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The Uniqueness of the Complete Norm Topology in Complete Normed Nonassociative Algebras

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A theorem on uniqueness of the complete norm topology for complete normed nonassociative algebras is proved. This theorem contains the well-known one by Johnson for associative Banach algebras and the recent analogous result by Aupetit for Banach–Jordan algebras.

INTRODUCTION

This paper contains an unexpected theorem of uniqueness of the complete algebra norm topology in general nonassociative algebras which implies the Johnson associative result [10]. The proof of this theorem consists of an ingenuous adaptation of the recent Aupetit proof [2] that the separating ideal for a homomorphism from a Banach (associative) algebra onto another one lies in the Jacobson radical. We summarize this adaptation: first we observe that the Aupetit proof remains true for a larger class of noncomplete normed associative algebras, the class of full subalgebras of Banach algebras. Then to each nonassociative algebra A we associate an associative algebra, called the full multiplication algebra of A (definition in Section 1), in such a way that when A is a complete normed nonassociative algebra then its full multiplication algebra is a full subalgebra of a suitable Banach algebra. Isomorphisms between nonassociative algebras induce isomorphisms between their corresponding full multiplication algebras. Thus in the complete normed case the above-mentioned extension of the Aupetit result can be applied to the induced isomorphism between the full multiplication algebras, giving nontrivial information which yields almost directly to our theorem.

Up to now the problem of uniqueness of the complete algebra norm topology in general nonassociative algebras has not been solved. Also we think that it has not been formally posed because for nonassociative algebras

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there is no concept of radical with the same algebraic relevance as that of Jacobson in the associative case (only for the finite-dimensional case is there a satisfactory concept of a general nonassociative radical; see [1, 9] and our Remark 1.6 below) and such that one could expect a complete normed nonassociative algebra with zero radical to have unique complete algebra norm topology. The proof of our theorem leads in a natural way to a suitable definition of radical for any nonassociative algebra A which we call the weak radical of A (Definition 1.5(iii)). The theorem states that a complete normed nonassociative algebra with zero weak radical has unique complete algebra norm topology. In the associative case the weak radical is contained in the Jacobson radical so Johnson's theorem follows immediately. Since we do not know whether or not the two radicals agree it may be that even in the associative case our theorem is stronger than that of Johnson. We think that this is the main problem which remains open in this paper.

There are some known particular results related to the problem of uniqueness of the complete algebra norm topology in nonassociative algebras. The particularity is due to one or more of the following facts:

—restriction to some of the most familiar classes of nonassociative algebras,

—the assumption on the general complete normed nonassociative algebra that a suitable radical wider than the weak radical is zero,

—additional analytic requirements.

Since all of these results are consequences of our theorem we discuss them in detail in the corollaries.

1. The Theorem

If X and Y are normed spaces and F is a linear mapping from X into Y we denote by $S(F)$ (the separating subspace for F) the set of those y in Y for which there is a sequence $\{x_n\}$ in X such that $0 = \lim\{x_n\}$ and $y = \lim\{F(x_n)\}$. When X and Y are normed algebras and F is a homomorphism with dense range then $S(F)$ is a two-sided ideal of Y so we call it the separating ideal for F.

As usual if a is an element of an associative normed algebra A the real number $\inf\{\|a^n\|^{1/n} : n \in N\}$ will be called the spectral radius of a and will be denoted by $r(a)$. Since the mapping $a \to r(a)$ from A into R is upper-semicontinuous and for complex A and fixed a and b in A the mapping $\lambda \to r(a + \lambda b)$ from $\mathbb{C}$ into R is subharmonic (apply the Vesentini theorem [30] to the completion of A) it follows that most of the proof of Theorem 1 in [2] remains true in the noncomplete case. So we have (see the proof of Theorem 1 in [2]):

LEMMA 1.1. *Let A and B be normed associative complex algebras and F be a linear mapping from A onto B such that the inequality $r(F(a)) \leqslant r(a)$ holds for all a in A. Then $r(b) = 0$ for all b in the separating subspace for F.*

For general noncomplete normed associative algebras the spectral radius has no algebraic significance. So isomorphisms between normed associative algebras may not preserve the spectral radius. However, there is a large class of noncomplete normed associative algebras for which the spectral radius has the same property as in the complete case; that is, the spectral radius of an element in the algebra is the maximum of the moduli of the numbers in the spectrum. To make this concrete, consider the following

DEFINITION 1.2. A subalgebra A of an associative algebra B is called a full subalgebra of B if A contains the quasiinverses of its elements that are quasiregular in B.

Trivial examples of full subalgebras are the left, right, and two-sided ideals. It is clear that if A is a full subalgebra of an associative complex algebra B and a is an element in A then the spectrums of a relative to A and B agree. So if B is a Banach algebra and the full subalgebra A is considered as a normed algebra with the restriction of the norm of B, then for any a in A the number $r(a)$ is the maximum of the moduli of the numbers in the spectrum of a (relative to A).

