webs T_j, $1 \le j \le h$, such that $t^* \in T_j$, for $1 \le t \le r(j)$, and given $i \in I_j, j < h$, if $i^* = (m_1, \ldots, m_{r-1}, m_r, \ldots, m_{q_i})$ then for each $r \le q_i$ there is a $w \in I_{j+1}$ such that $w^* = (m_1, \ldots, m_{r-1}, n_r, \ldots, n_u) \in T_{j+1}$ and $m_r < n_r$.
- The pairwise disjoint sets $A_{ij} \in \mathcal{A}, i \in I_j,\ j \le h$, such that if $(n_1, n_2, \ldots, n_p) \in T_j$ and $A_j = X - \cup\{A_{ir}, i \in I_r, 1 \le r \le j\}$ then $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(A_j, \mathcal{A})$.
- The measures $u_{ij} \in U_{i^*}^0$, $i \in I_j$, $j \le h$, such that

$$|u_{ij}(A_{ij})| > j, \qquad \sum \left\{|u_{ij}(A_{lk})|: l \in I_k, 1 \le k \le j-1\right\} \le 1.$$

In the step $h+1$ for each $w^* = (m_1, \ldots, m_{r-1}, m_r, m_{r+1}, \ldots, m_{q_w})$, $1 \le w \le r(h) = s$, and each natural $r \le q_w$ we obtain an element $(m_1, \ldots, m_{r-1}, n_r, n_{r+1}, \ldots, n_v) \in T_h$ such that $m_r < n_r$. The elements so obtained will be denoted by $(s+1)^*, (s+2)^*, \ldots, t^*$. Since $m_r < n_r$ the sequence $\{1^*, 2^*, \ldots, s^*, (s+1)^*, (s+2)^*, \ldots, t^*, \ldots\}$ obtained in this induction will be a v-web. If $I_{h+1} = \{1, 2, \ldots, s, s+1, \ldots, t\}$, then the cardinality of I_{h+1} is greater than or equal to $h+1$, since, by induction, $h \le r(h) = s$ and $s < t = r(h+1)$. If we apply Lemma 2.5 with $\alpha = h+1$, $A = A_h$, $T = T_h$, $x_1, x_2, \ldots, x_n$ equal to the characteristic functions of A_{ij}, $i \in I_j$, $1 \le j \le h$, and $(n_1^i n_2^i \ldots n_{p(i)}^i) = i^*$, $i \in I_{h+1}$, we obtain:

- t pairwise disjoint subsets $A_{l,h+1} \in \mathcal{A}$, $1 \le l \le t$, contained in A_h,
- t linear forms $u_{l,h+1} \in U_{l^*}^0$, $1 \le l \le t$, such that for each l

$$|u_{l,h+1}(A_{l,h+1})| > h+1, \qquad \sum \left\{|u_{l,h+1}(A_{ij})|: i \in I_j, 1 \le j \le h\right\} \le 1, \tag{2.1}$$

- some v-web T_{h+1}, with $\{1^*, 2^*, \ldots, s^*, (s+1)^*, \ldots, t^*\} \subset T_{h+1} \subset T_h$ such that if $(n_1, n_2, \ldots, n_p) \in T_{h+1}$ then $U_{n_1 n_2 \ldots n_p}$ is not a q-neighbourhood of zero in $l_0^\infty(A_h - \cup\{A_{i,h+1}: 1 \le i \le t\}, \mathcal{A})$.

The induction ends by writing $A_{h+1} = A_h - \cup\{A_{i,h+1}: 1 \le i \le t\}$. In the induction we have taken $m_r < n_r$. This implies that $\{i^* \in T: i \in \mathbb{N}\}$ verifies condition (1) of Definition 2.1, and therefore it is a v-web.

Let us now denote by $1^\wedge = (1,1)$, $2^\wedge = (1,2)$, $3^\wedge = (2,1)$, $4^\wedge = (3,1), \ldots,$ the elements of $\mathbb{N}^2$ following the diagonal order. Our next task will be to obtain a contradiction with some pairwise disjoint elements $B_{ij} \in \mathcal{A}$, and some measures $v_{ij} \in U_{i^*}^0$, $1 \le i, j < \infty$, such that $v_{ij}(B_{ij}) > j$, $|v_{ij}(\bigcup\{B_{mn}: (m,n) < (i,j)\})| < 1$, and $|v_{ij}|(\bigcup\{B_{mn}: (m,n) > (i,j)\}) < 1$. These elements will be drawn out from the previous A_{mn} and u_{mn}, by applying a new induction.

