Journal of Differential Equations **185**, 225–250 (2002)
doi:10.1006/jdeq.2001.4162

# On a Class of Semilinear Elliptic Equations in $R^n$

Soohyun Bae[1] and Tong Keun Chang

*Department of Mathematics, Yonsei University, Seoul 120-749, Republic of Korea*

Received May 21, 2001

We establish that for $n \geqslant 3$ and $p > 1$, the elliptic equation $\Delta u + K(x)u^p = 0$ in $\mathbf{R}^n$ possesses separated positive entire solutions of infinite multiplicity, provided that a locally Hölder continuous function $K \geqslant 0$ in $\mathbf{R}^n \setminus \{0\}$, satisfies $K(x) = O(|x|^\sigma)$ at $x = 0$ for some $\sigma > -2$, and $K(x) = c|x|^{-2} + O(|x|^{-n}[\log |x|]^q)$ near $\infty$ for some constants $c > 0$ and $q > 0$. In the radial case $K(x) = \frac{|x|^l}{1+|x|^\tau}$ with $l > -2$ and $\tau \geqslant 0$, or $K(x) = \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}}$ with $\lambda > 0$, we investigate separation phenomena of positive radial solutions, and show that if $n$ and $p$ are large enough, the equation possesses a positive radial solution with initial value $\alpha$ at 0 for each $\alpha > 0$ and a unique positive radial singular solution among which any two solutions do not intersect. 

*Key Words:* semilinear elliptic equations; separated positive solutions; infinite multiplicity; singular solutions.

## 1. INTRODUCTION

In this paper, we study the elliptic equation

$$\Delta u + K(x)u^p = 0, \tag{1.1}$$

where $n \geqslant 3$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $p > 1$, and $K$ is a locally Hölder continuous function in $\mathbf{R}^n \setminus \{0\}$. By an entire solution of (1.1), we mean a positive weak solution of (1.1) in $\mathbf{R}^n$ satisfying (1.1) pointwise in $\mathbf{R}^n \setminus \{0\}$.

There have been many works on (1.1) which occurs frequently in Riemannian geometry and mathematical physics. We refer the interested readers to [3, 4, 11–13, 17–19] and the references therein.

The fundamental questions are about the multiplicity of positive solutions and their characteristic properties. The equation shows different frames according to many cases of $K$ and the exponent $p$. This paper presents

[1]This research was supported by Korea Research Foundation Grant KRF-2001-005-D00009.

sufficient conditions verifying *infinite multiplicity* of positive solutions, and explains solutions in terms of *separation*. In such perspective, we review related works as follows. It was Ni [17] in 1982 who first studied (1.1) systematically. In case $|K(x)| \leqslant C(1 + |x|^l)$ for some $l < -2$, Ni showed that (1.1) with $p > 1$ possesses infinitely many positive solutions bounded away from 0. From his proof of this problem, we emphasize that any two solutions among them do not intersect (or are separated). In [6], Gui treated the opposite case $K$ behaving like $|x|^l$ at $\infty$ for $l > -2$, and established infinite multiplicity of separated positive solutions for (1.1) under the following conditions:

(K1) $K \geqslant 0$ is a locally Hölder continuous function in $\mathbf{R}^n \setminus \{0\}$,

(K2) $K(x) = O(|x|^\sigma)$ at $x = 0$ for some $\sigma > -2$,

and an integral condition controlling the deviation of $K$ at $\infty$ from $c|x|^l$ for some $c > 0$, $l > -2$ when $p \geqslant p(n, l)$, where

$$p_c = p_c(n, l) = \begin{cases} \frac{(n-2)^2 - 2(l+2)(n+l) + 2(l+2)\sqrt{(n+l)^2 - (n-2)^2}}{(n-2)(n-10-4l)} & \text{if } n > 10 + 4l, \\ \infty & \text{if } n \leqslant 10 + 4l. \end{cases}$$

Moreover, each solution $u$ satisfies the following asymptotic behavior:

$$\lim_{|x| \to \infty} |x|^m u(x) = L,$$

where $m = \frac{l+2}{p-1}$ and

$$L = L(n, p, l, c) = \left[ m(n - 2 - m)\frac{1}{c} \right]^{\frac{1}{p-1}}. \qquad (1.2)$$

Recently, motivated by a work [2] on infinite multiplicity for the inhomogeneous equation

$$\Delta u + u^p + f = 0,$$

Bae *et al.* [1] studied again (1.1) with $p \geqslant p_c(n, l)$, and improved Gui's result. In particular, infinite multiplicity of separated positive solutions for (1.1) is verified when $K$ satisfies (K1), (K2) and the following condition:

$$K(x) = c|x|^l + O(|x|^{-d}) \qquad \text{at } |x| = \infty$$

for some $d > n - \lambda_2(n, p, l) - m(p + 1)$, where

$$\lambda_2 = \lambda_2(n, p, l) = \frac{(n - 2 - 2m) + \sqrt{(n - 2 - 2m)^2 - 4(l + 2)(n - 2 - m)}}{2}.$$

The quadratic polynomial $P(z) = z^2 - (n-2-2m)z + (l+2)(n-2-m)$ has two positive real roots $\lambda_1 \leqslant \lambda_2$ if and only if $n > 10 + 4l$ and $p \geqslant p_c$. These two numbers $\lambda_1$, $\lambda_2$ play important roles in describing the asymptotic behavior at $\infty$ of solutions in case $K(x) = c|x|^l$ (see [7, 10]).

On the basis of the above-mentioned Ni's result and the observation, $p_c(n,l) \to 1$ as $l \to -2$, it is assumed that (1.1) possesses separated positive solutions of infinite multiplicity when $K(x)$ has a similar behavior to $c|x|^{-2}$ at $\infty$. The first objective of this paper is to study this borderline problem, and to establish infinite multiplicity for any $p > 1$. Before stating our result, we summarize two known facts concerning the case. In [12], Li and Ni established that if a nonnegative radial function $K$ in $\mathbf{R}^n$, satisfies $K(r) = O(r^\sigma)$ at $r = 0$ for some $\sigma \geqslant 0$, $r^2 K(r) \to c > 0$ as $r \to \infty$, and

$$\limsup_{r\to\infty} r(\log r)^2 [r^2 K(r)]_r < \frac{cp}{(n-2)(p-1)},$$

then there exists $\alpha^* > 0$ such that for each $\alpha \in (0, \alpha^*]$, (1.1) with $1 < p < (n+2+2\sigma)/(n-2)$ has a positive radial solution $u_\alpha$ with $u_\alpha(0) = \alpha$. For the nonradial case, Gui [5] proved

Theorem A. *If $C_1 \leqslant (1+|x|)^2 K(x) \leqslant C_2$ for some $C_2 \geqslant C_1 > 0$ and $K(x) = c|x|^{-2} + O(|x|^{-d})$ at $|x| = \infty$ for some $d > 2$, then* (1.1) *with $p > 1$ possesses infinitely many separated positive entire solutions with the asymptotic behavior*

$$\lim_{|x|\to\infty} (\log|x|)^{1/(p-1)} u(x) = L, \tag{1.3}$$

*where*

$$L = L(n, p, -2, c) = \left[\frac{n-2}{(p-1)c}\right]^{\frac{1}{p-1}}. \tag{1.4}$$

In the former, $K$ is radially symmetric while in the latter, $K$ is positive. Moreover, $K$ has no singularity at the origin in both cases. Without these conditions on $K$, we establish infinite multiplicity for (1.1) through an analysis on asymptotic behavior near infinity.

Theorem 1.1. *Let $p > 1$. If $K$ satisfies* (K1), (K2) *and*

$$K(x) = c|x|^{-2} + O(|x|^{-n}[\log|x|]^q), \tag{1.5}$$

*near $|x| = \infty$ for some constants $c > 0$ and $q > 0$. Then,* (1.1) *possesses infinitely many separated positive entire solutions with the asymptotic behavior* (1.3). *In case $K$ is radial, there exist separated positive radial solutions $u_\alpha$ indexed continuously by all small initial data $\alpha > 0$.*

The question, whether $[\log|x|]^q$ in (1.5) can be replaced by the form $|x|^q$ with $0<q<n-2$, is not yet answered.

To prove Theorem 1.1, we make use of the particular barrier method initiated by Gui [5, 6] and modified in [1, 2]. In order to activate this method efficiently, one needs detailed information on the asymptotic behavior of the difference of positive solutions to the specific equation (1.1) with $K(x) = c|x|^{-2}$ near $\infty$. The first step is to investigate the asymptotic behavior. Afterwards, by employing Green's identity, we construct infinitely many pairs of super- and sub-solutions of the given equation (1.1). By standard techniques showing the existence of positive solutions, Theorem 1.1 is verified.

