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Hoffman polynomials of nonnegative irreducible matrices and strongly connected digraphs ☆

Yaokun Wu [a,b,*], Aiping Deng [c]

[a] *Department of Mathematics, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China*

[b] *College of Advanced Science and Technology, Dalian University of Technology, Dalian 116024, China*

[c] *Department of Mathematics, Dong Hua University, Shanghai 200051, China*

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Abstract

For a nonnegative $n \times n$ matrix A, we find that there is a polynomial $f(x) \in \mathbb{R}[x]$ such that $f(A)$ is a positive matrix of rank one if and only if A is irreducible. Furthermore, we show that the lowest degree such polynomial $f(x)$ with $\operatorname{tr} f(A) = n$ is unique. Thus, generalizing the well-known definition of the Hoffman polynomial of a strongly connected regular digraph, for any irreducible nonnegative $n \times n$ matrix A, we are led to define its Hoffman polynomial to be the polynomial $f(x)$ of minimum degree satisfying that $f(A)$ is positive and has rank 1 and trace n. The Hoffman polynomial of a strongly connected digraph is defined to be the Hoffman polynomial of its adjacency matrix. We collect in this paper some basic results and open problems related to the concept of Hoffman polynomials.

AMS classification: 05C50; 15A24

Keywords: Perron eigenvalue; Perron eigenvector; Perron pair; Matrix equation; Tensor product; Elementary equivalence; Split; Amalgamation; Harmonic digraph

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* Corresponding author. Address: Department of Mathematics, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China.

E-mail address: ykwu@sjtu.edu.cn (Y. Wu).

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1. Introduction

We consider finite digraphs admitting loops and multiple arcs and we view graphs as symmetric digraphs. For a digraph Γ, let $A(\Gamma)$ be its adjacency matrix. For any integer $n > 0$ we write J_n for the $n \times n$ matrix of all ones and $\mathbf{j}_n$ the $n \times 1$ vector of all ones, respectively, and we will omit the subscript n where unambiguous. Hoffman's Theorem is a well-known result in algebraic graph theory and has been collected into many textbooks; see [3, Proposition 3.2], [4, p. 271], [9, Theorem 5.1.3, 5.3.1], [13, Theorem 3.7], [36, Theorem 31.13], and [42, Theorem 8.6.23]. It goes as follows:

Theorem 1.1 [27]

(i) *There exists a polynomial $f(x)$ such that*

$$f(A(\Gamma)) = J \tag{1}$$

if and only if the digraph Γ is strongly connected and regular.

(ii) *For a strongly connected r-regular digraph Γ, the unique polynomial of least degree satisfying Eq. (1) is $H_\Gamma(x) = \frac{|V(\Gamma)|q(x)}{q(r)}$ where $(x-r)q(x)$ is the minimal polynomial of $A(\Gamma)$.*

(iii) *The valence r of the strongly connected regular digraph Γ is the greatest real root of $H_\Gamma(x) = |V(\Gamma)|$.*

In light of Theorem 1.1, for any strongly connected regular digraph Γ the unique polynomial $f(x)$ of lowest degree satisfying $J = f(A(\Gamma))$ is called its Hoffman polynomial [9]. There has been a great deal of work concerning this interesting concept.

For instance, Dress and Stevanović [15] establish some Hoffman-type identities for the class of harmonic and semiharmonic graphs. Teranishi generalizes Hoffman identities to non-regular graphs through the use of the Laplacian [40]. Hou and Tian present some generalizations of Hoffman identities by means of main eigenvalues [28].

Another direction is the computation of the Hoffman polynomial of the tensor product of a cycle and a De Bruijn digraph, which is done by Comellas et al. [10] in their course of calculating the spectra of wrapped butterfly digraphs. Using the same technique as [10], Comellas and Mitjana [11] obtain the Hoffman polynomial and then the spectrum of any cycle prefix digraph. Wang et al. determine the Hoffman polynomials and spectra of some more general regular strongly connected digraphs using a much more direct approach [41].

Much work about Hoffman polynomials is carried out in the guise of solving matrix equations (see Sections 3, 7 and 8). This includes the work of enumeration, representation and classification of strongly regular graphs and strongly regular digraphs, which corresponds to degree two Hoffman polynomials; see our Example 8.2 and [24,30]; [36, Chapter 21] and references therein. For more on the study of digraphs with Hoffman polynomials x^k or $x^k + x^{k+\ell}$, we refer to [33,43–45,47,48].

Refs. [20,43] discuss some type of Hoffman polynomials whose corresponding digraphs will always have a line digraph structure. We also mention that it is reported in [46,49] that both the wrapped butterfly digraphs and the De Bruijn digraphs are characterized by their Hoffman polynomials and some simple rank condition.

Along the lines of research mentioned above, the present authors are concerned with those polynomials that send a nonnegative irreducible matrix to a positive rank one matrix. Note that each positive rank one matrix can be written as the product of a positive column vector and a positive row vector. Indeed, we shall show that for any digraph Γ with adjacency matrix A, there is a polynomial $f(x) \in \mathbb{R}[x]$ such that $f(A) = \xi\zeta^\top$ for two positive column vectors ζ and ξ if and only if Γ is strongly connected and ζ and ξ are respectively left and right Perron eigenvectors of A. Parallel to the definition of Hoffman polynomials of strongly connected regular digraphs [27] and the construction of Hoffman-type identities by Dress and Stevanović [15], the previously asserted fact motivates us to introduce the Hoffman polynomial for any irreducible nonnegative matrix, including the adjacency matrix of a strongly connected digraph. This paper is to address some simple results around this extended definition of Hoffman polynomial. We will generalize some corresponding results on Hoffman polynomials of strongly connected regular digraphs to not necessarily regular ones and we will also present some open problems.

This paper is organized as follows. The definition and some basic properties of Hoffman polynomials are established in Section 2. In Section 3 we point out the connection between some types of matrix equations and the Hoffman polynomials. Section 4 is devoted to the Hoffman polynomials of the tensor products of two digraphs. In Section 5 we take up the relationship between the Hoffman polynomials of two matrices which are elementarily equivalent. Then in Section 6 we deal with a class of special digraphs which are specified by some Hoffman-type identities. We continue to consider some questions on Hoffman polynomials with at most two terms in Section 7. Finally, we close this paper in Section 8 by collecting miscellaneous results which illustrate how the concept of Hoffman polynomial can be recognized in the literature.

2. Hoffman polynomial

First we introduce some elementary notation. As usual, let $\mathbb{R}^n$ denote the set of real column vectors of dimension n and let $\mathrm{Mat}_n(\mathbb{R})$ ($\mathrm{Mat}_n(\mathbb{R}^+)$) denote the set of (nonnegative) real $n \times n$ matrices. We use the notation I for the identity matrix. For a matrix A, let $\mathrm{Sp}(A)$ represent the set of eigenvalues of A and $m_A(x)$ represent the minimal polynomial of A. Sometimes we write $A > 0$ ($A \geqslant 0$) if A is a positive (nonnegative) matrix. For a square nonnegative integer matrix A, $\Gamma(A)$ stands for the digraph which has A as its adjacency matrix. For an irreducible nonnegative matrix A, the Perron–Frobenius theory tells us that A has a unique positive eigenvalue λ_A, which has algebraic multiplicity one and will be called the *Perron eigenvalue* of A. Accordingly, we define

$$q_A(x) = \frac{m_A(x)}{x - \lambda_A} \tag{2}$$

and

$$H_A(x) = \frac{nq_A(x)}{q_A(\lambda_A)}, \tag{3}$$

where n is the size of A. We also recall the fact that any irreducible nonnegative matrix A possesses a positive right eigenvector and a positive left eigenvector, which are both unique up to multiplication by a positive scalar and will be referred to as a *right Perron eigenvector* and a *left Perron eigenvector* of A, respectively, and the associated eigenvalue of a Perron eigenvector is just the Perron eigenvalue λ_A. If $\zeta^\top$ and ξ are respectively left and right Perron eigenvectors of A, then we call (ζ, ξ) a *Perron pair* of A. For a given digraph Γ, let $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$ denote its vertex set, arc set and adjacency matrix, respectively. If Γ is a strongly connected digraph, we know that $A(\Gamma)$ is a nonnegative irreducible matrix and thus we often do not distinguish strongly connected digraphs from nonnegative irreducible integer matrices. For instance, this allows us to refer to the Perron eigenvalue and the minimal polynomial of Γ and use the notation λ_Γ and $m_\Gamma(x)$, respectively.

We now come to some facts on irreducible matrices.

Theorem 2.1. *Let $\zeta, \xi \in \mathbb{R}^n$ and $A \in \mathrm{Mat}_n(\mathbb{R}^+)$. If $\zeta, \xi > 0$ and there is a polynomial $f(x) \in \mathbb{R}[x]$ such that $f(A) = \xi\zeta^\top$, then A is irreducible and (ζ, ξ) is a Perron pair of A.*

Proof. If A is reducible, then all of the powers of A will have certain fixed positions occupied by zeros and thus $f(A)$ cannot be a positive matrix. Therefore, the first claim comes from the fact that $\xi\zeta^\top$ is a positive matrix. Clearly, any Perron eigenvector of A is still a positive eigenvector of $f(A)$ and hence a Perron eigenvector of $f(A)$. Observe that $\zeta^\top$ is a left Perron eigenvector of $\xi\zeta^\top$ and ξ a right Perron eigenvector of $\xi\zeta^\top$. The remaining claims now follow from the uniqueness of Perron eigenvectors of a nonnegative irreducible matrix. □

Theorem 2.2. *Assume that A is a nonnegative irreducible matrix with a Perron pair (ζ, ξ). Then there is a polynomial $f(x) \in \mathbb{R}[x]$ such that $f(A) = \xi\zeta^\top$.*

Proof. Recall that $m_A(x)$ is the monic polynomial of the lowest degree that annihilates A. On the one hand, this gives

$$q_A(A) \neq 0. \tag{4}$$

On the other hand, it implies

$$(A - \lambda_A I)q_A(A) = 0 \tag{5}$$

and

$$q_A(A)(A-\lambda_A I)=0. \tag{6}$$

The Perron–Frobenius theory says that λ_A is a simple root of the characteristic polynomial of A and it has $\zeta^\top$ and ξ as its corresponding left eigenvector and right eigenvector, respectively. Thus, we deduce from Eq. (5) that all columns of $q_A(A)$ are multiples of ξ and derive from Eq. (6) that all rows of $q_A(A)$ are multiples of $\zeta^\top$. This shows that $q_A(A)$ is a multiple of $\xi\zeta^\top$. Considering Eq. (4), we can further assert that $q_A(A)=\alpha\xi\zeta^\top$ for some $\alpha\in\mathbb{R}\backslash\{0\}$. The proof is now complete by putting $f(x)=\alpha^{-1}q_A(x)$. $\square$

We pause here to introduce three sets of polynomials for any nonnegative irreducible $n\times n$ matrix A:

$$\mathscr{F}_1(A)=\{f(x): f(A)>0,\ \operatorname{rank} f(A)=1,\ f(\lambda_A)=n\}, \tag{7}$$

$$\mathscr{F}_2(A)=\{f(x): f(A)=\xi\zeta^\top,\ \zeta,\xi>0,\ \zeta^\top\xi=n\}, \tag{8}$$

$$\mathscr{F}_3(A)=\{f(x): f(A)>0,\ \operatorname{rank} f(A)=1,\ \operatorname{tr} f(A)=n\}. \tag{9}$$

Lemma 2.3. $\mathscr{F}_1(A)=\mathscr{F}_2(A)=\mathscr{F}_3(A)$.

