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Cohen–Macaulay Local Rings of Maximal Embedding Dimension

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Let $(R, \mathbf{m})$ be a d-dimensional local Cohen-Macaulay ring of multiplicity e. If v denotes the embedding dimension of R, $v \leqslant e + d - 1$, [1]. If $v = d$ or $d + 1$ much of the structure of R, in terms of such measuring devices as the Hilbert function, Betti numbers etc., is completely determined by e. We show here that the same is true for d-dimensional local Cohen–Macaulay rings of embedding dimension $v = e + d - 1$ and d-dimensional local Gorenstein rings of embedding dimension $e + d - 2$. An investigation of these rings was begun [16] and [17] where properties of the associated graded rings were given.

Inspiration for some of the results in this paper comes from the paper [18] of Wahl which gives equations defining rational surface singularities and certain elliptic singularities. The local rings at these singularities are examples of the rings under discussion here. I wish to thank J. Wahl for suggesting that the investigations begun in [16] and [17] could be carried further.

1. Statement of the Main Results and Examples

Some measures of the singularity of a local ring $(R, \mathbf{m})$ are the Hilbert function $H_R(n) = \dim_k \mathbf{m}^n/\mathbf{m}^{n+1}$ where $k = R/\mathbf{m}$, the Poincaré series $P_{R,k}(t) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^R(k, k)t^i$ and, if R is presented as a quotient S/I of a regular local ring S, the Poincaré series $P_{S,R}(t) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^S(R, k)t^i$. We record here explicit calculation of these measures for d-dimensional local Cohen–Macaulay rings of embedding dimension $e + d - 1$ and for d-dimensional local Gorenstein rings of embedding dimension $e + d - 2$. The main results will be proved in Sections 3 and 4. Section 5 contains some information on the behavior of these rings under blowing up.

First, some remarks about notation. If $(R, \mathbf{m})$ is a local ring, k will denote the residue field $R/\mathbf{m}$ and grR will denote the associated graded ring: $grR = R/\mathbf{m} \oplus \mathbf{m}/\mathbf{m}^2 \oplus \cdots$. If a is a nonzero element of R with $a \in \mathbf{m}^s \backslash \mathbf{m}^{s+1}$ then we

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will say that a has order s. $\bar{a}$ will denote the initial form of a in grR, i.e., $\bar{a}$ is the image $a + \mathbf{m}^s/\mathbf{m}^{s+1}$ in grR. $\bar{a}$ has degree s. If L is a filtered R-module, $L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$, grL will denote the associated graded module $grL = L/L_1 \oplus L_1/L_2 \oplus \cdots$. Unless the contrary is stated, the filtration on an ideal I of R will be the induced filtration $I_n = I \cap \mathbf{m}^n$. As binomial coefficients will appear in some formulas we will use the conventions: $\binom{i}{j} = 1$ if $j = 0$, $\binom{i}{j} = 0$ if $j \neq 0$ and $i < j$ or if $j < 0$.

THEOREM 1. *Let* $(R, \mathbf{m})$ *be a d-dimensional local Cohen–Macaulay ring of multiplicity e and embedding dimension* $v = e + d - 1$.

(i) $H_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-2}{n}$ *for all* $n \geqslant 0$, *if* $d > 0$. *For* $d = 0$, $H_R(0) = 1$, $H_R(1) = e - 1$ *and* $H_R(n) = 0$ *for* $n > 1$.

(ii) $P_{R,k}(t) = (1 + t)^d \sum_{i=0}^{\infty} (e - 1)^i t^i$.

(iii) *If* $R = S/I$ *with* $(S, \mathbf{p})$ *a regular local ring and* $I \subseteq \mathbf{p}^2$, *then* $\operatorname{Tor}_i^S(R, k) = 0$ *for* $i > e - 1$ *and* $\dim \operatorname{Tor}_i^S(R, k) = i\binom{e}{i+1}$ *for* $i = 1,..., e - 1$.

COROLLARY. *With* $R = S/I$ *as in* (iii) *above, I is generated by* $\binom{e}{2}$ *elements of order* 2 *in S.* $grR = grS/grI$ *where grI is generated by the* $\binom{e}{2}$ *initial forms of a minimal generating set for I.*

THEOREM 2. *Let* $(R, \mathbf{m})$ *be a d-dimensional local Gorenstein ring of multiplicity* $e > 3$ *and embedding dimension* $v = e + d - 2$.

(i) $H_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-3}{n}$ *for all* $n \geqslant 2$ *if* $d > 0$. *If* $d = 0$, $H_R(0) = 1$, $H_R(1) = e - 2$, $H_R(2) = 1$ *and* $H_R(n) = 0$ *for* $n > 2$.

(ii) $P_{R,k}(t) = (1 + t)^d \sum_{i=0}^{\infty}(e - 2)^i t^i / 1 + \sum_{i=0}^{\infty}(e - 2)^i t^{i+2}$.

(iii) *If* $R = S/I$ *with* $(S, \mathbf{p})$ *a regular local ring and* $I \subset \mathbf{p}^2$, *then* $\operatorname{Tor}_i^S(R, k) = 0$ *for* $i > e - 2$, $\dim \operatorname{Tor}_{e-2}^S(R, k) = 1$ *and*

$$\dim \operatorname{Tor}_i^S(R, k) = \frac{i(e - i - 2)}{e - 1} \binom{e}{i + 1}, \qquad i = 1,..., e - 3.$$

COROLLARY. *With* $R = S/I$ *as in* (iii) *above, I is generated by* $e(e - 3)/2$ *elements of order* 2 *in S.* $grR = grS/grI$ *where grI is generated by the* $e(e - 3)/2$ *initial forms of a minimal generating set for I. Thus if* $R = S/I$ *is a d-dimensional local Gorenstein ring with embedding dimension* $v = e + d - 2$ *and* $(S, \mathbf{p})$ *is a regular local ring with* $I \subset \mathbf{p}^2$, *then R is a complete intersection if and only if* $e = 3$ *or* 4 *if and only if grR is a complete intersection.*

We now give some examples exhibiting the diversity of rings under discussion. In all of the examples the dimension can be increased by adjunction of analytic indeterminates without changing the relation on the embedding dimension. In Examples 1–6 below, k denotes a field.

0. Any regular local ring satisfies $v = e + d - 1$.

1. Let X be an indeterminate and e an integer $\geqslant 1$.

$k[[X^e, \ldots, X^{2e-1}]]$ is a 1-dimensional local domain with $v = e$.

2. Let X and Y be indeterminates, and s an integer $\geqslant 1$.

$R = k[[(X, Y)^s]]$ is a 2-dimensional normal (cf. [6]) domain with $e = s$ and $v = s + 1$. R is defined by the 2×2 minors of the matrix of indeterminates $\binom{z_1 z_2 \cdots z_s}{z_2 z_3 \cdots z_{s+1}}$. R is an example of a rational surface singularity.

