Using dimensional reduction for hadronic collisions

Adrian Signer [a,*], Dominik Stöckinger [b]

[a] *Institute for Particle Physics Phenomenology, Durham University, Durham DH1 3LE, UK*
[b] *Institut für Kern- und Teilchenphysik, TU Dresden, D-01062 Dresden, Germany*

Abstract

We discuss how to apply regularization by dimensional reduction for computing hadronic cross sections at next-to-leading order. We analyze the infrared singularity structure, demonstrate that there are no problems with factorization, and show how to use dimensional reduction in conjunction with standard parton distribution functions. We clarify that different versions of dimensional reduction with different infrared and factorization behaviour have been used in the literature. Finally, we give transition rules for translating the various parts of next-to-leading order cross sections from dimensional reduction to other regularization schemes.

© 2008 Elsevier B.V. All rights reserved.

PACS: 11.10.Gh; 11.15.-q; 12.38.Bx; 12.60.Jv

1. Introduction

Recently progress on the understanding of regularization by dimensional reduction has been achieved in three directions. First, a mathematically consistent definition avoiding the problem found in Refs. [1,2] was formulated, and a succinct method to check the symmetry properties of dimensional reduction was developed [3], leading to the verification of supersymmetry in important cases at the two-loop level [4]. Second, explicit calculations demonstrated how dimensional reduction can be applied to multiloop calculations and how renormalization has to be carried out in a non-supersymmetric context [5]. This provides the basis of transition rules between various definitions of parameters such as α_s or m_b and is useful to derive the GUT-scale values of these

---

* Corresponding author.
  E-mail address: adrian.signer@durham.ac.uk (A. Signer).

parameters from the experimental values [6]. Third, an obstacle in the application of dimensional reduction to hadronic processes was removed [7] by the resolution of the factorization problem of dimensional reduction found in Refs. [8–10].

The purpose of the present article is to further elaborate on the application of dimensional reduction to hadronic processes. In Ref. [7] we restricted ourselves to the case considered in Ref. [8], the real corrections to the process $gg \to t\bar{t}$, and showed that, despite first appearances, in the collinear limit these real corrections factorize into products of splitting functions and leading-order cross sections.

Here we will consider real and virtual NLO QCD corrections to arbitrary hadronic $2 \to (n-2)$ processes with massless or massive partons. We will discuss the infrared singularity structure and the associated regularization-scheme dependence of all these corrections, provide transition rules between the schemes and show that all singularities factorize. In this way we show that the framework of dimensional reduction is completely consistent with factorization, and we show how this scheme can be used to compute hadronic processes in practice.

One of the main points of this article is the distinction of two different versions of dimensional reduction that have been used in the literature. One of the reasons why the factorization problem of Refs. [8–10] has remained unsolved for so long is that these two versions have mainly been applied by two different communities. The version used in Refs. [8–10] is the same as the one defined in Refs. [3,11,12] and is the one mainly used in the context of supersymmetry. The version used in Refs. [13–15], which was denoted by DR and is actually equivalent to the four-dimensional helicity (FDH) scheme [16] at one-loop, is mainly used in the context of QCD. For the latter version, the infrared singularity structure and transition rules have already been derived [13–15]. We denote these two versions by DRED and FDH. They differ in their treatment of external particles, in a way analogous to the difference of the "conventional" and "'t Hooft–Veltman" versions of dimensional regularization, CDR and HV.

In the main part of the present article we will provide results and transition rules for all these four regularization schemes, keeping in mind that the results for the FDH, CDR and HV schemes can already be found in Refs. [13–15], while the results for DRED are new. The results for the infrared singularity structure in CDR, HV, FDH and in DRED are developed in Sections 2 and 3, respectively. The practical application of DRED and transition rules are discussed in Section 4. Appendix A provides explicit results for all relevant splitting functions, and in Appendix B we provide three explicit examples of NLO computations in DRED.

1.1. Elements and scheme dependences of hadronic cross sections

We consider n-parton processes with up to two hadrons in the initial state at next-to-leading (NLO) order in QCD. The partons can be either massless quarks q or gluons g or massive partons such as heavy quarks Q, gluinos, or squarks. In our equations we will restrict ourselves to the most interesting case of two initial-state partons as the simpler cases can be obtained by straightforward modifications. The cross sections of such processes can be written as

$$d\sigma\big(H_1(K_1)H_2(K_2) \to a_3 \dots a_n\big) = \sum_{a_1,a_2} \int_0^1 dx_1 \, f_{a_1/H_1}(x_1) \int_0^1 dx_2 \, f_{a_2/H_2}(x_2)$$
$$\times d\hat{\sigma}\big(a_1(x_1 K_1)a_2(x_2 K_2); a_3 \dots a_n\big), \qquad (1)$$

where $H_{1,2}$ are the initial-state hadrons, $K_{1,2}$ their momenta and a_i $(i = 3, \dots, n)$ the final-state partons. The sums run over all possible flavours of the initial-state partons $a_{1,2}$ of the hard

partonic cross section $d\hat{\sigma}$, and $f_{a_i/H_i}(x)$ denote the appropriate parton distribution functions (PDF). In the computation of hadronic cross sections, three scheme choices have to be made: the choice of the renormalization scheme, the factorization scheme, and the regularization scheme.

The functional dependence of the hard cross section $d\hat{\sigma}$ on input parameters like α_s, particle masses, etc., depends on the renormalization and the factorization schemes.

The choice of the renormalization scheme is equivalent to a precise definition of the input parameters entering the computation, in particular of α_s, particle masses and other coupling constants. Once it has been fixed and renormalization has been carried out accordingly, all off-shell Green functions are finite and unambiguously defined. Changing the renormalization scheme changes both the functional form of $d\hat{\sigma}$ and the numerical values of the input parameters, such that $d\hat{\sigma}$ is renormalization-scheme independent up to terms which are formally of higher order than NLO. Common renormalization schemes are the $\overline{\text{MS}}$-scheme for α_s, the $\overline{\text{DR}}$-scheme for supersymmetric parameters, or the on-shell scheme for masses.[1] In the following we are not concerned with the renormalization-scheme dependence and assume that some renormalization scheme has been fixed.

The choice of the factorization scheme is equivalent to a precise definition of the parton distribution functions. Both the functional form of the hard cross section and the numerical values of the parton distribution functions depend on this choice, but this dependence cancels in the full hadronic cross section up to terms which are formally of higher order than NLO. The major part of the following considerations is independent of the factorization scheme, and we simply assume that some factorization scheme has been fixed. At the end we will specialize to the important case of the $\overline{\text{MS}}$-factorization scheme. In that case, the parton distribution functions f_{a_i/H_i} can be taken for instance from the well-known MRST or CTEQ sets [17,18].

Our main focus is the influence of the regularization scheme (RS). After removing the regularization, all quantities appearing in Eq. (1) are RS independent, but the hard partonic cross section is a sum of RS dependent parts. It is commonly written as

$$d\hat{\sigma} = d\sigma_{\text{RS}}^{\text{Born}} + d\sigma_{\text{RS}}^{\text{real}} + d\sigma_{\text{RS}}^{\text{virt}} + d\sigma_{\text{RS}}^{\text{coll}}, \tag{2}$$

where the RS dependence is explicitly indicated. The lowest-order, or Born cross section $d\sigma^{\text{Born}}$ is finite, and in the limit where the regularization is removed its RS dependence vanishes. The three NLO contributions are the real and virtual corrections $d\sigma^{\text{real}}$, $d\sigma^{\text{virt}}$ and the collinear counterterm $d\sigma^{\text{coll}}$, which subtracts initial-state collinear singularities. All NLO contributions involve collinear and/or soft singularities and depend on the RS in their finite and their divergent parts. There are no ultraviolet singularities and associated RS dependences in $d\sigma^{\text{virt}}$, because these are eliminated by renormalization and by fixing the renormalization scheme.

1.2. Variants of dimensional regularization and dimensional reduction

In all dimensional schemes space–time is continued from 4 to D dimensions, where $D = 4 - 2\epsilon$ is an arbitrary complex number. In this way momentum integrals become well-defined and ultraviolet and infrared singularities appear as $1/\epsilon^k$-poles as $\epsilon \to 0$. Gluon fields are treated differently in dimensional regularization and dimensional reduction. In the former, gluons are treated as D-dimensional as well; in the latter, gluons are treated as 4-dimensional.

---

[1] We stress that, although the $\overline{\text{MS}}$- and $\overline{\text{DR}}$-schemes have originally been defined with reference to specific regularization schemes, they can be realized in the context of any regularization scheme.

Both choices have certain advantages. The purely D-dimensional treatment of all objects leads to simpler expressions, but it breaks supersymmetry owing to the different number of degrees of freedom of the gluon and the gluino. The 4-dimensional treatment of the gluon is better compatible with supersymmetry and it is more amenable to helicity methods, which are commonly used to simplify QCD higher-order computations.

In order to formulate the two schemes one needs to distinguish three spaces:

- The original 4-dimensional space (4S).
- The formally D-dimensional space for momenta and momentum integrals. This space is actually an infinite-dimensional vector space with certain D-dimensional properties [19,20], and is sometimes called "quasi-D-dimensional space" (QDS). The space 4S is therefore a subspace of QDS.
- The formally 4-dimensional space for e.g. gluons in dimensional reduction. This space has to be a superspace of QDS in order for the dimensionally reduced theory to be gauge invariant. Hence it cannot be identified with the original 4S—it can only be constructed as a "quasi-4-dimensional space" (Q4S) [3,21] with certain 4-dimensional properties. In practice the distinction between Q4S and 4S often does not matter, but it is important in the definition of the different versions of dimensional reduction, see below, and to avoid the inconsistency uncovered in Ref. [2].

These three spaces are characterized by their metric tensors, which we denote by $g^{\mu\nu}$ (for Q4S), $\hat{g}^{\mu\nu}$ (for QDS), and $\bar{g}^{\mu\nu}$ (for 4S). The dimensionalities of the spaces are expressed by the following equations:

$$g^{\mu\nu} g_{\mu\nu} = 4, \qquad \hat{g}^{\mu\nu} \hat{g}_{\mu\nu} = D = 4 - 2\epsilon, \qquad \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} = 4. \tag{3}$$

The following projection relations express that 4S is a subspace of QDS and QDS is a subspace of Q4S:

$$g^{\mu\nu} \hat{g}_{\nu}{}^{\rho} = \hat{g}^{\mu\rho}, \qquad g^{\mu\nu} \bar{g}_{\nu}{}^{\rho} = \bar{g}^{\mu\rho}, \qquad \hat{g}^{\mu\nu} \bar{g}_{\nu}{}^{\rho} = \bar{g}^{\mu\rho}. \tag{4}$$

It is useful to introduce the orthogonal complement to QDS. This is a $(4 - D = 2\epsilon)$-dimensional space with metric tensor $\tilde{g}^{\mu\nu}$, which satisfies

$$g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}, \tag{5}$$

$$\tilde{g}^{\mu\nu} \tilde{g}_{\mu\nu} = 4 - D = 2\epsilon, \tag{6}$$

$$g^{\mu\nu} \tilde{g}_{\nu}{}^{\rho} = \tilde{g}^{\mu\rho}, \qquad \hat{g}^{\mu\nu} \tilde{g}_{\nu}{}^{\rho} = 0, \qquad \bar{g}^{\mu\nu} \tilde{g}_{\nu}{}^{\rho} = 0. \tag{7}$$

Within this framework it is now possible to precisely state the calculational rules of dimensional regularization and dimensional reduction. Since momenta are always treated in D dimensions, it only needs to be specified how gluons (or other vector fields) are treated. More precisely, it needs to be specified which metric tensors are used in gluon propagator numerators and in gluon polarization sums. Particularly important for the understanding of factorization is the treatment of gluon polarization sums in squared matrix elements. Without regularization, such polarization sums can be written as

$$\sum_{\text{pols}} \epsilon^{\mu} \epsilon^{\nu *} = -\bar{g}^{\mu\nu} + \frac{n^{\mu} k^{\nu} + k^{\mu} n^{\nu}}{(nk)} - \frac{n^2 k^{\mu} k^{\nu}}{(nk)^2}, \tag{8}$$

Table 1
Treatment of internal and external gluons in the four different RS, i.e. prescription for which metric tensor is to be used in propagator numerators and polarization sums. For the definition of "internal" and "external" see text.

	CDR	HV	FDH	DRED
Internal gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
External gluon	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$g^{\mu\nu}$

where k is the gluon momentum and n is a gauge vector such that $nk \neq 0$. With regularization, the metric tensor in this polarization is replaced by either $g^{\mu\nu}$, $\hat{g}^{\mu\nu}$, $\tilde{g}^{\mu\nu}$, or $\bar{g}^{\mu\nu}$.

It is not strictly necessary to regularize all gluons. Only gluons that appear inside a divergent loop or phase space integral ("internal") need to be regularized; for all other gluons ("external") regularization is optional. The precise definitions of "internal/external" in this context are as follows: "Internal gluons" are defined as either virtual gluons that are part of a one-particle irreducible loop diagram or, for real correction diagrams, gluons in the initial or final state that are collinear or soft. "External gluons" are defined as all other gluons.

Now, since external gluons do not have to be treated in the same way as internal ones, it is in fact possible to distinguish two variants of each regularization. The two variants of dimensional regularization are:

- CDR ("conventional dimensional regularization"): Here internal and external gluons (and other vector fields) are all treated as D-dimensional.
- HV ("'t Hooft–Veltman scheme"): Internal gluons are treated as D-dimensional but external ones are treated as strictly 4-dimensional.

Note that the above definition of internal gluons in phase space integrals is necessary for unitarity but leads to complications in the treatment of phase space integrals in schemes where internal and external gluons are treated differently. The two analogous variants of dimensional reduction are:

- DRED ("original/old dimensional reduction"): Internal and external gluons are all treated as quasi-4-dimensional.
- FDH ("four-dimensional helicity scheme"): Internal gluons are treated as quasi-4-dimensional but external ones are treated as strictly 4-dimensional.

Table 1 illustrates these four schemes.

Note that the version of dimensional reduction denoted by DR e.g. in Refs. [13,15] is equivalent to FDH at the one-loop level (see e.g. Refs. [13,22]).[2] The infrared properties of the three schemes CDR, HV, DR (or equivalently FDH) have been studied and compared in Ref. [15] and found to be consistent with factorization.

An apparent inconsistency between dimensional reduction and factorization has been identified in Refs. [8,10], but in these references the version DRED has been used. In Ref. [7] it was found that factorization holds as expected in DRED if external quasi-4-dimensional gluons are decomposed into D-dimensional gauge fields and $(4 - D)$-dimensional "ϵ-scalars", which are

---

[2] In Ref. [23] a two-loop definition of the FDH scheme has been given. In what follows we will only use the one-loop definition and the one-loop equivalence of FDH and DR.

treated as separate partons. Technically, this decomposition amounts to replacing

$$
\begin{aligned}
\mathcal{M}_{\mathrm{DRED}}(\dots g \dots) &= \mathcal{M}_{\mathrm{DRED}}(\dots \hat{g} \dots) + \mathcal{M}_{\mathrm{DRED}}(\dots \tilde{g} \dots) \\
&= \sum_{\check{g} \in \{\hat{g}, \tilde{g}\}} \mathcal{M}_{\mathrm{DRED}}(\dots \check{g} \dots),
\end{aligned}
\tag{9}
$$

for squared matrix elements, where the different gluon types g, $\hat{g}$, $\tilde{g}$ are denoted by the same symbols as the associated metric tensors. The algebraic expressions for the partonic processes involving g, $\hat{g}$, or $\tilde{g}$ are defined by the values of the corresponding gluon polarization sums. These read

$$
g: \quad \sum_{\mathrm{pols}} \epsilon^\mu \epsilon^{\nu*} \to -g^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{(nk)} - \frac{n^2 k^\mu k^\nu}{(nk)^2},
\tag{10a}
$$

$$
\hat{g}: \quad \sum_{\mathrm{pols}} \epsilon^\mu \epsilon^{\nu*} \to -\hat{g}^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{(nk)} - \frac{n^2 k^\mu k^\nu}{(nk)^2},
\tag{10b}
$$

$$
\tilde{g}: \quad \sum_{\mathrm{pols}} \epsilon^\mu \epsilon^{\nu*} \to -\tilde{g}^{\mu\nu}.
\tag{10c}
$$

Eq. (9) follows trivially from Eq. (5).

The decomposition of gluons into their D-dimensional and ϵ-scalar part in DRED is also relevant for the renormalization of UV divergences. In order to make Green functions with external ϵ-scalars finite, the renormalization constants for ϵ-scalar couplings in general have to be different from the corresponding gluon couplings. For example, in pure QCD, the couplings α_s and α_e for the quark–antiquark–gluon and the quark–antiquark–ϵ-scalar vertices receive different counterterms $\delta\alpha_s \neq \delta\alpha_e$ even if $\alpha_s = \alpha_e$ at tree level [5,24].

1.3. Splittings in the four schemes

An essential part of the RS dependence of NLO contributions is related to the RS dependence of the splittings $i \to jk$ of one parton i into two collinear partons j, k. The RS dependence of real corrections is related to the splitting functions $P_{i \to jk}^{\mathrm{RS}}$; the RS dependence of virtual corrections is related to constants $\gamma_{\mathrm{RS}}(i)$ [13], which in turn can be derived from the $P_{i \to jk}^{\mathrm{RS}}$ via unitarity [15]. In this section we explain the RS dependence of the splitting functions and correspondingly of the $\gamma_{\mathrm{RS}}(i)$. The full results can be found in Appendix A. Fig. 1 shows the most interesting case of a gluon splitting into two collinear gluons. According to the definition given above, the two collinear gluons j and k are treated as "internal", and the virtual gluon i as "external". The appropriate treatment of the gluons in the four RS can be read off from Table 1 and is displayed in the figure. Two simple observations allow an easy comparison of the four cases.