Proof. $f(A)>0$ together with $\operatorname{rank} f(A)=1$ is equivalent to $f(A)=\xi\zeta^\top$ for some column vectors $\zeta,\xi>0$. By Theorem 2.1, ξ must be a right Perron eigenvector of A. Thus the first equality is a result of $f(\lambda_A)\xi=f(A)\xi$ whereas the second equality follows from $\operatorname{tr}\xi\zeta^\top=\operatorname{tr}\zeta^\top\xi=\zeta^\top\xi$. $\square$

Lemma 2.3 illustrates that $\mathscr{F}_1(A)$, $\mathscr{F}_2(A)$ and $\mathscr{F}_3(A)$ are simply three different representations of the same set, which we will then call $\mathscr{F}(A)$. Note that when referring to $\mathscr{F}(A)$ later, we will freely use any of these three descriptions.

Theorem 2.4. *Let A be a nonnegative irreducible matrix of order n. Then all polynomials in $\mathscr{F}(A)$ are multiples of $H_A(x)$ and $H_A(x)$ is the unique polynomial belonging to $\mathscr{F}(A)$ of lowest degree.*

Proof. First note that $q_A(\lambda_A)\neq 0$ as λ_A is a simple root of $m_A(x)$. According to the proof of Theorem 2.2, $H_A(A)=\xi\zeta^\top$ where (ζ,ξ) is a Perron pair of A. This says that $H_A(A)$ is positive and of rank one. Also, it is trivial to see that $H_A(\lambda_A)=n$, which then proves that $H_A(x)\in\mathscr{F}_1(A)=\mathscr{F}(A)$.

Next, for any $f(x)\in\mathscr{F}(A)=\mathscr{F}_2(A)$ we know that $f(A)$ is a multiple of $H_A(A)$, taking into account Theorem 2.1. This demonstrates that $(x-\lambda_A I)f(x)$ annihilates A and hence $H_A(x)\mid f(x)$. Therefore, $H_A(x)$ is of lowest degree in $\mathscr{F}(A)$. If $f(x)\in\mathscr{F}_1(A)$ has the same degree with $H_A(x)$, then we know $f(x)=cH_A(x)$

for some constant c. Clearly, Eq. (7) implies that $c = 1$ and thus the conclusion is proved. $\square$

Remark 2.5. Let A be an $n \times n$ nonnegative irreducible matrix. Then by Theorem 2.4, we know that $\mathscr{F}(A) = \{\frac{g(x)}{g(\lambda_A)} H_A(x) : \ g(x) \in \mathbb{R}(x), g(A)H_A(A) > 0$ or $g(A)H_A(A) < 0\}$. Especially, when A has a constant line sum, $\mathscr{F}(A)$ becomes $\{\frac{g(x)}{g(\lambda_A)} H_A(x) : g(x) \in \mathbb{R}(x), g(\lambda_A) \neq 0\}$ and thus $\mathbb{R}\mathscr{F}(A)$ is just the ideal generated by $H_A(x)$ in $\mathbb{R}(x)$. It is interesting to investigate if any similar thing can be said about $\mathscr{F}(A)$ when A does not have a constant line sum.

Theorem 2.6. *For any $n \times n$ nonnegative irreducible matrix A, its Perron eigenvalue λ_A is the greatest real root of $H_A(x) = n$.*

Proof. Theorem 2.4 says that $H_A(x) \in \mathscr{F}_1(A)$, establishing the fact $H_A(\lambda_A) = n$. Now suppose $\mu > \lambda_A$ and we turn to prove $|H_A(\mu)| > |H_A(\lambda_A)| = n$, which will surely end the proof. Our task further reduces to deducing $|q_A(\mu)| > |q_A(\lambda_A)|$. Let $q_A(x) = \prod_{\lambda \in \mathrm{Sp}(A)\setminus\{\lambda_A\}} (x - \lambda)^{\ell_\lambda(A)}$. Because λ_A is the Perron eigenvalue of A, for each $\lambda \in \mathrm{Sp}(A)\setminus\{\lambda_A\}$ it follows that $|\lambda| < \lambda_A$ and hence $|\mu - \lambda| > |\lambda_A - \lambda|$. The proof is then finished by noting that $|q_A(\mu)| = \prod_{\lambda \in \mathrm{Sp}(A)\setminus\{\lambda_A\}} |\mu - \lambda|^{\ell_\lambda(A)} > \prod_{\lambda \in \mathrm{Sp}(A)\setminus\{\lambda_A\}} |\lambda_A - \lambda|^{\ell_\lambda(A)} = |q_A(\lambda_A)|$. $\square$

The *period* of a nonnegative square matrix A is the greatest common divisor of those integers $k \geqslant 1$ for which $\mathrm{tr} A^k > 0$. The matrix $\xi\zeta^\top$ appearing in the definition of $\mathscr{F}_2(A)$ is related to the matrix A as follows.

Theorem 2.7 [35, Exercise 4.5.14]. *Let A be an $n \times n$ nonnegative irreducible matrix with Perron eigenvalue λ and period p. Let (ζ, ξ) be a Perron pair of A which is normalized so that $\zeta^\top \xi = n$. Then we have*

$$\lim_{k \to +\infty} \left(I + \frac{A}{\lambda} + \cdots + \frac{A^{p-1}}{\lambda^{p-1}} \right) \frac{A^k}{\lambda^k} = \frac{\xi\zeta^\top}{n}.$$

We also remark that when A is a (0, 1) irreducible matrix the matrix $\xi\zeta^\top$ is involved with the measure of maximal entropy for a subshift of finite type associated with A [31, p. 166].

So far, for a nonnegative irreducible matrix A we have found that the polynomial $H_A(x)$ defined by Eq. (3) does play some interesting role. We will call it the *Hoffman polynomial* of A. Correspondingly, the *Hoffman polynomial* of a strongly connected digraph Γ is defined to be the Hoffman polynomial of its adjacency matrix and is denoted by $H_\Gamma(x)$. This extends the definition of Hoffman polynomials for strongly connected regular digraphs, as can be seen from the following generalization of Theorem 1.1.

Theorem 2.8. *Let Γ be a finite digraph on n vertices and $A = A(\Gamma)$. Then the following hold.*

(i) *Γ is strongly connected if and only if there is a polynomial $f(x) \in \mathbb{R}[x]$ such that $f(A)$ has rank one and is positive, namely if and only if $\mathscr{F}(A) \neq \emptyset$.*
(ii) *When Γ is strongly connected, $H_\Gamma(x)$ is the unique polynomial of lowest degree in $\mathscr{F}(A)$.*
(iii) *When Γ is strongly connected, $H_\Gamma(x) = n$ has λ_A as its greatest real root.*

Proof. Follows directly from Theorems 2.1, 2.2, 2.4 and 2.6. □

Example 2.9. Although the minimal polynomial and the Hoffman polynomial are closely related concepts, they are not uniquely determined by each other. By way of example, consider the adjacency matrix B of the binary De Bruijn digraph $B(2, 3)$ [47]. Since $B^3 = J$, both B and $B \bigotimes B$ have x^3 as their Hoffman polynomial (Example 4.4). But the minimal polynomials of B and $B \bigotimes B$ are different. Indeed, they even have different Perron eigenvalues. In the other direction, we look at the matrix A described in Example 5.16 below. We see that x^3 is the common minimal polynomial of A and B while $H_A(x) = \frac{5}{8}x^3 \neq x^3 = H_B(x)$.

The next result summarizes the relationship among $H_A(x)$, $m_A(x)$ and the size n of a nonnegative irreducible matrix A.

Proposition 2.10. *Let A be an $n \times n$ nonnegative irreducible matrix. Then the following hold*:

(i) *$H_A(x)$ is determined by $m_A(x)$ and n;*
(ii) *n is determined by $H_A(x)$ and $m_A(x)$;*
(iii) *$m_A(x)$ is determined by n and $H_A(x)$.*

Proof. (i) $H_A(x)$ is defined by $q_A(x)$, λ_A and n. But $q_A(x)$ and λ_A are uniquely determined by $m_A(x)$.

(ii) Note that $n = H_A(\lambda_A)$ and λ_A is the largest real root of $m_A(x)$.

(iii) By Theorem 2.6, we know that λ_A is determined by $H_A(x)$ and n. We also note that $q_A(x)$ is the monic polynomial obtained by dividing $H_A(x)$ by its leading coefficient. Since $m_A(x) = (x - \lambda_A)q_A(x)$, we conclude that $H_A(x)$ together with n uniquely determines $m_A(x)$. □

There are two types of very natural questions concerning the concept of Hoffman polynomials. The first is how to determine the Hoffman polynomials of given irreducible matrices or strongly connected digraphs. This also includes the question of determining the relationship between Hoffman polynomials of different matrices (digraphs) related in various ways. Our work in Sections 3, 4 and 5 is along this direction. The dual question is to decide for a given polynomial all those matrices (digraphs) which have it as their Hoffman polynomials. This includes the existence question, construction question, classification question and enumeration question. Clearly, this research has much to do with solving matrix equations. We will carry some elementary discussion on this question in Sections 6, 7 and 8.

3. Matrix equation

Suppose we are given two positive vectors ζ and ξ of length n such that $\zeta^\top \xi = n$. Then for a given polynomial f, consider the matrix equation

$$f(A) = T \tag{10}$$

for an unknown $n \times n$ nonnegative integer matrix A, where $T = \xi\zeta^\top$. In some situations, the task of solving Eq. (10) turns out to be the same as determining all digraphs which have $f(x)$ as their Hoffman polynomials. The main theme of this section is to investigate the form of $f(x)$ which guarantees that $\deg f(x) = \min_{g(x)\in\mathscr{F}(A)} \deg g(x)$ holds for each solution A to Eq. (10). This kind of result has appeared in characterizing Kautz digraphs [43].

Theorem 3.1. *Let Γ be a digraph on n vertices with adjacency matrix A and $\zeta \in \mathbb{R}^n$ a positive vector satisfying $\mathbf{j}^\top\zeta = n$. If A is a solution to Eq.* (10) *where $T = \mathbf{j}\zeta^\top$ and $n > 1 + \lambda_A + \cdots + \lambda_A^{\deg f - 1}$, then $H_\Gamma(x) = f(x)$. The same result holds when $T = \mathbf{j}\zeta^\top$ is replaced by $T = \zeta\mathbf{j}^\top$.*

Proof. We only prove the first reading. By Theorem 2.1, we know that A has $\mathbf{j}$ as a right Perron eigenvector and thus A^i has constant row sum λ_A^i for each nonnegative integer i. This asserts that each row of A^i has at most λ_A^i nonzero entries. But each row of a positive $n \times n$ matrix has exactly n positive entries. Consequently, due to the assumption that $n > 1 + \lambda_A + \cdots + \lambda_A^{\deg f - 1}$, we have $\deg g \geqslant \deg f$ for any polynomial $g(x)$ with $g(A) > 0$. Making use of Theorem 2.4 then completes the proof. □

Corollary 3.2. *Let $f(x)$ be a polynomial with nonnegative coefficients and whose leading coefficient is not less than* 1. *Suppose Γ is a digraph whose adjacency matrix A satisfies $f(A) = J$. If Γ is not a cycle, then $H_\Gamma(x) = f(x)$.*

Proof. By Theorem 1.1, Γ is regular. As it is not a cycle, we can assume that it has constant degree $\lambda_A = \lambda > 1$. It is easy to see that Γ has a total of $n = f(\lambda)$ vertices. Say the degree of $f(x)$ is k. Then we have $n = f(\lambda) \geqslant \lambda^k > 1 + \lambda_A + \cdots + \lambda_A^{k-1}$ and so Theorem 3.1 gives the assertion. □

4. Tensor product

In this section we discuss the computation of the Hoffman polynomial of the tensor product of two nonnegative irreducible matrices. Given two digraphs Γ_1 and Γ_2, their *tensor product* $\Gamma_1 \otimes \Gamma_2$ is defined to be the digraph with $V(\Gamma_1 \otimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$ and $E(\Gamma_1 \otimes \Gamma_2)$ has $(x_1, x_2)(y_1, y_2)$ as an element of multiplicity $m_1 m_2$, where m_i is the multiplicity of (x_i, y_i) in $E(\Gamma_i)$, $i = 1, 2$. That is, $A(\Gamma_1 \otimes \Gamma_2) = A(\Gamma_1) \otimes A(\Gamma_2)$.