3. Let X, Y and Z be indeterminates.

$$R = k[X, Y, Z]_{(X,Y,Z)}/(Z^2 - X^4 - Y^4)$$

is a 2-dimensional normal domain with non-normal quadratic transform $R[Y/X, Z/X]_{(X, Y/X, Z/X)}$ satisfying $v = 3 = e + 1$. R is not a rational singularity.

4. Let X be an indeterminate and e an integer > 2.

$k[[X^e, \ldots, X^{2e-2}]]$ is a 1-dimensional local Gorenstein domain with $v = e - 1$.

5. Let X_1, X_2, X_3 and X_4 be indeterminates.

$k[[(X_1, X_2, X_3)^3]]$ is a 3-dimensional normal Gorenstein local ring (cf. [9]), with $e = 9$ and $v = 10 = e + d - 2$. $k[[(X_1, X_2, X_3, X_4)^2]]$ is a 4-dimensional normal Gorenstein local ring, (cf. [9]), with $e = 8$ and $v = 10 = e + d - 2$.

6. Let V be a finite dimensional vector space over k and let $(\, , \,)$ be a non-degenerate symmetric bilinear form on V. This form determines a 0-dimensional local Gorenstein ring R with embedding dimension $e(R) - 2$ as follows. Let $R = k \oplus V \oplus ks$ with multiplication $s \cdot s = V \cdot s = 0$, $v \cdot w = (v, w) \cdot s$ for v, w in V and multiplication by the first component just ordinary multiplication. We will see later that grR, for any 0-dimensional local Gorenstein ring R of embedding dimension $e - 2$, is of this form.

2. Preliminary Remarks on Graded Rings, Graded Resolutions and Hilbert Functions

In this section we recall some facts to be used in the sequel about the inter-dependence of certain Hilbert functions and Betti numbers, and some Artin–Rees conditions which allow passage from a resolution of an ideal to a graded resolution of the form ideal or vice versa. In addition, we state some change of rings theorems and some facts about reductions and graded rings which will also be needed.

Let $\bar{S} = k[X_1, \ldots, X_0]$ be a polynomial ring over the field k and let $\bar{I}$ be a homogeneous ideal. Let

$$0 \to \bar{F}_r \to \bar{F}_{r-1} \to \cdots \to \bar{F}_1 \to \bar{S} \to \bar{S}/\bar{I} \to 0, \qquad (*)$$

with $\bar{F}_j$ a free $\bar{S}$-module, be a minimal resolution for $\bar{S}/\bar{I}$. (*) induces exact sequences

$$0 \to (\bar{F}_r)_n \to (\bar{F}_{r-1})_n \to \cdots \to (\bar{F}_1)_n \to (\bar{S})_n \to (\bar{S}/\bar{I})_n \to 0 \qquad (*)_n$$

of the homogeneous components of degree n. These are exact sequences of finite dimensional vector spaces over k so we have that

$$H_{\bar{S}/\bar{I}}(n) = \dim_k(\bar{S}/\bar{I})_n = \sum_{i=0}^{r} (-1)^i \dim(\bar{F}_i)_n$$

with $\bar{F}_0 = \bar{S}$. Let the generators $Z_{i1}, \ldots, Z_{ib_i}$ of $\bar{F}_i$ have degrees $a_{i1}, \ldots, a_{ib_i}$ respectively. Then, since $(1-t)^{-q} = \sum_{n=0}^{\infty} \binom{n+q-1}{q-1} t^n$, we get

$$\sum_{n=0}^{\infty} H_{\bar{S}/\bar{I}}(n)t^n = (1-t)^{-q} - (t^{a_{11}} + \cdots + t^{a_{1b_1}})(1-t)^{-q} + \cdots + (-1)^r(t^{a_{r1}} + \cdots + t^{a_{rb_r}})(1-t)^{-q}. \qquad (2.1)$$

The resolution (*) is said to be *pure* if $\bar{F}_j \simeq \bar{S}(-a_j)^{b_j}$ where $\bar{S}(-i)_n = \bar{S}_{-i+n}$ and the a_j are positive integers with $a_1 < a_2 < \cdots < a_r$. Since (*) is assumed to be minimal, "pure" means that the generators Z_{jp} of $\bar{F}_j$ all have the same degree a_j.

If (*) is a pure resolution of $\bar{S}/\bar{I}$ we get, from (2.1),

$$\sum_{n=0}^{\infty} H_{\bar{S}/\bar{I}}(n)t^n = (1-t)^{-q} \sum_{i=0}^{r} (-1)^i b_i t^{a_i}. \qquad (2.2)$$

Suppose, conversely, that $\bar{S}/\bar{I}$ has a minimal resolution (*) and that there are positive integers $a_1 < a_2 < \cdots < a_r$ such that the Hilbert function $H_{\bar{S}/\bar{I}}(n)$ satisfies (2.2) with $b_i = \operatorname{rank} \bar{F}_i$. Then it is clear that (*) is a pure resolution for $\bar{S}/\bar{I}$.

Now let $(S, \mathbf{p})$ be a local ring. We will say that a finitely generated S-module E is $\mathbf{p}$-filtered if for all large i, $E_{i+1} = \mathbf{p}E_i$. Let E and F be finitely generated $\mathbf{p}$-filtered S-modules. A homomorphism $f: E \to F$ is *strict* if $f(E_i) = f(E) \cap F_i$ for every $i \geqslant 0$. "Strict" means that the grading which $f(E)$ has as a quotient of E is the same as the induced grading which $f(E)$ gets as a submodule of F. Strict homomorphisms preserve exactness upon passage to the associated graded modules.

(2.3) LEMMA. *Let $E \xrightarrow{f} F \xrightarrow{h} H$ be an exact sequence of finitely generated $\mathbf{p}$-filtered S-modules. If f and h are strict homomorphisms then*

$$grE \xrightarrow{grf} grF \xrightarrow{grh} grH$$

is exact.

Proof. cf. [10; chapter 9, Thm. 9].

We will apply (2.3) to obtain conditions which will allow passage from a minimal resolution to a minimal associated graded resolution.

(2.4) COROLLARY. *Let $(S, \mathbf{p})$ be a local ring and I an ideal. Let*

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_j \xrightarrow{d_j} F_{j-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} S \longrightarrow S/I \longrightarrow 0$$

be a minimal free resolution for S/I over S. Let $K_j = d_j(F_j) =$ kernel d_{j-1}. If, for each $j \geqslant 1$, there is a positive integer q_j such that $\mathbf{p}^n K_j = K_j \cap \mathbf{p}^{q_j+n} F_{j-1}$ for all $n \geqslant 0$, then the F_j can be $\mathbf{p}$-filtered so that

$$0 \longrightarrow grF_r \longrightarrow \cdots \longrightarrow grF_j \xrightarrow{grd_j} grF_{j-1} \longrightarrow \cdots \longrightarrow grF_1 \xrightarrow{grd_1} grS \longrightarrow grS/I \longrightarrow 0$$

is a pure minimal free resolution of grS/I over grS, where $grF_j \cong grS(-q_1 - \cdots - q_j)$. Thus, in particular, the initial forms of a minimal generating set for I minimally generate grI.

