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Closed Form Solution of the Diffusion Transport Equation in Multiple Scattering

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Abstract—The solution of the Yang transport equation of the multiple scattering theory of charged particles is discussed. The method of Lie groups is utilized for the purpose of the construction of the solution of this equation. The fundamental solution of the Yang equation is provided as the closed form expression valid in the neighborhood of the center of coordinates.

Keywords—Transport theory, Diffusion, Yang equation, Lie groups.

1. INTRODUCTION AND NOTATION

This paper is devoted to the derivation of the approximate solution of the equation describing the distribution of scattering charged particles under the small angle approximation. We construct the fundamental solution for the three-dimensional Yang equation [1,2] using the method of Lie groups approach [3] to the analysis of partial differential equations. The probabilistic insight into properties of the solution of this equation allows us to find first the relevant group of transformations and then to calculate formulas for its fundamental solution.

We use the coordinate system with axes Ox, Oy, Oz where Ox, Oy lie in a horizontal plane and Oz points down and is perpendicular to the plane Oxy. A beam of charged particles moving in the direction Oz enters the medium at the point O. We denote by t the actual path length of a particle's evolution in 3d space, by $\overrightarrow{r} = (x, y, z)$ particle's position and by $\Omega = (\sin\theta\cos\varphi, \sin\theta\cos\varphi, \cos\theta)$ the unit vector in the direction of the velocity of a particle. The description of the particle's direction of motion in the plane perpendicular to Oz axis can then be represented approximately as $(\theta_x, \theta_y) = (\theta\cos\varphi, \theta\cos\varphi)$ and the distribution of particles by the position (x, y, z) and the direction (θ_x, θ_y) of the velocity is described by the function

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$f(t, x, y, z, \theta_x, \theta_y)$ that satisfies the following equation:

$$\frac{\partial f}{\partial t} = -\theta_x \frac{\partial f}{\partial x} - \theta_y \frac{\partial f}{\partial y} - \left[1 - \frac{(\theta_x^2 + \theta_y^2)}{2}\right] \frac{\partial f}{\partial z} + \frac{D}{2}\left(\frac{\partial^2 f}{\partial \theta_x^2} + \frac{\partial^2 f}{\partial \theta_y^2}\right). \tag{1}$$

Let us notice that equation (1) describes the diffusional, small angle approximation of the process of multiple scattering of charged particles [1,2] and that it shows also strong affinity to semicontinuous Boltzman equation of nonlinear kinetic theory [4,5]. Introducing instead of variables $(t, x, y, z, \theta_x, \theta_y)$ new variables $(t, x, y, \theta_x, \theta_y, \epsilon)$ where $\epsilon = (t - z)$ is the excess path length and denoting the new function by the same letter $f(t, x, y, \theta_x, \theta_y, \epsilon)$, we will come to the following 3d Yang equation [1,2]:

$$\frac{\partial f}{\partial t} = -\theta_x \frac{\partial f}{\partial x} - \theta_y \frac{\partial f}{\partial y} - \frac{1}{2}\left(\theta_x^2 + \theta_y^2\right) \frac{\partial f}{\partial \epsilon} + \frac{D}{2}\left(\frac{\partial^2 f}{\partial \theta_x^2} + \frac{\partial^2 f}{\partial \theta_y^2}\right), \tag{2}$$

with initial condition

$$f|_{t=0} = \delta(x)\delta(y)\delta(\theta_x)\delta(\theta_y)\delta(\epsilon), \qquad f|_{\epsilon=0} = 0. \tag{3}$$

Let us replace D by 1 and denote by $u(t, x, y, \theta_x, \theta_y, \epsilon)$ the solution of equations (2),(3). The solution of (2),(3) with arbitrary D will be then

$$f\left(t, x, y, \theta_x, \theta_y, \epsilon\right) = D^3 u\left(Dt, Dx, Dy, \theta_x, \theta_y, D\epsilon\right). \tag{4}$$