PROPOSITION 1.3. *Let A and B be full subalgebras of suitable complex Banach algebras and F be a homomorphism from A onto B. Then the separating ideal for F is contained in the Jacobson radical of B.*

Proof. Since F is a homomorphism, for each a in A we have $\mathrm{sp}(B, F(a)) \subset \mathrm{sp}(A, a)$ and using that A is a full subalgebra of a Banach algebra we obtain

$$r(F(a)) \leqslant \sup\{|\lambda| : \lambda \in \mathrm{sp}(B, F(a))\} \leqslant \sup\{|\lambda| : \lambda \in \mathrm{sp}(A, a)\} = r(a).$$

So we can apply our Lemma 1.1 to obtain that $r(b) = 0$ for all b in $S(F)$. Now using that B is a full subalgebra of a Banach algebra it follows that $\mathrm{sp}(B, b) = \{0\}$, which implies that b has a quasiinverse in B. Thus $S(F)$ is a quasiinvertible ideal of B so it is contained in the Jacobson radical of B.

The following corollary is not needed in what follows but it seems to be of interest in its own right because it is an associative extension of Johnson's theorem and one of the few results on automatic closeability of partially defined operators between Banach algebras (see [20] for a related result). For it we use the customary terminology for partially defined linear operators between Banach spaces. Thus a partially defined homomorphism between Banach algebras will be a partially defined linear operator (say ϕ)

whose domain (Dom(ϕ)) is a subalgebra of the first algebra and such that $\phi(ab) = \phi(a)\,\phi(b)$ for all a, b in Dom(ϕ).

COROLLARY 1.4. *Let ϕ be a partially defined homomorphism between complex Banach algebras A and B. Assume that the domain of ϕ is a full subalgebra of A, that the range of ϕ is B, and that B has zero Jacobson radical. Then ϕ is closeable.*

Proof. Straightforward from Proposition 1.3.

Notation. For a vector space X we denote by $L(X)$ the associative algebra of all linear mappings from X into X.

For an element a in a nonassociative algebra A we denote by L_a (resp.: R_a) the element in $L(A)$ defined by $L_a(x) = ax$ (resp.: $R_a(x) = xa$) for all x in A and we denote by L_A and R_A the sets $L_A = \{L_a: a \in A\}$, $R_A = \{R_a: a \in A\}$.

DEFINITIONS 1.5. (i) Since the intersection of full subalgebras of an associative algebra A is another full subalgebra of A it follows that for any nonempty subset S of A there is a smallest full subalgebra of A which contains S. This subalgebra will be called *the full subalgebra of A generated by S*.

(ii) Now let A be a nonassociative algebra. The full subalgebra of $L(A)$ generated by $L_A \cup R_A$ will be called *the full multiplication algebra of A* and will be denoted by $FM(A)$.

(iii) Consider the set $W(A)$ of those elements a in A for which L_a and R_a belong to the Jacobson radical of $FM(A)$. $W(A)$ is a subspace of A so it contains a largest subspace invariant under the algebra of operators $FM(A)$. This last subspace, which is clearly a two-sided ideal of A, will be called *the weak radical of A* and denoted by w-Rad(A).

Remark 1.6. Albert gave a concept of radical for finite-dimensional nonassociative algebras and proved that such an algebra has zero radical if and only if it is either the zero algebra or a direct sum of ideals which are simple algebras (see [1; 9, pp. 1090–1091]). In the finite-dimensional case, our weak radical is close to the Albert radical. In fact from Albert's definition it can be easily shown that the weak radical of any finite-dimensional nonassociative algebra is contained in the Albert radical. On the other hand, there is a finite-dimensional nonassociative algebra A whose Albert radical is an associative field [1, Sect. 6]. If e denotes the unit of this field then $e \notin$ w-Rad(A) since for every a in w-Rad(A) there is a b in w-Rad(A) such that $a + b - ba = 0$ (see the proof of Proposition 2.3 below). This shows that the inclusion of the weak radical in the Albert radical may be strict. For our pathological algebra A we have in fact that w-Rad$(A) = \{0\}$ because w-Rad(A) is a proper ideal of a field.

The following lemma is included only in order to make possible the extension of the main theorem to the case of real algebras. As in the associative case (see [5, Definition 13.1]) any nonassociative real algebra A can be considered as a real subalgebra of a nonassociative complex algebra $A_{\mathbb{C}}$ which satisfies $A_{\mathbb{C}} = A \oplus iA$ and is called the complexification of A. For associative A, Rad(A) will denote the Jacobson radical of A.

LEMMA 1.7. *Let A be a nonassociative real algebra. Then* w-Rad$(A_{\mathbb{C}}) \cap A \subset$ w-Rad(A).