Proof. We filter the F_j to make d_j strict. Filter F_1 as follows: $(F_1)_n = \mathbf{p}^{n-q_1} F_1$, where $\mathbf{p}^j = S$ for $j \leqslant 0$. Then the generators of F_1 have order q_1 and the map $d_1 : F_1 \to S$ is strict since

$$d_1((F_1)_n) = d_1(\mathbf{p}^{n-q_1} F_1) = p^{n-q_1} d_1 F_1 = \mathbf{p}^{n-q_1} I = I \cap \mathbf{p}^n = d_1 F_1 \cap \mathbf{p}^n, \quad \text{for all } n \geqslant 0.$$

Suppose that

$$grF_j \xrightarrow{grd_j} grF_{j-1} \longrightarrow \cdots \longrightarrow grS \longrightarrow grS/I \longrightarrow 0$$

is exact and F_j has the filtration: $(F_j)_n = \mathbf{p}^{n-q_1-\cdots-q_j} F_j$. Filter F_{j+1} as follows: $(F_{j+1})_n = \mathbf{p}^{n-q_1-\cdots-q_{j+1}} F_{j+1}$. The generators of F_{j+1} have degree $q_1 + \cdots + q_{j+1}$ and the map d_{j+1} is strict since

$$\begin{aligned} d_{j+1}((F_{j+1})_n) &= d_{j+1}(\mathbf{p}^{n-q_1-\cdots-q_{j+1}} F_{j+1}) = \mathbf{p}^{n-q_1-\cdots-q_{j+1}} K_{j+1} \\ &= K_{j+1} \cap \mathbf{p}^{n-q_1-\cdots-q_j} F_j \\ &= K_{j+1} \cap (F_j)_n = d_{j+1} F_{j+1} \cap (F_j)_n \qquad \text{for} \quad n \geqslant 0. \end{aligned}$$

On the other hand, in [18], Wahl shows that a pure minimal resolution of grS/I over grS can be lifted to a minimal resolution of S/I over S.

(2.5) LEMMA [18]. *Let $(S, \mathbf{p})$ be a local ring and $E \subseteq F$ finitely generated S-modules. Give F the $\mathbf{p}$-adic filtration and E the induced filtration: $E_n = E \cap \mathbf{p}^n F$ so that $grE \subseteq grF$. If grE is generated by homogeneous elements of degree q, then*

$E \cap \mathbf{p}^{n+q} = \mathbf{p}^n E$ for all $n \geqslant 0$. $e_1, \ldots, e_t$ minimally generate E if and only if $\bar{e}_1, \ldots, \bar{e}_t$ minimally generate grE.

(2.6) Corollary [18]. *Let $(S, \mathbf{p})$ be a local ring and let I be an ideal of S such that S/I has homological dimension r over S. If*

$$\bar{\mathscr{F}}: 0 \to \bar{F}_r \to \cdots \to \bar{F}_j \xrightarrow{\bar{d}_j} \bar{F}_{j-1} \to \cdots \to \bar{F}_1 \to grS \to grS/I \to 0$$

is a pure minimal free resolution for grS/I over grS with, say, $\bar{F}_j = grS(-a_j)$, then

$$\mathscr{F}: 0 \to F_r \to \cdots \to F_j \xrightarrow{d_j} F_{j-1} \to \cdots \to F_1 \to S \to S/I \to 0$$

is a minimal free resolution for S/I over S, where F_j has the filtration $(F_j)_n = \mathbf{p}^{n-a_j}F_j$ and $grF_j = \bar{F}_j$ and $grd_j = \bar{d}_j$ for $j = 1, \ldots, r$, i.e., $gr(\mathscr{F}) = \bar{\mathscr{F}}$.

(2.7) Corollary. *Let $(S, \mathbf{p})$ be a local ring and I an ideal of S such that S/I has homological dimension r over S. If*

$$\bar{\mathscr{F}}: 0 \to \bar{F}_r \to \cdots \to \bar{F}_j \xrightarrow{\bar{d}_j} \bar{F}_{j-1} \to \cdots \to \bar{F}_1 \to grS \to grS/I \to 0$$

is a minimal free resolution for grS/I over grS with $b_j = \operatorname{rank} \bar{F}_j$, and if there are positive integers $a_1 < \cdots < a_r$ such that the Hilbert function $H_{grS/I}(n)$ satisfies (2.2), then $\bar{\mathscr{F}}$ is a pure minimal resolution for grS/I over grS so that the conclusion of (2.6) holds.

Proof. This follows from the remark following (2.2).

Since we will be working with Cohen–Macaulay local rings it is often possible to reduce questions to the case of dimension zero by dividing by a maximal resular sequence. For later reference we state two results which allow us to do this.

(2.8) Proposition [5]. *Let $(R, \mathbf{m})$ be a local ring and x a nonzero divisor in $\mathbf{m} \setminus \mathbf{m}^2$. Let $R^* = R/xR$. Then*

$$P_{R,k}(t) = (1 + t)\, P_{R^*,k}(t).$$

(2.9) Proposition [2]. *Let $(S, \mathbf{p})$ be a local ring and I an ideal. Let x be a nonzero divisor in $\mathbf{p}$, $S^* = S/xS$ and $I^* = (I, x)/xS$. If*

$$\mathscr{F}^*: 0 \to F_r^* \to \cdots \to F_j^* \xrightarrow{d_j^*} F_{j-1}^* \to \cdots \to F_1^* \to S^* \to S^*/I^* \to 0$$

is a minimal free resolution of S^*/I^* over S^*, then $\mathscr{F}^*$ can be lifted to a minimal free resolution

$$\mathscr{F}: 0 \to F_r \to \cdots \to F_j \xrightarrow{d_j} F_{j-1} \to \cdots \to F_1 \to S \to S/I \to 0$$

of S/I over S where $F_j^* = F_j \otimes S^*$ and $d_j^* = d_j \otimes S^*$.

Remark. The same result holds if S is a graded ring with unique maximal homogeneous ideal $\mathbf{p}$, and I and x are homogeneous.

The proofs of Theorems 1 and 2 depend upon the relationship between embedding dimension and minimal reductions of the maximal ideal. Proofs of the facts which are given below can be found in [11, 16, 17].