In case $K$ is radial, we obtain a continuous family of separated positive radial solutions in proving Theorem 1.1 as a by-product, if initial data are small enough. The next question is whether these separation phenomena are valid up to $\infty$. More generally, we study this question under a monotonicity assumption on $K(r)$. The initial value problem for positive radial solutions is

$$u'' + \frac{n-1}{r}u' + K(r)u^p = 0, \qquad u(0) = \alpha > 0. \tag{1.6}$$

This has a unique solution $u \in \mathbf{C}^2((0,\varepsilon)) \cap \mathbf{C}([0,\varepsilon))$ for $\varepsilon > 0$ small under the following condition:

(K) $K$ is a nonnegative radial function in $\mathbf{C}((0,\infty))$ with $K \not\equiv 0$ and

$$\int_0 rK(r)\,dr < \infty.$$

(See [18, Propositions 4.1 and 4.2].) We denote the unique solution by $u_\alpha(r)$ and call $u_\alpha$ a *slowly decaying solution* if $u_\alpha(r) > 0$ on $(0,\infty)$ and $r^{n-2}u_\alpha(r) \to \infty$ as $r \to \infty$. Note that under (K), $r^{n-2}u_\alpha(r)$ is increasing as $r$ increases if $u_\alpha > 0$ on $(0,\infty)$. The structure of **Type S** is as follows:

(1.6) has a slowly decaying solution $u_\alpha(r)$ for every $\alpha > 0$.

For this structure, Ni and Yotsutani showed the following in [18, Theorem 6].

THEOREM B. *Let* $p \geqslant \frac{n+2+2l}{n-2}$ *with* $l > -2$. *If* $K$ *satisfies* (K), $K(r) = O(r^l)$ *at* $r = 0$ *and* $(r^{-l}K(r))' \leqslant 0, \not\equiv 0$ *in* $(0,\infty)$, *then for every* $\alpha > 0$, (1.6) *has a positive solution* $u_\alpha$ *on* $(0,\infty)$ *and*

$$\int_0^\infty K u_\alpha^p r^{n-1}\,dr = \infty.$$

*Moreover, if* $p > \frac{n+2+2l}{n-2}$ *and* $(r^{-l}K(r))' \equiv 0$, *then this still holds.*

For solution structures including various types, we refer the interested readers to [9, 18, 20, 21]. To specify asymptotic behavior at $\infty$, $r^{-l}K(r) \to c > 0$ as $r \to \infty$, can be added to the assumptions of Theorem B, with the result that by [10, Theorem 1], each solution $u_\alpha$ satisfies

$$\lim_{r\to\infty} r^m u_\alpha(r) = L(n, p, l, c). \tag{1.7}$$

As the simplest form, the Lane–Emden equation or Emden–Fowler equation from astrophysics:

$$\Delta u + c|x|^l u^p = 0, \tag{1.8}$$

in $\mathbf{R}^n$, where $l > -2$, $p > \frac{n+2+2l}{n-2}$ and $c > 0$, gives an insight into the structure of Type S. As seen in Theorem B, (1.8) has the structure of Type S. Let $v_\alpha$ be a positive radial solution $v_\alpha$ with $v_\alpha(0) = \alpha$ for each $\alpha > 0$. Then, $v_\alpha$ satisfies (1.7). Furthermore, *any two positive radial solutions of* (1.8) *cannot intersect each other if and only if* $p \geqslant p_c(n, l)$ *with* $l > -2$. (See Propositions 3.5 and 3.7 in [19].) Therefore, we expect that if $p \geqslant p_c(n, l)$, then the structure, under proper conditions on $K(r)$, is of **Type SS**:

(1.6) has the structure of Type S, and any two positive solutions are separated.

There have been several similar studies on separation structure (see [1, 2, 5–7, 16]). The structure of Type SS is closely related with the stability of positive solutions, which are translated as positive steady states of the corresponding parabolic equations. In [7, 8], Gui *et al.* observed that the critical exponent $p = p_c(n, 0)$ is the dividing line of "instability" in case $p < p_c(n, 0)$ and "stability" in case $p \geqslant p_c(n, 0)$ in a certain sense.

In a recent paper [16], Liu *et al.* studied the structure of Type SS and proved the following

THEOREM C. *Let* $p > p_c(n, l)$ *with* $l > -2$. *Suppose that* $K \in \mathbf{C}^1((0, \infty))$ *satisfies*

$$\frac{d}{dr}(r^{-l}K(r)) \leqslant 0, \qquad r \in (0, \infty)$$

*and*

$$\lim_{r\to 0} r^{-l}K(r) = k_0 > 0, \qquad \lim_{r\to\infty} r^{-l}K(r) = c > 0.$$

*Then*, (1.6) *has the structure of Type SS. Moreover, there is a unique singular solution* $U(r)$ *that for each positive radial solution* $u_\alpha$ *of* (1.6), $r^m u_\alpha(r) \to L(n, p, l, c)$ *as* $r \to \infty$ *and* $r^m U(r) \to L(n, p, l, k_0)$ *as* $r \to 0$,

$$u_\alpha(r) < U(r) \leqslant \frac{L(n, p, l, 1)}{[r^2 K(r)]^{\frac{1}{p-1}}}. \tag{1.9}$$

The second objective of this paper is to include the case $p = p(n, l)$ and to search for the structure of Type SS when $K(x) = \frac{|x|^l}{1+|x|^\tau}$ for some $l > -2$ and $\tau \geqslant 0$, or $K(x) = \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}}$ for some $\lambda > 0$. Our result covers these two cases.

THEOREM 1.2. *Let $p \geqslant p_c(n, l)$ with $l > -2$. Assume that K satisfies* (K) *and $r^{-l}K(r)$ is non-increasing in $r \in (0, \infty)$. Then,* (1.6) *has the structure of Type SS and possesses a singular solution $U(r)$ as the monotone limit of positive regular solutions $u_\alpha(r)$ satisfying* (1.9).

Importantly, the structure of Type SS is established without the convergence of $r^{-l}K(r)$ to a positive constant as $r$ tends to $\infty$. For example, the case in which $r^{-l}K(r)$ behaves like $r^{-\tau}$ at $\infty$ for some $\tau \geqslant 0$, can be solved. Moreover, (1.9) provides upper bounds of the family $\{u_\alpha\}$ on compact regions in $(0, \infty)$, which lead immediately to the existence of a positive radial singular solution. Theorem 1.2 can be applied directly to the typical equation

$$\Delta u + \frac{|x|^l}{1+|x|^\tau} u^p = 0 \qquad \text{in } \mathbf{R}^n,$$

where $l > -2$ and $\tau \geqslant 0$ as follows:

COROLLARY 1.3. *Let $l > -2$, $\tau \geqslant 0$ and $p \geqslant p_c(n, l) (> \frac{n+2+2l}{n-2})$. Then, the equation*

$$u'' + \frac{n-1}{r} u' + \frac{r^l}{1+r^\tau} u^p = 0, \qquad u(0) = \alpha > 0, \tag{1.10}$$

*has the structure of Type SS and possesses a unique singular solution $U(r)$ that for every $\alpha > 0$,*

$$u_\alpha(r) < U(r) \leqslant \left[\frac{1+r^\tau}{r^{l+2}}\right]^{\frac{1}{p-1}} L(n, p, l, 1)$$

*and*

$$\lim_{r \to 0} r^{\frac{l+2}{p-1}} U(r) = L(n, p, l, 1).$$

*Moreover,*

$$\lim_{r \to \infty} r^{\frac{l-\tau+2}{p-1}} u_\alpha(r) (\text{or } U(r)) = L(n, p, l-\tau, 1) \qquad \text{if } l - \tau > -2,$$

$$\lim_{r \to \infty} (\log r)^{\frac{1}{p-1}} u_\alpha(r) (\text{or } U(r)) = L(n, p, -2, 1) \qquad \text{if } l - \tau = -2,$$

*and at* $\infty$,

$$u_\alpha(r)(or\ \ U(r)) - c_\alpha(or\ \ c_\infty) \sim \begin{cases} r^{2+l-\tau} & \text{if } \ -2 > l-\tau > -n, \\ r^{2-n}\log r & \text{if } \ l-\tau = -n, \\ r^{2-n} & \text{if } \ l-\tau < -n, \end{cases}$$

*where* $\lim_{r\to\infty} u_\alpha(r)(or\ \ U(r)) = c_\alpha(or\ \ c_\infty) > 0$. *Here, "$f \sim g$ at $\infty$" means that there exist two positive constants $C_1, C_2$ such that $C_1 g \leqslant f \leqslant C_2 g$ near $\infty$.*

Uniqueness part of singular solutions was proved in [16] while the asymptotic behaviors in Corollary 1.3 can be proved by some results in [10, 12]. The precise statements to guarantee these two parts in Corollary 1.3 shall be given in Section 3.