First, the projection of a D-dimensional onto a strictly 4-dimensional parent gluon does not change the structure of the result of the splitting functions. And second, the result in DRED should be decomposed according to Eq. (9) into four splittings $\hat{g} \to \hat{g}\hat{g}$, $\hat{g} \to \tilde{g}\tilde{g}$, $\tilde{g} \to \hat{g}\tilde{g}$, $\tilde{g} \to \tilde{g}\hat{g}$.[3] Then the result in CDR is identical to the DRED result for $\hat{g} \to \hat{g}\hat{g}$, and all scheme differences can be explained in the following way:

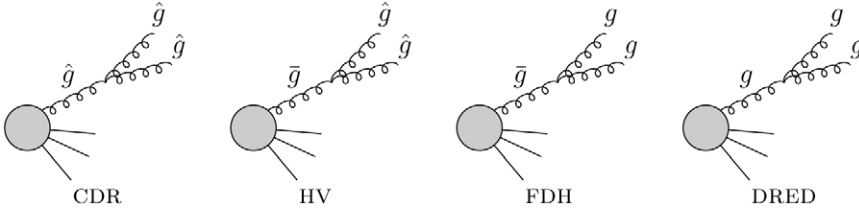
[3] Splittings involving an odd number of $\tilde{g}$ vanish.

Fig. 1. Gluon splitting into two collinear gluons in the four schemes, indicating the appropriate treatment of each gluon.

- The splitting $g \to gg$ is identical in the CDR and HV schemes. Formally, this is expressed in the equality

$$P_{g^* \to gg}^{< \mathrm{CDR}}(z) = P_{g^* \to gg}^{< \mathrm{HV}}(z) = P_{\hat{g}^* \to \hat{g}\hat{g}}^{< \mathrm{DRED}}(z) \tag{11}$$

  for the splitting functions defined for $z < 1$.
- In the FDH scheme the outgoing gluons are treated as quasi-4-dimensional. The resulting additional term can be interpreted as being due to the splitting $\hat{g} \to \tilde{g}\tilde{g}$ as already discussed in Ref. [15]. Hence,

$$P_{g^* \to gg}^{< \mathrm{FDH}}(z) = P_{\hat{g}^* \to \hat{g}\hat{g}}^{< \mathrm{DRED}}(z) + P_{\hat{g}^* \to \tilde{g}\tilde{g}}^{< \mathrm{DRED}}(z). \tag{12}$$

- In the DRED scheme the parent gluon is also treated as quasi-4-dimensional, and therefore the two additional splittings $\tilde{g} \to \hat{g}\tilde{g}$ and $\tilde{g} \to \tilde{g}\hat{g}$ are possible. In the spirit of our discussion around Eq. (9) we do not combine the DRED splitting functions into a single one.

The splitting functions involving quarks are related in a similar way. Via unitarity, the RS dependence of the constants $\gamma(i)$ follows from the splitting functions [15] and can thus be explained in an analogous way:

- The $\gamma(i)$ in CDR and HV are the same,

$$\gamma_{\mathrm{CDR}}(i) = \gamma_{\mathrm{HV}}(i) \quad \text{for } i \in \{g, q\}. \tag{13}$$

- The additional terms in the FDH scheme are due to the splittings $\hat{g} \to \tilde{g}\tilde{g}$ and $q \to q\tilde{g}$:

$$\gamma_{\mathrm{FDH}}(g) = \gamma_{\mathrm{HV}}(g) - \int_0^1 dz\, z\, P_{\hat{g} \to \tilde{g}\tilde{g}}^{< \mathrm{DRED}}(z), \tag{14}$$

$$\gamma_{\mathrm{FDH}}(q) = \gamma_{\mathrm{HV}}(q) - \int_0^1 dz\, z\, \big[P_{q \to q\tilde{g}}^{< \mathrm{DRED}}(z) + P_{q \to \tilde{g}q}^{< \mathrm{DRED}}(z) \big]. \tag{15}$$

- In DRED one has to distinguish γ constants for $\hat{g}$, $\tilde{g}$, and q. The ones corresponding to $\hat{g}$ and q are the same as the ones in the FDH scheme; the one for $\tilde{g}$ is related to the additional splittings $\tilde{g} \to \hat{g}\tilde{g}$ and $\tilde{g} \to \tilde{g}\hat{g}$:

$$\gamma_{\mathrm{DRED}}(\hat{g}) = \gamma_{\mathrm{FDH}}(g), \tag{16}$$

$$\gamma_{\mathrm{DRED}}(q) = \gamma_{\mathrm{FDH}}(q), \tag{17}$$

$$\gamma_{\text{DRED}}(\tilde{g}) = -\int\limits_0^1 dz\, z\, \frac{(1-z)}{(1-z)_+} \Big[P_{\tilde{g}\to\tilde{g}\hat{g}}^{<\,\text{DRED}}(z)$$

$$+ P_{\tilde{g}\to\tilde{g}\tilde{g}}^{<\,\text{DRED}}(z) + 2N_F P_{\tilde{g}\to q\bar{q}}^{<\,\text{DRED}}(z) \Big]. \tag{18}$$

These relations form the basis for understanding the RS dependence of NLO contributions and in particular the difference between DRED and the other schemes. In the subsequent sections we will see that additional RS dependences arise from the crossing of the splitting functions to initial-state parton splitting and from the RS dependence of the LO matrix element.

2. CDR, HV, FDH

Our starting point is the decomposition, Eq. (2), of the hard partonic cross section, and we are mainly interested in the RS dependence of the separate terms contributing to $d\hat{\sigma}$. In this section we will restrict ourselves to the well-known cases of CDR, HV and FDH. The RS dependence of quantities will be indicated by a subscript RS∗, the star reminding us that we consider CDR, HV and FDH, but not (yet) DRED.

2.1. Born term

We consider the partonic process

$$a_1(p_1)a_2(p_2) \to a_3(p_3)\ldots a_n(p_n), \tag{19}$$

where a_i and p_i denote the flavour and the momentum of parton i respectively. The l-loop correction to the RS dependent squared matrix element for the process given in Eq. (19) is denoted by $\mathcal{M}_{\text{RS}*}^{(l)}(a_1(p_1), a_2(p_2); a_3(p_3)\ldots a_n(p_n))$ or by $\mathcal{M}_{\text{RS}*}^{(l)}(a_1\ldots a_n)$ for short. For the cross section we need the averaged squared matrix elements

$$\big\langle \mathcal{M}_{\text{RS}*}^{(l)}(a_1, a_2; \ldots a_n) \big\rangle = \frac{1}{2s_{12}} \frac{1}{\omega_{\text{RS}*}(a_1)\omega_{\text{RS}*}(a_2)} \mathcal{M}_{\text{RS}*}^{(l)}(a_1, a_2; \ldots a_n), \tag{20}$$

where $\omega_{\text{RS}*}(a_i)$ denotes the RS dependent number of degrees of freedom of a parton with flavour a_i and $s_{12} \equiv 2(p_1 \cdot p_2)$ in the case of massless incoming partons.

The Born cross section is obtained by integrating the squared and averaged tree-level matrix element over the $(n-2)$ parton phase space $d\Phi_{n-2}(p_1, p_2; p_3\ldots p_n)$ multiplied by a measurement function for an infrared-safe quantity and a symmetry factor. The latter two are always implicitly understood in our notation and we simply write

$$d\sigma_{\text{RS}*}^{(0)}(a_1\ldots a_n) = \int d\Phi_{n-2}(p_1\ldots p_n) \big\langle \mathcal{M}_{\text{RS}*}^{(0)}(a_1\ldots a_n) \big\rangle. \tag{21}$$

The RS dependence in Eq. (21) is due to $\mathcal{O}(\epsilon)$ terms in $\mathcal{M}_{\text{RS}*}^{(0)}$. Since we consider an infrared-finite quantity, the phase-space integration does not introduce any poles. Therefore, we can take the limit $\epsilon \to 0$ and

$$d\sigma^{\text{Born}}(a_1\ldots a_n) \equiv \big[d\sigma_{\text{RS}*}^{(0)}(a_1\ldots a_n) \big]_{D\to 4} \tag{22}$$

is RS independent, as indicated by the absence of the subscript RS∗.

2.2. Virtual corrections

For the virtual corrections we need $\mathcal{M}^{(1)}_{RS*}$, the interference terms of the one-loop amplitude and the tree-level amplitude. The structure of the singular terms of $\mathcal{M}^{(1)}_{RS*}$ is well known [14,25]. For the fully renormalized matrix element, it is given by

$$
\mathcal{M}^{(1)}_{RS*}(a_1 \dots a_n) = \frac{\alpha_s}{2\pi} c_\Gamma \left[\mathcal{M}^{(0)}_{RS*}(a_1 \dots a_n) \left(-\frac{1}{\epsilon} \sum_i \gamma_{RS*}(a_i) \right) \right.
$$
$$
\left. + \sum_{i,j} \mathcal{V}(i,j) \mathcal{M}^{ij}_{RS*}(a_1 \dots a_n) + \mathcal{M}^{(1)}_{NS}(a_1 \dots a_n) \right],
\tag{23}
$$

where the sums i and j are over all initial or final state partons and we introduced

$$
c_\Gamma \equiv (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = \left(\frac{e^{\gamma_E}}{4\pi} \right)^{-\epsilon} \left(1 - \frac{\epsilon^2 \pi^2}{12} + \mathcal{O}(\epsilon^3) \right).
\tag{24}
$$

The soft and collinear poles are contained in the terms proportional to $\mathcal{M}^{(0)}_{RS*}$ and $\mathcal{M}^{ij}_{RS*}$. The latter are the colour-linked Born squared matrix elements introduced in Ref. [26] and correspond to the square of the colour-correlated tree amplitudes with a $\mathbf{T}_i \cdot \mathbf{T}_j$ insertion, used in Ref. [27]. If particles i and j are massless, we have

$$
\mathcal{V}(i,j) = -\frac{1}{2\epsilon^2} \operatorname{Re}\left(-\frac{s_{ij}}{\mu^2} \right)^{-\epsilon}
\tag{25}
$$

and Eq. (23) reduces to the well-known expression for the singularities of one-loop QCD amplitudes [25]. If one or both of the particles i, j are massive, these expressions have to be generalized [14], but the structure of the singularities remains as in Eq. (23).

The RS dependence of Eq. (23) is contained in the constants $\gamma_{RS*}(a_i)/\epsilon$, as well as in $\mathcal{M}^{(0)}_{RS*}$ and $\mathcal{M}^{ij}_{RS*}$. The remaining term, $\mathcal{M}^{(1)}_{NS}(a_1 \dots a_n)$, is in general very complicated, but is finite and, after taking the limit $D \to 4$, RS independent.

The essential, non-trivial part of the RS dependence is due to the $\gamma_{RS*}(a_i)/\epsilon$ terms. These are closely related to collinear singularities due to self-energy insertions on external legs and depend on the flavour a_i of leg i. The RS dependence of γ_{RS*} has been given in Ref. [13]. Via unitarity it is related to parton splittings $a_i \to$ anything [15], as summarized in Section 1.3.

In the present paper we determine γ_{RS*} by insisting that the sum rules hold in all RS to all orders in ϵ. This is a slightly different approach compared to Ref. [13] and simply amounts to a shift of finite terms between $\mathcal{M}^{(1)}_{NS}$ and the γ_{RS*} terms in Eq. (23). Neglecting $\mathcal{O}(\epsilon^2)$ terms, we find

$$
\gamma_{CDR}(g) = \gamma_{HV}(g) = \frac{\beta_0}{2} + \epsilon \frac{T_F N_F}{3}; \qquad \gamma_{CDR}(q) = \gamma_{HV}(q) = \frac{3C_F}{2} + \epsilon \frac{C_F}{2},
$$
$$
\gamma_{FDH}(g) = \frac{\beta_0}{2} + \epsilon \frac{2T_F N_F - N_c}{6}; \qquad \gamma_{FDH}(q) = \frac{3C_F}{2}
\tag{26}
$$

with $\beta_0 = (11 N_c - 4 T_F N_F)/3$ and $T_F = 1/2$. For heavy quarks the result is RS independent [14] and we have $\gamma(Q) = C_F$.

The final virtual corrections $d\sigma^{virt}_{RS*}$ are obtained as

$$
d\sigma^{virt}_{RS*}(a_1 \dots a_n) = \int d\Phi_{n-2}(p_1 \dots p_n) \langle \mathcal{M}^{(1)}_{RS*}(a_1 \dots a_n) \rangle.
\tag{27}
$$

Since this phase space integration does not give rise to any pole in ϵ, taking the limit $D \to 4$ in $\mathcal{M}_{\text{NS}}^{(1)}$ is justified.

2.3. Real corrections

For the real corrections $d\sigma_{\text{RS}*}^{\text{real}}$ to the partonic process $a_1 a_2 \to a_3 \ldots a_n$ we have to consider contributions from all $2 \to (n-1)$ processes that are obtained by a split of any of the outgoing partons. We have to evaluate the corresponding squared matrix elements $\mathcal{M}_{\text{RS}}^{(0)}(a_1, a_2; \bar{a}_3 \ldots \bar{a}_{n+1})$ and integrate them over the $(n-1)$ parton phase space $d\Phi_{n-1}(p_1, p_2; p_3 \ldots p_{n+1})$

$$d\sigma_{\text{RS}*}^{\text{real}} = \sum_{\bar{a}_i} \int d\Phi_{n-1}(p_1, p_2; p_3 \ldots p_{n+1}) \langle \mathcal{M}_{\text{RS}*}^{(0)}(a_1, a_2; \bar{a}_3 \ldots \bar{a}_{n+1}) \rangle. \tag{28}$$

In Eq. (28) we denote by $\bar{a}_i$, $i \in \{3 \ldots n+1\}$ the flavour of the outgoing partons, and as indicated by $\sum_{\bar{a}_i}$, we have to sum over all relevant processes.

As is well known, the matrix elements can develop singularities in regions of the phase space where a parton becomes soft or two partons become collinear. The integration over $d\Phi_{n-1}(p_1 \ldots p_{n+1})$ in this region then results in $1/\epsilon^2$ and $1/\epsilon$ poles. Thus the RS dependence of the matrix elements which manifests itself in the $\mathcal{O}(\epsilon)$ terms of $\mathcal{M}_{\text{RS}}^{(0)}$ results in differences in the $\mathcal{O}(1/\epsilon)$ and in the finite terms of the real corrections.[4]

In order to deal with the phase-space integration at NLO one often uses either phase-space slicing [28] or subtraction [26,29], and several general procedures have been developed [27,30, 31]. For our purposes it is sufficient to know that they all rely on the same main points. Using the results given below it will be obvious how any of these procedures can be applied in the context of DRED.

The first point is that in any of the singular regions the matrix elements take a simple form and can be written as a factor containing the kinematic singularity times a reduced (colour-linked) tree-level matrix element, associated with a $2 \to (n-2)$ process. Secondly, the phase space is factorized according to $d\Phi_{n-1}(p_1 \ldots p_{n+1}) = d\Phi_{n-2}(p_1' \ldots p_n') d\Phi_{\text{rad}}$. The factor with the kinematic singularity is integrated analytically over $d\Phi_{\text{rad}}$, producing the poles in analytic form. These poles will be multiplied by the reduced (colour-linked) matrix element and are to be integrated over a slightly modified $(n-2)$ parton phase space $d\Phi_{n-2}(p_1' \ldots p_n')$. It is therefore not surprising that the real corrections have a similar structure as the virtual corrections, Eqs. (23) and (27). We will now look at all three potentially singular regions in turn.

Soft region: In the limit where gluon g_k (or another massless gauge boson) becomes soft we have

$$\mathcal{M}_{\text{RS}*}^{(0)}(a_1, a_2; \ldots g_k(p_k) \ldots \bar{a}_{n+1}) \overset{p_k \to 0}{=} g_s^2 \sum_{i,j} \frac{s_{ij}}{s_{ik} s_{jk}} \mathcal{M}_{\text{RS}*}^{ij}(a_1 \ldots a_n), \tag{29}$$

where we introduced $g_s^2 = 4\pi\alpha_s$. It is understood that the set of partons $\{a_3 \ldots a_n\}$ is equal to the set $\{\bar{a}_3 \ldots g_k \ldots \bar{a}_{n+1}\}$ with g_k removed. Similar comments apply to the analogous equations below. The phase space integration of Eq. (29) leads to $\int d\Phi_{\text{rad}} s_{ij}/(s_{ik} s_{jk})$ and results in poles that

---

[4] Note that in the HV and FDH schemes real soft and/or collinear gluons have to be treated as "internal", i.e. in the same way as gluons in a closed loop but differently from observed, "external" gluons. This is the only source of the RS dependence of the real corrections in these schemes.

cancel the corresponding poles in $\mathcal{V}(i, j)$. Due to the measurement function implicitly included in $d\Phi_{n-2}$, the remaining integration does not result in any singularities. The scheme dependence enters only through $\mathcal{M}_{\mathrm{RS}*}^{ij}$ and after summation over all real processes trivially cancels between the real and virtual corrections.