Let $(R, \mathbf{m})$ be a d-dimensional Cohen–Macaulay local ring with $R/\mathbf{m}$ infinite. Then grR has a system of parameters consiting of elements of degree 1; in other words, there are elements $x_1, \ldots, x_d$ in R such that $\mathbf{m}^{r+1} = (x_1, \ldots, x_d)\mathbf{m}^r$ for some $r \geqslant 0$. Such a system of elements is called a minimal reduction for $\mathbf{m}$. We will use the notation $\mathbf{x}$ to stand either for the sequence of elements $x_1, \ldots, x_d$ or for the ideal generated by $x_1, \ldots, x_d$. The context will make it clear.

If the embedding dimension of R is $e + d - 1$ then there are elements $\mathbf{x} = x_1, \ldots, x_d$ such that $\mathbf{m}^2 = \mathbf{xm}$. If R is Gorenstein of embedding dimension $e + d - 2$ then there are elements $\mathbf{x} = x_1, \ldots, x_d$ such that $\mathbf{m}^3 = \mathbf{xm}^2$. In both cases the initial forms $\bar{x}_1, \ldots, \bar{x}_d$ in grR form a regular sequence so that $gr(R/(x_1, \ldots, x_d)R) \cong grR/(\bar{x}_1, \ldots, \bar{x}_d)\, grR$. This means that the Hilbert function for R can be computed from the Hilbert function for $R^* = R/\mathbf{x}R$ as follows. Let $H_R{}^0(n) = H_R(n) = \dim_k \mathbf{m}/\mathbf{m}^2$. Define $H_R{}^j(n) = \sum_{i=0}^{n} H^{j-1}(i)$. Then, with the above hypotheses,

$$H_R(n) = H^d_{R^*}(n) \qquad \text{for all} \quad n \geqslant 0.$$

(2.10) In the case where R is Cohen–Macaulay of embedding dimension $e + d - 1$, grR is Cohen–Macaulay and $H_{R^*}(1) = e - 1$ and $H_{R^*}(n) = 0$ for $n > 1$.

(2.11) In the case where R is Gorenstein of embedding dimension $e + d - 2$, grR is Gorenstein and $H_{R^*}(0) = 1$, $H_{R^*}(1) = e - 2$, $H_{R^*}(2) = 1$ and $H_{R^*}(n) = 0$ for $n > 2$.

3. Cohen-Macaulay Local Rings of Embedding Dimension $e + d - 1$

Let $(R, \mathbf{m})$ be a d-dimensional local Cohen–Macaulay ring of embedding dimension $v = e + d - 1$. For the proof of Theorem 1 we may assume that $(R, \mathbf{m})$ has infinite residue field $R/\mathbf{m} = k$. For if u is an indeterminate, the ring $R(u) = R[u]_{\mathbf{m}R[n]}$ is faithfully flat over R. Consequently, the dimensions of the vector spaces to be computed for Theorem 1 remain the same under the change of rings $R \to R(u)$.

The proof of Theorem 1 follows from the fact that with $R/\mathbf{m}$ infinite, $v = e + d - 1$ implies that there are d elements $\mathbf{x} = x_1, \ldots, x_d$ in R with $\mathbf{m}^2 = \mathbf{xm}$.

Proof of Theorem 1. Let $R^* = R/\mathbf{x}R$ and $\mathbf{m}^* = \mathbf{m}/\mathbf{x}R$. R^* is a zero dimensional local ring of embedding dimension $e - 1$.

(i) By (2.10) we have

$$H_{R^*}(n) = H^0_{R^*}(n) = \begin{cases} 1 & \text{if } n = 0 \\ e - 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

and $H_R(n) = H^d_{R^*}(n)$. Using the relation $H^j_{R^*}(n) = H^j_{R^*}(n-1) + H^{j-1}_{R^*}(n)$, we may use double induction on n and d to get

$$H_R(n) = H^d_{R^*}(n) = \binom{n+d-2}{n-1} e + \binom{n+d-2}{n}$$

$$\text{for all} \quad n \geqslant 0, \quad \text{if } d > 0.$$

(ii) As R is Cohen–Macaulay we may apply (2.8) to get $P_{R,k}(t) = (1+t)^d P_{R^*,k}(t)$. Since $(\mathbf{m}^*)^2 = 0$, $\mathbf{m}^*$ is a vector space over k and, as is well-known, $P_{R^*,k}(t) = \sum_{i=0}^{\infty}(e-1)^i$. For

$$\operatorname{Tor}_i^{R^*}(k, k) \cong \operatorname{Tor}_{i-1}^{R^*}(\mathbf{m}^*, k) \cong \operatorname{Tor}_{i-1}^{R^*}(k^{e-1}\ k) \cong \operatorname{Tor}_{i-1}^{R^*}(k, k)^{e-1}, \quad \text{for } i \geqslant 1.$$

Explicitly we have,

$$\dim \operatorname{Tor}_i{}^R(k, k) = \sum_{j=0}^{i} (e-1)^{i-j} \binom{d}{j} \tag{3.2}$$

and

$$\dim \operatorname{Tor}_i{}^R(k, k) = \sum_{j=0}^{d} (e-1)^{i-j} \binom{d}{j} \qquad \text{for} \quad i > d.$$

(iii) Now assume that $R = S/I$ with $(S, \mathbf{p})$ a regular local ring and $I \subsetneq \mathbf{p}^2$. The elements $x_1, \ldots, x_d$ with $\mathbf{m}^2 = \mathbf{x}\mathbf{m}$ are images of elements $\tilde{x}_1, \ldots, \tilde{x}_d$ in S which are part of a minimal basis for $\mathbf{p}$. By (2.9) a minimal resolution for R over S can be obtained by lifting a minimal resolution for $R^* = R/\mathbf{x}R$ over $S^* = S/\mathbf{x}S$. But $R^* = S^*/(\mathbf{p}^*)^2$. $\mathbf{p}^{*2}$ can be realized as the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} y_1^* & & \cdots & y_{e-1}^* & 0 \\ 0 & y_1^* & \cdots & y_{e-2}^* & y_{e-1}^* \end{pmatrix}$$

where $\mathbf{p}^* = (y_0^*, \ldots, y_{e-1}^*)$. Thus the Eagon–Northcott complex [4] gives a minimal resolution for R^* over S^*. In this case the Eagon–Northcott complex has the form

$$\mathscr{F}: 0 \to F_{e-1} \to F_{e-2} \to \cdots \to F_1 \to S^* \to S^*/(\mathbf{p}^*)^2 \to 0$$

where $F_i = K_{i+1} \otimes \Phi_{i-1}$ for $i = 1, \ldots, e-1$ with $\mathscr{K} = (K_i)$ the exterior S^*-algebra on $Z_1, \ldots, Z_e$ and $\Phi = (\Phi_i)$ the symmetric S^*-algebra on $Y_1, Y_2,$

and differential determined by the minors of the matrix. Thus F_i is a free S^*-module on $i\binom{e}{i+1}$ generators for $i = 1,\ldots, e-1$. It follows that $\operatorname{Tor}_i^S(R, k) = 0$ for $i > e-1$ and $\dim \operatorname{Tor}_i^S(R, k) = i\binom{e}{i+1}$ for $i = 1,\ldots, e-1$. This concludes the proof of Theorem 1.