Suppose that we are interested only in the particles' positions and directions of the particles' velocities. In this case, the evolution of density distribution function is described by $v(t, x, y, \theta_x, \theta_y) = \int_0^\infty u(t, x, y, \theta_x, \theta_y, \epsilon)\, d\epsilon$ that satisfies the following Fermi equation:

$$\frac{\partial v}{\partial t} = -\theta_x \frac{\partial v}{\partial x} - \theta_y \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 v}{\partial \theta_x^2} + \frac{\partial^2 v}{\partial \theta_y^2}\right), \tag{5}$$

with the initial condition

$$v|_{t=0} = \delta(x)\delta(y)\delta(\theta_x)\delta(\theta_y). \tag{6}$$

Equations (5),(6) have the following solution:

$$v(t, x, y, \theta_x, \theta_y) = \frac{3}{\pi^2 t^4} \exp\left(-\frac{2t^2\left(\theta_x^2 + \theta_y^2\right) + 6\left(x^2 + y^2\right) - 6t\left(x\theta_x + y\theta_y\right)}{t^3}\right). \tag{7}$$

In the future, we will need the function $V(x, y, \theta_x, \theta_y) = v(1, x, y, \theta_x, \theta_y)$ that is given by the following formula:

$$V\left(x, y, \theta_x, \theta_y\right) = \frac{3}{\pi^2} \exp\left(-2\left(\theta_x^2 + \theta_y^2\right) - 6\left(x^2 + y^2\right) + 6\left(x\theta_x + y\theta_y\right)\right). \tag{8}$$

The function $V(x, y, \theta_x, \theta_y)$ is the joint probability density distribution in variables $(x, y, \theta_x, \theta_y)$ for all particles with path length 1.

2. 3D YANG EQUATION AND ITS SOLUTION

Now we return back to the 3d Yang equation for the function $u(t, x, y, \theta_x, \theta_y, \epsilon)$,

$$\frac{\partial u}{\partial t} = -\theta_x \frac{\partial u}{\partial x} - \theta_y \frac{\partial u}{\partial y} - \frac{1}{2}\left(\theta_x^2 + \theta_y^2\right) \frac{\partial u}{\partial \epsilon} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial \theta_x^2} + \frac{\partial^2 u}{\partial \theta_y^2}\right), \tag{9}$$

with initial condition

$$u|_{t=0} = \delta(x)\delta(y)\delta(\theta_x)\delta(\theta_y)\delta(\epsilon), \qquad u|_{\epsilon=0} = 0, \tag{10}$$

where x, y, θ_x, θ_y are not restricted, but $\epsilon \geq 0$ because it denotes excess path length. Equations (9),(10) admit the following group of similarity transformations:

$$t^* = \gamma^2 t, \qquad x^* = \gamma^3 x, \qquad \theta^* = \gamma\theta, \qquad \epsilon^* = \gamma^4\epsilon, \qquad u^* = \gamma^{-12}u. \tag{11}$$

From this, we derive the identity for $u(t, x, \theta, \epsilon)$,

$$u(t, x, y, \theta_x, \theta_y, \epsilon) = \gamma^{12} u\left(\gamma^2 t, \gamma^3 x, \gamma^3 y, \gamma\theta_x, \gamma\theta_y, \gamma^4\epsilon\right),$$

and replacing γ by $t^{-1/2}$, we obtain

$$\begin{aligned} u(t, x, y, \theta_x, \theta_y, \epsilon) &= t^{-6} u\left(1, t^{-3/2}x, t^{-3/2}y, t^{-1/2}\theta_x, t^{-1/2}\theta_x, t^{-2}\epsilon\right) \\ &= t^{-6} U\left(t^{-3/2}x, t^{-3/2}y, t^{-1/2}\theta_x, t^{-1/2}\theta_x, t^{-2}\epsilon\right), \end{aligned}$$

where $U(x, y, \theta_x, \theta_y, \epsilon) = u(1, x, y, \theta_x, \theta_y, \epsilon)$. The function $U(x, y, \theta_x, \theta_y, \epsilon)$ is the probability distribution function of $x, y, \theta_x, \theta_y, \epsilon$ for all particles with path length equal to 1. Substituting the previous expression for function $u(t, x, y, \theta_x, \theta_y, \epsilon)$ into 3d Yang equation, we will get the following equation for function $U(x, y, \theta_x, \theta_y, \epsilon)$:

$$\begin{aligned} &\left(\frac{\partial^2 U}{\partial\theta_x^2} + \frac{\partial^2 U}{\partial\theta_y^2}\right) + (3x - 2\theta_x)\frac{\partial U}{\partial x} + (3y - 2\theta_y)\frac{\partial U}{\partial y} \\ &+ \left(4\epsilon - \theta_x^2 - \theta_y^2\right)\frac{\partial U}{\partial \epsilon} + \theta_x\frac{\partial U}{\partial x} + \theta_y\frac{\partial U}{\partial y} + 12U = 0, \end{aligned} \tag{12}$$

with initial condition

$$U|_{\epsilon=0} = 0, \qquad \int U(x, y, \theta_x, \theta_y, \epsilon)\, dx\, dy\, d\theta_x\, d\theta_y\, d\epsilon = 1. \tag{13}$$