Final-state collinear region: In the limit where two outgoing partons $\bar{a}_k$ and $\bar{a}_l$ become collinear we have

$$
\mathcal{M}_{\mathrm{RS}*}^{(0)}\big(a_1, a_2; \ldots \bar{a}_l(p_l) \ldots \bar{a}_k(p_k) \ldots \bar{a}_{n+1}\big)
$$
$$
\overset{p_k \| p_l}{=} \frac{2g_s^2}{s_{kl}} P_{(kl)^* \to kl}^{< \mathrm{RS}*}(z) \mathcal{M}_{\mathrm{RS}*}^{(0)}\big(a_1, a_2; \ldots a_{(kl)}(p_k + p_l) \ldots a_n\big). \tag{30}
$$

As detailed in Appendix A, $P_{(kl)^* \to kl}^{< \mathrm{RS}*}$ is the RS dependent splitting function defined for $z < 1$ with $p_k \to z(p_k + p_l)$ and $p_l \to (1 - z)(p_k + p_l)$. In Eq. (30) the flavours $\bar{a}_k$ and $\bar{a}_l$ are fixed. This uniquely determines the flavour of the parent parton $a_{(kl)}$. It is understood that if the split is flavour forbidden we set $P_{(kl)^* \to kl}^{< \mathrm{RS}*} = 0$. To avoid a proliferation of subscripts, we denote the flavour of the partons in the splitting functions simply by (kl), etc., rather than $a_{(kl)}$. The parent parton is slightly off shell as indicated by the notation $(kl)^*$.

Contrary to the soft limit, in the collinear limit there are two sources of RS dependence. Apart from the trivial dependence through $\mathcal{M}_{\mathrm{RS}*}^{(0)}(a_1 \ldots a_n)$, the prefactor $P_{(kl)^* \to kl}^{< \mathrm{RS}*}$ is also RS dependent. Its RS dependence can be found in Section 1.3 and in Appendix A. Since the z dependence in Eq. (30) is entirely in the prefactor, the integration $\int d\Phi_{\mathrm{rad}} P_{(kl)^* \to kl}^{< \mathrm{RS}*}/s_{kl}$ can be performed separately. The terms related to the collinear singularities due to the splitting of parton $a_i = a_{(kl)}$ schematically can be written as

$$
d\sigma_{\mathrm{RS}*}^{\mathrm{real}, i}(a_1 \ldots a_i \ldots a_n) = -\frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} d\sigma_{\mathrm{RS}*}^{(0)}(a_1 \ldots a_i \ldots a_n)
$$
$$
\times \sum_{a_k} \int dz\, \Theta\!\left(z - \frac{1}{2}\right) \frac{(1-z)}{(1-z)_+} P_{i^* \to kl}^{< \mathrm{RS}*}(z), \tag{31}
$$

where the sum over all possible splittings, $\sum_{a_k}$, is due to the sum over the relevant real processes, Eq. (28). Note that in the sum $a_k \in \{g, q, \bar{q}\}$ a sum over the N_F massless quark flavours is implicitly understood. After this sum the integrand is symmetric with respect to $z \leftrightarrow 1 - z$. Hence the integration can be restricted to the region $z > 1/2$ and the potential singularity at $z = 1$ is regularized with the usual +prescription. The z integration in Eq. (31) results in a RS dependence of the singular prefactor that multiplies the Born term. In fact, the factor in the second line of Eq. (31) is equal to $(-\gamma_{\mathrm{RS}*}(a_i))$. Therefore, after summing up the contributions of all final state partons, $i \in \{3, \ldots, n\}$ these terms precisely cancel the singularity and the RS dependence of those virtual terms displayed in the first line of Eq. (23) that are associated with outgoing partons, $i \geqslant 3$ [15].

Initial-state collinear region: Finally we turn to the case of an outgoing parton $\bar{a}_k$ becoming collinear to the incoming parton a_1 (or a_2). There are some important differences with respect to Eq. (30). To start with, the collinear limit has to be written with the spin/colour summed/averaged matrix elements $\langle \mathcal{M}_{\mathrm{RS}*}^{(0)} \rangle$. With $p_k \to (1 - z)p_1$ the collinear limit is given by

$$
\langle \mathcal{M}_{\mathrm{RS}*}^{(0)}\big(a_1(p_1), a_2; \ldots \bar{a}_k(p_k) \ldots \bar{a}_{n+1}\big)\rangle
$$
$$
\overset{p_k \| p_1}{=} \frac{2g_s^2}{s_{1k}} P_{1 \to (1k)^* k}^{< \mathrm{RS}*}(z) \langle \mathcal{M}_{\mathrm{RS}*}^{(0)}\big(a_{(1k)}(zp_1), a_2; \ldots a_n\big)\rangle. \tag{32}
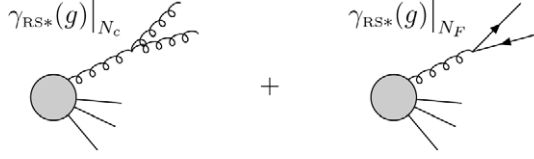$$

Fig. 2. Illustration of Eq. (31) for the case where a_i is an outgoing splitting gluon. The sum over all relevant real processes, $\sum_{a_k}$, gives rise to two contributions. The one on the left (right) results in the N_c (N_F) part of $\gamma_{\mathrm{RS}*}(g)$.
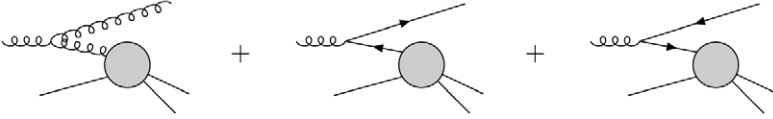

Fig. 3. Illustration of Eq. (35) for an incoming splitting gluon. The sum over all relevant real processes, $\sum_{a_k}$, gives rise to three terms.

Contrary to Eq. (30), the z dependence in Eq. (32) is not restricted to the prefactor and, therefore, the z-integration results in a more complicated structure. In Eq. (32) the flavours a_1 and $\bar{a}_k$ are fixed and uniquely determine the flavour of parton $a_{(1k)}$, which is slightly off shell. The splitting functions of Eqs. (30) and (32), with initial/final-state off-shell parton, are related by a crossing relation

$$P_{l \to (lk)^* k}^{< \mathrm{RS}*}(z) = (-1)^{\#f+1} \frac{\omega_{\mathrm{RS}}(a_{(lk)})}{\omega_{\mathrm{RS}*}(a_l)} z P_{(lk)^* \to lk}^{< \mathrm{RS}*}\left(\frac{1}{z}\right), \tag{33}$$

where $\#f$ denotes the number of crossed fermions. We remark that the well-known crossing symmetry $P_{l \to (lk)^* k}^{< \mathrm{CDR}} = P_{l^* \to (lk)k}^{< \mathrm{CDR}}$ does not hold in all RS, and in general we have

$$P_{l \to (lk)^* k}^{< \mathrm{RS}*}(z) = P_{l^* \to (lk)k}^{< \mathrm{RS}*}(z) + \Delta_{l \to (lk)k}^{\mathrm{RS}*}(z) \tag{34}$$

with $\Delta^{\mathrm{FDH}} \neq 0$ and $\Delta^{\mathrm{HV}} \neq 0$. The explicit form of Δ^{FDH} and Δ^{HV} can easily be found using Eq. (33) and the results in Appendix A. However, as will be discussed in Section 2.4, the distinction made in Eq. (34) is ultimately not required.

Using Eqs. (34) and (A.20) to express the collinear limit, Eq. (32), in terms of the full splitting functions $P_{1 \to (1k)k}^{\mathrm{RS}*}$ and summing over all relevant real processes, we can write the initial state collinear term for parton 1 schematically as

$$\begin{aligned} d\sigma_{\mathrm{RS}*}^{\mathrm{real},1}(a_1 \ldots a_n) = \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \Bigg[& \gamma_{\mathrm{RS}*}(a_1) \, d\sigma_{\mathrm{RS}*}^{(0)}(a_1(p_1), a_2; \ldots a_n) \\ & - \sum_{a_k} \int dz \left(P_{1 \to (1k)k}^{\mathrm{RS}*} + \Delta_{1 \to (1k)k}^{\mathrm{RS}*} \right) d\sigma_{\mathrm{RS}*}^{(0)}(a_{(1k)}(zp_1), a_2; \ldots a_n) \Bigg]. \end{aligned} \tag{35}$$

The first term on the r.h.s. of Eq. (35) is due to the $\delta(1-z)$ term present in Eq. (A.20). Together with the corresponding term for the second incoming parton, $d\sigma_{\mathrm{RS}*}^{\mathrm{real},2}$, this results in a term that precisely cancels the singularity and the RS dependence of those virtual terms displayed in the first line of Eq. (23) that are associated with incoming partons $i \leqslant 2$. The remaining terms given in the second line on the r.h.s. of Eq. (35) are associated with collinear counterterms.

2.4. Collinear counterterm

In the sum of the virtual and real corrections, $d\sigma_{\mathrm{RS}*}^{\mathrm{virt}} + d\sigma_{\mathrm{RS}*}^{\mathrm{real}}$, all singularities and RS dependences cancel, apart from the terms given in the last line of Eq. (35). These are cancelled by the collinear counterterm $d\sigma_{\mathrm{RS}*}^{\mathrm{coll}}$. While the divergent parts of the collinear counterterms are completely determined, there is some freedom in how to specify the finite parts of $d\sigma_{\mathrm{RS}*}^{\mathrm{coll}}$. Any specific choice of the finite parts of $d\sigma_{\mathrm{RS}*}^{\mathrm{coll}}$ is equivalent to the definition of a particular factorization scheme. Leaving the factorization scheme open, we can write

$$
\begin{aligned}
&d\sigma_{\mathrm{RS}*,\mathrm{FS}}^{\mathrm{coll}}(a_1, a_2; \dots a_n) \\
&= \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \sum_{a_k} \int dz \big[\big(P_{1\to ik}^{\mathrm{RS}*}(z) + \Delta_{1\to ik}^{\mathrm{RS}*}(z) + \epsilon X_{1\to ik}^{\mathrm{FS}}(z) \big) d\sigma_{\mathrm{RS}*}^{(0)}\big(a_i(zp_1), a_2(p_2); \dots a_n\big) \\
&\quad + \big(P_{2\to ik}^{\mathrm{RS}*}(z) + \Delta_{2\to ik}^{\mathrm{RS}*}(z) + \epsilon X_{2\to ik}^{\mathrm{FS}}(z) \big) d\sigma_{\mathrm{RS}*}^{(0)}\big(a_1(p_1), a_i(zp_2); \dots a_n\big) \big],
\end{aligned}
\tag{36}
$$

where the sum is over all possible splittings of the incoming partons, and the index i is defined in analogy to Eqs. (30) and (31). The $X_{l\to ik}^{\mathrm{FS}}$ are the finite (i.e. ϵ-independent), RS independent terms which define the factorization scheme. The formulas in Sections 2.2 and 2.3 show that in this way the hard (subtracted) partonic cross section

$$
\begin{aligned}
&d\hat\sigma_{\mathrm{FS}}(a_1 \dots a_n) \\
&= \big[d\sigma_{\mathrm{RS}*}^{\mathrm{Born}}(a_1 \dots a_n) + d\sigma_{\mathrm{RS}*}^{\mathrm{virt}}(a_1 \dots a_n) + d\sigma_{\mathrm{RS}*}^{\mathrm{real}}(a_1 \dots a_n) + d\sigma_{\mathrm{RS}*,\mathrm{FS}}^{\mathrm{coll}}(a_1 \dots a_n) \big]_{D\to 4}
\end{aligned}
\tag{37}
$$

is finite and RS independent as indicated by the absence of the subscript RS. In this approach the most natural factorization scheme would correspond to setting all $X^{\mathrm{FS}} = 0$. In principle, this particular factorization scheme is as good as any other, and it could be used in practice in Eq. (1) in conjunction with parton distribution functions $f_{a_i/H}$ determined in the same scheme. In practice, however, parton distribution functions such as the standard MRST or CTEQ sets [17,18] are mainly available in the $\overline{\mathrm{MS}}$ factorization scheme, which is different.

The $\overline{\mathrm{MS}}$ factorization scheme is defined by using RS = CDR and replacing the square bracket in Eq. (36) by

$$
\big[\big[P_{1\to ik}^{\mathrm{CDR}}(z) \big]_{D\to 4} d\sigma_{\mathrm{CDR}}^{(0)}\big(a_i(zp_1), a_2; \dots a_n\big) + \{1 \leftrightarrow 2\} \big].
\tag{38}
$$

The minimal subtraction procedure corresponds to setting $\epsilon \to 0$ in the splitting functions $P_{1\to ik}^{\mathrm{CDR}}$. Thus, even if we use CDR (and even though $\Delta_{1\to ik}^{\mathrm{CDR}} = 0$) the $\overline{\mathrm{MS}}$ scheme does not correspond to $X^{\mathrm{FS}} = 0$ but to

$$
\epsilon X_{1\to ik}^{\overline{\mathrm{MS}}}(z) = -P_{1\to ik}^{\overline{\mathrm{MS}}\epsilon}(z),
\tag{39}
$$

where $P_{1\to ik}^{\overline{\mathrm{MS}}\epsilon}(z) \equiv P_{1\to ik}^{\mathrm{CDR}}(z) - [P_{1\to ik}^{\mathrm{CDR}}(z)]_{D\to 4}$ denote the $\mathcal{O}(\epsilon)$ terms of the splitting functions in CDR.

In the evaluation of the real corrections, Eq. (35), as well as in the collinear counterterm, Eq. (36), we made the distinction between $P_{l^*\to ik}^{<\mathrm{RS}*}$ and $P_{l\to i^*k}^{<\mathrm{RS}*}$, the splitting functions appropriate for an outgoing and incoming split, respectively. In a general RS, these two splitting functions differ as indicated in Eq. (34), hence the presence of the Δ terms in Eqs. (35) and (36). While the expressions given in Eqs. (35) and (36) are those that naturally arise in the calculation, we note that the Δ terms cancel in the sum of $d\sigma_{\mathrm{RS}*}^{\mathrm{real}} + d\sigma_{\mathrm{RS}*,\mathrm{FS}}^{\mathrm{coll}}$. Thus we can drop $\Delta_{1\to(1k)k}^{\mathrm{RS}*}$ in

Eq. (35) (and the corresponding term in $d\sigma_{\text{RS}*}^{\text{real},2}$) if we also drop $\Delta_{1\to ik}^{\text{RS}*}$ (and $\Delta_{2\to ik}^{\text{RS}*}$) in Eq. (36). Ultimately, the distinction between $P_{l*\to ik}^{<\text{RS}*}$ and $P_{l\to i*k}^{<\text{RS}*}$ is not needed.

3. Dimensional reduction

In this section we show how the structure described in Section 2 can be generalized to include DRED. As mentioned in the introduction, the key point is to split the gluon into a D-dimensional gluon $\hat{g}$ and a $(4-D)$-dimensional ϵ-scalar $\tilde{g}$ by setting $g = \hat{g} + \tilde{g}$ and using Eq. (9). Often it is sufficient to perform this split for only one gluon as in Eq. (9); in general, if a process contains #g gluons, $g_{i_1} \ldots g_{i_{\#g}}$, we can decompose the matrix element in DRED into $2^{\#g}$ terms according to

$$\mathcal{M}_{\text{DRED}}(\ldots g_{i_1} \ldots g_{i_{\#g}} \ldots) = \sum_{\check{g}_{i_1} \in \{\hat{g}, \tilde{g}\}} \cdots \sum_{\check{g}_{i_{\#g}} \in \{\hat{g}, \tilde{g}\}} \mathcal{M}_{\text{DRED}}(\ldots \check{g}_{i_1} \ldots \check{g}_{i_{\#g}} \ldots). \tag{40}$$

We consider the two partons $\hat{g}$ and $\tilde{g}$ to be two different partons and consequently regard the r.h.s. of Eq. (40) as a sum over the squared matrix elements of $2^{\#g}$ different processes. To bring our notation in line with the previous section, we will write Eqs. (9) and (40) as

$$\mathcal{M}_{\text{DRED}}(a_1 \ldots a_i \ldots a_n) = \sum_{\check{a}_i} \mathcal{M}_{\text{DRED}}(a_1 \ldots \check{a}_i \ldots a_n), \tag{41}$$

$$\mathcal{M}_{\text{DRED}}(a_1 \ldots a_n) = \sum_{\{\check{a}\}} \mathcal{M}_{\text{DRED}}(\check{a}_1 \ldots \check{a}_n), \tag{42}$$

respectively, where it is understood that if $a_i = g$ we sum over the two terms $\check{a}_i \in \{\hat{g}, \tilde{g}\}$ whereas if $a_i = q$ there is only one term in the sum $\check{a}_i \in \{q\}$.

For the spin summed/averaged matrix elements the relation equivalent to Eq. (42) reads

$$\langle \mathcal{M}_{\text{DRED}}(a_1, a_2; \ldots a_n) \rangle = \sum_{\{\check{a}\}} \frac{\omega_{\text{DRED}}(\check{a}_1)}{\omega_{\text{DRED}}(a_1)} \frac{\omega_{\text{DRED}}(\check{a}_2)}{\omega_{\text{DRED}}(a_2)} \langle \mathcal{M}_{\text{DRED}}(\check{a}_1, \check{a}_2; \ldots \check{a}_n) \rangle. \tag{43}$$

The explicit expressions for $\omega_{\text{RS}}(a_i)$ are given in Appendix A in Eq. (A.19).