If $R = S/I$ is as in Theorem 1(iii), we have by Theorem 1 that I is generated by $\binom{e}{2}$ elements. The following lemma shows that these generators all have order 2.

(3.4) LEMMA. *Let $(S, \mathbf{p})$ be a local ring and I an ideal contained in $\mathbf{p}^r$ for some $r > 0$. Let $\mathbf{x} = x_1,\ldots, x_t$ be elements in $\mathbf{p}$ which form a regular sequence* mod I. *If $\mathbf{p}^r \subseteq \mathbf{x}\mathbf{p}^{r-1} + I$, then $\mathbf{p}^j I = I \cap \mathbf{p}^{r+j}$ for $j \geqslant 0$.*

Proof. By induction on j, it follows that $\mathbf{p}^{r+j} = \mathbf{x}^{j+1}\mathbf{p}^{r-1} + \mathbf{p}^j I$. Thus, $I \cap \mathbf{p}^{r+j} = I \cap \mathbf{x}^{j+1}\mathbf{p}^{r-1} + \mathbf{p}^j I = \mathbf{p}^j I + I \cap \mathbf{x}^{j+1}\mathbf{p}^{r-1} \subseteq \mathbf{p}^j I + I \cap \mathbf{x}^{j+1} S = \mathbf{p}^j I + \mathbf{x}^{j+1} I \subseteq \mathbf{p}^j I$.

Remark. There is an analogous formulation of the above lemma for a homogeneous ideal I in a graded ring S with unique maximal homogeneous ideal $\mathbf{p}$. Observe that we do need some hypotheses on S/I. For example, let $I = (XY - Y^2, Y^3)$ in $k[[X, Y]]$, k a field. Then $(X, Y)^2 \subseteq (X)(X, Y) + I$ but $Y^3 \in I \cap (X, Y)^3 \backslash (X, Y)I$. Note that $XY^2 = Y(XY - Y^2) - Y^3 \in I$.

(3.5) COROLLARY. *Let $(R, \mathbf{m})$ be a d-dimensional local Cohen–Macaulay ring of embedding dimension $e + d - 1$. Assume that $R = S/I$ with $(S, \mathbf{p})$ a regular local ring and $I \subseteq \mathbf{p}^2$. Then I is generated by $\binom{e}{2}$ elements of order 2 in S. $grR = grS/grI$ where grI is generated by the initial forms of a minimal generating set for I.*

Proof. The first statement follows from (3.3) and (3.4) since we may assume that $R/\mathbf{m}$ is infinite to count the number of elements in a minimal basis for I. The second statement follows from (3.4) and (2.3) but it is even easier to notice that with $R^* = R/\mathbf{x}R$ as in the proof of Theorem 1, $grR^* \simeq grR/(\bar{x}_1,\ldots, \bar{x}_d)\, grR$ and grR^* is a zero dimensional local ring of maximal embedding dimension $e - 1$. Applying Theorem 1 and the first statement of this Corollary to grR^* we see that $(grI, \bar{x}_1,\ldots, \bar{x}_d)/(\bar{x}_1,\ldots, \bar{x}_d) \simeq grI/(\bar{x}_1,\ldots, \bar{x}_d)\, grI$ is generated by $\binom{e}{2}$ homogeneous forms of degree 2.

(3.4) begins the process of grading the resolution of $R = S/I$ over S by the method described in (2.4) but to carry this further we would need more information about the differential in the Eagon–Northcott resolution. Instead we use the method of (2.6) or (2.7).

(3.6) PROPOSITION. *Let $(R, \mathbf{m})$ be a d-dimension local Cohen–Macaulay ring of embedding dimension $e + d - 1$. Assume that $R = S/I$ with $(S, \mathbf{p})$ a regular local ring and $I \subseteq \mathbf{p}^2$. Then the ith syzygies, $1 < i \leqslant e - 1$, in a minimal resolution for R over S are generated by $i\binom{e}{i+1}$ elements of order 1. A minimal resolution for R* *over S can be graded to give a minimal associated graded resolution for grR over grS.*

Proof. We may assume that $R/\mathfrak{m}$ is infinite. Let $S^* = S/\mathbf{x}S$, $R^* = R/\mathbf{x}R$ and $I^* = (I, \mathbf{x})/\mathbf{x}S$, where $\mathbf{x} = \tilde{x}_1, \ldots, \tilde{x}_d$ are elements of S mapping to the minimal reduction $\mathbf{x}$ of $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathbf{x}\mathfrak{m}$. We look at a minimal resolution $\bar{\mathscr{F}}$ for grR over grS. By the graded version of (2.9), such a resolution can be lifted from a minimal resolution $\bar{\mathscr{F}}^*$ for grR^* over grS^*. Now grR^* is a 0-dimensional local ring of embedding dimension $e - 1$, hence, as we have seen, the Eagon–Northcott complex is a minimal resolution for grR^* over grS^*. That this resolution is pure can be seen directly by looking at the degrees of the generators, or, using the method of (2.7), since we know the Hilbert function $H_{grR^*}(n)$, by checking that relation (2.2) holds with $a_i = i + 1$ for $i = 1, \ldots, e - 1$. The degree of the generators of the ith syzygies in $\bar{\mathscr{F}}$ is $a_i - a_{i-1}$ for $i = 2, \ldots, e - 1$.

4. Gorenstein Local Rings of Embedding Dimension $e + d - 2$

The Gorenstein local rings of embedding dimension $e + d - 1$ have multiplicity at most 2 and consequently $v = d$ or $d + 1$ [17]. Thus the maximal embedding dimension of a Gorenstein local ring R of multiplicity $e > 2$ is $e + d - 2$. If R has multiplicity 3 then $v = d + 1$ and $\hat{R}$, the completion of R, is a hypersurface. Theorem 2 gives the structure of Gorenstein local rings of embedding dimension $v = e + d - 2$ and multiplicity $e > 3$.

As for the proof of Theorem 1 we may assume that $R/\mathfrak{m}$ is infinite. The proof of Theorem 2 is based on the fact that a d-dimensional local Gorenstein ring $(R, \mathfrak{m})$ with $R/\mathfrak{m}$ infinite and embedding dimension $e + d - 2$ has a minimal reduction $\mathbf{x} = x_1, \ldots, x_d$ for $\mathfrak{m}$ with $\mathfrak{m}^3 = \mathbf{x}\mathfrak{m}^2$. Let $R^* = R/\mathbf{x}R$ and $\mathfrak{m}^* = \mathfrak{m}/\mathbf{x}R$. R^* is a zero dimensional local Gorenstein ring of embedding dimension $e - 2$. Parts of the proof of Theorem 2 are more involved than the proof of Theorem 1, so we handle each part separately.