Proof. The real algebra $L(A)$ can and will be identified in an obvious way with the full real subalgebra of $L(A_{\mathbb{C}})$, whose elements are the linear operators on $A_{\mathbb{C}}$ which leave A invariant. For $a \in A$ the multiplication operators by a on A are then identified with the multiplication operators by a on $A_{\mathbb{C}}$ so the symbols L_a, R_a have an unambiguous meaning. $FM(A_{\mathbb{C}}) \cap L(A)$ is clearly a full subalgebra of $L(A)$ which contains $L_A \cup R_A$ so we have (1) $FM(A) \subset FM(A_{\mathbb{C}}) \cap L(A)$. In fact $FM(A)$ is a full real subalgebra of $FM(A_{\mathbb{C}})$. It follows that Rad$(FM(A_{\mathbb{C}})) \cap FM(A)$ is a quasi-invertible ideal of $FM(A)$. Therefore Rad$(FM(A_{\mathbb{C}})) \cap FM(A) \subset$ Rad$(FM(A))$. This inclusion together with the definition of the weak radical gives w-Rad$(A_{\mathbb{C}}) \cap A \subset W(A)$. It remains to show that the subspace w-Rad$(A_{\mathbb{C}}) \cap A$ of A is invariant under $FM(A)$. This is a consequence of the fact that w-Rad$(A_{\mathbb{C}})$ is invariant under $FM(A_{\mathbb{C}})$ and the inclusion (1).

Remark 1.8. For any Banach space X let $BL(X)$ be the Banach algebra of all continuous linear mappings from X into X. By the Banach isomorphism theorem $BL(X)$ is a full subalgebra of $L(X)$. So if A is a complete normed nonassociative algebra we have in view of the obvious inclusion $L_A \cup R_A \subset BL(A)$ that $FM(A) \subset BL(A)$.

PROPOSITION 1.9. *Let A and B be complete normed nonassociative algebras and T be an isomorphism from A onto B. Then the separating ideal for T is included in the weak radical of B.*

Proof. Assume first that A and B are complex algebras. Consider the isomorphism $G \to TGT^{-1}$ from $L(A)$ onto $L(B)$. Since T is onto and $TL_aT^{-1} = L_{T(a)}$, $TR_aT^{-1} = R_{T(a)}$ for all a in A, our isomorphism maps $L_A \cup R_A$ onto $L_B \cup R_B$, so $FM(A)$ onto $FM(B)$. $FM(A)$ and $FM(B)$ are full subalgebras of the Banach algebras $BL(A)$ and $BL(B)$, respectively (see the previous remark) and the mapping $\tilde{T}\colon G \to TGT^{-1}$ is an isomorphism from $FM(A)$ onto $FM(B)$, so Proposition 1.3 applies and gives $S(\tilde{T}) \subset$

$\mathrm{Rad}(FM(B))$. For b in $S(T)$ we have easily that L_b and R_b belong to $S(\tilde{T})$, so to $\mathrm{Rad}(FM(B))$. Thus $S(T) \subset W(B)$ and the proof will be concluded by showing that $S(T)$ is invariant under $FM(B)$. To this end let $H \in FM(B)$ and let $G \in FM(A)$ be such that $\tilde{T}(G) = H$. If $\{a_n\}$ is a sequence in A with $\{a_n\} \to 0$ and $\{T(a_n)\} \to b$ we have $\{G(a_n)\} \to 0$ and $\{T(G(a_n))\} = \{H(T(a_n))\} \to H(b)$ so $H(b) \in S(T)$ as required.

Now assume that A and B are real algebras. Then as in the associative case [5, Proposition 13.3] $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are in a natural way complete normed nonassociative complex algebras. Also T is extended to an isomorphism $\hat{T}$ from $A_{\mathbb{C}}$ onto $B_{\mathbb{C}}$. Applying the complex case of our proof we obtain $S(\hat{T}) \subset \text{w-Rad}(B_{\mathbb{C}})$. But clearly $S(T) \subset S(\hat{T}) \cap B$ so $S(T) \subset \text{w-Rad}(B)$ by the previous lemma.

Now our main result follows from the closed graph theorem:

THEOREM 1.10. *Let A be a complete normed nonassociative algebra and assume that the weak radical of A is zero. Then A has a unique complete algebra norm topology.*

2. THE COROLLARIES

In this section we show that the weak radical of a nonassociative algebra is contained in various other previously defined radicals. Thus from our main theorem we obtain particular results on uniqueness of norm topology in nonassociative algebras. In this way most of the known theorems about the topic appear and other new results are obtained.