In 1986, Batt *et al.* [3] proposed the equation

$$\Delta u + \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}} u^p = 0 \qquad \text{in } \mathbf{R}^3, \tag{1.11}$$

where $\lambda > 0$. This model with $\lambda = 2$ was formulated in 1930 by Matukuma to describe the dynamics of globular cluster of stars in $\mathbf{R}^3$. Here, $u > 0$ represents the gravitational potential, $\varrho = -\frac{1}{4\pi}\Delta u$ is the density and $\int_{\mathbf{R}^3} \varrho(x)\, dx$ is the total mass. Since the globular cluster has the radial symmetry, positive radial entire solutions are of particular interest. On Matukuma equation, it is known that if $1 < p < 5$, then $u_\alpha$ is a slowly decaying solution only for small $\alpha > 0$, while if $p \geqslant 5$, the structure is of Type S. (See Theorems 5 and 6 in [18].) We remark a consequence of Theorem 1.1: For $p > 1$ and $\lambda > 0$, any two positive radial solutions of (1.11) are separated if initial data are small enough. Applying Theorem 1.2 to (1.11), we establish the following

Corollary 1.4. *Let* $\lambda > 0$ *and* $p \geqslant p_c(n, \lambda-2)(> 1 + \frac{2\lambda}{n-2})$. *Then, the equation*

$$u'' + \frac{n-1}{r}u' + \frac{r^{\lambda-2}}{(1+r^2)^{\lambda/2}} u^p = 0, \qquad u(0) = \alpha > 0, \tag{1.12}$$

*has the structure of Type SS and possesses a unique singular solution $U(r)$ that for every $\alpha > 0$,*

$$u_\alpha(r) < U(r) \leqslant \left[\frac{(1+r^2)^{\frac{\lambda}{2}}}{r^\lambda}\right]^{\frac{1}{p-1}} L(n,p,\lambda-2,1),$$

$$\lim_{r\to 0} r^{\frac{\lambda}{p-1}}\, U(r) = L(n,p,\lambda-2,1),$$

*and*

$$\lim_{r\to\infty} (\log r)^{\frac{1}{p-1}} u_\alpha(r)(or\ \ U(r)) = L(n, p, -2, 1).$$

We observe that in $\mathbf{R}^3$, if $0<\lambda<\frac{1}{4}$ and $p\geqslant p_c(3, \lambda - 2)$, (1.12) has the structure of Type SS and moreover, possesses a unique singular solution. The additional assumption $p > 1 + \lambda$ implies that the mass sum on the unit ball is finite.

This paper is organized as follows. The asymptotic behavior of positive radial solutions of (1.1) with $K(x) = c|x|^{-2}$ near $\infty$ is studied in Section 2 and then, we prove Theorem 1.1 and apply multiplicity results to Riemannian geometry. In Section 3, we prove Theorem 1.2 and make several remarks.

## 2. INFINITE MULTIPLICITY

In this section, we consider the case that $K(x)$ behaves like $|x|^{-2}$ at $\infty$. Before studying the nonradial case, we analyze the radial case in detail. Then, we proceed similar arguments as in [1, 2, 6] in order to prove infinite multiplicity for general cases. It will turn out that the asymptotic behavior of the difference of two positive radial solutions of

$$\Delta u + c|x|^{-2}u^p = 0 \tag{2.1}$$

near $\infty$ for some $c > 0$, plays a central role in establishing infinite multiplicity for (1.1).

2.1. In this subsection, we consider the asymptotic behavior of positive radial solutions of (2.1). We first recall the following asymptotic behavior (see [10, Lemma 5.1]).

LEMMA 2.1. *Let $p > 1$, $c > 0$ and $u$ be a positive radial solution of* (2.1). *If*

$$\lim_{r\to\infty} (\log r)^{1/(p-1)}u(r) = L,$$

*then*

$$u(r) = \frac{L}{(\log r)^{1/(p-1)}} - \frac{pL\log(c\log r)}{(p-1)^2(n-2)(\log r)^{p/(p-1)}} + o\left(\frac{1}{(\log r)^{p/(p-1)}}\right), \tag{2.2}$$

*near $\infty$, where $L = L(n, p, -2, c)$ is given by* (1.4).

Another ingredient is that any two positive radial solutions of (2.1) do not intersect infinitely. To prove this, we need Lemma 4.1 in [2].

LEMMA 2.2. *Suppose that $W$ satisfies*

$$W'' + \Phi(t)W' + \Psi(t)W \leqslant (or \geqslant)\ 0$$

*in* $[T, +\infty)$, *where* $\Phi > 0$ *and*

$$\Psi(t) \leqslant \tfrac{1}{4}\,\Phi^2(t) + \tfrac{1}{2}\Phi'(t).$$

*Then, $W$ does not change sign for $t$ large.*

The difference of two positive radial solutions of (2.1) displays the following asymptotic behavior.

PROPOSITION 2.3. *Let $p > 1$ and $v_1$, $v_2$ be two positive radial solutions of* (2.1). *Suppose that*

$$\lim_{r\to\infty} (\log r)^{1/(p-1)} v_1(r) = L(n, p, -2, c) = \lim_{r\to\infty} (\log r)^{1/(p-1)} v_2(r). \quad (2.3)$$

*Then*,

$$\lim_{r\to\infty} (\log r)^d [v_2(r) - v_1(r)] = 0$$

*for any $d > 0$.*

*Proof.* Set $W(t) := V_2(t) - V_1(t)$, $t = \log r$, where $V_i(t) = v_i(r)$, $i = 1, 2$. Then, by (2.2)

$$W(t) = o(t^{-p/(p-1)}) \qquad \text{at} + \infty \quad (2.4)$$

and thus,

$$\int^{+\infty} W^2(s)\, ds < \infty. \quad (2.5)$$

Moreover, $W$ satisfies

$$W_{tt} + (n-2)W_t + h(t)W = 0, \quad (2.6)$$

where

$$h(t) := \begin{cases} c\frac{V_2^p - V_1^p}{V_2 - V_1} & \text{if } V_2(t) \neq V_1(t), \\ pcV_1^{p-1} & \text{if } V_2(t) = V_1(t). \end{cases}$$

Since $pc \min\{V_1, V_2\}^{p-1} \leqslant h(t) \leqslant pc \max\{V_1, V_2\}^{p-1}$, we have from (2.3),

$$\lim_{t\to+\infty} th(t) = \frac{p(n-2)}{p-1}. \quad (2.7)$$

By Lemma 2.2, $W$ does not change sign for $t$ large. We may assume that $W(t)>0$ on $[T,+\infty)$ for some $T>0$. From (2.6), we have $W_t(t)\leqslant e^{-(n-2)(t-T)}W_t(T)$ for $t>T$. Since $v_i(r)$ goes to 0 as $r$ tends to $\infty$, $v_i$ and $V_i$ are decreasing eventually near $\infty$ and $+\infty$, respectively. Hence, near $+\infty$,

$$h_t=\frac{pc(V_{2t}V_2^{p-1}-V_{1t}V_1^{p-1})-pc(V_{2t}-V_{1t})\xi^{p-1}}{V_2-V_1}\leqslant 0 \tag{2.8}$$

for some $V_1\leqslant\xi\leqslant V_2$. Multiplying (2.6) by $W_t$ and integrating over $[\tau, t]$ for $\tau\geqslant T$ large enough that (2.8) holds for $t\geqslant\tau$, we have

$$-\left[\frac{1}{2}\,W_s^2+\tfrac{1}{2}\,h(s)W^2(s)\right]_\tau^t=\int_\tau^t\left[(n-2)W_s^2(s)-\frac{1}{2}h_s(s)W^2(s)\right]ds,$$

which combined with (2.4), (2.7) and (2.8) implies

$$\int_\tau^{+\infty}W_s^2<\infty,\quad \int_\tau^{+\infty}(-h_s)W^2<\infty. \tag{2.9}$$

Then,

$$\lim_{t\to+\infty}W_t^2(t)=0 \tag{2.10}$$

and

$$\frac{1}{2}\,W_t^2(\tau)+\frac{1}{2}\,h(\tau)W^2(\tau)=\int_\tau^{+\infty}\left[(n-2)W_s^2(s)-\frac{1}{2}h_s(s)W^2(s)\right]ds. \tag{2.11}$$

Multiplying (2.6) by $W$ and integrating over $[\tau,+\infty)$, we obtain from (2.4) and (2.10),

$$W(\tau)W_t(\tau)+\frac{n-2}{2}\,W^2(\tau)=\int_\tau^{+\infty}[h(s)W^2(s)-W_s^2(s)]\,ds. \tag{2.12}$$

Again, multiplying (2.6) by $tW_t$ and integrating over $[\tau,+\infty)$, we observe that

$$\int_\tau^{+\infty}sW_s^2<\infty,\qquad \int_\tau^{+\infty}(-sh_s)W^2<\infty. \tag{2.13}$$

Moreover, it follows that

$$\lim_{t\to+\infty}tW_t^2(t)=0 \tag{2.14}$$

since by (2.4) and (2.7),

$$\lim_{t\to+\infty}th(t)W^2(t)=0.$$

Then,

$$\tfrac{1}{2}\,\tau W_t^2(\tau) + \tfrac{1}{2}\,\tau h(\tau) W^2(\tau)$$
$$= \int_\tau^{+\infty} \left[(n-2)s - \frac{1}{2}\right] W_s^2(s)\,ds - \frac{1}{2}\int_\tau^{+\infty} [sh_s(s) + h(s)] W^2(s)\,ds. \quad (2.15)$$

Multiplying (2.6) by $tW$, integrating over $[\tau, +\infty)$, and using (2.4), (2.10) and (2.13), we have

$$\tau W(\tau) W_t(\tau) - \frac{1}{2}\,W^2(\tau) + \frac{n-2}{2}\tau W^2(\tau)$$
$$= \int_\tau^{+\infty} \left[\left\{sh(s) - \frac{n-2}{2}\right\} W^2(s) - sW_s^2(s)\right] ds. \quad (2.16)$$