Proof of Theorem 2(i). (i) of Theorem 2 follows immediately from (2.11). We have $H_R(n) = H^d_{R^*}(n)$ and

$$H^0_{R^*}(n) = \begin{cases} 1 & \text{if } n = 0 \\ e - 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n > 2. \end{cases}$$

It follows, by double induction on n and d, that if $d > 0$,

$$H_R(n) = \binom{n + d - 2}{n - 1} e + \binom{n + d - 3}{n} \qquad \text{for} \quad n \geqslant 2.$$

(ii) of Theorem 2 follows from a recent result of Avramov and Levin [8] and of Rahbar-Rochandel [13] on the computation of the Betti numbers $\dim \mathrm{Tor}_i^R(k, k)$ for a zero dimensional local Gorenstein ring R under the change of rings $R \to R_{\mathrm{soc}}$, where $R_{\mathrm{soc}} = R/\mathrm{socle}\, R$, and socle R is the annihilator of $\mathbf{m}$.

(4.1) THEOREM [8, 13]. *Let $(R, \mathbf{m})$ be a zero dimensional local Gorenstein ring with residue field k and embedding dimension greater than one. Then,*

$$P_{R_{\mathrm{soc}},k}(t) = P_{R,k}(t) + t^2 P_{R,k}(t)\, P_{R_{\mathrm{soc}},k}(t).$$

Proof of Theorem 2(ii). We use (2.8) to reduce the computation to that of $P_{R^*,k}(t)$. We have $(\mathbf{m}^*)^2 = \mathrm{socle}\, R^* = sR^*$ for some s in $(\mathbf{m}^*)^2$. $R^*_{\mathrm{soc}} = R^*/sR^*$ is a zero dimensional local ring of embedding dimension $e - 2$ with maximal ideal of square zero, i.e., R^*_{soc} is a zero dimensional local ring of maximal embedding dimension $e(R^*_{\mathrm{soc}}) - 1$ as $e(R^*_{\mathrm{soc}}) = e - 1$. Hence, as in the proof of of Theorem 1(ii), $P_{R^*_{\mathrm{soc}},k}(t) = \sum_{i=0}^{\infty}(e-2)^i t^i$. Thus, by (4.1),

$$P_{R^*,k}(t) = \sum_{i=0}^{\infty} (e-2)^i t^i / 1 + t^2 \sum_{i=0}^{\infty} (e-2)^i t^i$$

and

$$P_{R,k}(t) = (1+t)^d \sum_{i=0}^{\infty} (e-2)^i t^i / 1 + \sum_{i=0}^{\infty} (e-2)^i t^{i+2}.$$

Remark. It is well-known that the homology of the Koszul complex $K(\mathbf{y})$ on the generators $\mathbf{y} = y_1, \ldots, y_n$ of the maximal ideal of a zero dimensional local Gorenstein ring $(R, \mathbf{m})$ is a Poincaré algebra, i.e., $H_i(K(\mathbf{y})) \cdot H_{n-i}(K(\mathbf{y})) = H_n(K(\mathbf{y})) \cong \mathrm{socle}\, R$. If R has embedding dimension $e - 2$, this is the only non-trivial product, i.e., $H_i(K(\mathbf{y})) \cdot H_j(K(\mathbf{y})) = 0$ for $i + j < n$.

To prove Theorem 2(iii) we will use Wahl's method, cf. (2.6), for lifting resolutions from grR to R. To begin we need to know that grI is generated by elements of degree 2.

(4.2) LEMMA. *Let $(S, \mathbf{p})$ be a local ring of embedding dimension q. Let I be an ideal properly contained in $\mathbf{p}^2$ with S/I Gorenstein. Let $\mathbf{x} = x_1, \ldots, x_t$, $t < q - 1$, be a subset of a minimal basis for $\mathbf{p}$ such that $\mathbf{x}$ is a maximal regular sequence* mod I. *If $\mathbf{p}^3 \subseteq \mathbf{x}\mathbf{p}^2 + I$, then $\mathbf{p}^3 = \mathbf{x}\mathbf{p}^2 + \mathbf{p}I$.*

Proof. Let $x_1, \ldots, x_t, y_1, \ldots, y_r$, $r > 1$, minimally generate $\mathbf{p}$. The Gorenstein hypothesis provides an element s in $\mathbf{p}^2 \backslash \mathbf{p}^3$ such that $\mathbf{p}^2 = (s, \mathbf{x}\mathbf{p}, I)$. For we have $(I, \mathbf{x}) : \mathbf{p} = (s, \mathbf{x}, I)$ for some $s \in \mathbf{p}^2 \backslash \mathbf{p}^3$. But $\mathbf{p}^2 \cdot \mathbf{p} \subseteq (I, \mathbf{x})$ so we have $\mathbf{p}^2 \subseteq (s, \mathbf{x}, I)$ and, by the hypothesis on $\mathbf{x}$, we have $\mathbf{p}^2 = (s, \mathbf{x}\mathbf{p}, I)$. We may assume that $s = y_1^2$ or $s = y_1 y_2$. We need to show that $s\mathbf{p} \subseteq \mathbf{x}\mathbf{p}^2 + \mathbf{p}I$. We have that

$s\mathbf{p} \subseteq \mathbf{xp} + I$. Note that we may assume that $y_1 y_j \in \mathbf{xp} + I$ for $j \neq 1, 2$. For $y_1 y_j = cs + \eta + i$ with $\eta \in \mathbf{xp}$ and $i \in I$. If $c \in \mathbf{p}$, $y_1 y_j \in \mathbf{xp} + I$. If $c \notin \mathbf{p}$, then $y_1(y_j - cy_h) \in \mathbf{xp} + I$, where $h = 1$ or 2, and we may replace y_j by $y_j - cy_h$. Consequently, $sy_j \in \mathbf{xp}^2 + \mathbf{p}I$ for $j \neq 1, 2$.

Case 1. $s = y_1^2$. For this case we may assume, just as above, in addition that $y_1 y_2 \in \mathbf{xp} + I$ so that $sy_j = y_1^2 y_j \in \mathbf{xp} + \mathbf{p}I$ for $j \neq 1$. Now there is some l such that $y_2 y_l = cs + \eta + i$ with $\eta \in \mathbf{xp}$, $i \in I$ and $c \notin \mathbf{p}$. Otherwise $y_2\mathbf{p} \subseteq \mathbf{xp} + I$ so $y_2 \in (s, \mathbf{x}, I)$, a contradiction. Thus $y_2 y_l = cs + \eta + i$ with $c \notin \mathbf{p}$ and $sy_1 = y_1^3 = c^{-1} y_2 y_l y_1 - c^{-1}\eta y_1 - c^{-1} i y_1 \in \mathbf{xp} + \mathbf{p}I$.