DEFINITIONS 2.1. (i) The following extension of the Jacobson radical has been used in order to find a structure theory for some particular classes of nonassociative algebras (see [12, 13]). Consider a maximal modular left ideal M of a nonassociative algebra A (definition as in the associative case [5, Definition 9.1]) and let P be the largest two-sided ideal of A contained in M. Two-sided ideals P obtained in this way are called primitive ideals of A. Notice that this definition of primitive ideal agrees with the usual one in the associative case (see [5, Definition 24.11, Proposition 24.12(i)]). The radical of A is defined as the intersection of all primitive ideals of A and is denoted by $\mathrm{Rad}(A)$ (with the usual convention that $\mathrm{Rad}(A) = A$ if there are no primitive ideals of A).

(ii) The strong radical of a nonassociative algebra A is defined as the intersection of all maximal modular two-sided ideals of A and is denoted by s-Rad(A).

(iii) The annihilator of a nonassociative algebra A is defined as the set $\{a \in A: L_a = R_a = 0\}$ and is denoted by $\mathrm{An}(A)$.

(iv) The concept of Jacobson radical for associative algebras and the McCrimmon radical [16] for Jordan algebras can be unified if we consider the class of noncommutative Jordan algebras (definition in [24, p. 141]) which includes alternative (in particular, associative) and Jordan algebras. For a and b in a noncommutative Jordan algebra define the quasiproduct $a \circ b$ by $a \circ b = a + b - ab$. If for a in A there is a b in A such that the equalities $a \circ b = b \circ a = 0$ and $(a \circ a) \circ b = b \circ (a \circ a) = a$ are verified, a is said to be a quasiinvertible element of A. A subset of A is said to be quasiinvertible if all of its elements are quasiinvertible. The McCrimmon radical of A is defined as the largest quasiinvertible two-sided ideal of A and is denoted by M-Rad(A). Its existence was proved in [16] for commutative A. For an arbitrary noncommutative Jordan algebra A the existence of the McCrimmon radical may be obtained as follows: the algebra A^+ (the same vector space as that of A with product $a \circ b = \frac{1}{2}(ab + ba)$) is a Jordan algebra with the same quasiinvertible subsets as those of A [15, Theorem 2.2] so, clearly, the largest two-sided ideal of A contained in the McCrimmon radical of A^+ is the desired McCrimmon radical of A. The specialization of the concept of the McCrimmon radical to alternative algebras is due to Smiley [28]. Also for an alternative algebra A the equalities Rad(A) = M-Rad(A) = M-Rad(A^+) are true [33, 18].

LEMMA 2.2. *Let A be a nonassociative algebra and Q be a two-sided ideal of A. Assume that for any q in Q there is an a in A such that $q + a - aq = 0$. Then Q is contained in the radical of A.*

Proof. Suppose, to the contrary, that M is a maximal modular left ideal of A such that Q is not included in M. Then $Q + M = A$ by the maximality of M. Choose $q \in Q$, $m \in M$ such that $q + m$ is a right modular unit for M. Then q is also a right modular unit for M. Let a be in A such that $q + a - aq = 0$. We have $q = aq - a \in M$, a contradiction (see [5, Proposition 9.2(ii)]).

PROPOSITION 2.3. *For any nonassociative algebra A we have that* $\mathrm{An}(A) \subset$ w-Rad$(A) \subset$ Rad$(A) \subset$ s-Rad(A). *Also if A is a noncommutative Jordan algebra then* w-Rad$(a) \subset$ M-Rad$(A) \subset$ Rad(A).

Proof. The inclusion $\mathrm{An}(A) \subset W(A)$ is clear. So to conclude $\mathrm{An}(A) \subset$ w-Rad(A) it is enough to prove that An(A) is invariant under $FM(A)$. But the set $P = \{F \in L(A) : F(\mathrm{An}(A)) = \{0\}\}$ is a left ideal (so a full subalgebra) of $L(A)$ which contains $L_A \cup R_A$. Therefore $FM(A) \subset P$, that is, $FM(A)(\mathrm{An}(A)) = \{0\}$, which is more than the invariance of An(A) under $FM(A)$.

By definition of the weak radical, if q is an element in w-Rad(A) then R_q has a quasiinverse (say T) in $FM(A)$. Write $a = T(q) - q$. Then $q + a - aq =$

$T(q) - T(q)q + q^2 = (T + R_q - R_q T)(q) = 0$. Now the inclusion w-Rad$(A) \subset$ Rad(A) follows from Lemma 2.2.

It is easy to see that maximal modular two-sided ideals are primitive ideals so Rad$(A) \subset$ s-Rad(A).