We will start by taking $B_{1^\wedge} = B_{11} = A_{11}$ and $v_{1^\wedge} = v_{11} = u_{11}$. Now we split the family $\{\mathcal{B}_j = \{A_{ij}: i \in I_j\}: j > 1\}$ into infinitely many $\mathcal{C}_n$, $n \in \mathbb{N}$, such that each $\mathcal{C}_n$ contains infinitely many $\mathcal{B}_j$. Since v_{11} is a bounded measure there must be a family $\mathcal{C}_p$ such that the variation of v_{11} in $\cup\{C: C \in \mathcal{C}_p\}$ is less than 1. We will denote this family by $\mathcal{D}_1$.

Let us suppose that we have determined the $B_{i^\wedge}$, $v_{i^\wedge}$ and the families $\mathcal{D}_i$, for $1 \le i \le k-1$, such that if $i^\wedge = (m,n)$ then:

- $B_{i^\wedge} = B_{mn} = A_{mh(i)}$, with $h(i-1) < h(i)$, and $\mathcal{D}_i \subset \mathcal{D}_{i-1}$ for $2 \le i \le k-1$,
- $\mathcal{D}_i$ is the union of infinitely many families $\mathcal{B}_j = \{A_{pj}: p \in I_j\}$, with $j > h(i)$,
- $v_{i^\wedge} = v_{mn} = u_{mh(i)}$ verifies that $|v_{i^\wedge}|(\cup\{B: B \in \mathcal{D}_i\}) < 1$, for $1 \le i \le k-1$.

Suppose that $k^{\wedge} = (r, s)$. From the fact that $\mathcal{D}_{k-1}$ contains infinitely many $\mathcal{B}_j = \{A_{ij}\colon i \in I_j\}$, where the cardinality of I_j is greater than or equal to j, it follows that there must be some $A_{rh(k)}$ in $\mathcal{D}_{k-1}$ such that $h(k-1) < h(k)$. Then, we define $B_{k^{\wedge}} = B_{rs} = A_{rh(k)}$ and $v_{k^{\wedge}} = v_{rs} = u_{rh(k)}$.

Now we split the family $\{\mathcal{B}_j = \{A_{ij}\colon i \in I_j\} \subset \mathcal{D}_{k-1}, j > h(k)\}$ into finitely many subfamilite as above and denote by $\mathcal{D}_k$ one of these subfamilies such that the variation of $v_{k^{\wedge}}$ in $\mathcal{D}_k$ is less than 1. In this way the inductive process supplies the families $\{B_{rs}\colon (r,s) \in \mathbb{N}^2\}$, $\{v_{rs}\colon (r,s) \in \mathbb{N}^2\}$, and $\{\mathcal{D}_k\colon k \in \mathbb{N}\}$, with $v_{rs} \in U^0_{r*}$ and

$$|v_{rs}|\big(\cup\{B_{mn}\colon (m,n) > (r,s)\}\big) \le |v_{k^{\wedge}}|\big(\bigcup\{B\colon B \in \mathcal{D}_k\}\big) \le 1. \quad (2.2)$$

From (2.1) it follows that for every $(r,s) \in \mathbb{N}^2$,

$$|v_{rs}(B_{rs})| > s, \qquad |v_{rs}\big(\cup\{B_{mn}\colon (m,n) < (r,s)\}\big)| < 1. \quad (2.3)$$

If we settle $B = \cup\{B_{ij}\colon (i,j) \in \mathbb{N}^2\}$ then, applying Proposition 2.2 to the v-web $\{s^*\colon s \in \mathbb{N}\}$, we conclude that there must be some s^* such that $e(B) \in E_{s^*}$. Therefore there exists a $\lambda > 0$ such that $e(B) \in \lambda U_{s^*}$. Since $v_{sj} \in U^0_{s^*}$ we deduce that $|v_{sj}(B)| < \lambda$, for every $j \in \mathbb{N}$.