We stress that while the split $g = \hat{g} + \tilde{g}$ is conceptually simple, it seems to complicate practical computations. As we will see in the later sections, however, in an explicit computation of a physical process in DRED it is only required at a very limited number of steps. In particular, it will turn out that no PDF for the unphysical ϵ-scalar $\tilde{g}$ will be required.

In the present section we will use the split to understand the infrared structure of matrix elements in DRED. In fact, it is straightforward to see that Eqs. (23), (30), and (32) for the collinear singularities of virtual and real corrections hold in the same form in DRED for the individual processes with split partons, i.e. if we replace RS* → DRED and $a_i \to \check{a}_i$ in these equations. However, our main interest are the infrared properties and RS dependences of matrix elements for full gluons; therefore we will carry out the sums over $\{\check{a}_i\}$ wherever possible.

3.1. Born term

The full tree-level matrix element in DRED is equal to the one in FDH and HV and can be obtained from the CDR result simply by setting $D \to 4$

$$\mathcal{M}_{\text{DRED}}^{(0)}(a_1 \dots a_n) = \mathcal{M}_{\text{HV}}^{(0)}(a_1 \dots a_n) = \mathcal{M}_{\text{FDH}}^{(0)}(a_1 \dots a_n)$$
$$= \left[\mathcal{M}_{\text{CDR}}^{(0)}(a_1 \dots a_n) \right]_{D \to 4}. \tag{44}$$

The Born cross section in DRED can be obtained from Eq. (44) and satisfies

$$d\sigma_{\text{DRED}}^{(0)}(a_1 \dots a_n) = d\sigma_{\text{HV}}^{(0)}(a_1 \dots a_n) = d\sigma_{\text{FDH}}^{(0)}(a_1 \dots a_n). \tag{45}$$

3.2. Virtual corrections

The structure of the virtual corrections in DRED is analogous to Eq. (23) for each individual DRED process with split partons $\check{a}_i$. Hence, by summing over all processes as in Eq. (42) we obtain

$$\mathcal{M}_{\text{DRED}}^{(1)}(a_1 \dots a_n) = \sum_{\{\check{a}\}} \frac{\alpha_s}{2\pi} c_\Gamma \left[\mathcal{M}_{\text{DRED}}^{(0)}(\check{a}_1 \dots \check{a}_n) \left(-\frac{1}{\epsilon} \sum_i \gamma_{\text{DRED}}(\check{a}_i) \right) \right.$$
$$\left. + \sum_{i,j} \mathcal{V}(i,j) \mathcal{M}_{\text{DRED}}^{ij}(\check{a}_1 \dots \check{a}_n) + \mathcal{M}_{\text{NS}}^{(1)}(\check{a}_1 \dots \check{a}_n) \right]. \tag{46}$$

The non-trivial structure of Eq. (46) is most essential for the γ terms, since $\gamma_{\text{DRED}}(\hat{g}) \neq \gamma_{\text{DRED}}(\tilde{g})$. In fact, as discussed in Section 1.3 and Appendix A, the γ for $\hat{g}$ and q match the ones in the FDH scheme, while the one for $\tilde{g}$ is new and different:

$$\gamma_{\text{DRED}}(\hat{g}) = \gamma_{\text{FDH}}(g) = \frac{\beta_0}{2} + \epsilon \frac{2 T_F N_F - N_c}{6}, \tag{47a}$$

$$\gamma_{\text{DRED}}(q) = \gamma_{\text{FDH}}(q) = \frac{3 C_F}{2}, \tag{47b}$$

$$\gamma_{\text{DRED}}(\tilde{g}) = 2 N_c - T_F N_F. \tag{47c}$$

On the other hand, $\mathcal{V}(i,j)$ is not affected if a $\hat{g}$ is replaced by a $\tilde{g}$, and the Born terms in the last line on the r.h.s. of Eq. (46) can be combined in a straightforward way. Hence one can immediately obtain the result for either the fully or the partially combined process, $\mathcal{M}_{\text{DRED}}^{(1)}(a_1 \dots a_n)$ or $\mathcal{M}_{\text{DRED}}^{(1)}(\check{a}_1, \check{a}_2; a_3 \dots a_n)$, if desired. In particular, the DRED result for the process involving only full gluons satisfies

$$\mathcal{M}_{\text{DRED}}^{(1)}(a_1 \dots a_n) = \frac{\alpha_s}{2\pi} c_\Gamma \left[\sum_i \sum_{\check{a}_i} \mathcal{M}_{\text{DRED}}^{(0)}(a_1 \dots \check{a}_i \dots a_n) \left(-\frac{1}{\epsilon} \gamma_{\text{DRED}}(\check{a}_i) \right) \right.$$
$$\left. + \sum_{i,j} \mathcal{V}(i,j) \mathcal{M}_{\text{DRED}}^{ij}(a_1 \dots a_n) + \mathcal{M}_{\text{NS}}^{(1)}(a_1 \dots a_n) \right], \tag{48}$$

where $\mathcal{M}_{\text{DRED}}^{ij}(a_1 \dots a_n) = \mathcal{M}_{\text{FDH}}^{ij}(a_1 \dots a_n)$ and $\mathcal{M}_{\text{NS}}^{(1)}(a_1 \dots a_n)$ is the RS independent term appearing also in Eq. (23).

We recall that $\mathcal{M}_{\text{DRED}}^{(1)}$ denotes the fully renormalized one-loop matrix element and Eq. (46) does not contain any ultraviolet singularities. In DRED this implies that off-shell Green functions are finite also if external $\tilde{g}$ are present. This requires that couplings involving $\hat{g}$ and couplings involving $\tilde{g}$ in general renormalize differently [24]. As a result, the renormalization procedure in non-supersymmetric theories can be slightly more involved in DRED. We refer to Appendix B for examples.

3.3. Real corrections

The calculation of the real corrections in DRED follows the same pattern as in the other schemes discussed in Section 2.3. However, there are some important differences which we will consider for the three singular regions in turn.

Soft region: In close analogy to Eq. (29), we have to consider the limit of the real matrix element when a gluon becomes soft. In DRED this soft gluon can be either a $\hat{g}$ or a $\tilde{g}$. The soft limit is governed by eikonal factors of the form $p_i^\mu p_j^\nu/(s_{ik}s_{jk})$, contracted with the corresponding polarization sum of the soft gluon $\breve{g}_k$. The polarization sums in Eq. (10) then show that a soft $\hat{g}_k$ leads to the same limit as a soft full gluon g_k, while a soft ϵ-scalar $\tilde{g}_k$ leads to zero. Hence,

$$\mathcal{M}_{\text{DRED}}^{(0)}\big(a_1, a_2; \ldots g_k(p_k) \ldots \bar{a}_{n+1}\big) \overset{p_k \to 0}{=} g_s^2 \sum_{i,j} \frac{s_{ij}}{s_{ik}s_{jk}} \mathcal{M}_{\text{DRED}}^{ij}(a_1 \ldots a_n), \tag{49}$$

$$\mathcal{M}_{\text{DRED}}^{(0)}\big(a_1, a_2; \ldots \tilde{g}_k(p_k) \ldots \bar{a}_{n+1}\big) \overset{p_k \to 0}{=} 0. \tag{50}$$

In analogy to the behaviour of $\mathcal{V}(i, j)$ in the virtual corrections, the soft limit Eq. (49) does not require the split $g_k = \hat{g}_k + \tilde{g}_k$. Thus, with respect to the soft limit, DRED is equivalent to HV and FDH.

Final-state collinear region: Here the split $g = \hat{g} + \tilde{g}$ is essential. According to the main result of Ref. [7], the key equation for the collinear limit, Eq. (30), has to be modified in DRED if the parent parton $a_{(kl)}$ is a gluon. In this case the flavour of the parent parton is not uniquely determined by $\bar{a}_k$ and $\bar{a}_l$ and we have to sum over the two possibilities $\breve{a}_{(kl)} \in \{\hat{g}, \tilde{g}\}$. Thus, in DRED Eq. (30) becomes

$$\mathcal{M}_{\text{DRED}}^{(0)}\big(a_1, a_2; \ldots \bar{a}_l(p_l) \ldots \bar{a}_k(p_k) \ldots \bar{a}_{n+1}\big)$$
$$\overset{p_k \| p_l}{=} \frac{2g_s^2}{s_{kl}} \sum_{\breve{a}_{(kl)}} P_{(kl)^* \to kl}^{< \text{DRED}}(z) \, \mathcal{M}_{\text{DRED}}^{(0)}\big(a_1, a_2; \ldots \breve{a}_{(kl)}(p_k + p_l) \ldots a_n\big). \tag{51}$$

We remark that Eq. (51) leads us to consider splitting functions with $\breve{a}_{(kl)} = \hat{g}$ and $\breve{a}_{(kl)} = \tilde{g}$ in DRED. They are given in Appendix A. Eq. (51) remains true if all partons a_j are replaced by $\breve{a}_j$, however, only for $\breve{a}_i = \breve{a}_{(kl)}$ the split is strictly required. Thus, Eq. (31) is modified to

$$d\sigma_{\text{DRED}}^{\text{real},i}(a_1 \ldots a_i \ldots a_n) = -\frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \sum_{\breve{a}_i} d\sigma_{\text{DRED}}^{(0)}(a_1 \ldots \breve{a}_i \ldots a_n)$$
$$\times \sum_{a_k} \int dz \, \Theta\left(z - \frac{1}{2}\right) \frac{(1-z)}{(1-z)_+} P_{i^* \to kl}^{< \text{DRED}}(z). \tag{52}$$

This is illustrated in Fig. 4, where we consider the particularly interesting case of an outgoing splitting gluon. For each choice $\breve{a}_i \in \{\hat{g}, \tilde{g}\}$ we have to sum over all splittings $a_k \in \{g, q, \bar{q}\}$. Thus, the two terms of Fig. 2 become the four terms of Fig. 4.

Initial-state collinear region: As for the final-state collinear region, the key point is the necessary split $g = \hat{g} + \tilde{g}$ in the factorization of the collinear limit. In DRED, Eq. (32) has to be generalized to
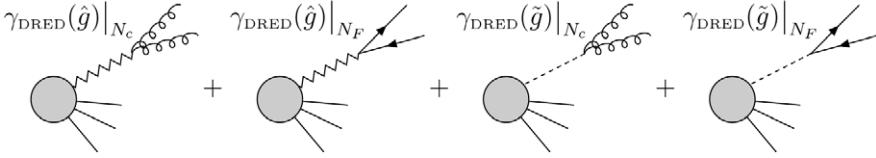
Fig. 4. Illustration of Eq. (52) for the case of an outgoing splitting gluon. The sum over all relevant real processes, $\sum_{a_k}$, together with the sum due to the split $\breve{a}_i \in \{\hat{g}, \tilde{g}\}$ gives rise to four terms, resulting in the N_c and N_F parts of $\gamma_{\text{DRED}}(\hat{g})$ and $\gamma_{\text{DRED}}(\tilde{g})$ respectively. Gluons g and (anti)quarks are drawn as usual. Dashed lines represent $\tilde{g}$ and $\hat{g}$ is represented by a zigzag line.
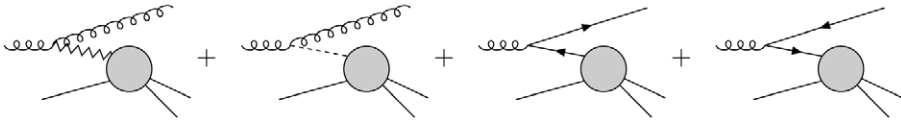

Fig. 5. Illustration of Eq. (54) for an incoming splitting gluon with partons represented as in Fig. 4. The sum over all relevant real processes, $\sum_{a_k}$, together with the sum due to the split $\breve{a}_{(1k)} \in \{\hat{g}, \tilde{g}\}$ gives rise to four terms.

$$
\begin{aligned}
&\langle \mathcal{M}^{(0)}_{\text{DRED}}\big(a_1(p_1), a_2; \ldots \bar{a}_k(p_k) \ldots \bar{a}_{n+1}\big)\rangle \\
&\quad \overset{p_k \| p_1}{=} \frac{2g_s^2}{s_{1k}} \sum_{\breve{a}_{(1k)}} P^{<\,\text{DRED}}_{1\to(1k)^*k}(z) \langle \mathcal{M}^{(0)}_{\text{DRED}}\big(\breve{a}_{(1k)}(zp_1), a_2; \ldots a_n\big)\rangle.
\end{aligned}
\tag{53}
$$

Again, as far as the collinear limit is concerned, not all gluons in Eq. (53) have to be split. Only for the virtual parton $\breve{a}_{(1k)}$ the split is essential.

Note that in DRED the crossed splitting functions satisfy the crossing relation Eq. (34) without Δ terms. Hence, the initial state collinear term for parton 1 can be written as

$$
\begin{aligned}
d\sigma^{\text{real},1}_{\text{DRED}}(a_1 \ldots a_n) = \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \Bigg[&\sum_{\breve{a}_1} \gamma_{\text{DRED}}(\breve{a}_1)\, d\sigma^{(0)}_{\text{DRED}}\big(\breve{a}_1(p_1), a_2; \ldots a_n\big) \\
&- \sum_{a_k} \sum_{\breve{a}_{(1k)}} \int dz\, P^{\text{DRED}}_{1\to(1k)k}\, d\sigma^{(0)}_{\text{DRED}}\big(\breve{a}_{(1k)}(zp_1), a_2; \ldots a_n\big) \Bigg].
\end{aligned}
\tag{54}
$$

As illustrated in Fig. 5, in the sum over $\breve{a}_{(1k)}$ in Eq. (54) it is essential that we treat $\hat{g}$ and $\tilde{g}$ as separate partons, whereas $a_k \in \{g, q, \bar{q}\}$. For an incoming gluon, the three terms of Fig. 3 are generalized in DRED to the four terms of Fig. 5.

3.4. Collinear counterterm

The collinear counterterm in DRED can now be constructed in the same way as in the other schemes. Generalizing Eq. (36) to DRED we can write

$$
\begin{aligned}
&d\sigma^{\text{coll}}_{\text{DRED,FS}}(a_1, a_2; \ldots a_n) \\
&= \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \sum_{a_k} \sum_{\breve{a}_i} \int dz \Big[\big(P^{\text{DRED}}_{1\to ik}(z) + \epsilon X^{\text{FS}}_{1\to ik}(z)\big) d\sigma^{(0)}_{\text{DRED}}\big(\breve{a}_i(zp_1), a_2(p_2); \ldots a_n\big) \\
&\quad + \big(P^{\text{DRED}}_{2\to ik}(z) + \epsilon X^{\text{FS}}_{2\to ik}(z)\big) d\sigma^{(0)}_{\text{DRED}}\big(a_1(p_1), \breve{a}_i(zp_2); \ldots a_n\big) \Big].
\end{aligned}
\tag{55}
$$

The $X^{\text{FS}}_{l \to ik}$ involving partons $\breve{a}_i \in \{\hat{g}, \tilde{g}\}$ appearing here are defined in terms of the functions appearing in Eq. (36) as

$$X^{\text{FS}}_{l \to \breve{g}k} = \frac{\omega_{\text{DRED}}(\breve{g})}{\omega_{\text{DRED}}(g)} X^{\text{FS}}_{l \to gk}. \tag{56}$$

With these definitions, the results of this section, Eqs. (48) and (54), show that the hard partonic cross section

$$\begin{aligned}
d\hat{\sigma}_{\text{FS}}&(a_1 \dots a_n) \\
&= \left[d\sigma^{\text{Born}}_{\text{DRED}}(a_1 \dots a_n) + d\sigma^{\text{virt}}_{\text{DRED}}(a_1 \dots a_n) + d\sigma^{\text{real}}_{\text{DRED}}(a_1 \dots a_n) \right. \\
&\quad \left. + d\sigma^{\text{coll}}_{\text{DRED,FS}}(a_1 \dots a_n) \right]_{D \to 4}
\end{aligned} \tag{57}$$

is equal to the one in the other schemes given in Eq. (37). This shows in particular that it is possible to realize the $\overline{\text{MS}}$ factorization scheme in DRED in the same way as in CDR, HV, or the FDH scheme. In order to make this result explicit we close the section by providing the full form of the appropriate collinear counterterm, valid in all RS,

$$\begin{aligned}
d\sigma^{\text{coll}}_{\text{RS},\overline{\text{MS}}}&(a_1, a_2; \dots a_n) \\
&= \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \sum_{a_k} \sum_{\breve{a}_i} \int dz \left[\left(P^{\text{RS}}_{1 \to ik}(z) - P^{\overline{\text{MS}}\epsilon}_{1 \to ik}(z) \right) d\sigma^{(0)}_{\text{RS}}\left(\breve{a}_i(zp_1), a_2; \dots \right) \right. \\
&\quad \left. + \{1 \leftrightarrow 2\} \right],
\end{aligned} \tag{58}$$

where the sum $\sum_{a_k}$ runs over $a_k \in \{g, q, \bar{q}\}$ in all RS, whereas the sum $\sum_{\breve{a}_i}$ runs over in $\breve{a}_i \in \{g, q, \bar{q}\}$ in CDR, HV, FDH, and over $\breve{a}_i \in \{\hat{g}, \tilde{g}, q, \bar{q}\}$ in DRED. Also, in DRED we define[5]

$$P^{\overline{\text{MS}}\epsilon}_{1 \to \breve{g}k}(z) \equiv \frac{\omega_{\text{DRED}}(\breve{g})}{\omega_{\text{DRED}}(g)} P^{\overline{\text{MS}}\epsilon}_{1 \to gk}(z) \tag{59}$$

in analogy to Eq. (56).