Integrating (2.11) over $[\tau_1, +\infty)$ such that for $t \geqslant \tau_1$, $W(t) > 0$ and $h_t(t) \leqslant 0$, and using (2.5), (2.8) and (2.9), we have

$$\frac{1}{2}\int_{\tau_1}^{+\infty} [W_t^2(\tau) + h(\tau) W^2(\tau)]\,d\tau$$
$$= \int_{\tau_1}^{+\infty} \int_\tau^{+\infty} \left[(n-2) W_s^2(s) - \frac{1}{2}\,h_s(s) W^2(s)\right] ds\,d\tau < \infty, \quad (2.17)$$

which implies

$$\int_{\tau_1}^{+\infty} \int_\tau^{+\infty} W_s^2(s)\,ds\,d\tau < \infty, \qquad \int_{\tau_1}^{+\infty} \int_\tau^{+\infty} [-h_s(s)] W^2\,ds\,d\tau < \infty. \quad (2.18)$$

Integrating (2.12) over $[\tau_1, +\infty)$, we have

$$\frac{1}{2}\,W^2(\tau_1) - \frac{n-2}{2}\int_{\tau_1}^{+\infty} W^2(\tau)\,d\tau$$
$$= \int_{\tau_1}^{+\infty} \int_\tau^{+\infty} [W_s^2(s) - h(s) W^2(s)]\,ds\,d\tau, \quad (2.19)$$

which combined with (2.5) and (2.18) implies

$$\int_{\tau_1}^{+\infty} \int_\tau^{+\infty} h(s) W^2(s)\,ds\,d\tau < \infty. \quad (2.20)$$

Integrating (2.15) over $[\tau_1, +\infty)$, we have from (2.5), (2.7), (2.13) and (2.20),

$$\int_{\tau_1}^{+\infty} \int_\tau^{+\infty} sW_s^2\,ds\,d\tau < \infty, \quad \int_{\tau_1}^{+\infty} \int_\tau^{+\infty} [-sh_s(s)] W^2\,ds\,d\tau < \infty \quad (2.21)$$

and

$$\frac{1}{2}\int_{\tau_1}^{+\infty}[\tau W_t^2(\tau)+\tau h(\tau)W^2(\tau)]\,d\tau$$
$$=\int_{\tau_1}^{+\infty}\int_{\tau}^{+\infty}\left[\left\{(n-2)s-\frac{1}{2}\right\}W_s^2(s)\right.$$
$$\left.-\frac{1}{2}\{sh_s(s)+h(s)\}W^2(s)\right]ds\,d\tau. \quad (2.22)$$

Integrating (2.16) over $[\tau_1, t]$, we have

$$\left[\frac{\tau}{2}\,W^2(\tau)\right]_{\tau_1}^{t}+\int_{\tau_1}^{t}\left[\frac{n-2}{2}\tau-1\right]W^2(\tau)\,d\tau$$
$$=\int_{\tau_1}^{t}\int_{\tau}^{+\infty}\left[\left\{sh(s)-\frac{n-2}{2}\right\}W^2(s)-sW_s^2(s)\right]ds\,d\tau.$$

By (2.4), (2.7) and (2.21),

$$\frac{\tau_1}{2}\,W^2(\tau_1)-\int_{\tau_1}^{+\infty}\left[\frac{n-2}{2}\tau-1\right]W^2(\tau)\,d\tau$$
$$=\int_{\tau_1}^{+\infty}\int_{\tau}^{+\infty}\left[sW_s^2(s)-\left\{sh(s)-\frac{n-2}{2}\right\}W^2(s)\right]ds\,d\tau, \quad (2.23)$$

which implies

$$\int_{\tau_1}^{+\infty}\int_{\tau}^{+\infty}W^2(s)\,ds\,d\tau<\infty. \quad (2.24)$$

Indeed,

$$\int_{\tau_1}^{+\infty}\int_{\tau}^{+\infty}\left\{sh(s)-\frac{n-2}{2}\right\}W^2(s)\,ds\,d\tau-\int_{\tau_1}^{+\infty}\left[\frac{n-2}{2}\tau-1\right]W^2(\tau)\,d\tau$$
$$=\int_{\tau_1}^{+\infty}\left\{sh(s)-\frac{n-2}{2}\right\}W^2(s)(s-\tau_1)\,ds-\int_{\tau_1}^{+\infty}\left[\frac{n-2}{2}\,s-1\right]W^2(s)\,ds$$
$$=\int_{\tau_1}^{+\infty}\{sh(s)-n+2\}sW^2(s)\,ds+\int_{\tau_1}^{+\infty}\left[1-\tau_1\left\{sh(s)-\frac{n-2}{2}\right\}\right]W^2(s)\,ds$$

and $sh(s)-n+2$ converges to $\frac{n-2}{p-1}$ as $s\to+\infty$. Then, (2.24) follows. Again, it follows from (2.17), (2.19), (2.22) and (2.23) that

$$\int_{\tau_2}^{+\infty}\int_{\tau_1}^{+\infty}\int_{\tau}^{+\infty}sW_s^2(s)\,ds\,d\tau\,d\tau_1<\infty$$

and

$$\int_{\tau_2}^{+\infty} \int_{\tau_1}^{+\infty} \int_{\tau}^{+\infty} W^2(s)\, ds\, d\tau\, d\tau_1 < \infty.$$

Iterating the above process, we obtain

$$\int_{\tau_d}^{+\infty} \int_{\tau_{d-1}}^{+\infty} \cdots \int_{\tau}^{+\infty} W_s^2(s)\, ds\, d\tau\, d\tau_1 \cdots d\tau_{d-1} < \infty \tag{2.25}$$

and

$$\int_{\tau_d}^{+\infty} \int_{\tau_{d-1}}^{+\infty} \cdots \int_{\tau}^{+\infty} W^2(s)\, ds\, d\tau\, d\tau_1 \cdots d\tau_{d-1} < \infty$$

for any positive integer $d$. Note that by Fubini's Theorem,

$$\int_{\tau_1}^{+\infty} \int_{\tau}^{+\infty} W_s^2(s)\, ds\, d\tau = \int_{\tau_1}^{+\infty} \int_{\tau_1}^{s} W_s^2(s)\, d\tau\, ds = \int_{\tau_1}^{+\infty} W_s^2(s)(s-\tau_1)\, ds < \infty.$$

Hence, applying this modification repeatedly to (2.25), we have the equivalent form

$$\int_{\tau_d}^{+\infty} s^d W_s^2\, ds < \infty.$$

Then, for $\tau$ large,

$$\begin{aligned} |W(\tau)| &= \left| \int_{\tau}^{+\infty} W_s(s)\, ds \right| = \left| \int_{\tau}^{+\infty} s^{-d}(s^d W_s)\, ds \right| \\ &\leqslant \left[ \int_{\tau}^{+\infty} s^{-2d}\, ds \right]^{1/2} \left[ \int_{\tau}^{+\infty} s^{2d} W_s^2\, ds \right]^{1/2} \\ &= \frac{C_d}{[(2d-1)\tau^{2d-1}]^{1/2}}, \end{aligned}$$

where

$$C_d = \left[ \int_{\tau}^{+\infty} s^{2d} W_s^2\, ds \right]^{1/2} < \infty.$$

That is,

$$\tau^{2(d-1)} |W(\tau)|^2 \leqslant \frac{C_d^2}{(2d-1)\tau}.$$

Therefore, we conclude that

$$\lim_{t\to+\infty} t^{d-1}W(t) = 0$$

for any integer $d \geqslant 1$, which completes the proof of Proposition 2.3. ∎

The assumptions on $K$ at $\infty$ in Theorem 1.1 comes from Proposition 2.3, which is one of the major elements to prove Theorem 1.1.

2.2. In order to prove infinite multiplicity for general cases, we first consider positive radial solutions of (1.6) with a radial function $K$. In particular, $K(r)$ satisfies the following condition:

(K3) $\int_1^\infty |K(r) - cr^{-2}| r^{n-1} (\log r)^{-a}\, dr < \infty$ for some $c > 0$, $a > 0$.

For our convenience, we fix a family $\{\bar{u}_\alpha\}$ of separated positive radial solutions of (1.6) indexed by $\alpha \in (0, \alpha^*]$ for some $\alpha^* > 0$ such that $\bar{u}_\alpha(0) = \alpha$, $\bar{u}_\alpha$ is monotonically increasing on $(0, \alpha^*]$ and

$$\lim_{r\to\infty} (\log r)^{1/(p-1)} \bar{u}_\alpha(r) = L(n, p, -2, c), \tag{2.26}$$

where $K$ is a smooth positive radial function $\bar{K}$ satisfying

$$\bar{K}(r) = \frac{1}{1+r^2} \qquad \text{for } 0 \leqslant r \leqslant 1$$

and

$$\bar{K}(r) = \frac{c}{r^2} \qquad \text{for } r \geqslant 2.$$

(See [5, Theorem 5.1 and Lemmas 5.3, 5.6] for the existence.) It follows from Proposition 2.3 that for each $\alpha \in (0, \alpha^*)$,

$$F_\alpha(r) \coloneqq \bar{u}_{\alpha^*}(r) - \bar{u}_\alpha(r) = o([\log r]^{-d}) \qquad \text{as } r \to \infty \tag{2.27}$$

for any $d > 0$.