This contradicts (2.2) and (2.3) by means of which $|v_{sj}(B)| = |v_{sj}(\bigcup\{B_{m,n}\colon (m,n) < (s,j)\}) + v_{sj}(B_{sj}) + v_{sj}(\bigcup\{B_{m,n}\colon (m,n) > (s,j)\})| > j - 2.$ ∎

THEOREM 2.7. *Let $\mathcal{W} = \{E_t\colon t \in T_\infty\}$ be a linear increasing web in $l_0^\infty(X, \mathcal{A})$. Then there exists a strand $\{E_{n_1 n_2 \dots n_i}\colon i \in \mathbb{N}\}$ such that every $E_{n_1 n_2 \dots n_i}$ is barrelled and dense in $l_0^\infty(X, \mathcal{A})$.*

Proof. Let us assume that each strand $\{E_{n_1 n_2 \dots n_i}\colon i \in \mathbb{N}\}$ of $\mathcal{W}$ contains some $E_{n_1 n_2 \dots n_i}$ which is not barrelled or not dense in $l_0^\infty(X, \mathcal{A})$. By an inductive process, we are going to obtain a v-web T such that none of the $E_t, t \in T$, are barrelled, in contradiction with Proposition 2.6.

From Valdivia's theorem of suprabarrelledness of $l_0^\infty(X, \mathcal{A})$ [17, Theorem 1], it follows that there exists a natural number b_1 such that for $n_1 \ge b_1$ every E_{n_1} is barrelled and dense in $l_0^\infty(X, \mathcal{A})$. We write $J_1 = \{n_1 \in \mathbb{N}\colon n_1 \ge b_1\}$. By the Amemiya–Kōmura property [1; 12, Corollary 8.2.12] given $n_1 \in J_1$ we have that there exists a $b_2 \in \mathbb{N}$ such that for $n_2 \ge b_2$ each E_{n_1, n_2} is dense in $l_0^\infty(X, \mathcal{A})$. But if the barrelled space F is dense in G then G is barrelled and, therefore, for each $a_1 \in J_1$, two cases can occur:

(1) There exists in $\mathbb{N}$ a cofinite subset N_{a_1} such that for every $(a_1, a_2) \in \{a_1\} \times N_{a_1}$ we have that E_{a_1, a_2} is non-barrelled. Then we write $I_{a_1} = a_1 \times N_{a_1}$ and $J_{a_1} = \emptyset$. The set I_{a_1} is infinite.

(2) There exists in $\mathbb{N}$ a cofinite subset M_{a_1} such that for every $(a_1, a_2) \in \{a_1\} \times M_{a_1}$ we have that E_{a_1, a_2} is barrelled and dense in $l_0^\infty(X, \mathcal{A})$. Then we write $I_{a_1} = \emptyset$ and $J_{a_1} = \{a_1\} \times M_{a_1}$. Now J_{a_1} is infinite.

We will conclude the first step of this induction by writing $I_2 = \cup\{I_{a_1}, a_1 \in J_{a_1}\}$ and $J_2 = \cup\{J_{a_1}, a_1 \in J_{a_1}\}$.

If $(a_1, a_2) \in J_2$ then by the aforementioned Amemiya–Kōmura property there exists a $b_3 \in \mathbb{N}$ such that for $n_3 \geq b_3$ each E_{n_1, n_2, n_3} is dense in $l_0^\infty(X, \mathcal{A})$, and then we may obtain I_3 and J_3 exactly as before, and we continue the induction in an obvious way. If some J_i were empty then the inductive process would be finite.

We have that $I = \cup\{I_n, n \in \mathbb{N}\}$ is non-empty, because if $I = \emptyset$ and $\alpha_1 \in J_1$, then we may determine a sequence $(\alpha_1, \alpha_2) \in J_2$, $(\alpha_1, \alpha_2, \alpha_3) \in J_3, \ldots,$ and then each $E_{\alpha_1, \alpha_2, \ldots, \alpha_i}$ would be barrelled and dense in $l_0^\infty(X, \mathcal{A})$, contradicting our initial hypothesis. Finally I verifies condition (1) of the v-web's definition, because, by construction, given $(a_1, \ldots, a_i, \ldots, a_n) \in I$, we have that the sets $J_1, J_{a_1, \ldots, a_i, 1 \leq i \leq n-2}$, and $I_{a_1, \ldots, a_{n-1}}$ are infinite. By construction I verifies the condition (2) of Definition 2.1; the condition (3) follows from the hypothesis on the strands of $\mathcal{W}$. Therefore, we would have a v-web I such that each E_t, $t \in I$ would be non-barrelled, contradicting Proposition 2.6.

Remark 2.8. It was proved in [2] that $l_0^\infty(X, \mathcal{A})$ is not totally barrelled [18, Definition 1]. This property was proposed as a open question in [18].