Now let A be a noncommutative Jordan algebra. The inclusion M-Rad$(A) \subset$ Rad(A) follows immediately from Lemma 2.2. As above for q in w-Rad(A), R_q has a quasiinverse T in $FM(A)$ and if we write $a = T(q) - q$ the equality (1) $q \circ a = 0$ is obtained. From (1), using that A is a flexible algebra (that is, $(xy)x = x(yx)$ for all x, y in A), we obtain $(I_A - R_q)(a \circ q) = q \circ a - (q \circ a)q = 0$ so (2) $a \circ q = 0$ since $I_A - R_q$ is an invertible operator on A (in fact $I_A - T$ is its inverse). From (2), the flexibility of A, and the fact that R_q commutes with L_{q^2} we obtain $(I_A - R_q)((q \circ q) \circ a - q) = 0$ so (3) $\varphi(q \circ q) \circ a = q$. Also from (1), the flexibility of A, and the commutation of R_q with R_{q^2} the equality (4) $a \circ (q \circ q) = q$ is obtained in an analogous way. From (1), (2), (3), and (4) it follows that q is quasiinvertible in A so w-Rad(A) is a quasiinvertible two-sided ideal of A and so w-Rad$(A) \subset$ M-Rad(A).

Remarks 2.4. (i) The weakest result obtained from the above proposition and our main theorem is that *complete normed nonassociative algebras with zero strong radical have a unique complete algebra norm topology*. This is considered as known by some people. The Rickart proof [22] for the associative case has been adapted to the case of Jordan algebras (see [4, 14]). Although we do not know a published proof for general nonassociative algebras, an easy proof can be given (without using the main result of this paper) by reducing as usual to the unital simple case and then by showing that the separating ideal for two complete algebra norms $\| \ \|_1$ and $\| \ \|_2$ on the unital simple algebra A does not contain the unit of A. This follows from the fact that

$$1 = r(I_A) \leqslant r(I_A - L_a) + r(L_a) \leqslant \|I_A - L_a\|_2 + \|L_a\|_1,$$

where I_A denotes the identity operator on A, a is an arbitrary element in A, and the spctral radius $r(\cdot)$ is evaluated unambiguously either in $BL(A, \| \ \|_1)$ or in $BL(A, \| \ \|_2)$ because both algebras are full subalgebras of $L(A)$.

(ii) Another consequence of Proposition 2.3 and our main theorem is that *complete normed nonassociative algebras with zero radical have a unique complete algebra norm topology*. This is, we think, a new result and we do not know a proof essentially different from the one given here. Although it can be considered formally as a general nonassociative extension of Johnson's theorem, it is of much less relevance in the general nonassociative context than in the special associative case. For if A is a nonassociative anticommutative algebra (in praticular a Lie algebra) we have easily that $\mathrm{Rad}(A)=A$ so the result is vacuous in this case. Also for Jordan algebras another, better natural extension of Johnson's result is known (see the following remark). The uniqueness of norm topology for a complete normed generalized accessible algebra A (see [13] for the definition) with zero radical can be seen by reducing as usual to the primitive case. Then A is either commutative or associative or is an algebra of octonions over its center (see [13, Theorem 1; 12]). In the first case the result follows from the above remark (note that for commutative A the equality $\mathrm{Rad}(A)=\text{s-Rad}(A)$ is true); in the second, from Johnson's theorem; and in the last, from the finite dimension of A because the center of A is a normed associative division algebra.

(iii) From Proposition 2.3 and Theorem 1.10 it follows also that *complete normed noncommutative Jordan algebras with zero McCrimmon radical have a unique complete algebra norm topology.* This result is essentially the theorem of uniqueness of norm topology for complete normed Jordan algebras given by Aupetit [2, Theorem 2]. The wider extent of our result is purely formal. For, if a complete normed noncommutative Jordan algebra A has zero McCrimmon radical, then A^+ is a complete normed Jordan algebra which has also zero McCrimmon radical (the inclusion $\text{M-Rad}(A)\subset\text{M-Rad}(A^+)$ for every noncommutative Jordan algebra A is in fact an equality in the complete normed case [6]) so Aupetit's theorem is applicable and A^+ has a unique complete algebra norm topology. Since every complete algebra norm on A is clearly a complete algebra norm on A^+, the uniqueness of the complete algebra norm topology on A follows immediately. In [21] the problem of uniqueness of norm topology for complete normed Jordan algebras with zero Topping radical (definition in [21, Sect. 3]) is attacked, successfully in some particular cases. It must be noted that the Topping radical agrees with the McCrimmon radical as a consequence of the results in [8].

(iv) The concept of full subalgebra for associative algebras is extended in an obvious way to the noncommutative Jordan context. Also the natural extension of Proposition 1.3 and Corollary 1.4, replacing Banach algebras by complete normed noncommutative Jordan algebras and the Jacobson radical by the McCrimmon radical, is valid.

The strongest result which we can obtain from Proposition 2.3 and our main theorem is vacuous in the case of anticommutative algebras. In fact every anticommutative algebra A is a noncommutative Jordan algebra satisfying that $\text{M-Rad}(A)=A$. In the rest of this section some other results are obtained which cover the anticommutative case. The first one follows from the next proposition. Following [24, p. 15], a nonassociative algebra A is said to be simple if there are a and b in A with $ab\neq 0$ and $\{0\}$ and A are the only two-sided ideals of A.