For any positive integer m, we use the shorthand $[m]$ for the set $\{1, \ldots, m\}$ throughout the paper. We use C_p to denote the directed cycle of length p. The least common multiple of two positive integers p and q is written as $[p, q]$. Recall that the *elementary divisors* of $A \in \mathrm{Mat}_n(\mathbb{C})$ are in one-to-one correspondence with the Jordan blocks of A. Here is a classical result on elementary divisors.

Theorem 4.1 [8, Theorem 4.6; 37, Theorem 1]. *Let $A,\ B \in \mathrm{Mat}_n(\mathbb{R})$. The complete list of elementary divisors of $A \otimes B$ in $\mathbb{C}[x]$ is as follows. To each pair consisting of an elementary divisor $(x-a)^p$ of A and an elementary divisor $(x-b)^q$ of B, there correspond the following elementary divisors of $A \otimes B$:*

(i) *When $a \neq 0$ and $b \neq 0$,*
$(x-ab)^{p+q-(2k-1)}$ *for $k \in [\min\{p, q\}]$;*

(ii) *When $a \neq 0$ and $b = 0$,*
x^q *(p times);*

(iii) *When $a = 0$ and $b \neq 0$,*
x^p *(q times);*

(iv) *When $a = 0$ and $b = 0$,*
x^k *(twice) for $k \in [\min\{p, q\} - 1]$;*
$x^{\min\{p,q\}}$ *($|p-q| + 1$ times).*

Theorem 4.1 translates immediately to corresponding properties of minimal polynomials, yielding the following.

Theorem 4.2. *Let $A, B \in \mathrm{Mat}_n(\mathbb{R})$. If*

$$m_A(x) = \prod_{i=0}^{s} (x - a_i)^{p_i} \quad \textit{and} \quad m_B(x) = \prod_{j=0}^{t} (x - b_j)^{q_j},$$

where $a_0 = 0,\ a_i \neq a_j$ for $0 \leqslant i \neq j \leqslant s$, and $b_0 = 0,\ b_i \neq b_j$ for $0 \leqslant i \neq j \leqslant t$, then

$$m_{A \otimes B}(x) = \prod_{k=0}^{u} (x - c_k)^{r_k},$$

where $c_0, \ldots, c_u$ *enumerate all different values taken by* $a_i b_j$, $0 \leqslant i \leqslant s, 0 \leqslant j \leqslant t$, $c_0 = 0$, $r_0 = \max\{p_0, q_0\}$, *and* $r_k = \max\{\ell + f - 1 : a_\ell b_f = c_k\}$ *for* $k \in [u]$.

Armed with Proposition 2.10 and Theorem 4.2, we arrive at the following.

Theorem 4.3. *Given two nonnegative irreducible matrices* A *and* B, *their Hoffman polynomials together with their sizes determine the Hoffman polynomial of* $A \otimes B$.

In the remaining part of this section, we apply Theorem 4.3 to the adjacency matrices of some strongly connected digraphs and list the computation results.

Example 4.4 [10,41]. Suppose Γ is a digraph with $A(\Gamma)^k = J_{r^k}$ for some $r > 0$. Then it follows from Corollary 3.2 that $H_\Gamma(x) = x^k$ and so $H_{C_p \otimes \Gamma}(x) = H_{C_p}\left(\frac{x}{r}\right) H_\Gamma(x)$.

Example 4.5 [41]. Let Γ be a digraph with $A(\Gamma)^k = J_{r^k+1} - I_{r^k+1}$ for some $r > 0$. Then Corollary 3.2 implies $H_\Gamma(x) = 1 + x^k$ and thus

$$H_{C_p \otimes \Gamma}(x) = \begin{cases} H_{C_p}\left(\frac{x}{r}\right) H_\Gamma(x) & \text{if } p|k; \\ \frac{r^k+1}{r^{[p,k]} - (-1)^{[p,k]}} (x^{[p,k]} - (-1)^{[p,k]}) H_{C_p}\left(\frac{x}{r}\right) & \text{if } p \nmid k. \end{cases}$$

Example 4.6. Suppose Γ and Σ are digraphs with $H_\Gamma(x) = \frac{n_1}{1 + r_1 + r_1^2}(1 + x + x^2)$ and $H_\Sigma(x) = \frac{n_2}{1 + r_2 + r_2^2}(1 + x + x^2)$, respectively, where n_i and r_i are some positive integers, $i \in [2]$. Then

$$H_{\Gamma \otimes \Sigma}(x) = \begin{cases} \frac{x^3 - 1}{r_1^3 r_2^3 - 1} H_\Gamma\left(\frac{x}{r_2}\right) H_\Sigma\left(\frac{x}{r_1}\right) & \text{if } r_1 \neq r_2; \\ \frac{n_1(x^3-1)}{r^6-1} H_\Sigma\left(\frac{x}{r}\right) = \frac{n_2(x^3-1)}{r^6-1} H_\Gamma\left(\frac{x}{r}\right) & \text{if } r_1 = r_2 = r. \end{cases}$$

Example 4.7. Let $A = A(\Gamma) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $m_A(x) = x(x - \sqrt{2})(x + \sqrt{2})$ and $H_A(x) = \frac{3}{4}x(x + \sqrt{2})$. A simple computation gives that $m_{C_3 \otimes \Gamma}(x) = x(x^3 - 2\sqrt{2})(x^3 + 2\sqrt{2})$ and so $H_{C_3 \otimes \Gamma}(x) = \frac{9}{48}x(x^2 + \sqrt{2}x + 2)(x^3 + 2\sqrt{2}) = \frac{3}{8}x(x^3 + 2\sqrt{2}) H_{C_3}\left(\frac{x}{\sqrt{2}}\right)$.

5. Elementary equivalence

Just as in Section 4, to facilitate the computation of Hoffman polynomials, we continue to investigate relationships between Hoffman polynomials of different irreducible matrices. We discuss in Section 5.1 the Hoffman polynomials of elementarily equivalent matrices. In Section 5.2 we study a special kind of elementary equivalence, namely that between a matrix and a splitting or an amalgamation of it. This provides us with a useful technique of reducing the computation of the Hoffman polynomial of a matrix to the calculation of the Hoffman polynomial of a matrix of smaller size. The digraph version of split and amalgamation operation is introduced in Section 5.3. At last, we present an interesting application of this computation technique in Section 5.4.

5.1. *Elementary equivalence of matrices*

Let A and B be nonnegative matrices. An *elementary equivalence* from A to B over $\mathbb{R}^+$ [35, Definition 7.2.1] is a pair (R, S) of rectangular nonnegative matrices satisfying

$$A = RS \quad \text{and} \quad B = SR.$$

In this case we write $(R, S) : A \approx_{\mathbb{R}^+} B$. Call a matrix *row-nontrivial* if it has no zero row and call a matrix *column-nontrivial* if it has no zero column.

Theorem 5.1. *Let A and B be two matrices such that $(R, S) : A \approx_{\mathbb{R}^+} B$ for a column-nontrivial nonnegative matrix R and a row-nontrivial nonnegative matrix S. If there is a polynomial $f(x) \in \mathbb{R}[x]$ such that*

$$f(A) = \xi\zeta^\top \text{ for some } \zeta, \xi > 0, \tag{11}$$

then

$$Bf(B) = yx^\top \text{ for some } x, y > 0. \tag{12}$$

Conversely, if we further assume that R is of full row rank and S is of full column rank, then Eq. (11) follows from Eq. (12).

Proof. Assume that Eq. (11) holds. Let $x^\top = \zeta^\top R$ and $y = S\xi$. We clearly have $x > 0$ and $y > 0$, as a result of our assumption on ζ, ξ, R and S. Now we can obtain Eq. (12) through the following calculation:

$$\begin{aligned} Bf(B) &= SRf(SR) && \text{by } B = SR \\ &= Sf(RS)R \\ &= Sf(A)R && \text{by } A = RS \\ &= S\xi\zeta^\top R && \text{by Eq. (11)} \\ &= yx^\top. \end{aligned}$$

For the converse direction, first note that applying Theorem 2.1 to Eq. (12) yields $By = \lambda_B y$ and $x^\top B = \lambda_B x^\top$. Consequently,

$$\lambda_B^2 yx^\top = Byx^\top B = SRyx^\top SR. \tag{13}$$

Next observe that a matrix of full row (column) rank must be row-nontrivial (column-nontrivial). This says that R is a row-nontrivial nonnegative matrix and S is a column-nontrivial nonnegative matrix. Thus we are allowed to derive from $x, y > 0$ that

$$\xi = \frac{1}{\lambda_B} Ry \quad \text{and} \quad \zeta^\top = \frac{1}{\lambda_B} x^\top S \tag{14}$$

are both positive vectors. We can now write down

$$\begin{aligned} Sf(A)R &= Sf(RS)R && \text{by } A = RS \\ &= SRf(SR) && \\ &= Bf(B) && \text{by } B = SR \\ &= yx^\top && \text{by Eq. (12)} \\ &= \frac{1}{\lambda_B^2} S(Ryx^\top S)R && \text{by Eq. (13)} \\ &= S\xi\zeta^\top R && \text{by Eq. (14).} \end{aligned} \tag{15}$$

By an appeal to the fact that S is of full column rank and R is of full row rank, we conclude from Eq. (15) that $f(A) = \xi\zeta^\top$, proving the result. □

The previous theorem will play a key role in our work of determining Hoffman polynomials. But to proceed, we had better first look at a simple result.

Lemma 5.2. *For any irreducible nonnegative matrix A, $x|H_A(x)$ if and only if* $\det A = 0$.