From the physical interpretation of the function $U(x, y, \theta_x, \theta_y, \epsilon)$, we can obtain the relation between the function $U(x, y, \theta_x, \theta_y, \epsilon)$ and previously found function $V(x, y, \theta_x, \theta_y)$,

$$\int_0^\infty U(x, y, \theta_x, \theta_y, \epsilon)\, d\epsilon = V(x, y, \theta_x, \theta_y). \tag{14}$$

Let us introduce a new unknown function $L(x, y, \theta_x, \theta_y, \epsilon)$ by the formula

$$U(x, y, \theta_x, \theta_y, \epsilon) = V(x, y, \theta_x, \theta_y)\, L(x, y, \theta_x, \theta_y, \epsilon). \tag{15}$$

The function $L(x, y, \theta_x, \theta_y, \epsilon)$ has the following physical interpretation: it is conditional probability distribution of ϵ given x, y, θ_x, θ_y. Therefore, L satisfies the conditions

$$\int_0^\infty L(x, y, \theta_x, \theta_y, \epsilon)\, d\epsilon = 1, \qquad L(x, y, \theta_x, \theta_y, \epsilon)\,|_{\epsilon=0} = 0. \tag{16}$$

Substituting $U(x, y, \theta_x, \theta_y, \epsilon)$ in its representation (15) into equation (12), we come to the following equation for $L(x, y, \theta_x, \theta_y, \epsilon)$:

$$\begin{aligned} &\frac{\partial^2 L}{\partial\theta_x^2} + \frac{\partial^2 L}{\partial\theta_y^2} + (3x - 2\theta_x)\frac{\partial L}{\partial x} + (3y - 2\theta_y)\frac{\partial L}{\partial y}(12x - 7\theta_x)\frac{\partial L}{\partial\theta_x} \\ &+ (12y - 7\theta_y)\frac{\partial L}{\partial\theta_y} + \left(4\epsilon - \theta_x^2 - \theta_y^2\right)\frac{\partial L}{\partial\epsilon} + 4L = 0 \end{aligned} \tag{17}$$

that should be solved with additional conditions (16).

Condition (16) means that the area under the curve $L = L(x, y, \theta_x, \theta_y, \epsilon) = L_{x,y,\theta_x,\theta_y}(\epsilon)$ is equal to 1. Let us construct a curve $L = L(x, y, \theta_x, \theta_y, \epsilon)$ from the curve $L = L(0, 0, 0, 0, \epsilon) = L_0(\epsilon)$ by the following geometric transformations. For 3d Yang equation, these transformations can be written as

$$(\epsilon, L_0) \to (\epsilon + \lambda, L_0), \qquad (\epsilon, L_0) \to \left(\lambda\epsilon, \frac{L_0}{\lambda}\right), \tag{18}$$

and they have a meaning of translation and hyperbolic rotation, both preserving the area under a curve. Heuristic motivation for using these two transformations is the following. The minimal possible value of ϵ is equal to $(t - \sqrt{t^2 - x^2 - y^2})$. If probabilistic model were exact, then the surface $\epsilon = t - \sqrt{t^2 - x^2 - y^2}$ would be separating the points in $(t, x, y, \theta_x, \theta_y, \epsilon)$ space where the function $u(t, x, y, \theta_x, \theta_y, \epsilon)$ is equal to 0 from the points where this function is greater than 0. This however, is inconsistent with similarity transformation (11). It is easy to see that if a point $(t, x, y, \theta_x, \theta_y, \epsilon)$ satisfies the equation $\epsilon = t - \sqrt{t^2 - x^2 - y^2}$, then the point $(t^*, x^*, y^*, \theta_x^*, \theta_y^*, \epsilon^*)$ given by (11) does not satisfy this equation. This is because Yang equation is only a small angle approximation to the exact equation. Under the assumption of small angle approximation (i.e., assuming that (x/t) and (y/t) are small), the equation $\epsilon = t - \sqrt{t^2 - x^2 - y^2}$ can be rewritten as

$$\epsilon = t - t\left(1 - \frac{1}{2}\left(\frac{x}{t}\right)^2 - \frac{1}{2}\left(\frac{y}{t}\right)^2\right) = \frac{x^2 + y^2}{2t}. \tag{19}$$