3. APPLICATIONS TO THE SPACE OF BOUNDED FINITELY ADDITIVE MEASURES

A subset M of $l_0^\infty(X, \mathcal{A})^* = ba(\mathcal{A})$ is said to be simply bounded in a subset $\mathcal{B}$ of $\mathcal{A}$ if, for every $A \in \mathcal{B}$, $\sup\{|\mu(A)|: \mu \in M\} < \infty$.

Our next result extends the Nikodym–Grothendieck boundedness theorem [4, VII].

THEOREM 3.1. *If $\mathcal{V} = \{\mathcal{A}_t: t \in T_\infty\}$ is an increasing web in the σ-algebra $\mathcal{A}$, there exists a strand $\{\mathcal{A}_{n_1 n_2 \ldots n_i}: i \in \mathbb{N}\}$ in $\mathcal{V}$ such that every family $\{\mu_s: s \in S\} \subset ba(\mathcal{A})$ which is simply bounded in an $\mathcal{A}_{n_1 n_2 \ldots n_i}$ verifies that it is bounded in $l_0^\infty(X, \mathcal{A})^*$.*

Proof. Let $\mathcal{W} = \{E_t: t \in T_\infty\}$ be the linear increasing web in $l_0^\infty(X, \mathcal{A})$ such that E_t is the linear hull of the characteristic functions $\{e(A): A \in \mathcal{A}_t\}$. By Theorem 2.7 there exists a strand $\{E_{n_1 n_2 \ldots n_i}: i \in \mathbb{N}\}$ such that every $E_{n_1 n_2 \ldots n_i}$ is barrelled and dense in $l_0^\infty(X, \mathcal{A})$.

Therefore if $\{\mu_s: s \in S\} \subset ba(\mathcal{A})$ is simply bounded in $\mathcal{A}_{n_1 n_2 \ldots n_i}$ then $\{\mu_s: s \in S\}$ is $\sigma(l_0^\infty(X, \mathcal{A})^*, E_{n_1 n_2 \ldots n_i})$-bounded. As $E_{n_1 n_2 \ldots n_i}$ is barrelled and dense in $l_0^\infty(X, \mathcal{A})$, it follows that $\{\mu_s, s \in S\}$ is equicontinuous, and, therefore, $\{\mu_s, s \in S\}$ is bounded in $l_0^\infty(X, \mathcal{A})^*$.

4. APPLICATIONS TO VECTOR MEASURES

From now onwards the word *space* will stand for a *real or complex locally convex Hausdorff space*. If E is a space, the topological dual of E will be denoted by E', as in [11, Sect. 15.9]. A space E is dual locally complete [16, Definition 1], if its weak topological dual, $E'(\sigma(E',E))$, is locally complete. A space E is Γ_r [15, Definition 1], (Λ_r) [16, Definition 2]) if given any quasi-complete (locally complete) subspace G of the weak algebraic dual of E such that G meets E' in a weak dense subspace, then $E' \subset G$. The B_r-complete spaces are Γ_r-spaces. Reflexive Banach spaces and Fréchet−Schwartz spaces provide some simple examples of Λ_r-spaces. When $(E, \mathcal{T})$ is a $\Gamma_r(\Lambda_r)$ space we will say that $\mathcal{T}$ is a $\Gamma_r(\Lambda_r)$ topology.

Using our Theorem 2.7 in [7], instead of [6, Theorem 1], we would obtain the following results:

PROPOSITION 4.1. *Let μ be a bounded finitely additive measure on $\mathcal{A}$ with values in a space E, and let $\mathcal{W} = \{E_t\colon t \in T_\infty\}$ be a linear increasing web in E such that every E_t has a Γ_r topology $\mathcal{T}_t$ finer than the topology induced by E.*

Then there exists a strand $\{E_{n_1 n_2 \ldots n_i}\colon i \in \mathbb{N}\}$ in $\mathcal{W}$ such that μ is a G-valued bounded finitely additive measure, G being $\bigcap\{E_{n_1 n_2 \ldots n_i}\colon i \in \mathbb{N}\}$ endowed with the initial topology corresponding to $\mathcal{T}_{n_1 n_2 \ldots n_i}$, $i \in \mathbb{N}$.

The next proposition extends a well-known result of J. Diestel and B. Faires [5, Theorem 1.1].