To prove the existence of separated positive radial solutions, which are indexed continuously by initial data, we need the following

Lemma 2.4. *Assume that* $K \not\equiv 0$ *holds* (K). *Suppose that there exist three solutions* $u_\alpha, u_\beta, u_\gamma$ *of* (1.6) *such that* $0 < u_\alpha < u_\beta < u_\gamma$ *in* $[0, \bar{R})$ *for some* $\bar{R} \in (0, \infty]$. *Then, for each* $\alpha < \delta < \beta$, (1.6) *possesses a positive radial solution* $u_\delta$ *in* $B_{\bar{R}}$ *satisfying*

$$0 < u_\alpha(r) < u_\delta(r) < u_\beta(r)$$

*for* $0 \leqslant r < \bar{R}$.

(See Lemma 2.5 in [1].) Now, we prove infinite multiplicity in the radial case.

PROPOSITION 2.5. *Let* $p > 1$. *Assume that* $K$ *satisfies* (K) *and* (K3) *for some* $c > 0$, $a > 0$. *Then, there exists a positive constant* $\gamma^* = \gamma^*(p, K)$ *such that for each* $\gamma \in (0, \gamma^*]$, (1.6) *possesses a positive radial solution* $u_\gamma$ *with* $u_\gamma(0) = \gamma$ *with the asymptotic behavior*

$$\lim_{r\to\infty} (\log r)^{1/(p-1)} u_\gamma(r) = L(n, p, -2, c) \tag{2.28}$$

*and no two of them can intersect.*

*Proof.* For each $0<\alpha<\alpha^*$, $F_\alpha > 0$ satisfies (2.27) for any $d > 0$, and

$$\Delta F_\alpha = -\bar{K}((\bar{u}_{\alpha^*})^p - \bar{u}_\alpha^p) \leqslant -p\bar{K}\bar{u}_\alpha^{p-1}F_\alpha.$$

For all $\gamma > 0$, there exists a unique positive solution $u_\gamma$ of (1.6) locally. First, we claim that for given $0<\beta<\alpha^*$, there exists $0<\bar{\gamma}<\beta$ such that for every $0<\gamma\leqslant\bar{\gamma}$, $u_\gamma<\bar{u}_\beta$ in $\overline{B(R_\gamma)}$ whenever $u_\gamma > 0$ in $B(R_\gamma)$ for some $R_\gamma > 0$.

Suppose that for any $0<\bar{\gamma}<\beta$, there exists $0<\tilde{\gamma}<\bar{\gamma}$ such that $u_{\tilde{\gamma}} > 0$ in $B(R_{\tilde{\gamma}})$, $w_{\tilde{\gamma}}(r) := \bar{u}_\beta(r) - u_{\tilde{\gamma}}(r) > 0$ on $[0, R_{\tilde{\gamma}})$ but $w_{\tilde{\gamma}}(R_{\tilde{\gamma}}) = 0$ for some $R_{\tilde{\gamma}} > 0$. Then, $w_{\tilde{\gamma}}$ satisfies

$$\Delta w_{\tilde{\gamma}} = -\bar{K}\bar{u}_\beta^p + Ku_{\tilde{\gamma}}^p$$

in $\overline{B(R_{\tilde{\gamma}})}$. Fix $\beta<\alpha<\alpha^*$. Applying Green's identity, we have

$$\begin{aligned}
0 &\leqslant \int_{B(R_{\tilde{\gamma}})} (w_{\tilde{\gamma}}\, \Delta F_\alpha - F_\alpha\, \Delta w_{\tilde{\gamma}}) \\
&\leqslant \int_{B(R_{\tilde{\gamma}})} \{-p\bar{K}\bar{u}_\alpha^{p-1} w_{\tilde{\gamma}} F_\alpha + \bar{K}\bar{u}_\beta^p F_\alpha - K u_{\tilde{\gamma}}^p F_\alpha\} \\
&\leqslant \int_{B(R_{\tilde{\gamma}})} \{-p\bar{K}\bar{u}_\alpha^{p-1} w_{\tilde{\gamma}} F_\alpha + p\bar{K}\bar{u}_\beta^{p-1} w_{\tilde{\gamma}} F_\alpha + (\bar{K} - K) u_{\tilde{\gamma}}^p F_\alpha\}
\end{aligned}$$

and

$$p\int_{B(R_{\tilde{\gamma}})} [\bar{u}_\alpha^{p-1} - \bar{u}_\beta^{p-1}]\bar{K} w_{\tilde{\gamma}} F_\alpha \leqslant \int_{B(R_{\tilde{\gamma}})} (\bar{K} - K) u_{\tilde{\gamma}}^p F_\alpha.$$

Since $\bar{u}_\beta > 0$ in $\mathbf{R}^n$ and $u_{\tilde{\gamma}} \leqslant \tilde{\gamma}$ on $[0, R_{\tilde{\gamma}}]$, we may assume that for small $\tilde{\gamma} > 0$, $R_{\tilde{\gamma}} > 1$ and $w_{\tilde{\gamma}} \geqslant \frac{1}{2}\bar{u}_\beta(1)$ in $B_1$. Hence, for small $\bar{\gamma} > 0$ and thus, for small

$0<\tilde{\gamma}\leqslant\bar{\gamma}$, we have

$$\frac{1}{2}p\bar{u}_\beta(1)\int_{B(1)}[\bar{u}_\alpha^{p-1}-\bar{u}_\beta^{p-1}]\bar{K}F_\alpha\leqslant\int_{B(R_{\tilde{\gamma}})}(\bar{K}-K)u_{\tilde{\gamma}}^pF_\alpha \tag{2.29}$$

$$\leqslant\int_{B(R_{\tilde{\gamma}})}(K-\bar{K})\_\bar{u}_\beta^pF_\alpha,$$

where $k_\pm=\max(\pm k,0)$. However, this is impossible because from (2.26), (K3), (2.27) with $d>a-\frac{p}{p-1}$, and the Dominated Convergence Theorem, the right-hand side of (2.29) goes to 0 as $\tilde{\gamma}\to 0$ while the left-hand side is a fixed positive constant, which verifies the claim. Therefore, there exists $0<\bar{\gamma}<\beta$ such that for all $0<\gamma\leqslant\bar{\gamma}$, $0<u_\gamma<\bar{u}_\beta$ in $B(R_\gamma)$.

Regarding $R_\gamma$ as the supremum of the set $\{R>0\,|\,u_\gamma>0 \text{ in } B_R\}$, we observe that $R_\gamma\to\infty$ as $\gamma\to 0^+$ because for $0\leqslant r<R_\gamma$,

$$\begin{aligned}u_\gamma(r)&=\gamma+\int_0^r u_\gamma'\,ds\\&=\gamma-\int_0^r\int_0^s\left(\frac{t}{s}\right)^{n-1}K(t)u_\gamma^p(t)\,dt\,ds\\&\geqslant\gamma-\gamma^p\int_0^{R_\gamma}t^{n-1}K(t)\left[\int_t^{R_\gamma}s^{1-n}\,ds\right]dt\\&\geqslant\gamma\left[1-\frac{\gamma^{p-1}}{n-2}\int_0^{R_\gamma}tK(t)\,dt\right].\end{aligned} \tag{2.30}$$

If $\liminf_{\gamma\to 0^+}R_\gamma<\infty$, then from (K) and (2.30), $u_\gamma(R_\gamma)>0$ for some $\gamma>0$ small, which contradicts the definition of $R_\gamma$. Moreover, it follows that for given $R>0$, there exists $0<\tilde{\gamma}<\bar{\gamma}$ such that for $0<\gamma<\tilde{\gamma}$, $u_\gamma>0$ in $B_R$.

For $0<\beta<\alpha^*$, let $I_\beta$ be the set of $0<\gamma<\bar{\gamma}(\beta)$ satisfying

$$\frac{p}{2}\int_{B(1)}\frac{[\bar{u}_\beta^{p-1}-u_\gamma^{p-1}]F_\beta}{1+|x|^2}>\int_{B(R_\gamma)}(K-\bar{K})_+u_\gamma^{p-1}F_\beta.$$

Then, $I_\beta\supset(0,\gamma_\beta)$ for some $\gamma_\beta>0$ since from (2.26), (K3) and (2.27) with $d>a-1$, the right-hand side goes to 0 as $\gamma\to 0$ by the Dominated Convergence Theorem while the left-hand side is bounded below a positive constant which is irrelevant to $\gamma$ when $\gamma>0$ is small.

It follows from (2.30) that there exists $0<\hat{\gamma}\leqslant\gamma_\beta$ such that for all $0<\gamma<\hat{\gamma}$, $R_\gamma>1$ and $u_\gamma(r)\geqslant\frac{3}{4}\gamma$ on $[0,1]$.