Proof. $\det A = 0$ is equivalent to $x|m_A(x)$. Since the Perron eigenvalue must be positive, Eq. (2) together with Eq. (3) shows that $x|m_A(x)$ if and only if $x|H_A(x)$. Combining these, the lemma follows. □

Theorem 5.3. *Let A be an irreducible nonnegative $n \times n$ matrix. If B is an $m \times m$ matrix such that $(R, S) : A \approx_{\mathbb{R}^+} B$ for a column-nontrivial nonnegative matrix R and a row-nontrivial nonnegative matrix S. Then the following hold.*

(i) *B is also an irreducible nonnegative matrix and $H_B(x)$ takes one of the following three values: $\frac{m\lambda_B}{n}\frac{H_A(x)}{x}$, $\frac{m}{n}H_A(x)$, or $\frac{m}{n\lambda_A}xH_A(x)$. (Note that $\lambda_A = \lambda_B$.)*

(ii) *If* $\det A \det B \neq 0$, *then $m = n$ and A and B are similar and hence $H_B(x) = H_A(x)$; while if* $\det A \det B = 0$, *we have*

$$H_B(x) = \begin{cases} \frac{m\lambda_B}{n}\frac{H_A(x)}{x}, & \textit{if } \det B \neq 0 \textit{ and } \det A = 0; \\ \frac{m}{n\lambda_A}xH_A(x), & \textit{if } \det B = 0 \textit{ and } \det A \neq 0. \end{cases}$$

(iii) *If we further assume that R is of full row rank and S is of full column rank, then*

$$H_B(x) = \begin{cases} H_A(x), & \textit{if } \det B \neq 0 \ (\textit{or equivalently, } m = n); \\ \frac{m}{n\lambda_A}xH_A(x), & \textit{if } \det B = 0 \ (\textit{or equivalently, } m > n). \end{cases}$$

Proof. Looking at Eq. (7), the first claim is immediate from Theorems 2.1, 2.4 and 5.1. In view of Lemma 5.2 and the first claim, we come to the second assertion as well. We now prove the last reading. Since R is of full row rank and S is of full column rank, we obtain that A is nonsingular. If $\det B \neq 0$, the assertion follows from claim (ii). If $\det B = 0$, then $f(x) = \frac{H_B(x)}{x}$ is a polynomial on account of Lemma 5.2. It then follows from the last assertion of Theorem 5.1 that $f(A)$ is a positive matrix of rank 1. Therefore, we can use Theorem 2.8 to deduce that

$$\deg H_A(x) \leqslant \deg f(x) = \deg H_B(x) - 1.$$

Finally, observe that the desired result follows from claim (i), completing the proof. □

Remark 5.4. It is known that if matrices A and B satisfy $A = RS$ and $B = SR$ for some matrices R and S, then $m_B(x)$ is one of $m_A(x)$, $xm_A(x)$, or $\frac{m_A(x)}{x}$ [2]. In particular, we have $\lambda_A = \lambda_B$ for such pair of matrices. But can we say more about which possibility will occur, $m_B(x) = m_A(x)$, $m_B(x) = xm_A(x)$, or $m_B(x) = \frac{m_A(x)}{x}$? In terms of Eqs. (2) and (3), Theorem 5.3 can give some answers to this question. Note that we will find several more relations similar to those in Theorem 5.3 later and this observation applies as well.

5.2. *Splitting and amalgamation of matrices*

To make the results obtained in last subsection more useful for practical calculation, we will introduce an important kind of elementary equivalence here.

A *row* (*column*) *amalgamation matrix* is a (0, 1) matrix with exactly one 1 in each column (row) and at least one 1 in each row (column). We mention that in symbolic dynamics, a row amalgamation matrix is called a division matrix while a column amalgamation matrix is named simply as an amalgamation matrix [35, Definitions 2.4.13, 8.2.4] and they both play important roles in the course of classifying dynamical systems. We write $\Lambda : [m] = \cup_{i=1}^{n} \Lambda_i$ for a *partition* Λ of $[m]$ into pairwise disjoint nonempty subsets $\Lambda_1, \ldots, \Lambda_n$. We define the *characteristic matrix* of Λ as the $n \times m$ matrix whose ith row is the characteristic vector of Λ_i over $[m]$; we will use the notation $\chi(\Lambda)$ for it. Clearly, a row amalgamation matrix is nothing but the characteristic matrix of a partition and a column amalgamation matrix is just the transpose of a row amalgamation matrix.

We use $B(i, j)$, $B(i, \cdot)$ and $B(\cdot, j)$ to represent the (i, j) entry, the ith row and the jth column of a matrix B, respectively. Suppose B is a matrix of m columns. A *good column partition* of B is a partition Λ of $[m]$ into pairwise disjoint nonempty subsets $\Lambda_1, \ldots, \Lambda_n$, such that for each $i \in [n]$ the columns of B with indexes in Λ_i are identical. We always have two extremal good column partitions: the *full column partition* Λ of B where $B(\cdot, u) = B(\cdot, v)$ if and only if $\{u, v\}$ lies in Λ_i for some $i \in [n]$, and the *trivial column partition* Λ where $n = m$ and each Λ_i is a singleton set. We can perform as follows a *row amalgamation operation* on an $m \times m$ matrix B based on any of its good column partitions, say $\Lambda : [m] = \cup_{i=1}^{n} \Lambda_i$. Pick a transversal of Λ, say $k_i \in \Lambda_i$, $i \in [n]$; then create an $m \times n$ matrix B/Λ by taking $(B/\Lambda)(\cdot, i) = B(\cdot, k_i)$; and finally produce the $n \times n$ matrix B^{Λ} by putting $B^{\Lambda}(i, \cdot) = \sum_{u \in \Lambda_i} (B/\Lambda)(u, \cdot)$. The matrix B/Λ which occurred in the middle of the above process is called the *reduced matrix* of B with respect to the good column partition Λ and the final output matrix B^{Λ} is called the *row amalgamation* of B with respect to Λ. A simple check yields

$$B = (B/\Lambda)\chi(\Lambda) \quad \text{and} \quad B^{\Lambda} = \chi(\Lambda)(B/\Lambda). \tag{16}$$

Furthermore, we have the following.

Lemma 5.5. *Let $A \in \mathrm{Mat}_n(\mathbb{R})$ and $B \in \mathrm{Mat}_m(\mathbb{R})$. Then A is a row amalgamation of B if and only if there are two matrices R and S such that $A = RS$, $B = SR$, and R is a row amalgamation matrix.*

Proof. The necessity follows plainly from Eq. (16). We now set up the backward implication. Suppose that $A = RS$ and $B = SR$ for some row amalgamation matrix $R_{n\times m}$ and rectangular matrix $S_{m\times n}$. We can assume that $R = \chi(\Lambda)$ for some partition Λ of $[m]$. Then $B = SR$ tells us that Λ is a good column partition of B and $S = B/\Lambda$. Since $A = RS$, we deduce that $A = B^{\Lambda}$, as desired. □

The row amalgamation matrix of B with respect to its full column partition is called the *full row amalgamation* of B, and is denoted by B_{fr} [31, p. 67]. Here we commit the abuse of notation by not distinguishing between permutation-similar matrices. Let $B_{f^0r} = B$ and $B_{f^1r} = B_{fr}$. Then the kth *full row amalgamation* of B, denoted by B_{f^kr}, is defined recursively to be the full row amalgamation of $B_{f^{k-1}r}$. Continue the procedure of forming full row amalgamations until no nontrivial good column partition can be found any more, we produce the *total row amalgamation* of B, which will be referred to as B_r later [31, p. 39]; [35, p. 426].

We now introduce the inverse of the row amalgamation operation. Let A be an $n \times n$ matrix and σ a decomposition of the rows of A such that each row $A(i, \cdot)$ is divided into σ_i rows, say $A(i, \cdot) = \sum_{j=1}^{\sigma_i} A_i^j$, $i \in [n]$. Let $m = \sum_{i=1}^{n} \sigma_i$ and consider a partition $\Lambda : [m] = \cup_{i=1}^{n} \Lambda_i$ such that Λ_i contains σ_i elements, say $\Lambda_i = \{u_i^1, \ldots, u_i^{\sigma_i}\}$, $i \in [n]$. We can do the *row splitting operation* on A according to $\pi = \{\sigma, \Lambda\}$ in two steps as follows: First construct an $m \times n$ matrix $A[\pi]$ by assigning A_i^j to be its u_i^jth row, $i \in [n]$, $j \in [\sigma_i]$; then get a matrix A_π by requiring $A_\pi(\cdot, u) = A_\pi(\cdot, i)$ for each $u \in \Lambda_i$ and $i \in [n]$. We say that $A[\pi]$ is an *expanded matrix* of A and A_π a *row splitting* of A, respectively. We have a straightforward observation that

$$A = \chi(\Lambda)A[\pi] \quad \text{and} \quad A_\pi = A[\pi]\chi(\Lambda). \tag{17}$$

Moreover, we assert

Lemma 5.6. *Let $A \in \mathrm{Mat}_n(\mathbb{R})$ and $B \in \mathrm{Mat}_m(\mathbb{R})$. Then B is a row splitting of A if and only if there are two matrices R and S such that $A = RS$, $B = SR$, and R is a row amalgamation matrix.*

Proof. The forward direction is guaranteed by Eq. (17). It suffices to consider the converse. Suppose that $A = RS$ and $B = SR$ for some row amalgamation matrix R and rectangular matrix S. From the definition of row amalgamation matrix, we know that $R = \chi(\Lambda)$ for some partition Λ of $[m]$. Furthermore, invoking the fact that $A = RS$ we are allowed to deduce $S = A[\pi]$, where $\pi = \{\Lambda, \sigma\}$ for some decomposition σ. Finally, $B = SR$ means that B is equal to A_π, completing the proof. □

Combining Lemmas 5.5 and 5.6, we can verify that the two operations, row splitting and row amalgamation, are really the inverses of each other, as illustrated in the next theorem.

Theorem 5.7. *Let $A \in \mathrm{Mat}_n(\mathbb{R})$ and $B \in \mathrm{Mat}_m(\mathbb{R})$. Then A is a row amalgamation of B with S being the expanded matrix if and only if B is a row splitting of A with S being the reduced matrix.*

To come back to Hoffman polynomials, we first frame a simple result from Eq. (16).

Lemma 5.8. *Let B be a nonnegative square matrix and A a row amalgamation of B. Then A and B are elementarily equivalent over $\mathbb{R}^+$.*

Corollary 5.9. *Let $B \in \mathrm{Mat}_m(\mathbb{R})$ be a nonnegative irreducible matrix and $A \in \mathrm{Mat}_n(\mathbb{R})$ a row amalgamation of B. Then A is still a nonnegative irreducible matrix and one of the following two cases holds:*

(i) $H_B(x) = \frac{m}{n} H_A(x)$;
(ii) $H_B(x) = \frac{m}{n\lambda_A} x H_A(x)$.

Proof. In view of Theorem 5.3 (i) and Lemma 5.8, it suffices to establish that $\deg H_A(x) \leqslant \deg H_B(x)$. Lemma 5.5 asserts that $A = RS$ and $B = SR$, where R is a row amalgamation matrix and S a reduced matrix of B. We know that $RR^\top = D$ is a diagonal matrix with positive diagonal entries. Now assume $f(x)$ is a polynomial such that $f(B)$ is positive and has rank one. Then we see that

$$\begin{aligned} f(A) &= f(RS) && \text{by } A = RS \\ &= f(RS)RR^\top D^{-1} && \text{by } RR^\top = D \\ &= Rf(SR)R^\top D^{-1} \\ &= Rf(B)R^\top D^{-1} && \text{by } B = SR. \end{aligned} \tag{18}$$

From Eq. (18) we can deduce that $f(A) > 0$, since $f(B) > 0$, $R \geqslant 0$ has no zero rows and $D^{-1} \geqslant 0$ has no zero columns; we also find that rank $f(A) = 1$. By Theorem 2.4, we infer that $\deg H_A(x) \leqslant \deg H_B(x)$, as was to be shown. $\square$

Remark 5.10. The property of the row amalgamation matrix R that we use in proving the preceding corollary is that R is nonnegative and that there exists a nonnegative matrix R' such that RR' is the inverse of a nonnegative matrix. It is interesting to determine the structure of such matrices.

Remark 5.11. Corollary 5.9 tells us that if we can find a nontrivial good column partition of a nonnegative irreducible matrix B, then the task of computing its Hoffman polynomial can more or less be reduced to the determination of the Hoffman polynomial of its row amalgamation matrix, which is of a smaller size. Note that it is possible that B has no nontrivial good column partition while we can find $\lambda \in \mathbb{R}$ such that $B + \lambda I$ is still nonnegative irreducible and has a nontrivial good column partition. Since the relationship between the Hoffman polynomials of B and $B + \lambda I$ is very clear, we can use such reduction tool like Corollary 5.9 on $B + \lambda I$ and thus still facilitate the computation of $H_B(x)$.