PROPOSITION 2.5. *Every simple nonassociative algebra has zero weak radical.*

Proof. Let A be a simple algebra. Since every subspace of A invariant under $FM(A)$ is a two-sided ideal of A it follows that $FM(A)$ is an irreducible algebra of operators on A so, in particular, $FM(A)$ has zero Jacobson radical. Thus we have $W(A) = \text{An}(A)$. But $\text{An}(A)$ is a two-sided ideal of A which cannot agree with A so $W(A) = \{0\}$ and $\text{w-Rad}(A) = \{0\}$.

In what follows we consider complete normed complex algebras A with an algebra involution such that, for any selfadjoint element a in A, L_a and R_a are *hermitian* elements in the unital Banach complex algebra $BL(A)$ (definition in [5, Definition 10.12]). Such an algebra will be called an *algebra with hermitian multiplication.* Associative examples of algebras with hermitian multiplication are the C^*-algebras and the associative H^*-algebras [5, Definition 34.6]. Nonassociative examples are, among others, the nonassociative generalizations of the above-mentioned ones. These are the noncommutative Jordan C^*-algebras [19] (which include the Jordan C^*-algebras of Kaplansky [32]) and the nonassociative H^*-algebras [6, 7] (which include Jordan H^*-algebras [31] and Lie H^*-algebras [26]), respectively. Notice that, in a very precise sense, noncommutative Jordan C^*-algebras are the widest nonassociative generalization of associative C^*-algebras [23]. Many other examples of nonassociative algebras with hermitian multiplication may be obtained from associative algebras, since if A is associative with hermitian multiplication, then A^+ and A^- (the algebra with the same vector space as A and product $(a, b) \to \frac{1}{2}(ab - ba)$) are algebras with hermitian multiplication. An interesting independent example is the Lie algebra of all derivations on a C^*-algebra (see Remark 2.8(iii) below).

LEMMA 2.6. *The weak radical of a complex algebra with algebra involution is a selfadjoint subset.*

Proof. For F in $L(A)$ define $F^* \in L(A)$ by (1) $F^*(a) = (F(a^*))^*$ for all a in A. Then $F \to F^*$ is a semilineal automorphism of $L(A)$ which leaves $FM(A)$ invariant (because $L_a^* = R_{a^*}$ and $R_a^* = L_{a^*}$) so also leaves $\text{Rad}(FM(A))$ invariant. Thus clearly $W(A)$ is a selfadjoint subset of A. Also from (1) it follows that if X is a subsapce of A invariant under $FM(A)$ then so is X^*. Now it is clear that $\text{w-Rad}(A)$ is a selfadjoint subset of A.

PROPOSITION 2.7. *Let A be an algebra with hermitian multiplication. Then* $\text{w-Rad}(A) = \text{An}(A)$.

Proof. Using the inclusion $\text{An}(A) \subset \text{w-Rad}(A)$ (Proposition 2.3) and the previous lemma it is enough to prove that every selfadjoint element in w-

Rad(A) lies in An(A). Let a be a selfadjoint element in w-Rad(A). Then, for all complex numbers z, zL_a is quasiinvertible in $FM(A)$, so also in $BL(A)$ (recall the inclusion $FM(A) \subset BL(A)$ in Remark 1.8). Thus $\mathrm{sp}(BL(A), L_a) = \{0\}$ and $r(L_a) = 0$. Since L_a is hermitian in $BL(A)$, a theorem by Sinclair (see [5, Theorem 11.17]) gives $L_a = 0$. Analogously $R_a = 0$ and a belongs to An(A).

Remarks 2.8. (i) From Proposition 2.7 and our main theorem it follows that *an algebra with hermitian multiplication and zero annihilator has a unique complete algebra norm topology.* This result contains known facts for noncommutative Jordan C^*-algebras [19, Proposition 2.3] and nonassociative H^*-algebras with zero annihilator [6] (see also [3]).

(ii) Let A be an associative algebra with hermitian multiplication and zero center (for example, the C^*-algebra of all compact operators on an infinite-dimensional complex Hilbert space). Then A^- is a Lie algebra with hermitian multiplication and zero annihilator. So A^- has a unique complete algebra norm topology.