Case 2. $s = y_1 y_2$. We assume, by Case 1, that $y_1^2, \ldots, y_r^2 \in \mathbf{xp} + I$. Thus $sy_1 = y_1^2 y_2 \in \mathbf{xp}^2 + \mathbf{p}I$ and $sy_2 = y_1 y_2^2 \in \mathbf{xp}^2 + \mathbf{p}I$.

(4.3) COROLLARY. *With the hypotheses as in* (4.2),

$$\mathbf{p}^j I = I \cap \mathbf{p}^{j+2} \qquad \textit{for all} \quad j \geqslant 0.$$

Proof. From (4.2), by induction on j, we have $\mathbf{p}^{j+2} = \mathbf{x}^j\mathbf{p}^2 + \mathbf{p}^j I$. So $I \cap \mathbf{p}^{j+2} = I \cap \mathbf{x}^j\mathbf{p}^2 + \mathbf{p}^j I = \mathbf{p}^j I + I \cap \mathbf{x}^j \mathbf{p}^2 \subseteq \mathbf{p}^j I + I \cap \mathbf{x}^j S = \mathbf{p}^j I + \mathbf{x}^j I \subseteq \mathbf{p}^j I$.

Proof of Theorem 2(iii). Let $\tilde{\mathbf{x}} = \tilde{x}_1, \ldots, \tilde{x}_d$ be elements of S mapping onto $\mathbf{x} = x_1, \ldots, x_d$ in R. Let $S^* = S/\tilde{\mathbf{x}}S$ and, as usual, $R^* = R/\mathbf{x}R$. Let

$$\mathscr{F}: 0 \to \bar{F}_{e-2} \to \cdots \to \bar{F}_j \xrightarrow{\bar{d}_j} \bar{F}_{j-1} \to \cdots \to \bar{F}_1 \to grS^* \to grR^* \to 0$$

be a minimal free graded resolution of the graded (local) ring grR^*. Since grR^* is Gorenstein, $\bar{F}_{e-2} \cong grS^*$. As Stanley observes in [15, Thm. 4.1] a result of Buchsbaum and Eisenbud [3] shows that there is a degree preserving pairing $\bar{F}_i \otimes \bar{F}_{e-2-i} \to \bar{F}_{e-2} \cong grS^*$ which induces an isomorphism

$$\bar{F}_i \cong \operatorname{Hom}_{grS^*}(\bar{F}_{e-2-i}, grS^*).$$

By (4.3) the generators of $\bar{F}_1$ are all of degree 2. This means that the degree g of the generator of $\bar{F}_{e-2}$ is at least e. Assume that $g = e$. From $\bar{F}_1 \otimes \bar{F}_{e-3} \to \bar{F}_{e-2}$ it follows that $\bar{F}_{e-3}$ has all of its generators in degree $e - 2$. Thus we have $\bar{F}_j \cong grS^*(-j-1)^{b_j}$ for $1 \leqslant j < e-2$ and $\bar{F}_{e-2} \cong grS^*(-e)$. Thus to conclude that grR^* has a pure resolution over grS^* it remains to show that $g = e$.

(4.4) LEMMA (cf. Proof of Thm. 4.1 in [15]). *Let $\bar{R} = \bar{S}/\bar{I}$ be a zero dimensional graded Gorenstein ring which is a quotient modulo homogeneous ideal $\bar{I}$ of a polynomial ring $\bar{S} = k[X_1, \ldots, X_q]$ over a field k. Let*

$$0 \to \bar{F}_q \to \bar{F}_{q-1} \to \cdots \to \bar{F}_1 \to \bar{S} \to \bar{R} \to 0$$

be a minimal resolution of $\bar{R}$. Then the degree of the generator of $\bar{F}_q$ is $c + q$, where $\bar{R}_c$ is the socle of $\bar{R}$.

Proof. Let $\bar{R} = \bar{R}_0 \oplus \bar{R}_1 \oplus \cdots \oplus \bar{R}_c$ with $\bar{R}_0 = k$. Since $\bar{R}$ is Gorenstein,

$$t^c \sum_{i=0}^{c} \dim_k \bar{R}_i(1/t)^i = \sum_{i=0}^{c} \dim_k \bar{R}_i t^i.$$

By (2.1) we have

$$\mathscr{X}(t) = \sum_{i=0}^{c} \dim_k \bar{R}_i t^i = (1-t)^{-q} - (t^{a_{11}} + \cdots + t^{a_{1b_1}})(1-t)^{-q} + \cdots$$
$$+ (-1)^q t^{a_{q1}}(1-t)^{-q},$$

where $a_{i1}, \ldots, a_{ib_i}$ are the degrees of the generators of $\bar{F}_i$. We have $\mathscr{X}(t) = t^c\mathscr{X}(1/t)$. But

$$t^c\mathscr{X}(1/t) = t^c(1-(1/t))^{-q} - t^c(1/t^{a_{11}} + \cdots + 1/t^{a_{1b_1}})(1-(1/t))^{-q} + \cdots$$
$$+ (-1)^q t^c(1/t^{a_{q1}})(1-(1/t))^{-q}$$
$$= (-1)^q t^{c+q}(1-t)^{-q} - (-1)^q t^{c+q}(1/t^{a_{11}} + \cdots + 1/t^{a_{1b_1}})(1-t)^{-q} + \cdots$$
$$+ (-1)^q (-1)^q t^{c+q}(1/t^{a_{q1}})(1-t)^{-q}.$$

The fact that the pairing $\bar{F}_i \otimes \bar{F}_{q-i} \to \bar{F}_q$ is degree presering means that $a_{jl} < a_{q1}$ for all $j < q$ and $1 \leqslant l \leqslant b_j$. Thus the last term in the expression of $t^c\mathscr{X}(1/t)$ above must be equal to $(1-t)^{-q}$, that is, $a_{q1} = c + q$.

Thus we have that grR^* has a pure resolution over grS^* and we can compute the Betti numbers of the resolution from (2.2). We have $q = e - 2 = r$ and $a_0 = 0$, $a_j = j + 1$ for $1 \leqslant j < 2$ and $a_{e-2} = e$. Explicitly,

$$(1-t)^{e-2}(1 + (e-2)t + t^2) = 1 - b_1t^2 + \cdots + (-1)^j b_j t^{j+1} + \cdots + (-1)^{e-2}t^e.$$

Comparing coefficients of t^{i+1} we get, for $1 \leqslant i < e - 2$

$$b_i = (e-2)\binom{e-2}{i} - \binom{e-2}{i+1} - \binom{e-2}{i-1}$$
$$= (e-2)\binom{e-2}{i} - \left[\binom{e-1}{i+1} + \binom{e-1}{i} - 2\binom{e-2}{i}\right]$$
$$= e\binom{e-2}{i} - \binom{e}{i+1} = i\binom{e}{i+1}\frac{e-i-2}{e-1}.$$

This completes the proof of Theorem 2 and the Corollary which follows the statement of Theorem 2 in Section 1. We have also proved the following.