It is a simple matter to derive from Corollary 5.9 the following result. It says that the knowledge of Hoffman polynomials may help us estimate how many times of row amalgamations we have to perform in order to reach one digraph from the other.

Corollary 5.12. *Let B be an $m \times m$ nonnegative irreducible matrix. Then*

$$H_B(x) = \ell x^k H_{B_r}(x)$$

for some nonnegative integer k and some $\ell > 0$. The parameter k is no larger than the number of full row amalgamations required to reach the matrix B_r from B.

Problem 5.13. Develop some criterion to tell which of the two cases described in Corollary 5.9 happens.

Problem 5.13 for a matrix B and its full row amalgamation B_{fr} is partially exploited below. Note that if B is nonsingular then $B = B_{fr}$, and thus nothing needs to be said on this case.

Corollary 5.14. *Let B be an $m \times m$ nonnegative irreducible matrix and S an $m \times n$ matrix consisting of all the distinct columns of B. If* $\det B = 0$ *and S is of full column rank, then*

$$H_B(x) = \frac{n}{m\lambda_B} x H_{B_{fr}}(x).$$

Proof. Recall that we have $S = B/\Lambda$ for a full column partition Λ of B and B_{fr} coincides with B^{Λ}. Let $A = B_{fr}$ and $R = \chi(\Lambda)$. Lemma 5.8 claims that $(R, S) : A \approx_{\mathbb{R}^+} B$. But S has full column rank by our assumption and R surely has full row rank. Thus, by examining Eqs. (2) and (3) in addition, we get the result from Theorem 5.3. $\square$

By symmetry, we can define *good row partition*, *column splitting*, and *column amalgamation* and so on in the most obvious way. To distinguish them from corresponding concepts on row splitting and row amalgamation, etc., we use c in place of r in the relevant notation. Sometimes we just use amalgamation to represent either column amalgamation or row amalgamation.

Example 5.15. Let $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. A direct computation gives $A_{fr} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $A_r = A_{f^2r} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $H_A(x) = \frac{\sqrt{2}}{2}x^2(x + \sqrt{2}) = \frac{2\sqrt{2}}{3}xH_{A_{fr}}(x) = x^2 H_{A_r}(x)$.

We see that $A_c = A_r$ in Example 5.15. But this is not always true as illustrated by the next example.

Example 5.16. Let $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. We have $A_{fc} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $A_{f^2c} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A_c = A_{f^3c} = (2)$. However, $A_c \neq A_r = A_{fr} = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Here $H_A(x) = \frac{5}{8}x^3 = \frac{5}{4}H_{A_r}(x) = \frac{5}{4}x^3 H_{A_c}(x)$.

Note that in Example 5.16, A_r cannot be amalgamated to be of a smaller size despite of the fact $x | H_{A_r}(x)$. But A_r is elementarily equivalent to A_{fc} since $A_r = SR$ and $A_{fc} = RS$, where

$$R = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here the matrix R is not an amalgamation matrix. This is in accordance with the fact that A_r does not have a nontrivial amalgamation. In light of the above observation, it seems interesting to seek some sort of partial converse of Theorem 5.3.

Problem 5.17. Let $A \in \mathrm{Mat}_n(\mathbb{R}^+)$ and $B \in \mathrm{Mat}_m(\mathbb{R}^+)$ be two irreducible matrices such that $H_B(x) = \frac{n}{m\lambda_A} x H_A(x)$. Discuss the form of $H_A(x)$ which ensures that B has either a nontrivial good column partition or a nontrivial good row partition. In what circumstance can we assert that such a pair A and B are elementarily equivalent?

Problem 5.18. If $H_B(x) = \frac{n}{m\lambda_A} x H_A(x)$, can we find a matrix which is a common column splitting or a common row splitting of A and B, or can we find a matrix which is a column splitting of A and a row splitting of B? Moreover, if these kinds of common splittings do exist, how can we construct from A and B their common splittings of the smallest size efficiently?

5.3. Splitting and amalgamation of digraphs

Having considered general nonnegative irreducible matrices, let us turn to those integer ones, namely strongly connected digraphs. To start things off, let us prepare some terminology.

If an arc e goes from a vertex u to a vertex v, we say that u is the *initial vertex* (or the *tail*) of e and v the *terminal vertex* (or the *head*) of e. The incidence structure of a digraph Γ is characterized by two maps from $E(\Gamma)$ to $V(\Gamma)$, the *tail operator* $\mathtt{i}_\Gamma$ which sends an arc to its initial vertex and the *head operator* $\mathtt{t}_\Gamma$ which sends an arc to its terminal vertex. A vertex is a *source* of Γ if it is not the terminal vertex of any arc of Γ; a vertex is a *sink* of Γ if it is not the initial vertex of any arc of Γ. The *initial incidence matrix* of Γ is the matrix P_Γ of dimension $|V(\Gamma)| \times |E(\Gamma)|$ such that

$$P_\Gamma(i,j) = \begin{cases} 1, & \text{if } v_i = \mathtt{i}_\Gamma(e_j), \\ 0, & \text{otherwise} \end{cases}$$

and the *terminal incidence matrix* of Γ is the matrix Q_Γ of dimension $|V(\Gamma)| \times |E(\Gamma)|$ such that

$$Q_\Gamma(i,j) = \begin{cases} 1, & \text{if } v_i = \mathtt{t}_\Gamma(e_j), \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, when the digraph Γ is clear from the context we will often eliminate it from the notation. Let $E_u = E(\Gamma)_u$ denote the set of arcs of Γ starting at a vertex $u \in V(\Gamma)$. An *out-partition* π of Γ is a partition of $E(\Gamma)$ into disjoint sets $E_u^1, \ldots, E_u^{\pi_u}$, $u \in V(\Gamma)$, where $\cup_{i=1}^{\pi_u} E_u^i = E_u$ for each $u \in V(\Gamma)$. Note that we do not require $\pi_u \geqslant 1$ as in [35, Definition 2.4.3] and so our definition differs from the usual one used in symbolic dynamics. But we also point out that $\pi_u = 0$ can happen only when $E_u = \emptyset$ and hence the two definitions coincide when restricted to digraphs without sinks. An out-partition π is said to be *discrete* if $|E_u^i| = 1$ for all $u \in V(\Gamma)$ and $i \in [\pi_u]$, and π is said to be *indiscrete* if $\pi_u = 1$ for all $u \in V(\Gamma)$.

The *out-splitting* of Γ corresponding to an out-partition π, denoted by Γ_π, has vertex set $\{u^i : u \in V(\Gamma), i \in [\pi_u]\}$ and arc set $\{e^i : e \in E(\Gamma), i \in [\pi_{\mathtt{t}_\Gamma(e)}]\}$ and the incidence structure is given by requiring that for any $e \in E(\Gamma), i \in [\pi_{\mathtt{t}_\Gamma(e)}]$ we have $\mathtt{i}_{\Gamma_\pi}(e^i) = \mathtt{i}_\Gamma(e)^j$ and $\mathtt{t}_{\Gamma_\pi}(e^i) = \mathtt{t}_\Gamma(e)^i$, where j is chosen such that $e \in E^j_{\mathtt{i}_\Gamma(e)}$. We also call a digraph an *out-splitting* of Γ if it is isomorphic to Γ_π for some out-partition π of Γ. It is easy to see that a digraph Σ is a out-splitting of Γ if and only if $A(\Sigma)$ is a row splitting of $A(\Gamma)$. Let $E^u = E(\Gamma)^u$ denote the set of arcs of Γ ending at a vertex $u \in V(\Gamma)$. An *in-partition* π of Γ is a partition of $E(\Gamma)$ into disjoint sets $E^u_1, \dots, E^u_{\pi^u}$, $u \in V(\Gamma)$, where $\cup_{i=1}^{\pi^u} E^u_i = E^u$ for each $u \in V(\Gamma)$. Parallel to the definition of discrete out-partition and out-splitting, we define in the most obvious manner the *(in)discrete in-partition* and *in-splitting*.

For a given digraph Σ, we call a partition Λ of $V(\Sigma)$, say $V(\Sigma) = \cup_{i=1}^m \Lambda_i$, an *in-good* partition if for any $i \in [m]$ and any $u, v \in \Lambda_i$ the multiset of initial vertices of E^u is the same as that of E^v. The *in-amalgamation* of Σ for an in-good partition Λ, denoted by Σ^Λ, is the digraph having vertex set $\{\Lambda_1, \dots, \Lambda_t\}$ and there are t arcs going from Λ_i to Λ_j if and only if for each $v \in \Lambda_j$, there exist t arcs in $E(\Sigma)$ with $\mathtt{i}_\Sigma(e) \in \Lambda_i$ and $\mathtt{t}_\Sigma(e) = v$. Similarly, we define *out-good* partition and *out-amalgamation* in the most obvious way.

The next result indicates the duality between out-splitting and in-amalgamation of digraphs.

Theorem 5.19. *Let Γ be a digraph.*

(i) *For any out-partition π of Γ, Γ is an in-amalgamation of Γ_π;*
(ii) *For any in-good partition Λ of $V(\Gamma)$, Γ is an out-splitting of Γ^Λ.*

Proof. (i) Suppose π is an out-partition of Γ such that for each $u \in V(\Gamma)$, E_u is partitioned into sets $E^1_u, \dots, E^{\pi_u}_u$. Let $\Lambda_u = \{u^i : u \in V(\Gamma), i \in [\pi_u]\}$. It is easy to check that $\Gamma \cong (\Gamma_\pi)^\Lambda$ where Λ is the partition $V(\Gamma_\pi) = \cup_{i=1}^m \Lambda_u$.

(ii) Suppose the in-good partition Λ is $V(\Gamma) = \cup_{i=1}^m \Lambda_i$. Set $\pi_{\Lambda_i} = |\Lambda_i|$ and label the vertices of Γ in Λ_i by $\Lambda_i^1, \dots, \Lambda_i^{\pi_i}$. In virtue of the definition of Γ^Λ, we know that the arcs from Λ_i to Λ_j in Γ^Λ are in bijective correspondence with the occurrences of vertices from Λ_i in the multiset of initial vertices of E^u, where u is any vertex in Λ_j. Let $E(\Gamma^\Lambda)^t_{\Lambda_i}, t \in [\pi_i]$, be those arcs going out of Λ_i which correspond to the vertex Λ_i^t in the above correspondence. We thus obtain an out-partition π of Γ^Λ which partition $E(\Gamma^\Lambda)_{\Lambda_i}$ to be $\cup_{t=1}^{\pi_i} E(\Gamma^\Lambda)^t_{\Lambda_i}$ for each $i \in [m]$. We can verify that $\Gamma \cong (\Gamma^\Lambda)_\pi$, ending the proof. $\square$

Example 5.20. Let Σ be the digraph depicted in Fig. 1(a). Let $\Lambda_1 = \{v_1, v_2\}$, $\Lambda_2 = \{v_3\}$, $\Lambda_3 = \{v_4\}$, $\Lambda_4 = \{v_5, v_6, v_7\}$ and $\Lambda_5 = \{v_8, v_9\}$. Then the in-amalgamation Σ^Λ for $\Lambda : V(\Sigma) = \cup_{i=1}^5 \Lambda_i$ is showed in Fig. 1(b). Now let $\Gamma = \Sigma^\Lambda$ whose arcs are labelled as in Fig. 1(b). Let $E^1_{\Lambda_1} = \{a, b\}$, $E^2_{\Lambda_1} = \emptyset$, $E^1_{\Lambda_2} = \{c\}$, $E^1_{\Lambda_3} = \{d\}$, $E^1_{\Lambda_4} =$

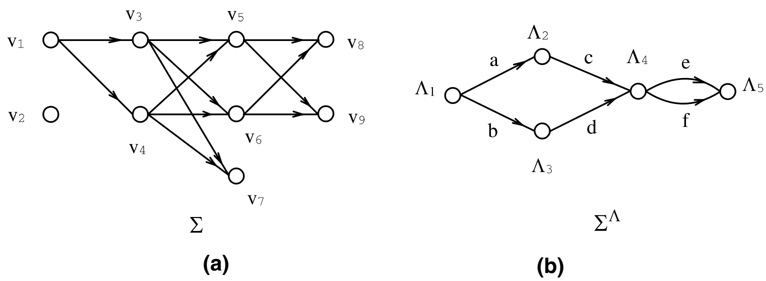

Fig. 1. Duality of out-splitting and in-amalgamation of digraphs.