(iii) Now let A be an arbitrary associative C^*-algebra. Since every derivation of A is continuous, the set D of all derivations of A has a natural complete normed Lie algebra structure as a closed subalgebra of the complete normed Lie algebra $(BL(A))^-$. For F in D let F^* be the new derivation of A defined by $F^*(a) = -(F(a^*)^*$ for all a in A. It is easy to see that the mapping $F \to F^*$ is an algebra involution on the Lie algebra D. Also if $F = F^*$ then a result by Sinclair [27, Remark 3.5] implies that F is a hermitian element in the unital Banach algebra $BL(A)$. Therefore the mapping $T \to \frac{1}{2}(FT - TF)$ from $BL(A)$ into $BL(A)$ is a hermitian operator on $BL(A)$. Now it is easily deduced that the mapping $G \to \frac{1}{2}(FG - GF)$ from D into D is also a hermitian operator on D. But this last operator is just the operator $L_F = -R_F$ on the Lie algebra D. Thus we have proved that D is an algebra with hermitian multiplication. Let G be in An(D). Then for any a in A we have $[G, L_a - R_a] = 0$ since $L_a - R_a$ belongs to D. So $L_{G(a)} - R_{G(a)} = [G, L_a - R_a] = 0$ and $G(a)$ belongs to the center of A. Now we have $[[G, L_a], L_a] = [L_{G(a)}, L_a] = 0$ and a theorem of Kleinecke (see [5, Proposition 18.13]) gives $r([G, L_a]) = 0$. Thus $r(L_{G(a)}) = 0$, $r(G(a)) = 0$, and $G = 0$. Hence D has zero annihilator, so a unique complete algebra norm topology. It seems that this result is new.

3. Continuity of Homomorphisms onto a Complete Normed Nonassociative Algebra

It would be desirable for every homomorphism from a complete normed nonassociative algebra A onto a complete normed nonassociative algebra B

with zero weak radical to be continuous. But this seems to be difficult because it is not clear that homomorphisms from A onto B induce homomorphisms from $FM(A)$ onto $FM(B)$. If we replace "weak radical" by any one of the radicals introduced in Section 2, the above problem has an affirmative answer (see Remark 3.4 below). This will follow from the next theorem. The only new analytic ingredient which we need for the proof is the following

LEMMA 3.1. *Let X and Y be Banach spaces, T a linear mapping from X onto Y, and F (resp.: G) a continuous linear operator on X* (resp. : Y). Assume that $TF = GT$. *Then $r(G) \leqslant r(F)$.*

Proof. By an easy complexification argument we can assume that X and Y are complex spaces. Let z be in $\mathrm{sp}(G)$ such that $|z| = r(G)$. Then $G - zI_Y$ lies in the boundary of the set of all invertible elements in $BL(Y)$ so the range of $G - zI_Y$ is not Y (see [22, p. 279]). Since $T(F - zI_X) = (G - zI_Y)\,T$ and T is onto it follows that the range of $F - zI_X$ is not X so $z \in \mathrm{sp}(F)$. Thus $r(G) = |z| \leqslant r(F)$ as required.

DEFINITION 3.2. Let A be a nonassociative algebra and let C be any subalgebra of $L(A)$ such that $L_A \cup R_A \subset C \subset FM(A)$. As in the definition of weak radical we can consider the largest C-invariant subspace of A consisting of elements a such that L_a and R_a lie in the Jacobson radical of C. This subspace will be called the C-radical of A and denoted by C-Rad(A). The *ultra-weak radical* of A (uw-Rad(A)) is defined as the sum of all the C-radicals of A when C runs through the set of all subalgebras of $L(A)$ satisfying $L_A \cup R_A \subset C \subset FM(A)$. Since the weak radical of A is a C-radical (take $C = FM(A)$) it follows that w-Rad$(A) \subset$ uw-Rad(A).

THEOREM 3.3. *Let T be a homomorphism from a complete normed nonassociative algebra A onto a complete normed nonassociative algebra B. Assume that the altra-weak radical of B is zero. Then T is continuous.*

Proof. By the above inclusion B has zero weak radical, so a unique complete algebra norm topology (Theorem 1.10). Thus it is enough to prove that $\mathrm{Ker}(T)$ is closed. Consider the couples (F, G) with F in $FM(A)$, G in $FM(B)$, and $TF = GT$. Let C (resp.: D) be the set of all G (resp.: F) which appear in these couples. It is easy to see that C (resp.: D) is a subalgebra of $FM(B)$ (resp.: $FM(A)$) including $L_B \cup R_B$ (resp.: $L_A \cup R_A$) and that $\hat{T}\colon F \to G$ is a homomorphism from D onto C satisfying $\hat{T}(L_a) = L_{T(a)}$ and $\hat{T}(R_a) = R_{T(a)}$ for all a in A. Let E be the closure in D of $\mathrm{Ker}(\hat{T})$ (recall that by Remark 1.8, $FM(A)$, so also D, is an algebra of continuous operators on A). Then $\hat{T}(E)$ is a two-sided ideal of C. For any element G in $\hat{T}(E)$ there is an F in D such that $\|F\| < 1$ and $TF = GT$. Therefore by Lemma 3.1 we have