We now claim that for small $0<\gamma<\hat{\gamma}$ so that $u_\gamma(r)\geqslant\frac{3}{4}\gamma$ for $0\leqslant r\leqslant 1$, there exists $0<\eta<\gamma$ entailing $u_\gamma>\bar{u}_\eta$ in $\mathbf{R}^n$. Suppose that there exists $0<\hat{\gamma}_1<\hat{\gamma}$

such that for each $0<\eta<\hat{\gamma}_1$, there exists $r_\eta>0$ satisfying $\hat{w}_\eta(r)=u_{\hat{\gamma}_1}(r)-\bar{u}_\eta(r)>0$ in $[0,r_\eta)$ and $\hat{w}_\eta(r_\eta)=0$. From Green's identity,

$$\begin{aligned}0 &\leqslant \int_{B(r_\eta)} (\hat{w}_\eta\,\Delta F_\beta - F_\beta\,\Delta\hat{w}_\eta)\\ &\leqslant \int_{B(r_\eta)} [-p\bar{K}\hat{w}_\eta\bar{u}_\beta^{p-1}F_\beta + Ku_{\hat{\gamma}_1}^p F_\beta - \bar{K}\bar{u}_\eta^p F_\beta]\end{aligned}$$

and

$$\begin{aligned}\int_{B(r_\eta)} p\bar{K}\hat{w}_\eta[\bar{u}_\beta^{p-1}-u_{\hat{\gamma}_1}^{p-1}]F_\beta &\leqslant \int_{B(r_\eta)} [p\bar{K}\hat{w}_\eta\bar{u}_\beta^{p-1} - \bar{K}(u_{\hat{\gamma}_1}^p - \bar{u}_\eta^p)]F_\beta\\ &\leqslant \int_{B(r_\eta)} (K-\bar{K})_+ u_{\hat{\gamma}_1}^p F_\beta.\end{aligned}$$

Since $\bar{u}_\eta$ is monotonically decreasing to 0 as $\eta$ decreases to 0 and thus $\bar{u}_\eta\to 0$ uniformly on $[0,R]$ for any fixed $R>0$, we may assume that $r_\eta>1$ and $\hat{w}_\eta(r)\geqslant\frac{3}{4}\hat{\gamma}_1-\bar{u}_\eta(r)\geqslant\frac{1}{2}\hat{\gamma}_1$ in $B_1$ if $\eta>0$ is small enough. Then, we have

$$\frac{p}{2}\int_{B(1)} \bar{K}[\bar{u}_\beta^{p-1}-u_{\hat{\gamma}_1}^{p-1}]F_\beta \leqslant \int_{B(R_\gamma)} (K-\bar{K})_+ u_{\hat{\gamma}_1}^{p-1}F_\beta,$$

which is impossible because $\hat{\gamma}_1\in I_\beta$. Therefore, for each $0<\beta<\alpha^*$, there exist $\beta>\gamma>\eta>0$ satisfying $\bar{u}_\eta<u_\gamma<\bar{u}_\beta$ in $\mathbf{R}^n$.

Repeating the above arguments, we find a decreasing sequence $\{u_{\gamma_i}\}$ of positive solutions of (1.6) such that there exists a positive decreasing sequence $\{\alpha_i\}$ going to 0 as $i\to 0$ with $0<\alpha_i<\alpha^*$ and

$$\bar{u}_{\alpha^*}>u_{\gamma_i}>\bar{u}_{\alpha_i}>u_{\gamma_{i+1}}>0 \qquad \text{in } \mathbf{R}^n$$

for each $i\geqslant 1$. By (2.26), every $u_{\gamma_i}$ has the asymptotic behavior (2.28). On the other hand, it follows from Lemma 2.4 that for every $\gamma_j>\gamma>\gamma_{j+1}, j\geqslant 2,\ u_\gamma$ exists globally and $u_{\gamma_j}>u_\gamma>u_{\gamma_{j+1}}$ in $\mathbf{R}^n$. Therefore, we conclude that there exists $\gamma^*>0$ such that for $0<\gamma\leqslant\gamma^*,\ u_\gamma>0$ in $\mathbf{R}^n$ and $u_\gamma$ is monotonic with respect to $\gamma$, which completes the proof. ∎

Considering the general case, we assume the following condition:

(K4) The infimum $K_1(r)$ and the supremum $K_2(r)$ of $K(x)$ on $\{x=(x_1,x_2)\colon |x_2|=r\}$ are continuous functions on $(0,\infty)$ and $\int_0 rK_2(r)\,dr<\infty$.

A direct application of Proposition 2.5 leads to the following assertion.

THEOREM 2.6. *Let $p > 1$ and $N \geqslant 3$. Assume that $K$ satisfies* (K1), (K2), (K4), *and for some constants $c > 0$ and $a > 0$,*

$$\int_1^\infty |K_i(r) - cr^{-2}| r^{N-1} (\log r)^{-a}\, dr < \infty, \qquad i = 1, 2, \tag{2.31}$$

*where* $K_1(r) := \inf_{|x_2|=r} K(x_1, x_2)$, $K_2(r) := \sup_{|x_2|=r} K(x_1, x_2)$. *Then,* (1.1) *possesses infinitely many positive entire solutions such that*

$$\lim_{|x_2| \to \infty} (\log |x_2|)^{1/(p-1)} u(x_1, x_2) = L(N, p, -2, c) \tag{2.32}$$

*uniformly in $x_1 \in \mathbf{R}^{n-N}$ and any two of them do not intersect.*

*Proof.* Applying Proposition 2.5 to $K_1$ and $K_2$, we have positive radial solution $w_1, w_2$ of $\Delta w + K_1 w^p = 0$ in $\mathbf{R}^N$ and positive radial solution $v_1, v_2$ of $\Delta v + K_2 v^p = 0$ in $\mathbf{R}^N$ satisfying

$$\bar{u}_{\alpha*} > v_1 > \bar{u}_{\alpha_1} > w_1 > \bar{u}_{\eta_1} > v_2 > \bar{u}_{\alpha_2} > w_2 \qquad \text{in } \mathbf{R}^N,$$

where $\bar{u}_{\alpha_1}, \bar{u}_{\eta_1}, \bar{u}_{\alpha_2}$ are solutions of (1.6) with $K = \bar{K}$. Since $\tilde{v}_i(x_1, x_2) := v_i(|x_2|)$ and $\tilde{w}_i(x_1, x_2) := w_i(|x_2|)$ are super-solutions and sub-solutions of (1.1) in $\mathbf{R}^n \setminus \{0\}$, respectively, by the standard super- and sub-solution method there exist solutions $u_i$ of (1.1) in $\mathbf{R}^n \setminus \{0\}$ such that

$$\tilde{v}_i \geqslant u_i \geqslant \tilde{w}_i, \qquad i = 1, 2.$$

Then, each $u_i$ is a weak solution of (1.1) in $\mathbf{R}^n$ and an entire solution in $\mathbf{C}^2(\mathbf{R}^n \setminus \{0\}) \cap \mathbf{C}(\mathbf{R}^n)$ (see [6, 17]). Repeating the above procedure, we construct infinitely many ordered positive entire solutions entailing the asymptotic behavior (2.32). ∎

From Theorem 2.6, Theorem 1.1 follows immediately as a typical case.

We interpret the result of Theorem 2.6 in the context of Riemannian geometry. Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $K$ be a given function. The scalar curvature problem is to find a metric $g_1$ on $M$ conformal to $g$ such that the corresponding scalar curvature to $g_1$ is $K$. The introduction of $u > 0$ by $g_1 = u^{4/(n-2)} g$, $n \geqslant 3$, brings out the equation

$$\frac{4(n-1)}{n-2} \Delta_g u - ku + K u^{\frac{n+2}{n-2}} = 0, \tag{2.33}$$

where $\Delta_g$ denotes the Laplace–Beltrami operator on $M$ in the $g$ metric and $k$ is the scalar curvature of $(M, g)$. If $M = \mathbf{R}^n$ and $g = \sum_{i=1}^n dx_i^2$ is the standard metric, then (2.33) reduces to

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \qquad \text{in } \mathbf{R}^n.$$

When $p = \frac{n+2}{n-2}$, Theorem 2.6 is translated as follows:

THEOREM 2.7. *Let $N \geqslant 3$. Assume that $K$ holds* (K1), (K2), (K4) *and* (2.31) *for some constants $c > 0$ and $a > 0$. Then, there exist infinitely many Riemannian metrics $g_1$ on $\mathbf{R}^n$ with the following properties*:

(i) *$K$ is the scalar curvature of $g_1$*;

(ii) *$g_1$ is conformal to the standard metric $g$ on $\mathbf{R}^n$*;

(iii) *$g_1$ is complete.*

## 3. STRUCTURE OF TYPE SS

In this section, we prove Theorem 1.2. For all $\alpha > 0$, we consider not only the existence of slow decaying solutions but also their separation properties. First, we make an interesting observation.

PROPOSITION 3.1. *Let $l > -2$ and $p > \frac{n+l}{n-2}$. Assume that $K$ satisfies* (K) *and*

$$\lim_{r\to 0} r^2 K(r) = 0. \tag{3.1}$$

*Then, every solution $u_\alpha$ of* (1.6) *with $u_\alpha(0) = \alpha > 0$ remains positive as long as the relation*

$$r^2 K(r) u_\alpha^{p-1}(r) < L^{p-1} \tag{3.2}$$

*holds from $r = 0$, where $L = L(n, p, l, 1)$ is given by* (1.2).