$\{e\}, E^2_{\Lambda_4} = \{f\}, E^3_{\Lambda_4} = \emptyset, E^1_{\Lambda_5} = \emptyset$ and $E^2_{\Lambda_5} = \emptyset$. We use π to denote this out-partition of Γ. It is easy to check that $\Gamma_\pi \cong \Sigma$.

When mentioning a partition of Γ later in this paper, we shall always mean an in-partition or an out-partition. For a partition π, depending on whether it is an in-partition or an out-partition, we will simply write $\Gamma(\pi)$ for Γ_π or Γ^π and $\pi(u)$ for π_u or π^u, for any $u \in V(\Gamma)$, correspondingly. We say that $\Gamma(\pi)$ is a *splitting* of Γ and Γ is an *amalgamation* of $\Gamma(\pi)$ [35, Definition 2.4.9].

The *line digraph* $L(\Gamma)$ of a digraph Γ has vertex set $E(\Gamma)$ and there is an arc from e_1 to e_2 if and only if $\mathtt{t}_\Gamma(e_1) = \mathtt{i}_\Gamma(e_2)$. For any positive integer m, the mth-iterated line digraph $L^m(\Gamma)$ of Γ is defined inductively by setting $L^0(\Gamma) = \Gamma$ and putting $L^m(\Gamma) = L(L^{m-1}(\Gamma))$ for $m \geqslant 1$.

Example 5.21. $\Gamma(\pi) \cong L(\Gamma)$ for any digraph Γ and its discrete partition π.

Example 5.22. $\Gamma(\pi) \cong \Gamma$ for any digraph Γ and its indiscrete partition π.

Let π be an out-partition of Γ. The *out-division matrix* P_π and the *out-arc matrix* Q_π are $|V(\Gamma)| \times |V(\Gamma_\pi)|$ matrices specified by

$$P_\pi(u, v^i) = \begin{cases} 1, & \text{if } u = v; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad Q_\pi(u, v^i) = |\{e \in E^i_v : \mathtt{t}_\Gamma(e) = u\}|$$

for any $u, v \in V(\Gamma), i \in [\pi_v]$ [35, Definition 2.4.11]. Correspondingly, if π is an in-partition of Γ, the *in-division matrix* P^π and *in-arc matrix* Q^π are matrices of dimension $|V(\Gamma)| \times |V(\Gamma^\pi)|$ such that

$$P^\pi(u, v_i) = \begin{cases} 1, & \text{if } u = v, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad Q^\pi(u, v_i) = |\{e \in E^v_i : \mathtt{i}_\Gamma(e) = u\}|$$

for any $u, v \in V(\Gamma), i \in [\pi^v]$. We are ready to give a description of the relationship between the adjacency matrices of a digraph and its amalgamation digraph.

Lemma 5.23. *Let Γ be a digraph.*

(i) *For any out-partition π of Γ, we have*

$$A(\Gamma) = P_\pi Q_\pi^\top \quad \text{and} \quad A(\Gamma_\pi) = Q_\pi^\top P_\pi. \tag{19}$$

(ii) *For any in-partition π of Γ, we have*

$$A(\Gamma) = Q^\pi P^{\pi\top} \quad \text{and} \quad A(\Gamma^\pi) = P^{\pi\top} Q^\pi. \tag{20}$$

Proof. The proof of [35, Theorems 2.4.12, 2.4.14] still works. □

We list below some basic properties of out(in)-division and out(in)-arc matrices associated with a partition of a digraph but skip their routine proofs.

Lemma 5.24. *Let Γ be a digraph.*

(i) *For any out-partition π of Γ, P_π is of full row rank if and only if $\pi_v \geqslant 1$ for all $v \in V(\Gamma)$ while $Q_\pi^\top$ is column-nontrivial if and only if Γ has no sources. Γ has a sink w if and only if* (a) *$\pi_w = 1$ and $Q_\pi^\top$ has a zero row*; *or* (b) *$\pi_w = 0$ and P_π has a zero row.*
(ii) *For any in-partition π of Γ, $P^{\pi\top}$ is of full column rank if and only if $\pi^v \geqslant 1$ for all $v \in V(\Gamma)$ while Q^π is row-nontrivial if and only if Γ has no sinks. Γ has a source w if and only if* (a) *$\pi^w = 1$ and Q^π has a zero column*; *or* (b) *$\pi^w = 0$ and $P^{\pi\top}$ has a zero column.*

Lemma 5.25. *Let Γ be a digraph, π_1 an out-partition and π_2 an in-partition of $E(\Gamma)$, respectively. If π_1 is discrete, then $P_{\pi_1} = P_\Gamma$ and $Q_{\pi_1} = Q_\Gamma$. If π_2 is discrete, then $Q^{\pi_2} = P_\Gamma$ and $P^{\pi_2} = Q_\Gamma$.*

In view of Example 5.21, Lemmas 5.23 and 5.25, we come to the following result, which has appeared many times in various literatures.

Corollary 5.26. *For any given digraph Γ, $A(\Gamma) = P_\Gamma Q_\Gamma^\top$ and $A(L(\Gamma)) = Q_\Gamma^\top P_\Gamma$.*

Theorem 5.27. *Let Γ and Σ be two digraphs. Then the following statements are equivalent*:

(i) *Σ is an out-splitting of Γ*;
(ii) *Γ is an in-amalgamation of Σ*;
(iii) *$A(\Sigma)$ is a row splitting of $A(\Gamma)$*;
(iv) *$A(\Gamma)$ is a row amalgamation of $A(\Sigma)$.*

Proof. The equivalence of (i) and (ii) comes from Theorem 5.19 and the equivalence of (iii) and (iv) can be seen from Theorem 5.7. Combining Lemmas 5.6, 5.23 and 5.24, we get that (i) and (iii) are equivalent. □

We remark that from Lemma 5.23 and the main result of [2], we have much knowledge of the relationship between the Jordan form, and hence many relevant parameters, of a digraph and its split digraph. Also note that using the notation of symbolic dynamics, Lemma 5.23 says that there is an elementary equivalence (over $\mathbb{Z}^+$) from $A(\Gamma)$ to $A(\Gamma(\pi))$ [35, Definition 7.2.1]. This again confirms that a digraph and its split digraph define conjugate dynamical systems and thus have equal parameters as long as it is a conjugate invariant, like the zeta function, the Bowen–Franks group, the Jordan form away from zero, some inverse limit spaces, the dimension group, and so on [31, Chapter 2]; [35, §6.3, §6.4, §7.4].

Our task below is to use Theorem 5.1 directly to find the relationship between the Hoffman polynomials of a digraph and its split digraph.

Corollary 5.28. *Let Γ be a strongly connected digraph and $\Gamma(\pi)$ a splitting of Γ for some partition π of Γ. Let $n = |V(\Gamma)|$ and $m = |V(\Gamma(\pi))|$. Then $H_{\Gamma(\pi)}(x)$ equals either $\frac{m}{n}H_\Gamma(x)$ or $\frac{m}{n\lambda_\Gamma}xH_\Gamma(x)$.*

Proof. It is a direct consequence of Corollary 5.3 and Lemmas 5.23 and 5.24. □

Theorem 5.29. *Let Γ be a strongly connected digraph with at least one arc. Let $\Gamma(\pi)$ be a splitting of Γ for some partition π of $E(\Gamma)$. If there exists at least one vertex $u \in V(\Gamma)$ with $|\pi(u)| > 1$ and if $Q_\pi^\top$ is of full column rank for π being an out-partition or Q^π is of full row rank for π being an in-partition, then*

$$H_{\Gamma(\pi)}(x) = \frac{m}{n\lambda_\Gamma}xH_\Gamma(x),$$

where $n = |V(\Gamma)|,\ m = |V(\Gamma(\pi))|$.

Proof. First note that Γ has neither sources nor sinks. Also observe that the condition that $|\pi(u)| > 1$ implies that there are identical rows in $A(\Gamma(\pi))$ if π is an out-partition and there are identical columns in $A(\Gamma(\pi))$ if π is an in-partition. Henceforth, we get $\det A(\Gamma(\pi)) = 0$, whichever case it is. The result is now immediate from a simple combination of Eqs. (2) and (3), Theorem 5.3, and Lemmas 5.23 and 5.24. □

Say that a digraph is *common* if it has a vertex whose in-degree is at least two or a vertex whose out-degree is at least two.

Corollary 5.30. *Let Γ be a strongly connected common digraph and π a discrete partition of Γ. Let $n = |V(\Gamma)|,\ m = |E(\Gamma)|$. Then we have*

$$H_{\Gamma(\pi)}(x) = \frac{m}{n\lambda_\Gamma}xH_\Gamma(x).$$

Proof. It suffices to check the case that π is a discrete out-partition. From the definition of a discrete out-partition we can find that $Q_\pi^\top$ is of full column rank. Hence the result comes directly from Theorem 5.29. $\square$

Since $\Gamma(\pi)$ is just $L(\Gamma)$ when π is discrete, we can rephrase that special case of Corollary 5.30 as below.

Corollary 5.31. *Let Γ be a strongly connected common digraph with $|V(\Gamma)| = n$ and $|E(\Gamma)| = m$. Then we have*

$$H_{L(\Gamma)}(x) = \frac{m}{n\lambda_\Gamma} x H_\Gamma(x).$$

Given a digraph Γ, we call a digraph the ith *full or total column* (*row*) *amalgamation* of Γ if its adjacency matrix equals $A(\Gamma)_{f^i c}$ or $A(\Gamma)_c$ ($A(\Gamma)_{f^i r}$ or $A(\Gamma)_r$). We denote the ith full and total column (row) amalgamation of Γ by $\Gamma_{f^i c}$ and Γ_c ($\Gamma_{f^i r}$ and Γ_r), respectively.

As reported in Example 5.16, the fact that $x|H_{A_c}(x)$ does not necessarily imply that the digraph $\Gamma(A_c)$ is a splitting of any digraph of smaller order. However, parallel to Problem 5.17, we want to know to which extent something in the opposite direction of Theorem 5.29 or Corollary 5.31 could be said, i.e., under what further assumptions can we deduce from $x|H_\Gamma(x)$ that Γ is a split digraph or even a line digraph of a digraph of a smaller size? Especially, we pose

Problem 5.32. Let k and ℓ be two positive integers and Γ a digraph with $H_\Gamma(x) = x^k + x^{k+\ell}$. Under which further assumption on Γ can we find a digraph Σ satisfying $\Gamma = L^k(\Sigma)$ and $H_\Sigma(x) = 1 + x^\ell$? Note that the assumption that Γ is regular does guarantee the existence of Σ as mentioned above [43].