The last equality gives for minimal possible excess path length formula $\epsilon = (x^2 + y^2)/(2t)$ that is consistent with similarity transformation (11). Returning back to the function $L_{x,y,\theta_x,\theta_y}(\epsilon)$ that describes conditional probability distribution of the projected excess path length for $t = 1$, we see that we should expect that $L_{x,y,\theta_x,\theta_y}(\epsilon)$ is not equal to 0 only when $\epsilon > (x^2 + y^2)/2$. So position of the support of the function $L_{x,y,\theta_x,\theta_y}(\epsilon)$ (by support, we mean the values of ϵ for which $L_{x,y,\theta_x,\theta_y}(\epsilon) \neq 0$) depends on x, y, θ_x, θ_y and we should include a translation in the transformation of $L_0(\epsilon)$. The second transformation in (18) is chosen for similar reasons. Again, in exact probabilistic model, the function $u(t, x, y, \theta_x, \theta_y, \epsilon)$ can have nonzero values only for $(t - \sqrt{t^2 - (x^2 + y^2)}) \le \epsilon \le t$, so not only the position but also the size of support of the function $L_{x,y,\theta_x,\theta_y}(\epsilon)$ changes with $(x, y, \theta_x, \theta_y)$ and the second transformation achieves exactly this change. The Yang equation is only a small angle approximation to the exact equation and the above described properties of support of $L_{x,y,\theta_x,\theta_y}(\epsilon)$ will not be satisfied for the exact model, nevertheless, it is important to include these properties explicitly in the construction of the approximate solution. Analytically, all this means is that we are trying to find $L(x, y, \theta_x, \theta_y, \epsilon)$ in the following form:

$$L(x, y, \theta_x, \theta_y, \epsilon) = b(x, y, \theta_x, \theta_y) \cdot L_0(b(x, y, \theta_x, \theta_y)\epsilon - a(x, y, \theta_x, \theta_y)), \tag{20}$$

with conditions for $a(x, y, \theta_x, \theta_y)$ and $b(x, y, \theta_x, \theta_y)$,

$$a(0,0,0,0) = 0, \qquad b(0,0,0,0) = 1.$$

In (20), L_0 is a function of one variable. Finding derivatives of $L(x, y, \theta_x, \theta_y, \epsilon)$ from (20) and replacing the combination $b(x, y, \theta_x, \theta_y)\epsilon - a(x, y, \theta_x, \theta_y)$ by a new variable that we again denote by ϵ, we will get the following equation for the function L_0:

$$\left(A\epsilon^2 + B\epsilon + C\right) L_0'' + (D\epsilon + E)\, L_0' + F L_0 = 0, \tag{21}$$

where A, B, C, D, E, F are given by the following formulas:

$$A = \frac{1}{b}\left(\left(\frac{\partial b}{\partial \theta_x}\right)^2 + \left(\frac{\partial b}{\partial \theta_y}\right)^2\right), \tag{22}$$

$$B = 2\frac{\partial b}{\partial \theta_x}\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right) + 2\frac{\partial b}{\partial \theta_y}\left(\frac{a}{b}\frac{\partial b}{\partial \theta_y} - \frac{\partial a}{\partial \theta_y}\right), \tag{23}$$

$$C = b\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right)^2 + b\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right)^2, \tag{24}$$

$$D = \frac{2}{b}\left(\frac{\partial b}{\partial \theta_x}\right)^2 + \frac{\partial^2 b}{\partial \theta_x^2} + \frac{2}{b}\left(\frac{\partial b}{\partial \theta_y}\right)^2 + \frac{\partial^2 b}{\partial \theta_x^2} + (3x - 2\theta_x)\frac{\partial b}{\partial x} + (3y - 2\theta_y)\frac{\partial b}{\partial y} + 2(12x - 7\theta_x)\frac{\partial b}{\partial \theta_x} + 2(12y - 7\theta_y)\frac{\partial b}{\partial \theta_y} + 4b, \tag{25}$$