*Proof.* Let $V(t) := r^m u_\alpha(r)$, $t = \log r$. Then, $V$ satisfies

$$V_{tt} + aV_t - L^{p-1}V + k(t)V^p = 0, \tag{3.3}$$

where $a = n - 2 - 2m$ and $k(t) := e^{-lt}K(e^t)$. It follows from (3.1) that

$$\lim_{t\to-\infty} k(t)V(t)^{p-1} = \lim_{r\to 0} r^2 K(r) u_\alpha^{p-1}(r) = 0$$

and thus, $kV^{p-1} < L^{p-1}$ near $-\infty$. Suppose that there exists $T$ such that $V$ is positive and $kV^{p-1} < L^{p-1}$ on $(-\infty, T)$, but $V(T) = 0$. Then, by (3.3), we have

$$V_{tt} + aV_t = (L^{p-1} - k(t)V^{p-1})V > 0 \quad \text{on } (-\infty, T). \tag{3.4}$$

Multiplying (3.4) by $e^{at}$ and integrating from $t$ to $T$, we obtain

$$\begin{aligned} e^{aT}V_t(T) &> e^{at}V_t(t) \\ &= mr^{a+m}u_\alpha(r) + r^{a+m+1}u'_\alpha(r) \end{aligned}$$

which goes to 0 as $r \to 0$ since

$$\begin{aligned} -r^{n-1}u'_\alpha(r) &= \int_0^r s^{n-1}K(s)u_\alpha^p(s)\,ds \\ &\leqslant \alpha^p r^{n-2}\int_0^r sK(s)\,ds \\ &< \infty \end{aligned}$$

and $ru'_\alpha(r) \to 0$ as $r \to 0$. Hence, we have $e^{aT}V_t(T) > 0$, a contradiction. ∎

If (3.2) is true on $[0, \infty)$, then $u_\alpha$ is a positive solution and $r^m u_\alpha(r)$ is strictly increasing as $r$ increases. In fact, the two conditions that $r^{-l}K(r)$ is non-increasing and $p \geqslant p_c(n, l)$, guarantee that this relation is satisfied in the entire space and (1.6) has the structure of Type SS.

THEOREM 3.2. *Let $p \geqslant p_c(n, l)$ with $l > -2$. Suppose that $K(r)$ satisfies* (K) *and $r^{-l}K(r)$ is non-increasing. Then, for each $0 < \alpha < \infty$,* (1.6) *possesses a slowly decaying solution $u_\alpha$ with $u_\alpha(0) = \alpha$ such that $r^m u_\alpha(r)$ is strictly increasing and* (3.2) *holds on* $[0, \infty)$, *where $L = L(n, p, l, 1)$.*

*Proof.* Condition (3.1) follows immediately from (K) and

$$\int_0^r sK(s)\,ds \geqslant \int_{r/2}^r s^{-l}K(s)s^{1+l}\,ds \geqslant \frac{1}{2+l}\left[1 - \frac{1}{2^{2+l}}\right]r^2K(r).$$

Let $\alpha > 0$ and $V(t) := r^m u_\alpha(r)$, $t = \log r$. Then, $V$ satisfies (3.3). Setting

$$T = \sup\{\tau \mid kV^{p-1} < L^{p-1} \text{ on } (-\infty, \tau)\},$$

we see by Proposition 3.1 that $V$ is positive on $(-\infty, T)$. Suppose that $T < +\infty$ and $k(T)V(T)^{p-1} = L^{p-1}$. By the proof of Proposition 3.1, $e^{at}V_t$ is strictly increasing on $(-\infty, T)$ and $V_t(t) > 0$ for $t \leqslant T$. We follow the argument in the proof of Proposition 3.7 in [19] to reach a contradiction. Let

$q(V) = V_t(t)$. Then, $q(V) > 0$ on $(0, [\frac{1}{k(T)}]^{1/(p-1)}L]$, $q(V) \to 0$ as $V \to 0^+$, and

$$\frac{dq}{dV} = -a + \frac{L^{p-1}V - k(t)V^p}{q}.$$

Therefore, for every $\mu > 0$, the line $q = \mu([\frac{1}{k(T)}]^{1/(p-1)}L - V)$ intersects the graph of $q(V)$. Let $(V_\mu, q(V_\mu))$ be the intersection with the smallest $V$-coordinate for each $\mu > 0$. Then, we have $\frac{dq}{dV} \geqslant -\mu$ at $V_\mu$ and

$$\frac{dq}{dV}(V_\mu) = -a + \frac{L^{p-1}V_\mu - k(t)V_\mu^p}{\mu([\frac{1}{k(T)}]^{1/(p-1)}L - V_\mu)}.$$

Since $k(t)$ is non-increasing, e.g., $k(t) \geqslant k(T)$ for $t \leqslant T$, we have

$$\begin{aligned} -\mu &\leqslant -a + \frac{k(T)V_\mu(\frac{1}{k(T)}L^{p-1} - V_\mu^{p-1})}{\mu([\frac{1}{k(T)}]^{1/(p-1)}L - V_\mu)} \\ &= -a + \frac{(p-1)k(T)V_\mu \bar{V}_\mu^{p-2}}{\mu} \qquad \text{for some } \bar{V}_\mu \in \left(V_\mu, \frac{L}{[k(T)]^{1/(p-1)}}\right) \\ &< -a + \frac{(p-1)L^{p-1}}{\mu}, \end{aligned}$$

i.e., for all $\mu > 0$,

$$\mu^2 - a\mu + (p-1)L^{p-1} > 0. \tag{3.5}$$

From (3.5) and $p \geqslant p_c > \frac{n+2+2l}{n-2}$, we observe that $a > 0$ and the determinant of the quadratic form in (3.5) is negative; $a^2 - 4(p-1)L^{p-1} < 0$ which, however, contradicts $p \geqslant p_c$. This shows that $kV^{p-1} < L^{p-1}$ on $(-\infty, +\infty)$ and (3.2) holds for $r > 0$. Consequently, $e^{at}V_t(t) > 0$ for all $t \in \mathbf{R}$. Therefore, $r^m u_\alpha(r)$ is strictly increasing and $u_\alpha$ is a slowly decaying solution. ∎

We are now ready to prove Theorem 1.2 (and the structure of Type SS). To obtain the separation property in Theorem C, Liu *et al.* multiplied two solutions by $r^m$ and took the ratio of them. This approach requires the strict inequality: $p > p_c(n, l)$. Taking the difference of two solutions multiplied by $r^m$ rather than the ratio, we circumvent this difficulty. Here, relation (3.2) is essentially employed.

*Proof of Theorem* 1.2. It follows from Theorem 3.2 that for each $\alpha > 0$, $u_\alpha$ is a slowly decaying solution. For $\alpha > 0$, let $V_\alpha(t) := r^m u_\alpha(r)$, $t = \log r$.

Setting $\Theta(t) := V_\beta(t) - V_\alpha(t)$ for $\beta > \alpha > 0$ given, we see that $\Theta$ is positive near $-\infty$ and satisfies

$$\Theta_{tt} + a\Theta_t + (p-1)L^{p-1}\Theta + G(t) = 0, \tag{3.6}$$

where

$$G(t) := -pL^{p-1}\Theta(t) + e^{-lt}K(e^t)(V_\beta^p - V_\alpha^p).$$

Suppose that there exists $T \in \mathbf{R}$ such that $\Theta(t) > 0$ on $(-\infty, T)$ and $\Theta(T) = 0$. It follows from (3.2) that for $t < T$,

$$\begin{aligned} G(t) &< -pL^{p-1}\Theta(t) + e^{-lt}K(e^t)\Theta(t)pV_\beta^{p-1} \\ &= -p\Theta(t)(L^{p-1} - e^{-lt}K(e^t)[r^m u_\beta(r)]^{p-1}) \\ &\leqslant 0. \end{aligned}$$

Let $q$ be a positive solution of the equation

$$q_{tt} + aq_t + (p-1)L^{p-1}q = 0 \tag{3.7}$$

such that $e^{at}(|q| + |q_t|) \to 0$ as $t \to -\infty$. Multiplying (3.6) by $q$, (3.7) by $\Theta$, and taking the difference, we have

$$(\Theta_t q - \Theta q_t)_t + a(\Theta_t q - \Theta q_t) + qG(t) = 0. \tag{3.8}$$

Multiplying (3.8) by $e^{at}$ and integrating over $(-\infty, T)$, we obtain

$$e^{aT}\Theta_t(T)q(T) = -\int_{-\infty}^{T} e^{as}q(s)G(s)\,ds > 0.$$

Thus, $\Theta_t(T) > 0$, which is impossible. Therefore, $V_\beta > V_\alpha$. Then, we conclude that $u_\beta > u_\alpha > 0$ in $\mathbf{R}^n$ for $\beta > \alpha > 0$, and (1.6) has the structure of Type SS.