5.4. An application

For their application to concurrent computation, Ho [23] is interested in the so-called *M-satisfiable* digraphs for integers *M*, namely digraphs whose adjacency matrices *A* satisfy $M \geqslant \max_{k\in\mathbb{Z}^+} \max_{i,j,j'} |A^k(i, j) - A^k(i, j')|$. The smaller the parameter *M* could take, the more uniform the task assignment corresponding to the digraph is. For example, if there is a unique walk of length *n* between any pair of ordered not necessarily distinct vertices [47], then this digraph must be 1-satisfiable. It is known that a digraph is *M*-satisfiable for some *M* if and only if its adjacency matrix *A* satisfies the matrix equation

$$A^m - A^n = \ell J \tag{21}$$

for some positive integers $m > n$ and ℓ [23]. Indeed, if $A(\Gamma) = A$ satisfies Eq. (21), then we can show that Γ is *M-satisfiable* for

$$M = \max_{k\in\{0,1,\ldots,m-1\}} \max_{i,j,j'} |A^k(i,j) - A^k(i,j')|. \tag{22}$$

Define a strongly connected digraph to be *generalized satisfiable* provided its Hoffman polynomial is a factor of a polynomial of the form $x^m - x^n$. Ho [23, Corollary 5] finds that a digraph is satisfiable if its line digraph is satisfiable. By virtue of Remark 2.5 and Corollary 5.28, a natural generalization of his result is that any amalgamation digraph or splitting digraph of a generalized satisfiable digraph must be generalized satisfiable. Note that the Decomposition Theorem [35, Theorem 7.1.2] says at this moment that this property specified by the Hoffman polynomials is a conjugacy invariant of the edge shift of the digraph.

Ho constructs a family of satisfiable digraphs Γ_k for each positive integer k. Moreover, Ho demonstrates that $A(\Gamma_k)$ is a solution to the equation $A^{2(k+1)} - A^{k+1} = 2^{k+1}J$. In order to use Eq. (22) to estimate how satisfiable Γ_k is, we want to find a matrix equation of the form of Eq. (21) with as low degree as possible for which $A(\Gamma_k)$ is a solution. Surprisingly, we can determine the minimum degree such polynomials by working out the tth full amalgamation digraph of Γ_k and then its Hoffman polynomial.

For any $k \geqslant 1$, the digraph Γ_k to be defined will be a 2-regular digraph on the set of $2^{k+1} - 1$ vertices, that is, $\{r\} \cup \{a_i, b_i | i \in [2^k - 1]\}$. We write $x \to \{y, z\}$ to refer to the operation of adding one arc from x to y and one arc from x to z. The incidence relation in Γ_k is built in the following procedure [23]:

R1. $r \to \{a_1, b_1\}$;
R2. $a_{2^i+j} \to \{a_{2^{i+1}+2j}, a_{2^{i+1}+2j+1}\},\ 0 \leqslant i \leqslant k-2,\ 0 \leqslant j \leqslant 2^i - 1$;
R3. $a_{2^{k-1}} \to \{r, b_1\}$;
R4. $a_{2^{k-1}+j} \to \{b_{2^k+2j}, b_{2^k+2j+1}\},\ 1 \leqslant j \leqslant 2^k - 1$;
R5. Swapping the roles of a and b, do R2, R3 and R4 once again.

Note that R1 and R2 define a complete binary tree of depth k and rooted at r.

Theorem 5.33. $H_{\Gamma_k}(x) = \frac{1}{2^{k-1}}x^{k-1}(1 + x + x^2 + \cdots + x^k)$.

Proof. Let $n = 2^{k+1} - 1$ and let $A = A(\Gamma_k)$, whose lines are indexed by $a_{2^k-1}, a_{2^k-2}, \ldots, a_1, r, b_1, \ldots, b_{2^k-2}, b_{2^k-1}$ in that order. Our strategy is to prove that $H_A(x) = \ell x^{k-1} H_{A_r}(x)$ for some $\ell \in \mathbb{R}$ and then turn to compute $H_{A_r}(x)$. The first goal is done in two steps, proving that A_r is obtained from A by performing $k-1$ full row amalgamations and that $H_{A_{f^{i-1}r}}(x) = \ell_i x H_{A_{f^i r}}(x)$ for some $\ell_i \in \mathbb{R},\ i \in [k-2]$.

To deduce $A_r = A_{f^{k-1}r}$, we proceed as follows. Note that there are $2(2^{k-1} - 1) = \frac{n-3}{2}$ pairs of vertices, namely $\{a_2, a_3\}, \{a_4, a_5\}, \ldots, \{a_{2^k-2}, a_{2^k-1}\}, \{b_2, b_3\}, \{b_4, b_5\}, \ldots, \{b_{2^k-2}, b_{2^k-1}\}$, each pair of them have a common set of in-neighbors. Hence $(\Gamma_k)_{fr}$ is obtained from Γ_k by merging each of those pairs of vertices and thus $(\Gamma_k)_{fr}$ is of order $n_1 = n - \frac{n-3}{2} = 2^k + 1$. $(\Gamma_k)_{f^2r}$ is obtained from $(\Gamma_k)_{fr}$ by merging $\frac{n-5}{2}$

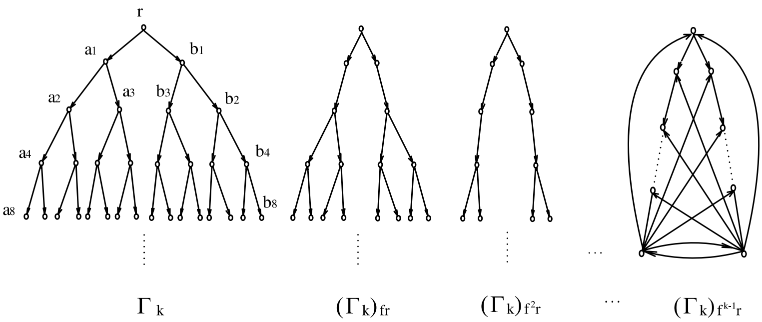

Fig. 2. In-amalgamation of Γ_k.

pairs of vertices each of which have the same set of in-neighbors in $(\Gamma_k)_{fr}$, and so on. This amalgamation process ends at $(\Gamma_k)_{f^{k-1}r}$ since there are no two vertices in $(\Gamma_k)_{f^{k-1}r}$ sharing the same set of in-neighbors. Note that $(\Gamma_k)_{f^ir}$, $i = 0, \dots, k-1$, all of them have constant in-degree 2, which should not be misunderstood from the local picture depicted in Fig. 2.

Next we prove that there is $\ell_i \in \mathbb{R}$ such that $H_{A_{f^{i-1}r}}(x) = \ell_i x H_{A_{f^ir}}(x)$ for each $i \in [k-1]$. Examining the above procedure of doing amalgamations, we get that for each $i \in [k-1]$, A_{f^ir} is of dimension $n_i = \frac{n_{i-1}+2i+1}{2} = 2^{k+1-i} + 2i - 1$ and $A_{f^{i-1}r} = S_iR_i$ and $A_{f^ir} = R_iS_i$ for some row amalgamation matrix R_i and reduced matrix S_i. By Corollary 5.14 we need only check that each S_i is of full column rank. For $i = 1$, we know that S_1 is obtained from A by deleting $2(2^{k-1} - 1)$ columns, say columns corresponding to $a_3, a_5, \dots, a_{2^k-1}$ and $b_3, b_5, \dots, b_{2^k-1}$. Picking rows corresponding to $a_{2^k-1}, a_{2^k-2}, \dots, a_{2^{k-1}}, r, b_{2^{k-1}}, \dots, b_{2^k-2}, b_{2^k-1}$, we find that S_1 has a submatrix $P_1 = \begin{pmatrix} & & I_{2^{k-1}-1} \\ & B & \\ I_{2^{k-1}-1} & & \end{pmatrix}$, where $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Since P_1 is nonsingular, we know that S_1 is of full column rank. Analogously, for each $i \in [k-1]$, there is a matrix P_i obtained from S_i by deleting some rows, which is permutation similar to $\begin{pmatrix} & & I_{2^{k-i}+i-2} \\ & B & \\ I_{2^{k-i}+i-2} & & \end{pmatrix}$. This verifies that S_i is of full column rank, as desired.

We now calculate $H_{A_r}(x)$. Consider an eigenvalue λ of A_r together with its eigenvector $\xi = (x_k, \dots, x_2, x_1, \gamma, y_1, y_2, \dots, y_k)^\top$. From $A_r\xi = \lambda\xi$ we read off that

$$\begin{cases} x_i = \lambda^{i-1}x_1, & i \in [k], \\ y_i = \lambda^{i-1}y_1, & i \in [k], \\ \gamma + \sum_{i=1}^k y_i = \lambda^k x_1, & \\ \gamma + \sum_{i=1}^k x_i = \lambda^k y_1. & \end{cases}$$

Consequently, we can find that A_r has an eigenvector $\mathbf{j}$ corresponding to eigenvalue 2 and has two independent eigenvectors

$$\xi_1 = (\omega^{k-1}, \ldots, \omega, 1, 2\omega^k, 1, \omega, \ldots, \omega^{k-1})$$

and

$$\xi_2 = (\omega^{k-1}, \ldots, \omega, 1, \omega^k, 0, 0, \ldots, 0),$$

corresponding to each $(k+1)$th root of unity $\omega \neq 1$. Noting $n_{k-1} = 2k+1$, the above observation shows that $\det(xI - A_r) = (x-2)(1+x+\cdots+x^k)^2$ and $m_{A_r}(x) = (x-2)(1+x+\cdots+x^k)$. Since the Perron eigenvalue of A_r is 2, we find that $H_{A_r}(x) = \ell_0(1+x+\cdots+x^k)$, for some $\ell_0 \in \mathbb{R}$.

Thus far, what we know is that $H_{\Gamma_k}(x) = \ell x^{k-1}(1+x+\cdots+x^k)$ for some $\ell \in \mathbb{R}$. Making use of Theorem 2.6 yields that $\ell = \frac{n}{2^{k-1}(1+2+\cdots+2^k)} = \frac{1}{2^{k-1}}$, the proof finished. $\square$

Here comes our promised application of the preceding computation of Hoffman polynomials.

Theorem 5.34. *For each $i \in \{0, 1, \ldots, k-1\}$, the ith full row amalgamation $(\Gamma_k)_{f^i r}$ of Γ_k is generalized satisfiable. Moreover, $g_i(x) = x^{k+1+i} - x^i$ is the unique polynomial of least degree satisfying*

(I) *$g_i(A_{f^i r})$ is positive and has rank 1; and*
(II) *$g_i(x)$ is of the form $x^p - x^q$ for some $p > q$.*

Proof. Assume that $i \in \{0, 1, \ldots, k-1\}$. The first claim comes directly from Remark 2.5 and Corollary 5.28. A more direct way to obtain it is to first see from the proof of Theorem 5.33 that $H_{(\Gamma_k)_{f^i r}}(x) = \eta_r x^{2k-1-r}(1+x+\cdots+x^k)$ for some $\eta_r \in \mathbb{R}$ and then notice $(x-1)(1+x+\cdots+x^k) = x^{k+1}-1$.

Now we consider the second assertion. It follows from the proof of Theorem 5.33 that $g_i(x)$ satisfies (I) and (II) for each i. Henceforth our task is to prove that $g_i(x)$ is of least degree among such polynomials and is unique. We prove this only for $i = k-1$, the other cases being similar.

Suppose that $f(x)$ is a polynomial satisfying conditions (I) and (II) for $i = k-1$. Using Theorems 2.4 and 5.33 we get $\deg f(x) \geqslant 2k-1$. If $\deg f(x) = 2k-1$, then from Remark 2.5, it follows that $f(A) = H_{\Gamma_k}(x)$ and so $f(x)$ fails to satisfy condition (II). This then gives $\deg f(x) \geqslant \deg g_{k-1}(x) = 2k$.