4. Applying DRED

In the previous two sections we discussed how the singularities and RS dependence between the various parts of Eq. (2) cancel, and we found that the subtracted partonic cross sections given in Eqs. (37) and (57) are finite and RS independent. This is precisely what we wanted to achieve. However, there is still one conceptual issue to be addressed.

The question is whether in the convolution of subtracted partonic cross sections with PDF we need to distinguish between $\hat{g}$ and $\tilde{g}$ in DRED. We will show that this is not the case. This will also entail that no PDF for finding an unphysical $\tilde{g}$ in a hadron will be required.

Once this issue is clarified, we will summarize our results and give transition rules between the various RS separately for all parts of the subtracted finite partonic cross sections.

---

[5] The Δ terms appearing in the HV and FDH schemes, see Eqs. (35) and (36), have been ignored. According to the remark at the end of Section 2.4 this is correct if the Δ terms are also ignored in the real corrections.

4.1. Parton distribution functions in DRED

The results (37) and (57) have been given without taking into account that the subtracted partonic cross sections have to be multiplied by PDF. In DRED it might seem natural to distinguish partonic cross sections with $\hat{g}$ or $\tilde{g}$ in the initial state and convolute them with different PDF. We will show that this is not required and that we can use Eq. (57) for initial state full gluons g, convoluted with just one PDF even in DRED. In particular, there is no need to introduce unphysical PDF for finding a $\tilde{g}$ in a hadron.

In the strict spirit of DRED it is correct to consider independent PDF for $\hat{g}$ and $\tilde{g}$ and write a hadronic cross section as a sum of the form

$$f_{\hat{g}/H} \otimes d\hat{\sigma}_{\mathrm{FS}}(\hat{g}_1 \ldots) + f_{\tilde{g}/H} \otimes d\hat{\sigma}_{\mathrm{FS}}(\tilde{g}_1 \ldots) + f_{q/H} \otimes d\hat{\sigma}_{\mathrm{FS}}(q_1 \ldots)$$
$$+ f_{\bar{q}/H} \otimes d\hat{\sigma}_{\mathrm{FS}}(\bar{q}_1 \ldots). \tag{60}$$

The partonic cross sections $d\hat{\sigma}(\breve{g}_1)$ can be constructed in the same way as Eq. (57). They are individually finite and satisfy

$$\sum_{\breve{a}_1} \sum_{\breve{a}_2} \frac{\omega_{\mathrm{DRED}}(\breve{a}_1)}{\omega_{\mathrm{DRED}}(a_1)} \frac{\omega_{\mathrm{DRED}}(\breve{a}_2)}{\omega_{\mathrm{DRED}}(a_2)} d\hat{\sigma}_{\mathrm{FS}}(\breve{a}_1, \breve{a}_2; \ldots a_n) = d\hat{\sigma}_{\mathrm{FS}}(a_1, a_2; \ldots a_n). \tag{61}$$

All the PDF in Eq. (60) would be obtained by performing a fit at one particular factorization scale μ_0 and then using Altarelli–Parisi equations to evolve them to any other scale μ.

The central point is that the unphysical PDF $f_{\tilde{g}/H}$ is of the order ϵ, and hence its contributions to both the hadronic cross section, Eq. (60), and to the evolution of the other PDF are of the order ϵ and thus negligible.

In order to prove this we start by noting that since in other regularization schemes one gluon PDF $f_{g/H}$ is sufficient it is possible to arrange the fit in DRED such that $f_{\tilde{g}/H}(\mu_0) = 0$ at the starting scale μ_0. The evolution is given by the Altarelli–Parisi equations, generalized to include $\tilde{g}$:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} f_{q/H}(z) \\ f_{\hat{g}/H}(z) \\ f_{\tilde{g}/H}(z) \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_z^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{q \to q} & P_{\hat{g} \to q} & P_{\tilde{g} \to q} \\ P_{q \to \hat{g}} & P_{\hat{g} \to \hat{g}} & P_{\tilde{g} \to \hat{g}} \\ P_{q \to \tilde{g}} & P_{\hat{g} \to \tilde{g}} & P_{\tilde{g} \to \tilde{g}} \end{pmatrix} \begin{pmatrix} f_{q/H}(\xi) \\ f_{\hat{g}/H}(\xi) \\ f_{\tilde{g}/H}(\xi) \end{pmatrix}, \tag{62}$$

where we have suppressed the μ dependence of α_s and $f_{a_i/H}$ and have used the short-hand notation $P_{i \to j} \equiv P_{i \to j(ij)}^{\mathrm{DRED}}(z/\xi)$. The evolution of $f_{\tilde{g}/H}(z)$ gets contributions from $P_{q \to \tilde{g}} \times f_{q/H}$, $P_{\hat{g} \to \tilde{g}} \times f_{\hat{g}/H}$ and $P_{\tilde{g} \to \tilde{g}} \times f_{\tilde{g}/H}$. They are $\mathcal{O}(\epsilon) \times 1$, $\mathcal{O}(\epsilon) \times 1$ and $1 \times \mathcal{O}(\epsilon)$, respectively, confirming that $f_{\tilde{g}/H}(\mu) = \mathcal{O}(\epsilon)$ for all scales μ. This in turn implies that the contribution to the evolution of $f_{q/H}$ and $f_{\hat{g}/H}$ due to $f_{\tilde{g}/H}$ is also $\mathcal{O}(\epsilon)$. The situation is in fact very similar to the contributions due to quarks if there were $N_F = \epsilon$ flavours. Finally, $d\hat{\sigma}_{\mathrm{FS}}(\breve{a}_1, \breve{a}_2; \ldots a_n)$ as appearing in Eqs. (60) and (61) is finite for all initial states $\breve{a}_1, \breve{a}_2$ separately. Hence the contribution of the unphysical PDF $f_{\tilde{g}/H}$ in Eq. (60) is $\mathcal{O}(\epsilon)$.

This confirms that DRED can be used throughout all parts of calculations for hadronic cross sections at one loop, without the need for unphysical PDF. It is correct to consider only one gluon PDF $f_{g/H}$ and the combined cross sections $d\hat{\sigma}(a_1, a_2; \ldots a_n)$ from Eq. (57) also in DRED. In other words, Eq. (1) is correct in all schemes, including DRED, if the sums over all parton types include only the full gluon g and quarks q, $\bar{q}$ (and possibly further, massive partons).

4.2. Summary of practical computations in DRED

Let us finally summarize how to do a next-to-leading order calculation in practice in DRED or any other scheme. The main point is that only for the ultraviolet renormalization and the collinear counterterm a split of g into $\hat{g} + \tilde{g}$ is required.

Virtual corrections: To obtain the virtual corrections we start by computing $\mathcal{M}^{(1)}_{\mathrm{RS}}(a_1 \dots a_n)$ with $a_i \in \{g, q, \bar{q}\}$. For the actual calculation of the one-loop diagrams we do not need to split the process into many different parts as in Eq. (46), but can compute directly with g. The structure of the ultraviolet counterterms depends on the RS and the symmetries of the underlying theory. If DRED is used in a supersymmetric context, even the counterterms can typically be computed without the split, and usual multiplicative renormalization is sufficient to generate the counterterms. Using HV or CDR in supersymmetric theories leads to the complication of non-multiplicative, supersymmetry-restoring counterterms. In non-supersymmetric theories, determining counterterms in DRED requires the split $g = \hat{g} + \tilde{g}$.

Once we have the renormalized one-loop matrix element, $\mathcal{M}^{(1)}_{\mathrm{RS}}(a_1 \dots a_n)$ we obtain $d\sigma^{\mathrm{virt}}_{\mathrm{RS}}$ by integration over the phase space. Again, a split of $d\sigma^{\mathrm{virt}}_{\mathrm{RS}}$ as in Eq. (61) is not required. The split $g = \hat{g} + \tilde{g}$ for external gluons becomes useful if we want to express the singularity structure or RS dependence in a simple way as done in the next subsection.

Real corrections: The real corrections in DRED can be obtained in a straightforward way by directly integrating the 4-dimensional tree-level matrix elements containing only 4-dimensional gluons g and (anti)quarks. Likewise, in CDR we have to integrate the D-dimensional tree-level matrix elements. Regarding the real corrections in HV and FDH we remind the reader of a subtlety related to unitarity (see also Ref. [15]). At first sight it might appear that there is no difference in $d\sigma^{\mathrm{real}}_{\mathrm{RS}}$ for schemes where the tree-level matrix elements are evaluated in four dimensions, e.g. between $d\sigma^{\mathrm{real}}_{\mathrm{HV}}$ and $d\sigma^{\mathrm{real}}_{\mathrm{FDH}}$. However, this is not correct. In order to maintain unitarity, in the singular regions initial and final state partons have to be treated in the same way as partons in a closed loop, i.e. as "internal". Thus in HV and FDH it is not correct to simply integrate the corresponding four-dimensional real tree-level matrix elements over the phase space. In particular, Eqs. (30) and (32) contain $\mathcal{O}(\epsilon)$ terms in FDH and HV and result in finite differences between $d\sigma^{\mathrm{real}}_{\mathrm{HV}}$ and $d\sigma^{\mathrm{real}}_{\mathrm{FDH}}$, even though the tree-level matrix elements agree.

In principle this procedure leads to the Δ terms in Eq. (35) because in the HV and FDH schemes incoming and outgoing splittings differ. As discussed in Section 2.4 it is possible to redefine the results by ignoring the Δ terms if the same is done in the collinear counterterms.

Collinear counterterm: The collinear counterterm is given by Eq. (58). It realizes the $\overline{\mathrm{MS}}$ factorization scheme independent of the RS used for the computation. For DRED, we stress that for the term given explicitly on the r.h.s. of Eq. (58) the partons $a_1 \dots a_n$ as well as a_k are never $\hat{g}$ or $\tilde{g}$ separately, but can always be combined to g. For the virtual parton $\breve{a}_i$, however, it is important to treat $\hat{g}$ and $\tilde{g}$ separately [7]. This requires the use of the splitting functions given in Eqs. (A.32) and (A.33). For the HV and FDH schemes, the Δ terms can be ignored in accordance with the computation of the real corrections.

4.3. Translation rules between different schemes

In the following we describe how the results in the various RS are related, making use of the split $g = \hat{g} + \tilde{g}$ as appropriate. We will focus on the virtual corrections; similar results for the

real corrections and the collinear counterterms can be trivially obtained from Eqs. (52), (54), (58). Starting from our renormalized result in CDR, $\mathcal{M}^{(1)}_{\text{CDR}}(a_1 \ldots a_n)$, written in the form of Eq. (23), we can obtain $\mathcal{M}^{(1)}_{\text{HV}}(a_1 \ldots a_n)$ simply by replacing the D-dimensional (colour-linked) Born terms by the corresponding 4-dimensional expressions. No further change is required, since $\gamma_{\text{CDR}}(a_i) = \gamma_{\text{HV}}(a_i)$.

The only difference between the HV and FDH scheme on the other hand does come from the differences of the γ_{RS} terms, which have been explained in Section 1.3 and quantitatively given in Eq. (26). For a process with $\#g$ gluons, $\#q$ massless (anti)quarks and $\#Q$ massive (anti)quarks the difference is

$$
\begin{aligned}
&\mathcal{M}^{(1)}_{\text{FDH}}(a_1 \ldots a_n) - \mathcal{M}^{(1)}_{\text{HV}}(a_1 \ldots a_n) \\
&= \frac{\alpha_s}{2\pi} \mathcal{M}^{(0)}_{\text{FDH}}(a_1 \ldots a_n) \sum_{x \in \{g,q,Q\}} \#x \left(\frac{\gamma_{\text{HV}}(a_x) - \gamma_{\text{FDH}}(a_x)}{\epsilon} \right) \\
&= \frac{\alpha_s}{2\pi} \mathcal{M}^{(0)}_{\text{FDH}}(a_1 \ldots a_n) \left[\#g \frac{N_c}{6} + \#q \frac{C_F}{2} \right].
\end{aligned}
\tag{63}
$$

In the second line the influence of the different γ_{RS} for all parton types is made explicit, in the third line the result in brought into a compact form.

The difference between FDH and DRED is obtained by taking the difference between Eqs. (48) and (23). We can bring it into a simple form by using that we can write the tree-level quantities in FDH in a DRED-like form,

$$
\mathcal{M}^{(0)}_{\text{FDH}}(a_1 \ldots a_i \ldots a_n) = \sum_{\breve{a}_i} \mathcal{M}^{(0)}_{\text{DRED}}(a_1 \ldots \breve{a}_i \ldots a_n).
\tag{64}
$$

The difference is then governed by the factors $\gamma_{\text{DRED}}(\hat{g}) - \gamma_{\text{FDH}}(g)$ and $\gamma_{\text{DRED}}(\tilde{g}) - \gamma_{\text{FDH}}(g)$. As explained in Section 1.3 the first of these vanishes. The second is non-zero,

$$
\gamma_{\text{DRED}}(\tilde{g}) - \gamma_{\text{FDH}}(g) = \frac{1 + \epsilon}{6} (N_c - 2 T_F N_F),
\tag{65}
$$

and is present for every $\tilde{g}$ in the initial or final state, see Eq. (48). Exploiting also that at leading order $\alpha_s = \alpha_e$, we obtain

$$
\begin{aligned}
&\mathcal{M}^{(1)}_{\text{DRED}}(a_1 \ldots a_n) - \mathcal{M}^{(1)}_{\text{FDH}}(a_1 \ldots a_n) \\
&= \frac{\alpha_s}{2\pi} \frac{\gamma_{\text{FDH}}(g) - \gamma_{\text{DRED}}(\tilde{g})}{\epsilon} \sum_{\{\breve{a}\}} \#\tilde{g}(\{\breve{a}\}) \mathcal{M}^{(0)}_{\text{DRED}}(\breve{a}_1 \ldots \breve{a}_n) \\
&= \frac{\alpha_s}{2\pi} \frac{1}{\epsilon} \frac{2 T_F N_F - N_c}{6} \sum_{j=1}^{\#g} \mathcal{M}^{(0)}_{\text{DRED}}(a_1 \ldots a_n) \Big|_{g_{i_j} \to \tilde{g}_{i_j}}.
\end{aligned}
\tag{66}
$$

Again, in the second line the influence of the different γ_{RS} relevant for $\tilde{g}$ is made explicit. In the third line we used that for processes with at least one $\tilde{g}$, i.e. with $\#\tilde{g}(\{\breve{a}\}) \geqslant 1$ we have $\mathcal{M}^{(0)}_{\text{DRED}} \sim \epsilon$; therefore we neglected the $\mathcal{O}(\epsilon)$ terms from Eq. (65). The notation

$\mathcal{M}_{\text{DRED}}^{(0)}(a_1 \ldots a_n)|_{g_{i_j} \to \tilde{g}_{i_j}}$ implies that all gluons except gluon i_j are 4-dimensional gluons. Thus in the final expression on the r.h.s. of Eq. (66) we sum over all processes where one 4-dimensional gluon g at a time is replaced by a $\tilde{g}$.

The transition (63) and (66) ignore $\mathcal{O}(\epsilon)$ terms and are given for pure QCD processes. However, they can easily be generalized to other processes, involving e.g. photons or massive partons, simply by using the corresponding explicit expressions for γ_{RS}.

5. Conclusions

The main result presented in this paper is that DRED can be used for the calculation of cross sections at NLO, even for processes with hadrons in the initial state. Problems related to factorization, as reported in the literature [8–10], can be avoided by taking into account the appropriate, generalized factorization in DRED [7]. We have shown explicitly how to use DRED together with an arbitrary factorization scheme. In particular, the conventional PDF [17,18] in the $\overline{\text{MS}}$-factorization scheme can be used. Also we have given explicit rules on how to transform separately the various parts of the hard partonic cross section, Eq. (2), from DRED to other RS. This completes the previously known set of transition rules between CDR, HV and FDH [13–15]. It is thus possible to use different RS for different parts of the calculation which might help simplifying the explicit computations. In this context we also reiterate the distinction between FDH and DRED. According to the definitions of the RS given in Section 1.2, at one loop FDH is equivalent to the scheme DR used e.g. in Refs. [13–15] but differs from DRED used e.g. in Refs. [3,11,12].

The salient feature of a consistent use of DRED is the split $g = \hat{g} + \tilde{g}$. In practice, this split does not significantly complicate calculations. It is needed mainly for the correct treatment of the collinear limit of squared matrix elements. Thus it affects the collinear counterterm and the phase-space integration over the singular, collinear region. The modifications regarding the former are shown in Eqs. (35) and (54). For the phase-space integration, the usual procedures have to be slightly modified. For the method presented in Ref. [30] for example, the collinear singularities in the real corrections are singled out using distributions, enforcing the collinear limit of the real matrix element squared. If this method is to be used together with DRED this simply means that the proper collinear limit, Eqs. (51) and (53), has to be taken. For the dipole subtraction method [27] additional dipoles with $\tilde{g}$ are required. These can be obtained making minor modifications of existing dipoles, similar to the corresponding adaptation to FDH [32].