$$\begin{aligned} E = {} & 2\frac{\partial b}{\partial \theta_x}\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right) + 2\frac{\partial b}{\partial \theta_y}\left(\frac{a}{b}\frac{\partial b}{\partial \theta_y} - \frac{\partial a}{\partial \theta_y}\right) + b\left(\frac{a}{b}\frac{\partial^2 b}{\partial \theta_x^2} - \frac{\partial^2 a}{\partial \theta_x^2}\right) \\ & + b\left(\frac{a}{b}\frac{\partial^2 b}{\partial \theta_y^2} - \frac{\partial^2 a}{\partial \theta_y^2}\right) + (3x - 2\theta_x)\, b\left(\frac{a}{b}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\right) \\ & + (3y - 2\theta_y)\, b\left(\frac{a}{b}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\right) + (12x - 7\theta_x)\, b\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right) \\ & + (12x - 7\theta_x)\, b\left(\frac{a}{b}\frac{\partial b}{\partial \theta_x} - \frac{\partial a}{\partial \theta_x}\right) + b^2\left(4\frac{a}{b} - \theta_x^2 - \theta_y^2\right), \end{aligned} \tag{26}$$

$$F = \frac{\partial^2 b}{\partial \theta_x^2} + \frac{\partial^2 b}{\partial \theta_y^2} + (3x - 2\theta_x)\frac{\partial b}{\partial x} + (3y - 2\theta_y)\frac{\partial b}{\partial y} + (12x - 7\theta_x)\frac{\partial b}{\partial \theta_x} + (12y - 7\theta_y)\frac{\partial b}{\partial \theta_y} + 4b. \tag{27}$$

Coefficients A, D, F are not independent, but satisfy a simple relation $D = F + 2A$, therefore, equation (21) can be rewritten in the following form:

$$\left(\left(A\epsilon^2 + B\epsilon + C\right) L_0' + \left(F\epsilon + E - B\right) L_0\right)' = 0. \tag{28}$$

Since L_0 and L_0' should decrease exponentially as $\epsilon \to \infty$, so

$$\left(A\epsilon^2 + B\epsilon + C\right) L_0' + \left(F\epsilon + E - B\right) L_0 = 0. \tag{29}$$

The function L_0 depends only on ϵ. Thus, it is possible to solve this equation only if it does not depend on x, θ_x, y, θ_y. Let us try to choose the functions $a(x, y, \theta_x, \theta_y)$, $b(x, y, \theta_x, \theta_y)$ in such a way that the above equation does not depend on x, y, θ_x, θ_y. From the property that $L_{x,y,\theta_x,\theta_y}(\epsilon)$ is not equal to 0 only when $\epsilon > (x^2 + y^2)/2$, we find that $a(x, y, \theta_x, \theta_y)$ should be chosen as

$$a(x, y, \theta_x, \theta_y) = \frac{1}{2} b(x, y, \theta_x, \theta_y)\left(x^2 + y^2\right). \tag{30}$$

For such $a(x, y, \theta_x, \theta_y)$, formulae for B, C, E become simpler (only these formulae contain $a(x, \theta)$),

$$B = C = 0, \tag{31}$$

$$E = -\left(\left(x - \theta_x\right)^2 + \left(y - \theta_y\right)^2\right) b^2 \tag{32}$$

Now equation (29) for L_0 takes the following form:

$$A\epsilon^2 L_0' + \left(F\epsilon + E\right) L_0 = 0. \tag{33}$$

This equation can be solved if expressions $E/A, F/A$ are constants, that is if they do not depend on x, y, θ_x, θ_y. Let us start with equation $E/A = C_1$. Substituting expressions (26) and (22) for E and A in $E/A = C_1$ and solving for b, we get formula

$$b(x, y, \theta_x, \theta_y) = \frac{-16 C_1}{\left(\left(x - \theta_x\right)^2 + \left(y - \theta_y\right)^2 + C_2(x, y)\right)^2}. \tag{34}$$