PROPOSITION 4.2. *Let μ be a finitely additive measure on $\mathcal{A}$ with values in a space E that has a web $\mathcal{W} = \{E_t\colon t \in T\}$, such that each E_t has a sequentially complete Γ_r topology $\mathcal{T}_t$, finer than that induced by E, under which it does not contain a copy of l^∞.*

If $u\mu$ is a countably additive measure for every u belonging to a weak total subset H of E', then there exists a strand $\{E_{n_1 n_2 \ldots n_i}\colon i \in \mathbb{N}\}$ in E such that μ is a G-valued countably additive vector measure, G being the vector space $\bigcap\{E_{n_1 n_2 \ldots n_i}\colon i \in \mathbb{N}\}$ endowed with the initial topology corresponding to $\mathcal{T}_{n_1 n_2 \ldots n_i}$, $i \in \mathbb{N}$.

The l^∞ condition of the preceding result may be avoided if we change the Γ_r by the Λ_r Valdivia spaces.

PROPOSITION 4.3. *Let μ be a finitely additive measure on $\mathcal{A}$ with values in a space E that has a web $\mathcal{W} = \{E_t\colon t \in T\}$, such that each E_t has a Λ_r topology $\mathcal{T}_t$, finer than that induced by E.*

If $u\mu$ is a countably additive measure for every u belonging to a weak total subset H of E', then there exists a strand $\{E_{n_1 n_2 \ldots n_i}\colon i \in \mathbb{N}\}$ in E such that μ

is a G-valued countably additive vector measure, G being the vector space $\bigcap\{E_{n_1 n_2 \dots n_i}\colon\ i \in \mathbb{N}\}$ *endowed with the initial topology corresponding to* $\mathcal{T}_{n_1 n_2 \dots n_i}$, $i \in \mathbb{N}$.

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REFERENCES

1. I. Amemiya and Y. Kōmura, Über nicht vollständige Montelräume, *Math. Ann.* **177** (1968), 273–277.
2. J. Arias de Reyna, $l_0^\infty(\Sigma)$ no es totalmente tonelado, *Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid* **79** (1980), 77–78.
3. M. de Wilde, "Closed Graph Theorem and Webbed Spaces," Pitman, London, 1978.
4. J. Diestel, "Sequences and Series in Banach Spaces," Springer-Verlag, Berlin/Heidelberg/New York, 1984.
5. J. Diestel and B. Faires, On Vector measures, *Trans. Amer. Math. Soc.* **198** (1975), 253–271.
6. J. C. Ferrando and M. López Pellicer, Strong barrelledness properties in $l_0^\infty(X, \mathcal{A})$ and bounded finitely additive measures, *Math. Ann.* **287** (1990), 727–736.
7. J. C. Ferrando and M. López Pellicer, A note on a theorem of J. Diestel and B. Faires, *Proc. Amer. Math. Soc.* **115** (1992), 1077–1081.
8. J. C. Ferrando, M. López Pellicer, and L. M. Sanchez Ruiz, Metrizable barrelled spaces, *in* "Pitman Res. Notes Math. Ser.," Vol. 332, Longman, Harlow/New York, 1995.
9. J. Ferrando and L. M. Sánchez Ruiz, A maximal class of spaces with strong barrelledness conditions, *Proc. Roy. Irish Acad. Sect. A* **92**, No. 1 (1992), 69–75.
10. J. Ferrando and L. M. Sánchez Ruiz, On a barrelled space of class $\mathfrak{N}_0$ and measure theory. *Math Scand.* **71** (1992), 96–104.
11. G. Köthe, "Topological Vector Spaces I," Springer-Verlag, Berlin/Heidelberg/New York, 1969.
12. P. Pérez Carreras and J. Bonet, Barrelled locally convex spaces, *in* "North-Holland Math. Stud.," Vol. 131, North-Holland, Amsterdam, 1987.
13. B. Rodríguez-Salinas, Sobre la clase del espacio tonelado $l_0^\infty(\Sigma)$, *Rev. Real Acad. Cienc. Exact. Fís. Natur.* 74, No. 5 (1980), 827–829.
14. S. A. Saxon, Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology, *Math. Ann.* **197** (1972), 87–106.
15. M. Valdivia, Sobre el teorema de la gráfica cerrada, *Collect. Math.* **22** (1971), 52–72.
16. M. Valdivia, Mackey convergence and the closed graph theorem, *Arch. Math.* **25** (1974), 649–656.
17. M. Valdivia, On certain barrelled normed spaces, *Ann. Inst. Fourier* **29** (1979), 39–56.
18. M. Valdivia and P. Pérez Carreras, On totally barrelled spaces, *Math. Z.* **178** (1981), 263–269.
19. M. Valdivia, Topics in locally convex spaces, *in* "North Holland Math. Stud." Vol. 67, North-Holland, Amsterdam, 1982.