In some cases, the split $g = \hat{g} + \tilde{g}$ is also required for the ultraviolet counterterms, since e.g. the couplings $\hat{g}q\bar{q}$ and $\tilde{g}q\bar{q}$ renormalize differently. This seems to be a disadvantage of DRED. On the other hand, one of the advantages of DRED is that in supersymmetric theories no supersymmetry-restoring counterterms are required (in many practical cases; for a recent discussion see Ref. [3]). In this case, also couplings with $\hat{g}$ and $\tilde{g}$ renormalize identically and renormalization is actually simpler in DRED than e.g. in CDR. This facilitates the use of the $\overline{\text{DR}}$-scheme for supersymmetric parameters which is used in a wide variety of calculations [33].

Thus, DRED is a RS which is well compatible with supersymmetry and which can be realized with minimal modifications compared to CDR and used for an arbitrary cross section at NLO. In the past, following the examples of e.g. Refs. [9,34], many predictions for supersymmetric processes at hadron colliders were calculated using CDR in spite of the required supersymmetry-restoring counterterms. In the future, similar calculations can alternatively be carried out using DRED, which can lead to simplifications with the present, better understanding of DRED.

Acknowledgements

Appendix A. Collinear limits and sum rules

In this appendix we study the collinear limit of squared matrix elements and derive the associated splitting functions and γ_{RS} terms. Even though most of the results presented here are well known, we repeat them for the reader's convenience and to fix our notation and conventions.

Following Ref. [15], we consider a slightly off-shell massless outgoing parton $a_i(p_i)$ that splits into massless on-shell partons $a_k(p_k)$ and $a_l(p_l)$. The momenta are parametrized as

$$p_k^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2(p \cdot n)}, \tag{A.1}$$

$$p_l^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{(1-z)}\frac{n^\mu}{2(p \cdot n)}, \tag{A.2}$$

with $p^2 = n^2 = (k_\perp \cdot p) = (k_\perp \cdot n) = 0$. The invariant mass of the incoming parton is $p_i^2 = 2(p_k \cdot p_l) = -k_\perp^2/(z(1-z))$ and vanishes in the collinear limit $k_\perp^\mu \to 0$.

To start with we consider the particularly interesting case where the parent parton is a gluon. Denoting by $\mathcal{A}_\mu^m$ the amplitude of the parent process, stripped of its polarization vector $\varepsilon^\mu(p_i)$, we can write the collinear limit of the full process as

$$\mathcal{M}_{RS}^{(0)}\big(1 \ldots g(p_k), g(p_l) \ldots n+1\big) \overset{p_k \| p_l}{=} \frac{4\pi\alpha_s}{p_k \cdot p_l}\delta_{mn}\mathcal{A}_\mu^m \mathcal{P}_{g_i^* \to g_k g_l}^{< \, RS\mu\nu}\mathcal{A}_\nu^{*n}, \tag{A.3}$$

where m and n are colour labels. After averaging over $k_\perp^\mu$, the operator $\mathcal{P}_{g_i^* \to g_k g_l}^{< \, RS\mu\nu}$ is proportional to the metric, which in accordance with Eq. (10) corresponds to the polarization sum $\sum \varepsilon^\mu \varepsilon^{*\nu}$. Explicitly we find

$$\mathcal{P}_{g^* \to gg}^{< \, CDR\mu\nu}(z) = -\hat{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right), \tag{A.4}$$

$$\mathcal{P}_{g^* \to gg}^{< \, HV\mu\nu}(z) = -\bar{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right), \tag{A.5}$$

$$\mathcal{P}_{g^* \to gg}^{< \, FDH\mu\nu}(z) = -\bar{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z} + \frac{2}{D-2}z(1-z)\right), \tag{A.6}$$

$$\mathcal{P}_{g^* \to gg}^{< \, DRED\mu\nu}(z) = -\hat{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z} + \frac{2}{D-2}z(1-z)\right)$$

$$\qquad\qquad - \tilde{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z}\right). \tag{A.7}$$

The interesting point is that in DRED we get a combination of $\hat{g}^{\mu\nu}$ and $\tilde{g}^{\mu\nu}$. Thus the collinear limit has to be written as a sum over two terms as in Eq. (30). In the spirit of DRED, we can further disentangle the splitting operator and write

$$\mathcal{P}_{g^*\to gg}^{<\,\mathrm{DRED}\,\mu\nu} = \mathcal{P}_{\hat{g}^*\to\hat{g}\hat{g}}^{<\,\mathrm{DRED}\,\mu\nu} + \mathcal{P}_{\hat{g}^*\to\tilde{g}\tilde{g}}^{<\,\mathrm{DRED}\,\mu\nu} + \mathcal{P}_{\tilde{g}^*\to\hat{g}\tilde{g}}^{<\,\mathrm{DRED}\,\mu\nu} + \mathcal{P}_{\tilde{g}^*\to\tilde{g}\hat{g}}^{<\,\mathrm{DRED}\,\mu\nu}$$

$$= -\hat{g}^{\mu\nu}(2N_c)\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right)$$

$$- \hat{g}^{\mu\nu}(2N_c)\frac{4-D}{D-2}z(1-z) - \tilde{g}^{\mu\nu}(2N_c)\frac{1-z}{z} - \tilde{g}^{\mu\nu}(2N_c)\frac{z}{1-z}. \qquad (A.8)$$

The splitting functions $P_{g^*\to gg}^{<\,\mathrm{RS}}$ can be read off of Eqs. (A.4)–(A.8) simply by dropping the polarization sum. Performing similar calculations for all other possible splits we find the following results:

$$P_{g^*\to gg}^{<\,\mathrm{CDR}} = P_{\hat{g}^*\to\hat{g}\hat{g}}^{<\,\mathrm{DRED}} = 2N_c\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right), \qquad (A.9)$$

$$P_{g^*\to gg}^{<\,\mathrm{FDH}} = P_{\hat{g}^*\to gg}^{<\,\mathrm{DRED}} = 2N_c\left(\frac{z}{1-z} + \frac{1-z}{z} + \frac{2}{D-2}z(1-z)\right), \qquad (A.10)$$

$$P_{g^*\to q\bar{q}}^{<\,\mathrm{CDR}} = P_{g^*\to q\bar{q}}^{<\,\mathrm{FDH}} = P_{\hat{g}^*\to q\bar{q}}^{<\,\mathrm{DRED}} = T_F\left(1 - \frac{4}{D-2}z(1-z)\right), \qquad (A.11)$$

$$P_{q^*\to qg}^{<\,\mathrm{CDR}} = P_{q^*\to q\hat{g}}^{<\,\mathrm{DRED}} = C_F\left(\frac{2z}{1-z} + \frac{D-2}{2}(1-z)\right), \qquad (A.12)$$

$$P_{q^*\to qg}^{<\,\mathrm{FDH}} = P_{q^*\to qg}^{<\,\mathrm{DRED}} = C_F\left(\frac{2z}{1-z} + (1-z)\right). \qquad (A.13)$$

The results for HV are always identical to the ones for CDR,

$$P_{i^*\to kl}^{<\,\mathrm{CDR}} = P_{i^*\to kl}^{<\,\mathrm{HV}}. \qquad (A.14)$$

The results particular to DRED are given by

$$P_{\hat{g}^*\to\tilde{g}\tilde{g}}^{<\,\mathrm{DRED}} = 2N_c\frac{4-D}{D-2}z(1-z), \qquad (A.15)$$

$$P_{\tilde{g}^*\to\hat{g}\tilde{g}}^{<\,\mathrm{DRED}} = 2N_c\left(\frac{1-z}{z}\right), \qquad (A.16)$$

$$P_{\tilde{g}^*\to q\bar{q}}^{<\,\mathrm{DRED}} = T_F, \qquad (A.17)$$

$$P_{q^*\to q\tilde{g}}^{<\,\mathrm{DRED}} = C_F\frac{4-D}{2}(1-z). \qquad (A.18)$$

The remaining splitting functions can be obtained by $P_{i^*\to kl}^{<\,\mathrm{RS}}(z) = P_{i^*\to lk}^{<\,\mathrm{RS}}(1-z)$. The splitting functions appropriate for the split of an incoming parton $P_{i\to k^*l}^{<\,\mathrm{RS}}(z)$ can be obtained through the crossing relation (33) with

$$\omega_{\mathrm{HV}}(g) = \omega_{\mathrm{FDH}}(g) = \omega_{\mathrm{DRED}}(g) = 2(N_c^2 - 1), \qquad \omega_{\mathrm{RS}}(q) = 2N_c,$$

$$\omega_{\mathrm{CDR}}(g) = \omega_{\mathrm{DRED}}(\hat{g}) = (D-2)(N_c^2 - 1),$$

$$\omega_{\mathrm{DRED}}(\tilde{g}) = (4-D)(N_c^2 - 1). \qquad (A.19)$$

We note that in CDR and in DRED there is no difference between the splitting functions for incoming and outgoing partons, i.e. Eq. (34) holds with all Δ terms equal to zero. Thus, Eqs. (A.15) and (A.16) and Eqs. (A.17) and (A.18) are not independent. For the HV and FDH scheme the Δ terms do not vanish. In the FDH scheme, for example, the different coefficient of the

three terms on the r.h.s. of Eq. (A.10) lead to $\Delta^{\mathrm{FDH}}_{g \to gg} \neq 0$. In the HV scheme, the different $\omega_{\mathrm{HV}}(g) \neq \omega_{\mathrm{CDR}}(g)$, together with Eq. (A.14), are the origin of the non-vanishing Δ^{HV} terms.

The splitting functions $P^{<\mathrm{RS}}_{i^* \to kl}$ are defined only for $z < 1$. We define the full splitting functions through the relation

$$P^{\mathrm{RS}}_{i \to kl}(z) \equiv \frac{(1-z)}{(1-z)_+} P^{<\mathrm{RS}}_{i^* \to kl}(z) + \delta_{ik} \gamma_{\mathrm{RS}}(a_i) \delta(1-z), \tag{A.20}$$

where we made use of the standard +prescription. The factors $\gamma_{\mathrm{RS}}(a_i)$ and thus $P^{\mathrm{RS}}_{i \to kl}$ are determined by requiring that the momentum sum rules

$$\int_0^1 dz\, z \big[P^{\mathrm{RS}*}_{g \to gg}(z) + 2N_F P^{\mathrm{RS}*}_{g \to q\bar{q}}(z) \big] = 0, \tag{A.21}$$

$$\int_0^1 dz\, z \big[P^{\mathrm{RS}*}_{q \to qg}(z) + P^{\mathrm{RS}*}_{q \to gq}(z) \big] = 0, \tag{A.22}$$

are satisfied in all the schemes CDR, HV, FDH, i.e. also taking into account terms of higher-order in ϵ if appropriate. Eq. (A.21) determines $\gamma_{\mathrm{RS}*}(g)$ and Eq. (A.22) determines $\gamma_{\mathrm{RS}*}(q)$. For DRED, the sum rules given in Eqs. (A.21) and (A.22) have to be generalized in an obvious way, since we also have to take into account $\tilde{g}$:

$$\int_0^1 dz\, z \big[P^{\mathrm{DRED}}_{\hat{g} \to \hat{g}\hat{g}}(z) + P^{\mathrm{DRED}}_{\hat{g} \to \tilde{g}\tilde{g}}(z) + 2N_F P^{\mathrm{DRED}}_{\hat{g} \to q\bar{q}}(z) \big] = 0, \tag{A.23}$$

$$\int_0^1 dz\, z \big[P^{\mathrm{DRED}}_{q \to qg}(z) + P^{\mathrm{DRED}}_{q \to gq}(z) \big]$$

$$= \int_0^1 dz\, z \big[P^{\mathrm{DRED}}_{q \to q\hat{g}}(z) + P^{\mathrm{DRED}}_{q \to q\tilde{g}}(z) + P^{\mathrm{DRED}}_{q \to \hat{g}q}(z) + P^{\mathrm{DRED}}_{q \to \tilde{g}q}(z) \big] = 0, \tag{A.24}$$

$$\int_0^1 dz\, z \big[P^{\mathrm{DRED}}_{\tilde{g} \to \tilde{g}\hat{g}}(z) + P^{\mathrm{DRED}}_{\tilde{g} \to \tilde{g}\tilde{g}}(z) + 2N_F P^{\mathrm{DRED}}_{\tilde{g} \to q\bar{q}}(z) \big] = 0. \tag{A.25}$$

As before, Eqs. (A.23) and (A.24) determine $\gamma_{\mathrm{DRED}}(\hat{g})$ and $\gamma_{\mathrm{DRED}}(q)$ respectively, while Eq. (A.25) determines $\gamma_{\mathrm{DRED}}(\tilde{g})$. The results are

$$\gamma_{\mathrm{CDR}}(g) = \frac{11N_c}{6} - \frac{(3D-8)T_F N_F}{3(D-2)}, \qquad\qquad \gamma_{\mathrm{CDR}}(q) = \frac{(10-D)C_F}{4}, \tag{A.26}$$

$$\gamma_{\mathrm{FDH}}(g) = \frac{(6D-13)N_c}{3(D-2)} - \frac{(3D-8)T_F N_F}{3(D-2)}, \qquad \gamma_{\mathrm{FDH}}(q) = \frac{3C_F}{2}, \tag{A.27}$$

$$\gamma_{\mathrm{DRED}}(\hat{g}) = \gamma_{\mathrm{FDH}}(g), \qquad\qquad\qquad\qquad \gamma_{\mathrm{DRED}}(q) = \gamma_{\mathrm{FDH}}(q), \tag{A.28}$$

$$\gamma_{\mathrm{DRED}}(\tilde{g}) = 2N_c - T_F N_F \tag{A.29}$$

and $\gamma_{\mathrm{HV}}(a_i) = \gamma_{\mathrm{CDR}}(a_i)$. Expanding these results and taking into account all terms to $\mathcal{O}(\epsilon)$ we obtain the results given in Eq. (26). We note that these results are also consistent with the quark-number conservation sum rule as well as with Eqs. (31) and (52). In FDH and DRED they also satisfy the supersymmetric relation

$$P_{g \to gg}^{\mathrm{FDH}} + 2N_F P_{g \to q\bar{q}}^{\mathrm{FDH}} = P_{q \to gq}^{\mathrm{FDH}} + P_{q \to qg}^{\mathrm{FDH}}, \tag{A.30}$$

$$P_{\hat{g} \to \hat{g}g}^{\mathrm{DRED}} + 2N_F P_{\hat{g} \to q\bar{q}}^{\mathrm{DRED}} = P_{q \to gq}^{\mathrm{DRED}} + P_{q \to qg}^{\mathrm{DRED}}, \tag{A.31}$$

if we set $N_c = C_F = 2T_F N_F$.

Finally we mention that the DRED splitting function used in Eqs. (54) and (55) are defined as

$$P_{g \to \hat{g}g}^{\mathrm{DRED}} \equiv \frac{\omega_{\mathrm{DRED}}(\hat{g})}{\omega_{\mathrm{DRED}}(g)} P_{\hat{g} \to \hat{g}\hat{g}}^{\mathrm{DRED}} + \frac{\omega_{\mathrm{DRED}}(\tilde{g})}{\omega_{\mathrm{DRED}}(g)} P_{\tilde{g} \to \hat{g}\tilde{g}}^{\mathrm{DRED}}, \tag{A.32}$$

$$P_{g \to \tilde{g}g}^{\mathrm{DRED}} \equiv \frac{\omega_{\mathrm{DRED}}(\hat{g})}{\omega_{\mathrm{DRED}}(g)} P_{\hat{g} \to \tilde{g}\hat{g}}^{\mathrm{DRED}} + \frac{\omega_{\mathrm{DRED}}(\tilde{g})}{\omega_{\mathrm{DRED}}(g)} P_{\tilde{g} \to \tilde{g}\hat{g}}^{\mathrm{DRED}}. \tag{A.33}$$

Using the explicit results above we find

$$P_{g \to \hat{g}*g}^{< \mathrm{DRED}} = 2N_c \left(\frac{D-2}{2} \frac{z}{1-z} + \frac{1-z}{z} + \frac{D-2}{2} z(1-z) \right), \tag{A.34}$$

$$P_{g \to \tilde{g}*g}^{< \mathrm{DRED}} = 2N_c \frac{4-D}{2} \left(\frac{z}{1-z} + z(1-z) \right), \tag{A.35}$$

$$P_{q \to \hat{g}*q}^{< \mathrm{DRED}} = C_F \left(\frac{2(1-z)}{z} + \frac{D-2}{2} z \right), \tag{A.36}$$

$$P_{q \to \tilde{g}*q}^{< \mathrm{DRED}} = C_F \frac{4-D}{2} z, \tag{A.37}$$

for the splitting functions used in Eq. (53).

Appendix B. Examples

B.1. $gg \to q\bar{q}$

The process $gg \to q\bar{q}$ with massless quarks has been computed long ago at one loop [35] in CDR and was one of the processes used to determine the relations between the HV and FDH scheme [13]. The one-loop matrix elements $\mathcal{M}_{\mathrm{RS}*}^{(1)}(g, g; q, \bar{q})$ were found to be related as given in Eq. (23). This and the related process with massive quarks was also at the centre of claims regarding problems with factorization in DRED [8–10]. The factorization issue related to the real corrections for these processes has been solved in Ref. [7]. Here we focus on some aspects related to issues with DRED, starting with the virtual corrections.