(4.5) COROLLARY. *Let* $(R, \mathbf{m})$ *be a d-dimensional local Gorenstein ring of embedding dimension* $e + d - 2$, $e > 3$. *Assume that* $R = S/I$ *with* $(S, \mathbf{p})$ *a regular local ring and* $I \subset \mathbf{p}^2$. *Then the ith syzygies in a minimal resolution for R over S are generated by* $i\binom{e}{i+1}(e - i - 2)/(e - 1)$ *elements of order* 1 *if* $1 < i < e - 2$. *The first and last syzygies are generated respectively by* $e(e - 3)/2$ *and* 1 *elements of order* 2.

As is well known, the socle of a zero dimensional local Gorenstein ring $(R, \mathbf{m})$ of embedding dimension $e - 2$ determines a non-degenerate symmetric bilinear form (,) on the k-vector space $V = \mathbf{m}/\mathbf{m}^2$. If $\bar{x}, \bar{y} \in \mathbf{m}/\mathbf{m}^2$, define $(\bar{x}, \bar{y}) = \bar{r}$, where $\bar{r}$ is the image in k of the element r given by the product $xy = rs$ where socle $R = sR$. Thus grR has the form $grR = k \oplus V \oplus ks$ as in Example 6 in Section 1. Hence, if $k = R/\mathbf{m}$ has characteristic $\neq 2$, we can find a basis for $\mathbf{m}/\mathbf{m}^2$ so that the matrix corresponding to (,) is diagonal and there are constants $c_{22}, \ldots, c_{e-2,e-2}$ in k so that $grR = k[X_1, \ldots, X_{e-2}]/\mathscr{X}$, with $\mathscr{X} = (X_iX_j; i \neq j,\ X_1^2 - c_{22}X_2^2, \ldots, X_1^2 - c_{e-2,e-2}X_{e-2}^2)$.

5. QUADRATIC TRANSFORMS

The quadratic transform of a local ring $(A, \mathbf{q})$ is the morphism

$$\operatorname{Proj} \bigoplus_{j=0}^{\infty} \mathbf{q}^j \xrightarrow{\varphi} \operatorname{Spec} A$$

obtained from the embedding $A \to \bigoplus_{j=0}^{\infty} \mathbf{q}^j$. Proj $\bigoplus_{j=0}^{\infty} \mathbf{q}^j$ is covered by the affine open sets Spec $A[\mathbf{q}/z_i]$, $i = 1, \ldots, n$, where $z_1, \ldots, z_n$ are a subset of a minimal basis for $\mathbf{q}$ which generate a $\mathbf{q}$-primary ideal. The morphism φ is an isomorphism except at $\mathbf{q}$. The fiber over $\mathbf{q}$ is Proj(grA). Hence it follows—and is part of the folklore of quadratic transforms—that the properties of Proj(grA) determine some of the properties of Proj $\bigoplus_{j=0}^{\infty} \mathbf{q}^j$. Algebraic proof of this fact, where the property in question is "being Cohen–Macaulay," can be found in [7, 12]. The same proof works for the property of "being Gorenstein."

(5.1) THEOREM (cf. [7, Thm. 4.11]). *Let* $(R, \mathbf{m})$ *be a local Cohen–Macaulay (resp. Gorenstein) ring of dimension* $d > 0$. *If grR is Cohen–Macaulay (resp. Gorenstein) then* Proj $\bigoplus_{j=0}^{\infty} \mathbf{m}^j$ *is Cohen–Macaulay (resp. Gorenstein).*

Proof. By the remarks above, (5.1) will follow if we show that grR Cohen–Macaulay (resp. Gorenstein) implies that the rings $R[\mathbf{m}/b]$ are Cohen–Macaulay (resp. Gorenstein) for any nonzero divisor $b \in \mathbf{m}$. Let $\mathscr{R} = R[\mathbf{m}t, 1/t]$. Then, [14], $\mathscr{R}/(1/t)\mathscr{R} \cong grR$, and grR is Cohen–Macaulay (resp. Gorenstein) if and only if $\mathscr{R}$ is Cohen–Macaulay (resp. Gorenstein) since $1/t$ is a nonzero divisor in the maximal homogeneous ideal of $\mathscr{R}$. (Here we use the fact that $\mathscr{R}$ is Cohen–Macaulay (resp. Gorenstein) if and only if $\mathscr{R}_{\mathscr{M}}$ is, where $\mathscr{M}$ is the maximal homogeneous ideal, $\mathscr{M} = (\mathbf{m}t, 1/t)\mathscr{R}$. But $\mathscr{R}[1/bt] = R[\mathbf{m}/b, 1/bt]$ is Cohen–Macaulay (resp. Gorenstein). Since b is a nonzero divisor in $\mathbf{m}$, bt is transcendental over $R[\mathbf{m}/b]$, and we have that $R[\mathbf{m}/b]$ is Cohen–Macaulay (resp. Gorenstein).

(5.2) COROLLARY. *If $(R, \mathbf{m})$ is a Cohen–Macaulay local ring of dimension $d > 0$ and embedding dimension $v = e + d - 1$, then* Proj $\bigoplus_{j=0}^{\infty} \mathbf{m}^j$ *is Cohen–Macaulay. If $(S, \mathbf{p})$ is a local Gorenstein ring of dimension $d > 0$ and embedding dimension $v = e + d - 2$, then* Proj $\bigoplus_{j=0}^{\infty} \mathbf{p}^j$ *is Gorenstein.*

Remarks. Note that if grR has a system of parameters $\bar{x}_1, \ldots, \bar{x}_d$ of degree 1, i.e., $x_1, \ldots, x_d$ is a minimal reduction for $\mathbf{m}$, then the d affine open sets Spec $R[\mathbf{m}/x_i]$ cover Proj $\bigoplus_{j=0}^{\infty} \mathbf{m}^j$.

The property $v = e + d - 1$ need not be preserved under quadratic transform. For example, with $d = 1$, if $R = k[[t^6, t^{10}, t^{11}, t^{14}, t^{15}, t^{19}]]$, k a field, then $R[\mathbf{m}/t^6] = k[[t^4, t^5, t^6]]$ so $v(R[\mathbf{m}/t^6]) \neq e(R[\mathbf{m}/t^6])$.

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