$r(G) \leqslant r(F) < 1$ so F (resp.: G) has a quasiinverse F^0 (resp.: G^0) in $BL(A)$ (resp.: $BL(B)$). F^0 (resp.: G^0) lies in $FM(A)$ (resp.: $FM(B)$) since $FM(A)$ (resp.: $FM(B)$) is a full subalgebra of $BL(A)$ (resp.: $BL(B)$). An easy calculation shows that $TF^0 = G^0T$, from which we deduce that G^0 belongs to C. Thus we have proved that $\hat{T}(E)$ is a quasiinvertible ideal of C so $\hat{T}(E) \subset \text{Rad}(C)$. If a is any element in the closure in A of $\text{Ker}(T)$ we have easily that L_a and R_a belong to E so $L_{T(a)}$ $(=\hat{T}(L_a))$ and $R_{T(a)}(=\hat{T}(R_a))$ belong to $\hat{T}(E)$. Thus $T(\overline{\text{Ker}(T)})$ is a subspace of B any element b of which satisfies that L_b and R_b belong to the radical of C. This together with the invariance of $T(\overline{\text{Ker}(T)})$ under C (which is easy to see) shows that $T(\overline{\text{Ker}(T)}) \subset C\text{-Rad}(B) \subset \text{uw-Rad}(B) = \{0\}$. So $\text{Ker}(T)$ is closed in A as required.

Remarks 3.4. (i) Let A be a nonassociative (resp.: noncommutative Jordan) algebra. The same argument as that in the proof of Proposition 2.3 shows that $C\text{-Rad}(A) \subset \text{Rad}(A)$ (resp.: $C\text{-Rad}(A) \subset \text{M-Rad}(A)$) for any subalgebra C of $L(A)$ such that $L_A \cup R_A \subset C \subset FM(A)$. Therefore uw-Rad$(A) \subset \text{Rad}(A)$ (resp.: uw-Rad$(A) \subset$ M-Rad(A)) and, by application of the above theorem, homomorphisms from a complete normed nonassociative algebra onto a complete normed nonassociative (resp.: noncommutative Jordan) algebra with zero radical (resp.: zero McCrimmon radical) are continuous. This improves the result by Aupetit in [2, Theorem 2].

(ii) Also, as in the proof of Proposition 2.5, every simple nonassociative algebra has zero ultraweak radical, so homomorphisms from a complete normed nonassociative algebra onto a complete normed simple nonassociative algebra are continuous.

We conclude this paper by listing some problems which arise in a natural way from our results.

PROBLEMS 3.5. (i) *Does the weak radical of an associative (resp.: Jordan) algebra agree with the Jacobson (resp.: McCrimmon) radical?* The answer is affirmative if the algebra is either associative and commutative or finite dimensional. The associative and Jordan cases are related because for an associative algebra A we can use our Proposition 2.3 and a result by McCrimmon [17] to obtain that w-Rad$(A) \subset$ w-Rad$(A^+) \subset$ M-Rad$(A^+) =$ Rad(A). Thus the Jordan algebra A^+ has weak radical equal to the McCrimmon radical when the associative algebra A has weak radical equal to the (Jacobson) radical. The weak radical of a noncommutative Jordan algebra does not agree in general with the McCrimmon radical as can deduced of the results in Section 2. Therefore the unification of a possible affirmative answer to the problem of the equality of weak and McCrimmon radicals for associative and Jordan algebras should be restricted to a smaller class of algebras, as, for example, the generalized standard algebras [25].

(ii) *Is every derivation of a complete normed nonassociative algebra with zero weak radical continuous*? Although every derivation of a Banach (associative) semisimple algebra is continuous [11] it seems to be unknown whether or not the separating ideal for a derivation of a Banach algebra lies in the Jacobson radical. If the answer to this question were affirmative and extensible to derivations of full subalgebras of Banach algebras then an argument very similar to that in the proof of Proposition 1.9 would show that the separating ideal of any derivation of a complete normed nonassociative algebra lies in the weak radical, giving an affirmative answer to our problem. Such an affirmative answer would imply that every Jordan derivation of a semisimple Banach algebra A is continuous so it is a derivation of A [27].

(iii) *Is the property* $\text{w-Rad}(A) = A$ *(*A*, a nonassociative algebra) a radical property in the precise sense of the word*? (see [29, pp. 11–12]). An affirmative answer to this problem would be the beginning of the development of a theory of the weak radical which could make easier an approach to other problems. If the answer were negative it would be desirable to find a radical property $\mathfrak{R}$ for nonassociative algebras such that for any nonassociative algebra A the inclusion $\mathfrak{R}(A) \subset \text{w-Rad}(A)$ is true and any complete normed $\mathfrak{R}$-semisimple nonassociative algebra has a unique complete algebra norm topology.

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