The calculation of the one-loop diagrams is straightforward and we stress once more that there is no need to disentangle g into $\hat{g} + \tilde{g}$ in the explicit calculation of the one-loop diagrams. The only issue in the computation of $\mathcal{M}_{\mathrm{DRED}}^{(1)}(g, g, q, \bar{q})$ is renormalization. For massless quarks we only have to consider coupling renormalization. In CDR, HV and FDH this simply amounts to adding the counterterm

$$\mathcal{M}_{\mathrm{RS}*}^{(1)}(g, g; q, \bar{q}) = \bar{\mathcal{M}}_{\mathrm{RS}*}^{(1)}(g, g, q, \bar{q}) + 2\delta Z_g^{\mathrm{RS}*} \mathcal{M}_{\mathrm{RS}*}^{(0)}(g, g; q, \bar{q}), \tag{B.1}$$
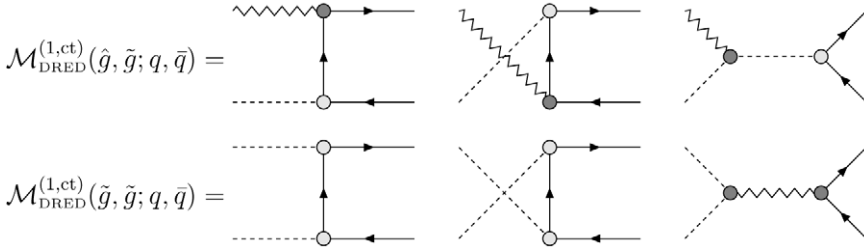
Fig. 6. Ultraviolet counterterm diagrams due to coupling renormalization $\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\hat{g}, \tilde{g}; q, \bar{q})$ (upper line) and $\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\tilde{g}, \tilde{g}; q, \bar{q})$ (lower line). Dark vertices represent counterterms $\delta Z_g^{\mathrm{DRED}}$ and bright vertices stand for counterterms $\delta \tilde{Z}_g^{\mathrm{DRED}}$.

where $Z_g^{\mathrm{RS}*}$ is the RS dependent coupling renormalization factor (in the $\overline{\mathrm{MS}}$ scheme). In DRED we have to split the counterterm contributions as

$$\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(g, g; q, \bar{q}) = \mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\hat{g}, \hat{g}; q, \bar{q}) + \mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\hat{g}, \tilde{g}; q, \bar{q})$$
$$+ \mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\tilde{g}, \hat{g}; q, \bar{q}) + \mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\tilde{g}, \tilde{g}; q, \bar{q}) \tag{B.2}$$

and renormalize all four parts on the r.h.s. separately. For this we need the coupling renormalization factors Z_g^{DRED} for the $\hat{g}q\bar{q}$ coupling and $\tilde{Z}_g^{\mathrm{DRED}}$ for the $\tilde{g}q\bar{q}$ coupling. They are well known [24] and read[6]

$$\delta Z_g^{\mathrm{DRED}} = \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} \frac{(-11 + \epsilon)N_c + 4T_F N_F}{6}, \tag{B.3}$$

$$\delta \tilde{Z}_g^{\mathrm{DRED}} = \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} \left(\frac{1}{2N_c} - \frac{3N_c}{2} + T_F N_F + \epsilon\ \mathrm{Finite} \right). \tag{B.4}$$

The finite $\mathcal{O}(\alpha_s)$ term in Eq. (B.3) is required because we use the $\overline{\mathrm{MS}}$ and not the $\overline{\mathrm{DR}}$ scheme. The divergent part of Eq. (B.4) is determined by requiring the cancellation of UV singularities in the off-shell $\tilde{g}\tilde{g}$ Green function. The finite $\mathcal{O}(\alpha_s)$ terms in Eq. (B.4) would have to be determined by a renormalization scheme. However, they will not affect the final result. As is to be expected, this allows us to perform the calculation without specifying a renormalization scheme for the unphysical gluons. The counterterm $\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\hat{g}, \hat{g}; q, \bar{q})$ is simply given by $2\delta Z_g^{\mathrm{DRED}} \mathcal{M}^{(0)}_{\mathrm{DRED}}(\hat{g}, \hat{g}; q, \bar{q})$. As illustrated in Fig. 6, the counterterm $\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\hat{g}, \tilde{g}; q, \bar{q})$ is given by $(\delta Z_g^{\mathrm{DRED}} + \delta \tilde{Z}_g^{\mathrm{DRED}}) \mathcal{M}^{(0)}_{\mathrm{DRED}}(\hat{g}, \tilde{g}; q, \bar{q})$, while $\mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(\tilde{g}, \tilde{g}; q, \bar{q})$ is not proportional to the corresponding tree-level amplitude. We explicitly verified that after renormalization the one-loop matrix element

$$\mathcal{M}^{(1)}_{\mathrm{DRED}}(g, g; q, \bar{q}) = \bar{\mathcal{M}}^{(1)}_{\mathrm{DRED}}(g, g; q, \bar{q}) + \mathcal{M}^{(1,\mathrm{ct})}_{\mathrm{DRED}}(g, g; q, \bar{q}) \tag{B.5}$$

in DRED is related to the other schemes as given in Eq. (66).

The calculation of the real matrix elements is trivial. They are simply the four-dimensional results, Eq. (44), and the corresponding real cross section can be obtained in DRED by integrating these matrix elements over the phase space. The only remaining and main issue is the factorization of the initial state collinear singularities. According to our discussion, Eq. (58), it is

---

[6] We have set $\alpha_s = \alpha_e$ in these results. This is allowed since we are working at one loop.

clear that we will have to add

$$
\begin{aligned}
&d\sigma_{\mathrm{DRED},\overline{\mathrm{MS}}}^{\mathrm{coll}}(g,g;q,\bar{q}) \\
&= \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon}\int dz\,\big[\big(P_{g\to\hat{g}g}^{\mathrm{DRED}}(z)-P_{g\to gg}^{\overline{\mathrm{MS}}\epsilon}(z)\big)\,d\sigma_{\mathrm{DRED}}^{(0)}\big(\hat{g}(zp_1),g(p_2);q,\bar{q}\big) \\
&\quad + \big(P_{g\to q\bar{q}}^{\mathrm{DRED}}(z)-P_{g\to q\bar{q}}^{\overline{\mathrm{MS}}\epsilon}(z)\big)\,d\sigma_{\mathrm{DRED}}^{(0)}\big(q(zp_1),g(p_2);g,q\big) \\
&\quad + \big(P_{g\to\bar{q}q}^{\mathrm{DRED}}(z)-P_{g\to\bar{q}q}^{\overline{\mathrm{MS}}\epsilon}(z)\big)\,d\sigma_{\mathrm{DRED}}^{(0)}\big(\bar{q}(zp_1),g(p_2);\bar{q},g\big) \\
&\quad + P_{g\to\tilde{g}g}^{\mathrm{DRED}}(z)\,d\sigma_{\mathrm{DRED}}^{(0)}\big(\tilde{g}(zp_1),g(p_2);q,\bar{q}\big)\big] + \{1\leftrightarrow 2\}.
\end{aligned}
\tag{B.6}
$$

The conversion to the $\overline{\mathrm{MS}}$ scheme requires the terms

$$
P_{g\to gg}^{\overline{\mathrm{MS}}\epsilon} \equiv P_{g\to gg}^{\mathrm{CDR}} - \big[P_{g\to gg}^{\mathrm{CDR}}\big]_{D\to 4} = \epsilon\frac{T_F N_F}{3}\delta(1-z),
\tag{B.7}
$$

$$
P_{g\to q\bar{q}}^{\overline{\mathrm{MS}}\epsilon} \equiv P_{g\to q\bar{q}}^{\mathrm{CDR}} - \big[P_{g\to q\bar{q}}^{\mathrm{CDR}}\big]_{D\to 4} = -\epsilon T_F 2z(1-z).
\tag{B.8}
$$

It is the last term on the r.h.s. of Eq. (B.6) that is non-standard and deserves special mention since it resolves the issue regarding the seemingly non-factorizing corrections in DRED.

Let us add a comment on why the factorization problem of Refs. [8–10] was found in the context of the process with massive quarks, discussed below, rather than the one with massless quarks. The reason is the fact that in the present, massless case, the DRED cross sections for the $\tilde{g}$ and $\hat{g}$ initial states happen to be equal,

$$
d\sigma_{\mathrm{DRED}}^{(0)}\big(\hat{g}(p_1),g(p_2);q,\bar{q}\big) = d\sigma_{\mathrm{DRED}}^{(0)}\big(\tilde{g}(p_1),g(p_2);q,\bar{q}\big).
\tag{B.9}
$$

Hence, in Eq. (B.6) one can combine the terms in the first and the last line to

$$
P_{g\to gg}^{\mathrm{DRED}}(z)\,d\sigma_{\mathrm{DRED}}^{(0)}\big(g(zp_1),g(p_2);q,\bar{q}\big),
\tag{B.10}
$$

and the process is seen to factorize even without distinguishing between $\hat{g}$ and $\tilde{g}$.

B.2. $gg\to Q\bar{Q}$

Problems to reconcile factorization with DRED were first mentioned in the context of this process with massive final state quarks [8]. As explained in Ref. [7] and the present paper, the factorization problem disappears if $\hat{g}$ and $\tilde{g}$ are treated as separate partons in formulas such as Eqs. (53) and (54) or in the last line of Eq. (B.6). The reason why the apparent problem has been found only in the massive process $gg\to Q\bar{Q}$ is not related to quark masses but to Eq. (B.9), which happens to hold in the massless case.

With this in mind, the massive process can be treated in the same way as the massless one. The only additional complication in the case of massive quarks is to consider the RS dependence of the mass renormalization Z_m^{RS} and external wave-function renormalization Z_Q^{RS} for massive quark lines. The RS dependence of these renormalization factors has been considered before (see e.g. Ref. [36]) and, using for example the pole scheme to define the mass of the heavy quark m, can be summarized as follows:

$$
Z_m^{\mathrm{CDR}} = Z_m^{\mathrm{HV}} = 1 + \frac{\alpha_s}{4\pi}\frac{c_\Gamma}{\epsilon}C_F\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\left(-\frac{3}{\epsilon}-4\right),
\tag{B.11}
$$

$$Z_m^{\text{DRED}} = Z_m^{\text{FDH}} = 1 + \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} C_F \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-\frac{3}{\epsilon} - 5\right), \tag{B.12}$$

and $Z_Q^{\text{RS}} = Z_m^{\text{RS}}$. As mentioned in the main text, $\gamma(Q) = C_F$ is RS independent [14]. We have verified by explicit calculation that using the RS dependent coupling renormalization and Eqs. (B.11) and (B.12) for the mass counterterms and wave-function renormalization, the RS dependence of the virtual corrections take the form as given in Eq. (46). To use DRED throughout in the calculation of this process we simply have to use the correct RS dependent collinear counterterm as given in Eq. (B.6) and fold the hard partonic cross sections with the standard PDF in the $\overline{\text{MS}}$ factorization scheme.

B.3. $gg \to h$

In this example we consider the production of a Higgs h through gluon fusion in a hadronic collision. While this process is relatively simple at one-loop it is complicated enough to illustrate all main points discussed in the main text. The interaction of the Higgs with gluons is given by the Lagrangian

$$\mathcal{L}_I = \frac{1}{2} g_h h \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}), \tag{B.13}$$

where the coupling g_h has mass dimension -1. In DRED we have to distinguish between the coupling for $\hat{g}\hat{g}h$, denoted by g_h and the coupling for $\tilde{g}\tilde{g}h$, denoted by $\tilde{g}_h$. At tree level the two couplings are the same, but they differ at higher orders.

The only process that contributes at tree level is $g(p_1)g(p_2) \to h$. The matrix elements are given by

$$\mathcal{M}_{\text{RS}}^{(0)}(\breve{g}, \breve{g}; h) = g_h^2 \omega_{\text{RS}}(\breve{g}) \frac{s_{12}^2}{4} \tag{B.14}$$

with $\breve{g} = g$ in CDR, HV and FDH and $\breve{g} \in \{\hat{g}, \tilde{g}\}$ for DRED and $\mathcal{M}_{\text{DRED}}^{(0)}(\tilde{g}, \hat{g}; h) = 0$.

For the calculation of the $\mathcal{O}(\alpha_s)$ corrections to $gg \to h$ the distinction between g_h and $\tilde{g}_h$ at tree level will be relevant for the renormalization. Importantly, however, for the non-trivial part of the explicit calculation of the virtual and real corrections we can set $g_h = \tilde{g}_h$ and we do not have to distinguish between $\hat{g}$ and $\tilde{g}$ in loop diagrams.

First we discuss the virtual corrections and how their RS dependences arise. The explicit calculation of the two non-vanishing one-loop diagrams in Feynman gauge results in the following unrenormalized one-loop matrix elements:

$$\bar{\mathcal{M}}_{\text{RS}}^{(1)}(\breve{g}_1, \breve{g}_2; h) = \mathcal{M}_{\text{RS}}^{(0)}(\breve{g}_1, \breve{g}_2; h) \frac{\alpha_s}{2\pi} c_\Gamma \left(-\frac{2N_c}{\epsilon^2}\right) \left|-\frac{s_{12}}{\mu^2}\right|^{-\epsilon} \tag{B.15}$$

where

$$\left|-\frac{s_{12}}{\mu^2}\right|^{-\epsilon} \equiv \operatorname{Re}\left(-\frac{s_{12}}{\mu^2}\right)^{-\epsilon} = \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} - \frac{\epsilon^2 \pi^2}{2} + \mathcal{O}(\epsilon^3). \tag{B.16}$$

In order to obtain the counterterms we only need to perform a renormalization transformation of the couplings, $g_h \to Z_{gh} g_h$, $\tilde{g}_h \to \tilde{Z}_{gh} \tilde{g}_h$. We use the $\overline{\text{MS}}$ scheme to define the renormalization constants in all RS. The results read

$$Z_{gh}^{\text{CDR}} = Z_{gh}^{\text{HV}} = 1 + \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} \frac{-11N_c + 4T_F N_F}{3}, \tag{B.17}$$

$$Z_{gh}^{\text{DRED}} = Z_{gh}^{\text{FDH}} = 1 + \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} \frac{(-11+\epsilon)N_c + 4T_F N_F}{3}, \tag{B.18}$$

$$\tilde{Z}_{gh}^{\text{DRED}} = 1 + \frac{\alpha_s}{4\pi} \frac{c_\Gamma}{\epsilon} (-4N_c + 2T_F N_F + \epsilon \text{ Finite}). \tag{B.19}$$

As expected, g_h renormalizes like the square of the strong coupling, and the difference between Z_{gh}^{CDR} and Z_{gh}^{DRED} is in agreement with the corresponding scheme difference of α_s [13,37], see also Ref. [24]. Writing $Z_{gh}^{\text{RS}} = 1 + \delta Z_{gh}^{\text{RS}}$ we have

$$\mathcal{M}_{\text{RS*}}^{(1)}(g,g;h) = \bar{\mathcal{M}}_{\text{RS*}}^{(1)}(g,g;h) + 2\,\delta Z_{gh}^{\text{RS*}} \mathcal{M}_{\text{RS*}}^{(0)}(g,g;h) \tag{B.20}$$

for CDR, HV and FDH. In the case of DRED we obtain

$$\begin{aligned}
\mathcal{M}_{\text{DRED}}^{(1)}(g,g;h) &= \bar{\mathcal{M}}_{\text{DRED}}^{(1)}(\hat{g},\hat{g};h) + 2\delta Z_{gh}^{\text{DRED}} \mathcal{M}_{\text{DRED}}^{(0)}(\hat{g},\hat{g};h) \\
&\quad + \bar{\mathcal{M}}_{\text{DRED}}^{(1)}(\tilde{g},\tilde{g};h) + 2\delta \tilde{Z}_{gh}^{\text{DRED}} \mathcal{M}_{\text{DRED}}^{(0)}(\tilde{g},\tilde{g};h).
\end{aligned} \tag{B.21}$$

Neglecting terms of $\mathcal{O}(\epsilon)$ this entails

$$\mathcal{M}_{\text{RS}}^{(1)}(g,g;h) = \frac{\alpha_s c_\Gamma}{2\pi} \mathcal{M}_{\text{RS}}^{(0)}(g,g;h) \left(-\frac{2N_c}{\epsilon^2} \left| -\frac{s_{12}}{\mu^2} \right|^{-\epsilon} - \frac{\beta_0}{\epsilon} + \Delta_{\text{RS}}^{\text{virt}} \right) \tag{B.22}$$

with $\Delta_{\text{CDR}}^{\text{virt}} = \Delta_{\text{HV}}^{\text{virt}} = 0$, $\Delta_{\text{FDH}}^{\text{virt}} = N_c/3$ and $\Delta_{\text{DRED}}^{\text{virt}} = 2N_F T_F/3$. Thus, after adding the counterterms, these expressions are in agreement with the general formula Eq. (46), with the finite RS independent part given by $\mathcal{M}_{\text{NS}}^{(1)}(g,g;h) = g_h^2(N_c^2 - 1)N_F s_{12}^2/6$. The scheme dependences also exemplify the formulas discussed in Section 4.3.

Turning to the calculation of the real corrections, we also have to take into account the processes with (anti)quarks in the initial state. The corresponding matrix elements in CDR are given by

$$\begin{aligned}
&\mathcal{M}_{\text{CDR}}^{(0)}(g,g;h,g) \\
&= g_h^2 4\pi \alpha_s N_c (N_c^2 - 1) \sum_{\text{cycl}} \left(\frac{(D-2)(s_{12}-s_{14})^2(s_{12}-s_{24})^2}{s_{12}s_{24}s_{14}} - 4s_{12} \right),
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
&\mathcal{M}_{\text{CDR}}^{(0)}(g,q;h,q) \\
&= g_h^2 4\pi \alpha_s T_F (N_c^2 - 1) \frac{(D-2)(s_{12}^2 + s_{14}^2) - 2(D-4)s_{12}s_{14}}{2s_{24}},
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
&\mathcal{M}_{\text{CDR}}^{(0)}(\bar{q},q;h,g) \\
&= g_h^2 4\pi \alpha_s T_F (N_c^2 - 1) \frac{(D-2)(s_{24}^2 + s_{14}^2) + 2(D-4)s_{24}s_{14}}{2s_{12}},
\end{aligned} \tag{B.25}$$

where the sum in Eq. (B.23) is over all cyclic permutations $\{p_1 \to p_2 \to -p_4\}$ and the corresponding matrix elements in HV, FDH and DRED can be obtained by setting $D \to 4$. Eqs. (B.24) and (B.25) are related by crossing.