We can also satisfy the equation $F/A = \text{const.}$ (though only approximately in the neighborhood of the point $x = y = \theta_x = \theta_y = 0$). To this end, it is enough to express $C_2(x, y)$ in the form

$$C_2(x,y) = \alpha + \beta\left(x^2 + y^2\right). \tag{35}$$

Substituting this function into equation $F/A = \text{const.}$, equating to zero the terms of degree one in both parts of this equation, and taking into account initial condition $b(0,0,0,0) = 0$, we get that $\alpha = \beta = 2$ and the final expression for $b(x, y, \theta_x, \theta_y)$ takes the form

$$b\left(x,y,\theta_x,\theta_y\right) = \frac{1}{\left[1 + (x^2+y^2) + \left((x-\theta_x)^2 + (y-\theta_y)^2\right)/2\right]^2} \tag{36}$$

and for $a(x, y, \theta_x, \theta_y)$,

$$a\left(x,y,\theta_x,\theta_y\right) = \frac{1}{2}\left(x^2+y^2\right) b\left(x,y,\theta_x,\theta_y\right). \tag{37}$$

With this choice of a and b, we have the following formula for F/A:

$$\frac{F}{A} = \frac{9}{2} + \frac{3}{2}\left(2x^2 + 2y^2 + (x-\theta_x)^2 + (y-\theta_y)^2\right), \tag{38}$$

with $E/A = -(1/4)$. We now drop the terms of the second order and will continue calculations as if $F/A = 9/2$ exactly. Equation (33) now becomes

$$\epsilon^2 L_0' + \left(\frac{9}{2}\epsilon - \frac{1}{4}\right) L_0 = 0. \tag{39}$$

Taking into account that the area under the curve $\epsilon \to L_0(\epsilon)$ should be equal to 1, equation (39) has the unique solution

$$L_0(\epsilon) = \frac{1}{240\sqrt{\pi}\epsilon^{9/2}} e^{-1/4\epsilon}. \tag{40}$$

Before writing the final formula, we will recall all the auxiliary formulas that we need

$$L = bL_0\left(b\epsilon - a\right),$$
$$U = \frac{3}{\pi^2}\exp\left\{-2\left(\theta_x^2+\theta_y^2\right) - 6\left(x^2+y^2\right) + 6\left(x\theta_x + y\theta_y\right)\right\}\cdot L,$$
$$u\left(t,x,y,\theta_x,\theta_y,\epsilon\right) = t^{-6} U\left(t^{-3/2}x, t^{-3/2}y, t^{-1/2}\theta_x, t^{-1/2}\theta_y, t^{-2}\epsilon\right),$$
$$f\left(t,x,y,\theta_x,\theta_y,\epsilon\right) = D^3 u\left(Dt, Dx, Dy, \theta_x, \theta_y, D\epsilon\right).$$

Consecutive substitutions of appropriate quantities in these formulas give us the final answer. We express it in terms of variables ρ, ψ, t, ϵ ($\rho, \psi, t-\epsilon$ are cylindrical coordinates of the point P that represents position of a moving particle) and θ, φ (θ, φ are spherical coordinates of Ω that represents direction of velocity of a moving particle). The relations between coordinates $t, \epsilon, \rho, \psi, \theta, \varphi$ and $t, x, y, \theta_x, \theta_y, \epsilon$ are

$$\rho^2 = x^2 + y^2, \qquad \theta^2 = \theta_x^2 + \theta_y^2,$$
$$\theta\rho\cos(\varphi - \psi) = x\theta_x + y\theta_y.$$

Thus, the final formula for $f(t, x, y, \theta_x, \theta_y, \epsilon)$ is

$$\begin{aligned} f(t,x,y,\theta_x,\theta_y,\epsilon) &= f(t,\rho,\theta,\varphi-\psi,\epsilon) \\ &= \frac{\sqrt{2}\left[Dt^3 + 3\rho^2/2 + t^2\theta^2/2 - t\theta\rho\cos\left(\varphi-\psi\right)\right]^7}{5\sqrt{\pi^5 D^{11} t^{27}}\left(2t\epsilon - \rho^2\right)^{9/2}} \\ &\quad \times \exp\left\{-\frac{2t^2\theta^2 + 6\rho^2 - 6t\rho\theta\cos(\varphi-\psi)}{Dt^3}\right\} \\ &\quad \times \exp\left\{-\frac{\left[Dt^3 + 3\rho^2/2 + t^2\theta^2/2 - t\theta\rho\cos(\varphi-\psi)\right]^2}{2Dt^3(2t\epsilon - \rho^2)}\right\}. \end{aligned} \tag{41}$$

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