Since any solution $u_\alpha$ has uniform bounds by (3.2) on any compact set in $(0, \infty)$, the existence of a singular solution of (1.6) follows in a standard method. We present a simple way to obtain the existence. Combining (3.2) and the fact that $r^{-l}K(r)$ is non-increasing, we have

$$\begin{aligned}
-u_\alpha'(r) &= \frac{1}{r^{n-1}} \int_0^r K(s)u_\alpha^p(s)s^{n-1}\,ds \\
&\leqslant \frac{L^p}{r^{n-1}} \int_0^r s^{n-1-\frac{2p}{p-1}} K(s)^{\frac{-1}{p-1}}\,ds \\
&\leqslant \frac{L^p}{r^{n-1}}\, r^{\frac{l}{p-1}}\, K(r)^{\frac{-1}{p-1}} \int_0^r s^{n-1-\frac{2p}{p-1}-\frac{l}{p-1}}\,ds \\
&= \frac{(p-1)L^p}{[(n-2)p-(n+l)][r^{p+1}K(r)]^{\frac{1}{p-1}}}.
\end{aligned}$$

Hence, $u_\alpha'$ is uniformly bounded on any compact subset of $(0,\infty)$ in $\alpha$ and consequently, $\{u_\alpha\}$ is equicontinuous on any compact subset. Since $u_\alpha$ is monotonically increasing, it follows from the Arzelà–Ascoli Theorem that $U(r) := \lim_{\alpha\to\infty} u_\alpha(r)$ is well-defined and continuous on $(0,\infty)$ and for each $\alpha > 0$,

$$u_\alpha(r) < U(r) \leqslant \frac{L(n,p,l,1)}{[r^2K(r)]^{\frac{1}{p-1}}}.$$

Let $B_{R,\rho} = \{\rho < r = |x| < R\}$. Consider the following boundary problem:

$$\Delta u + K(r)U^p = 0, \qquad u|_{\partial B_{R,\rho}} = U.$$

For each $\alpha > 0$, by the maximum principle, $u - u_\alpha > 0$ and thus, $u - U \geqslant 0$ in $B_{R,\rho}$. Letting $\phi_\varepsilon = \varepsilon e^r$, we have $\Delta(u - u_\alpha + \phi_\varepsilon) > 0$ in $B_{R,\rho}$ for any fixed $R, \rho$ and $\varepsilon$ if $\alpha$ is large enough. Letting $\alpha \to \infty$ and then $\varepsilon \to 0$, we have $u - U \leqslant 0$. Hence, $u = U$ in $B_{R,\rho}$ and $u = U$ on $(0,\infty)$. Therefore, $U$ is a singular solution of (1.6) and the proof of Theorem 1.2 is complete. ∎

*Remarks.* (a) Let $p > \frac{n+2+2\sigma}{n-2}$ with $\sigma > -2$. If $\lim_{r\to 0} r^{-\sigma}K(r) = k_0 > 0$ and

$$\int_0 \left[\frac{d}{dr}\,(r^{-\sigma}K(r))\right]_+ < \infty,$$

then (1.6) has at most one positive radial singular solution $U$ and if it exists,

$$\lim_{r\to 0} r^{\frac{\sigma+2}{p-1}}\, U(r) = L(n,p,\sigma,k_0).$$

(See Corollary 4.3 in [16].)

(b) Let $p > \frac{n+2+2l}{n-2}$ with $l > -2$. If $\lim_{r\to\infty} r^{-l}K(r) = c > 0$ and

$$\int^{\infty} \left[\frac{d}{dr}\left(r^{-l}K(r)\right)\right]_{+} < \infty,$$

then every positive radial solution $u$ of (1.6) near $\infty$ has the asymptotic behavior

$$\lim_{r\to\infty} r^{m}u(r) = L(n, p, l, c) \quad \text{or } 0.$$

(See Theorem 1 in [10].)

(c) Let $p > 1$. If $\lim_{r\to\infty} r^{2}K(r) = c > 0$ and

$$\int^{\infty} \left[\frac{d}{dr}\left(r^{2}K(r)\right)\right]^{+} < \infty,$$

then every positive radial solution $u$ of (1.6) near $\infty$ has the asymptotic behavior

$$\lim_{r\to\infty} (\log r)^{\frac{1}{p-1}} u(r) = L(n, p, -2, c) \quad \text{or } 0.$$

(See Theorem 2 in [10].)

(d) In Corollaries 1.3 and 1.4, the asymptotic behaviors when $K(r) \sim r^{d}$ at $\infty$ for some $d \geqslant -2$, follow easily from (a)–(c). For the case of $l - \tau < -2$ in Corollary 1.3, we have

$$u_{\alpha}(|x|) = c_{\alpha} + \frac{1}{(n-2)\omega_n} \int_{\mathbf{R}^n} \frac{|y|^{l}}{|x-y|^{n-2}(1+|y|^{\tau})} u_{\alpha}^{p}(y)\, dy,$$

where $\omega_n$ denotes the surface area of the unit sphere in $\mathbf{R}^n$. (See Lemmas 2.3, 2.6 and 2.8 in [12] with minor modifications.) Theorem 2.9 in [12] implies $c_{\alpha} > 0$. Indeed, if $c_{\alpha} = 0$, then $u(r) = O(r^{n-2})$ at $\infty$, a contradiction. Then, by Theorem 2.13 in [12], the desired asymptotic behavior in Corollary 1.3 is obtained. Since $u_{\alpha}$ is monotonically increasing to $U$, we see that by the Monotone Convergence Theorem,

$$U(|x|) = c_{\infty} + \frac{1}{(n-2)\omega_n} \int_{\mathbf{R}^n} \frac{|y|^{l}}{|x-y|^{n-2}(1+|y|^{\tau})} U^{p}(y)\, dy.$$

Since $p \geqslant p_c(n, l) > \frac{n+l}{n-2}$ and $r^{m}U(r)$ converges to $L(n, p, l, 1)$ at 0, $U$ also has the corresponding asymptotic behavior.

(e) Theorem B implies that (1.10) and (1.12) have the structure of Type S if $p \geqslant \frac{n+2+2l}{n-2}$ and $p \geqslant 1 + \frac{2\lambda}{n-2}$, respectively.

(f) For Eq. (1.11) and related topics, e.g., asymptotic behavior, radial symmetry, existence of a positive solution carrying a finite total mass, we refer the readers to [9, 11, 13–15, 18, 21].

## REFERENCES


1. S. Bae, T. K. Chang, and D. H. Pahk, Infinite multiplicity of positive entire solutions for a semilinear elliptic equation, *J. Differential Equations* **181** (2002), 367–387.
2. S. Bae and W.-M. Ni, Existence and infinite multiplicity for an inhomogeneous semilinear elliptic equation on $\mathbf{R}^n$, *Math. Ann.* **320** (2001), 191–210.
3. J. Batt, W. Faltenbacher, and E. Horst, Stationary spherically symmetric models in stellar dynamics, *Arch. Rational Mech. Anal.* **93** (1986), 159–183.
4. W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics, *Duke Math. J.* **52** (1985), 485–506.
5. C. Gui, Positive entire solutions of the equation $\Delta u + f(x, u) = 0$, *J. Differential Equations* **99** (1992), 245–280.
6. C. Gui, On positive entire solutions of the elliptic equation $\Delta u + K(x)u^p = 0$ and its applications to Riemannian geometry, *Proc. Roy. Soc. Edinburgh* **126A** (1996), 225–237.
7. C. Gui, W.-M. Ni, and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbf{R}^n$, *Comm. Pure Appl. Math.* **45** (1992), 1153–1181.
8. C. Gui, W.-M. Ni, and X. Wang, Further study on a nonlinear heat equation, *J. Differential Equations* **169** (2001), 588–613.
9. N. Kawano, E. Yanagida, and S. Yotsutani, Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, *Funkcial. Ekvac.* **36** (1993), 557–579.
10. Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbf{R}^n$, *J. Differential Equations* **95** (1992), 304–330.
11. Y. Li, On the positive solutions of the Matukuma equation, *Duke Math. J.* **70** (1993), 575–589.
12. Y. Li and W.-M. Ni, On conformal scalar curvature equation in $\mathbf{R}^n$, *Duke Math. J.* **57** (1988), 895–924.
13. Y. Li and W.-M. Ni, On the existence and symmetry properties of finite total mass solutions of the Matukuma equation, the Eddington equation and their generalizations, *Arch. Rational Mech. Anal.* **108** (1989), 175–194.
14. Y. Li and W.-M. Ni, On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in $\mathbf{R}^n$ I. Asymptotic behavior, *Arch. Rational Mech. Anal.* **118** (1992), 195–222.
15. Y. Li and W.-M. Ni, On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in $\mathbf{R}^n$. II: Radial symmetry, *Arch. Rational Mech. Anal.* **118** (1992), 223–243.
16. Y. Liu, Y. Li and Y. Deng, Separation property of solutions for a semilinear elliptic equation, *J. Differential Equations* **163** (2000), 381–406.
17. W.-M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry, *Indiana Univ. Math. J.* **31** (1982), 493–529.
18. W.-M. Ni and S. Yotsutani, Semilinear elliptic equations of Matukuma-type and related topics, *Japan J. Appl. Math.* **5** (1988), 1–32.

19. X. Wang, On Cauchy problems for reaction–diffusion equations, *Trans. Amer. Math. Soc.* **337** (1993), 549–590.
20. E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, *Arch. Rational Mech. Anal.* **124** (1993), 239–259.
21. E. Yanagida and S. Yotsutani, Global structure of positive solutions to equations of Matukuma type, *Arch. Rational Mech. Anal.* **134** (1996), 199–226.