We have now all the necessary matrix elements at hand to compute cross sections. In this example we restrict ourselves to the total cross section σ. The cancellation of singularities and the RS dependence is the same for any infrared-finite observable.

Defining $x \equiv M_h^2/s_{12} = p_3^2/s_{12}$ we write the leading-order cross section as

$$\sigma_{\text{RS}}^{(0)}(gg) \equiv \sigma_{\text{RS}}^{(0)}(g,g;h) = \frac{2\pi}{s_{12}} \delta(1-x) \langle \mathcal{M}_{\text{RS}}^{(0)}(g,g;h) \rangle. \tag{B.26}$$

Note that in DRED we have $\sigma_{\text{DRED}}^{(0)}(gg) = \sigma_{\text{DRED}}^{(0)}(\hat{g}g) = \sigma_{\text{DRED}}^{(0)}(\tilde{g}g)$, analogously to Eq. (B.9). This is another example of the special case discussed at the end of Ref. [7]. Obviously, the virtual corrections can be obtained by the same formula if we replace $\mathcal{M}_{\text{RS}}^{(0)}$ by $\mathcal{M}_{\text{RS}}^{(1)}$. Defining the function

$$S^{\text{virt}} \equiv \delta(1-x)\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}\left(-\frac{2N_c}{\epsilon^2}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} - \frac{\beta_0}{\epsilon} + \frac{7N_c}{6}\pi^2\right), \tag{B.27}$$

the explicit results read

$$\sigma_{\text{RS}}^{\text{virt}}(gg) = \frac{\alpha_s}{s_{12}}\langle\mathcal{M}_{\text{RS}}^{(0)}(g,g;h)\rangle\left(S^{\text{virt}} + \delta(1-x)\Delta_{\text{RS}}^{\text{virt}}\right). \tag{B.28}$$

To obtain the real corrections we parametrize the phase space with the help of the variable y defined such that

$$s_{14} = \frac{s_{12}}{2}(1-x)(1-y); \qquad s_{24} = \frac{s_{12}}{2}(1-x)(1+y). \tag{B.29}$$

The real corrections can then be written as

$$\sigma_{\text{RS}}^{\text{real}}(a_1 a_2) = \frac{(16\pi)^{\epsilon-1}}{\Gamma(1-\epsilon)} s_{12}^{-\epsilon}(1-x)^{1-2\epsilon}\int_{-1}^{1} dy\,(1-y^2)^{-\epsilon}\langle M_{\text{RS}}^{(0)}(a_1,a_2;h,a_3)\rangle \tag{B.30}$$

and evaluated using distribution identities like

$$\frac{1}{(1-x)^{1+2\epsilon}} = -\frac{1}{2\epsilon}\delta(1-x) + \frac{1}{(1-x)_+} - 2\epsilon\left(\frac{\ln(1-x)}{1-x}\right)_+ + \mathcal{O}(\epsilon^2)$$

$$\equiv -\frac{1}{2\epsilon}\delta(1-x) + I_+(x) - 2\epsilon L_+(x) + \mathcal{O}(\epsilon^2). \tag{B.31}$$

In CDR and DRED the integration is straightforward, while in HV and FDH the distinction between internal and external gluons as defined in Section 1.2 leads to a complication. As discussed there, in HV and FDH soft or collinear gluons have to be treated not as $\bar{g}$ but as $\hat{g}$ and g, respectively. In order to ensure this correct treatment, one can subtract the collinear limit of the integrand in Eq. (B.30) and integrate it separately, either explicitly or using e.g. the dipole formalism as in Ref. [15]. Alternatively, one can split the integrand using distribution identities for $(1+y)^{-1-\epsilon}$ similar to Eq. (B.31) and treat each term as appropriate.

Ultimately, defining the functions

$$S^{\text{real}}(gg) = N_c\left(\frac{e^{\gamma_E}}{4\pi}\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left(\frac{2}{\epsilon^2}\delta(1-x) - \frac{4}{\epsilon}\left(I_+(x) - x\left(2-x+x^2\right)\right)\right.$$

$$\left. + \frac{11}{3}(x-1)^3 + 8L_+(x)\left(x^2-x+1\right)^2 - \frac{\pi^2}{2}\delta(1-x)\right), \tag{B.32}$$

$$S^{\text{real}}(gq) = C_F\left(\frac{e^{\gamma_E}}{4\pi}\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left(\frac{2x-2-x^2}{\epsilon} + \frac{6x-x^2-3}{2}\right.$$

$$\left. - 2L_+(x)(1-x)\left(2x-2-x^2\right)\right), \tag{B.33}$$

the real cross section can be written as

$$\sigma_{\text{RS}}^{\text{real}}(gg) = \frac{\alpha_s}{s_{12}}\langle\mathcal{M}_{\text{RS}}^{(0)}(g,g;h)\rangle\left(S^{\text{real}}(gg) + \Delta_{\text{RS}}^{gg}\right), \tag{B.34}$$

$$\sigma_{\text{RS}}^{\text{real}}(gq) = \frac{\alpha_s}{s_{12}} \langle \mathcal{M}_{\text{RS}}^{(0)}(g, g; h) \rangle (S^{\text{real}}(gq) + \Delta_{\text{RS}}^{gq}), \tag{B.35}$$

with $\Delta_{\text{CDR}}^{gg} = \Delta_{\text{HV}}^{gg} = \Delta_{\text{DRED}}^{gg} = 0$, $\Delta_{\text{FDH}}^{gg} = -4N_c x^2(1-x)$, $\Delta_{\text{CDR}}^{gq} = \Delta_{\text{HV}}^{gq} = 0$ and $\Delta_{\text{FDH}}^{gq} = \Delta_{\text{DRED}}^{gq} = -C_F x^2$. The matrix element $\mathcal{M}_{\text{RS}}^{(0)}(\bar{q}, q; h, g)$, Eq. (B.25), will not produce any singularities upon integration over phase space. Thus $\sigma_{\text{RS}}^{\text{real}}(\bar{q}q)$ is finite.

The remaining ingredients needed for the hard partonic cross section are the collinear counterterms suitable for the $\overline{\text{MS}}$ factorization schemes. They are constructed according to Eq. (58) and read

$$
\begin{aligned}
& d\sigma_{\overline{\text{MS}},\text{RS}}^{\text{coll}}(gg) \\
& = \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \int dz \big[\big(P_{g\to\hat{g}g}^{\text{RS}}(z) - P_{g\to gg}^{\overline{\text{MS}}\epsilon}(z) \big) d\sigma_{\text{RS}}^{(0)} \big(\hat{g}(zp_1), g(p_2); h \big) \\
& \quad + P_{g\to\tilde{g}g}^{\text{RS}}(z) d\sigma_{\text{RS}}^{(0)} \big(\tilde{g}(zp_1), g(p_2); h \big) \big] + \{1 \leftrightarrow 2\},
\end{aligned} \tag{B.36}
$$

$$
\begin{aligned}
& d\sigma_{\overline{\text{MS}},\text{RS}}^{\text{coll}}(gq) \\
& = \frac{\alpha_s}{2\pi} \frac{c_\Gamma}{\epsilon} \int dz \big[\big(P_{q\to\hat{g}q}^{\text{RS}}(z) - P_{q\to gq}^{\overline{\text{MS}}\epsilon}(z) \big) d\sigma_{\text{RS}}^{(0)} \big(g(p_1), \hat{g}(zp_2); h \big) \\
& \quad + P_{q\to\tilde{g}q}^{\text{RS}}(z) d\sigma_{\text{RS}}^{(0)} \big(g(p_1), \tilde{g}(zp_2); h \big) \big],
\end{aligned} \tag{B.37}
$$

where

$$P_{q\to gq}^{\overline{\text{MS}}\epsilon} \equiv P_{q\to gq}^{\text{CDR}} - \big[P_{q\to gq}^{\text{CDR}} \big]_{D\to 4} = -\epsilon C_F z, \tag{B.38}$$

in addition to Eq. (B.7) is taking into account the conversion to the $\overline{\text{MS}}$ factorization scheme. In order to present the explicit results for the collinear counterterms we introduce the functions

$$S^{\text{coll}}(gg) = \frac{c_\Gamma}{\epsilon} 4N_c \big(x^2 I_+(x) + (1-x)(1+x^2) \big) + \frac{c_\Gamma}{\epsilon} \beta_0 \delta(1-x), \tag{B.39}$$

$$S^{\text{coll}}(gq) = \frac{c_\Gamma}{\epsilon} C_F \big(2 - 2x + x^2 \big). \tag{B.40}$$

The collinear counterterms can then be written as

$$\sigma_{\text{RS}}^{\text{coll}}(gg) = \frac{\alpha_s}{s_{12}} \langle \mathcal{M}_{\text{RS}}^{(0)}(g, g; h) \rangle \big(S^{\text{coll}}(gg) - \Delta_{\text{RS}}^{gg} - \Delta_{\text{RS}}^{\text{virt}} \delta(1-x) \big), \tag{B.41}$$

$$\sigma_{\text{RS}}^{\text{coll}}(gq) = \frac{\alpha_s}{s_{12}} \langle \mathcal{M}_{\text{RS}}^{(0)}(g, g; h) \rangle \big(S^{\text{coll}}(gq) - \Delta_{\text{RS}}^{gq} \big). \tag{B.42}$$

It is now easy to see that the subtracted, partonic cross sections

$$\hat{\sigma}(gg) \equiv \big[\sigma_{\text{RS}}^{\text{Born}}(gg) + \sigma_{\text{RS}}^{\text{virt}}(gg) + \sigma_{\text{RS}}^{\text{real}}(gg) + \sigma_{\text{RS}}^{\text{coll}}(gg) \big]_{D\to 4}, \tag{B.43}$$

$$\hat{\sigma}(gq) \equiv \big[\sigma_{\text{RS}}^{\text{real}}(gq) + \sigma_{\text{RS}}^{\text{coll}}(gq) \big]_{D\to 4}, \tag{B.44}$$

$$\hat{\sigma}(\bar{q}q) \equiv \big[\sigma_{\text{RS}}^{\text{real}}(\bar{q}q) \big]_{D\to 4}, \tag{B.45}$$

are finite and RS independent. In the sum all the RS dependent Δ_{RS} terms drop out, and in all RS we obtain the well-known result that can be found e.g. in Ref. [38].

To summarize the calculation in DRED: Both the one-loop diagrams and the real corrections and phase space integrals can be computed in a straightforward way, using only full gluons g. The split $g = \hat{g} + \tilde{g}$ has to be used in the evaluation of the collinear counterterms $d\sigma^{\text{coll}}$ and in the computation of the UV counterterms. In the end, the cross sections $\hat{\sigma}(gg)$, $\hat{\sigma}(gq)$, $\hat{\sigma}(q\bar{q})$, as well

as $\hat{\sigma}(qg)$, $\hat{\sigma}(g\bar{q})$, $\hat{\sigma}(\bar{q}g)$, $\hat{\sigma}(q\bar{q})$ are RS independent. The hadronic cross section is obtained by convoluting them with the standard parton distribution functions obtained in the $\overline{\text{MS}}$ factorization scheme.

References

[1] I. Jack, D.R.T. Jones, Regularisation of supersymmetric theories, in: G.L. Kane (Ed.): Perspectives on Supersymmetry, World Scientific, pp. 149–167, hep-ph/9707278.
[2] W. Siegel, Phys. Lett. B 94 (1980) 37.
[3] D. Stöckinger, JHEP 0503 (2005) 076, hep-ph/0503129.
[4] W. Hollik, D. Stöckinger, Phys. Lett. B 634 (2006) 63, hep-ph/0509298.
[5] R. Harlander, P. Kant, L. Mihaila, M. Steinhauser, JHEP 0609 (2006) 053, hep-ph/0607240;
    R.V. Harlander, D.R.T. Jones, P. Kant, L. Mihaila, M. Steinhauser, JHEP 0612 (2006) 024, hep-ph/0610206.
[6] R.V. Harlander, L. Mihaila, M. Steinhauser, Phys. Rev. D 76 (2007) 055002, arXiv: 0706.2953 [hep-ph].
[7] A. Signer, D. Stöckinger, Phys. Lett. B 626 (2005) 127, hep-ph/0508203.
[8] W. Beenakker, H. Kuijf, W.L. van Neerven, J. Smith, Phys. Rev. D 40 (1989) 54.
[9] W. Beenakker, R. Hopker, P.M. Zerwas, Phys. Lett. B 378 (1996) 159, hep-ph/9602378.
[10] J. Smith, W.L. van Neerven, Eur. Phys. J. C 40 (2005) 199, hep-ph/0411357.
[11] W. Siegel, Phys. Lett. B 84 (1979) 193.
[12] D.M. Capper, D.R.T. Jones, P. van Nieuwenhuizen, Nucl. Phys. B 167 (1980) 479.
[13] Z. Kunszt, A. Signer, Z. Trocsanyi, Nucl. Phys. B 411 (1994) 397, hep-ph/9305239.
[14] S. Catani, S. Dittmaier, Z. Trocsanyi, Phys. Lett. B 500 (2001) 149, hep-ph/0011222.
[15] S. Catani, M.H. Seymour, Z. Trocsanyi, Phys. Rev. D 55 (1997) 6819, hep-ph/9610553.
[16] Z. Bern, D.A. Kosower, Nucl. Phys. B 379 (1992) 451.
[17] A.D. Martin, W.J. Stirling, R.S. Thorne, G. Watt, Phys. Lett. B 652 (2007) 292, arXiv: 0706.0459 [hep-ph].
[18] CTEQ Collaboration: J. Pumplin, D.R. Stump, J. Huston, H.L. Lai, P. Nadolsky, W.K. Tung, JHEP 0207 (2002) 012, hep-ph/0201195.
[19] K.G. Wilson, Phys. Rev. D 7 (1973) 2911.
[20] J. Collins, Renormalization, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press.
[21] L.V. Avdeev, G.A. Chochia, A.A. Vladimirov, Phys. Lett. B 105 (1981) 272;
    L.V. Avdeev, A.A. Vladimirov, Nucl. Phys. B 219 (1983) 262.
[22] Z. Bern, L.J. Dixon, D.A. Kosower, Annu. Rev. Nucl. Part. Sci. 46 (1996) 109, hep-ph/9602280.
[23] Z. Bern, A. De Freitas, L.J. Dixon, H.L. Wong, Phys. Rev. D 66 (2002) 085002, hep-ph/0202271.
[24] I. Jack, D.R.T. Jones, K.L. Roberts, Z. Phys. C 62 (1994) 161, hep-ph/9310301.
[25] Z. Kunszt, A. Signer, Z. Trocsanyi, Nucl. Phys. B 420 (1994) 550, hep-ph/9401294.
[26] Z. Kunszt, D.E. Soper, Phys. Rev. D 46 (1992) 192.
[27] S. Catani, M.H. Seymour, Nucl. Phys. B 485 (1997) 291, hep-ph/9605323;
    S. Catani, M.H. Seymour, Nucl. Phys. B 510 (1998) 503, Erratum.
[28] W.T. Giele, E.W.N. Glover, Phys. Rev. D 46 (1992) 1980.
[29] R.K. Ellis, D.A. Ross, A.E. Terrano, Nucl. Phys. B 178 (1981) 421.
[30] S. Frixione, Z. Kunszt, A. Signer, Nucl. Phys. B 467 (1996) 399, hep-ph/9512328.
[31] W.T. Giele, E.W.N. Glover, D.A. Kosower, Nucl. Phys. B 403 (1993) 633, hep-ph/9302225;
    Z. Nagy, Z. Trocsanyi, Nucl. Phys. B 486 (1997) 189, hep-ph/9610498.
[32] J. Campbell, R.K. Ellis, F. Tramontano, Phys. Rev. D 70 (2004) 094012, hep-ph/0408158.
[33] J.A. Aguilar-Saavedra, et al., Eur. Phys. J. C 46 (2006) 43, hep-ph/0511344.
[34] W. Beenakker, R. Hopker, M. Spira, P.M. Zerwas, Nucl. Phys. B 492 (1997) 51, hep-ph/9610490.
[35] R.K. Ellis, J.C. Sexton, Nucl. Phys. B 269 (1986) 445.
[36] B. Kamal, Z. Merebashvili, A.P. Contogouris, Phys. Rev. D 51 (1995) 4808;
    B. Kamal, Z. Merebashvili, A.P. Contogouris, Phys. Rev. D 55 (1997) 3229, Erratum;
    B.W. Harris, E. Laenen, L. Phaf, Z. Sullivan, S. Weinzierl, Phys. Rev. D 66 (2002) 054024, hep-ph/0207055.
[37] G. Altarelli, G. Curci, G. Martinelli, S. Petrarca, Nucl. Phys. B 187 (1981) 461.
[38] S. Dawson, Nucl. Phys. B 359 (1991) 283.