It remains to prove the uniqueness of $g_{k-1}(x)$. If there is a polynomial, say $f(x) \neq g_{k-1}(x)$, of least degree satisfying conditions (I) and (II). Then we know that $f(x) = x^{2k-1} - x^n$ for some $n < 2k-1$. Without loss of generality, assume $n > k-1$. Now we have a polynomial $f(x) - g_{k-1}(x) = x^n - x^{k-1}$ which satisfies conditions (I) and (II) and has a degree smaller than $g_{k-1}(x)$. This contradicts with what we prove in the preceding paragraph and the proof is ended. $\square$

6. Harmonic digraph

We study in this section two classes of digraphs, the so-called harmonic digraphs and semiharmonic digraphs, whose symmetric digraph version is introduced by Dress and Stevanović [15]. A digraph Γ with adjacency matrix A is called *harmonic* if there is $\mu \in \mathbb{R}$ such that $A^2\mathbf{j} = \mu A\mathbf{j}$ and $\mathbf{j}^\top A^2 = \mu\mathbf{j}^\top A$ in which case Γ is also called *μ-harmonic*. A digraph Γ is called *semiharmonic* if $A^3\mathbf{j} = \mu A\mathbf{j}$ and $\mathbf{j}^\top A^3 = \mu\mathbf{j}^\top A$ for some $\mu \in \mathbb{C}$ in which case Γ is also called *μ-semiharmonic*. The ensuing two results are generalizations of corresponding ones of Dress and Stevanović [15].

Corollary 6.1. *Let Γ be a digraph without sinks or sources and let $A = A(\Gamma)$. There exists a polynomial $f(x)$ such that $f(A) = AJA$ if and only if Γ is a strongly connected harmonic digraph.*

Proof. First of all, we mention that the assumption that Γ has no sources or sinks implies that $\xi = A\mathbf{j}$ and $\zeta^\top = \mathbf{j}^\top A$ are both positive vectors.

Assume that there is a polynomial $f(x)$ such that $f(A) = AJA = \xi\zeta^\top$. As both ζ and ξ are positive, we squeeze information out of Theorem 2.1 that Γ is strongly connected and $A^2\mathbf{j} = \lambda_A A\mathbf{j}$ and $\mathbf{j}^\top A^2 = \lambda_A\mathbf{j}^\top A$, that is, Γ is λ_A-harmonic.

Conversely, assume that Γ is strongly connected and $A^2\mathbf{j} = \mu A\mathbf{j}$ and $\mathbf{j}^\top A^2 = \mu\mathbf{j}^\top A$ for some real number μ. Then both ξ and $\zeta^\top$ are Perron eigenvectors of A and therefore μ is the Perron eigenvalue of A. It is Theorem 2.2 now which guarantees the existence of a polynomial $f(x)$ for which $f(A) = \xi\zeta^\top = AJA$, completing the proof. □

The proof of the second result can be done almost word-for-word the same as that of Corollary 6.1.

Corollary 6.2. *Assume that Γ is a digraph without sinks or sources. Let $A = A(\Gamma)$, $\xi = A^2\mathbf{j} + \sqrt{\lambda}A\mathbf{j}$ and $\zeta^\top = \mathbf{j}^\top A^2 + \sqrt{\lambda}\mathbf{j}^\top A$, where λ is the spectral radius of A. Then there is a polynomial $f(x)$ such that $f(A) = \xi\zeta^\top$ if and only if Γ is a strongly connected λ-semiharmonic digraph.*

We include a result about the tensor product of two (semi)harmonic digraphs.

Theorem 6.3. *The tensor product of two (semi)harmonic digraphs is also a (semi)harmonic digraph.*

Proof. Let Γ and Σ be two harmonic digraphs with $A = A(\Gamma)$ and $B = A(\Sigma)$ such that $A^2\mathbf{j}_n = \lambda A\mathbf{j}_n$ and $B^2\mathbf{j}_m = \mu B\mathbf{j}_m$ for some positive number λ and μ. Then we have $(A \otimes B)^2\mathbf{j}_{nm} = (A \otimes B)^2(\mathbf{j}_n \otimes \mathbf{j}_m) = A^2\mathbf{j}_n \otimes B^2\mathbf{j}_m = \lambda\mu(A \otimes B)(\mathbf{j}_n \otimes \mathbf{j}_m) = \lambda\mu(A \otimes B)\mathbf{j}_{nm}$. In the same way, $\mathbf{j}_{nm}^\top(A \otimes B)^2 = \lambda\mu\mathbf{j}_{nm}^\top(A \otimes B)$. This shows that $\Gamma \otimes \Sigma$ is a $\lambda\mu$-harmonic digraph.

Similarly, the tensor product of a λ-semiharmonic digraph and a μ-semiharmonic digraph is a $\lambda\mu$-semiharmonic digraph. $\square$

Theorem 6.4. *Suppose that we have* (R, S): $A \approx_{\mathbb{R}^+} B$, *where* $\mathbf{j}^\top R = \mathbf{j}^\top$ *and* $S\mathbf{j} = \mathbf{j}$. *If* $\Gamma(A)$ *is* λ-(*semi*)*harmonic*, *then* $\Gamma(B)$ *is also* λ-(*semi*)*harmonic*.

Proof. Since $\Gamma(A)$ is λ-harmonic, $A^2\mathbf{j} = \lambda A\mathbf{j}$. It follows that $RSRS\mathbf{j} = \lambda RS\mathbf{j}$ because $A = RS$. Multiplying both sides of the above equation from left by S, we obtain that $B^2S\mathbf{j} = \lambda BS\mathbf{j}$ by considering that $B = SR$. Thus we have $B^2\mathbf{j} = \lambda\mathbf{j}$ as $S\mathbf{j} = \mathbf{j}$. A similar computation gives that $\mathbf{j}^\top B^2 = \lambda\mathbf{j}^\top B$. Hence $\Gamma(B)$ is λ-harmonic.

The proof for the case of semiharmonic digraph can be patterned on that of the previous case. $\square$

Corollary 6.5. *The line digraph of a* μ-(*semi*)*harmonic digraph is also a* μ-(*semi*)*harmonic digraph*.

Proof. It is notable that for a digraph Γ, both the transpose of its initial incidence matrix P_Γ and the terminal incidence matrix Q_Γ have a lone 1 in each column, i.e., $\mathbf{j}^\top P_\Gamma = \mathbf{j}^\top$ and $Q_\Gamma\mathbf{j} = \mathbf{j}$. Then Corollary 5.26 and Theorem 6.4 together yield the result. $\square$

Remark 6.6. The referee points out that the above results hold without the integrality assumption and suggests to pursue the generalizations of the existing results on harmonic digraphs to results for general nonnegative irreducible matrices.

The next two examples are presented by Grüneward and Dress.

Example 6.7. For $\lambda > 1$, let T_λ be the tree with one vertex v of degree $\lambda^2 - \lambda + 1$, while each neighbor of v has degree λ and all the remaining vertices have degree 1 [21]. By [21, Lemma 2.2] and [15, Theorem 1], if Γ is a strongly connected symmetric digraph with a vertex of degree not less than $\lambda^2 - \lambda + 1$ and $f(A(\Gamma)) = A(\Gamma)JA(\Gamma)$ for some polynomial $f(x)$, then $\Gamma \cong T_\lambda$.

Example 6.8. For two positive integers a and $k \geqslant 2$, let M_a^{2k} be the strongly connected symmetric digraph containing a cycle $(v_1, v_2, \ldots, v_{2k}, v_1)$ of length $2k$, v_{2i-1} having degree $2 + a$, v_{2i} having degree 2, and each neighbor of v_{2i-1} except v_{2i} having degree 1 [14]. By [14, Theorem 3.2] and [15, Theorem 2], if Γ is a strongly connected symmetric digraph with only one pair of cycles of length > 2 for which there are a positive number λ and a polynomial $f(x) \in \mathbb{R}(x)$ such that $f(A) = (A^2\mathbf{j} + \sqrt{\lambda}A\mathbf{j})(\mathbf{j}^\top A^2 + \sqrt{\lambda}\mathbf{j}^\top A)$, where $A = A(\Gamma)$, then $\Gamma \cong M_a^{2k}$ for some a and k.

7. Polynomial with at most two terms

In this section we restrict ourselves to the polynomials with at most two terms. The following theorem is an easy extension of [33, Theorem 1].

Theorem 7.1. *If for some $c \geqslant 1$, $c(x^k + 1)$ is the Hoffman polynomial of some digraph, then $c = 1$ and k is odd.*

Proof. Assume that A is the adjacency matrix of a digraph which has Hoffman polynomial $c(x^k + 1)$. Then $c(A^k + I) = T$ is a positive matrix with $\operatorname{rank} T = 1$ and $\operatorname{tr} T = n$, where n is the size of A. Since A^k is nonnegative, we deduce that $\frac{n}{c} - n = \operatorname{tr}(\frac{T}{c} - I) = \operatorname{tr} A^k \geqslant 0$, which is possible only if $c \leqslant 1$. But it is assumed that $c \geqslant 1$ and thus we get $c = 1$, as desired.

It remains to prove that k is odd. Assume otherwise, $k = 2h$ for some positive integer h. Observe that the eigenvalues of $T - I$ are $n - 1$, which is equal to λ_A^{2h}, with multiplicity 1 and -1 with multiplicity $n - 1$. Consequently, the eigenvalues of A^h are λ_A^h with multiplicity 1 and $\pm\sqrt{-1}$ with equal multiplicity. Since $\sqrt{-1}$ and $-\sqrt{-1}$ have to appear in pairs, this means that $\operatorname{tr} A^h = \lambda_A^h > 0$ and henceforth $\operatorname{tr} A^{2h} > 0$, contradicting with $\operatorname{tr} A^{2h} = \operatorname{tr}(T - I) = 0$. $\square$

Example 7.2. For $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, we have $H_A(x) = \frac{2}{5}x(x^2 + 1)$ and $\lambda_A = 2$.

Problem 7.3. Suppose l and k are two positive integers. Is there a $c > 1$ such that $cx^l(1 + x^k)$ is the Hoffman polynomial of some digraph?

Problem 7.4. Let $k > 1$ be an even number. Is $x^\ell(x^k + 1)$ the Hoffman polynomial of some digraph? Note that if $x^\ell(x^k + 1) = H_\Gamma(x)$ for an even integer k, then Γ cannot be regular [43, Theorem 2.1].

A *complete digraph* is a digraph whose adjacency matrix is $J - I$; while a *complete digraph with loops* is a digraph whose adjacency matrix is J.

Theorem 7.5. *If Γ is a symmetric digraph on n vertices such that between any two vertices a and b there are exactly $n + 1$ walks going from a to b whose length is either 1 or 2, then Γ is the complete digraph with loops.*

Proof. Let $A = A(\Gamma)$. Then what we know is that $A = A^\top$ and $A + A^2 = (n + 1)J_n$ and what we need to prove is $A = J_n$. First, we make use of Theorems 2.1 and 2.6

and Corollary 3.2 to find that A has constant line sum $n = \lambda_A$. Next, we deduce from $A = A^\top$ and $A + A^2 = (n+1)J_n$ that $A + AA^\top = (n+1)J_n$. Consequently, by looking at the diagonal of $(n+1)J_n$ we find that $\sum_{j=1}^{n}(A(i,j)^2 - A(i,j)) = \sum_{j=1}^{n} A(i,j)A(j,i) - \sum_{j=1}^{n} A(i,j) = (n+1-A(i,i)) - n \leqslant 1$, $i \in [n]$. But for a nonnegative integer x, $x^2 - x \leqslant 1$ can happen only if $x = 0$ or 1. Accordingly, we get that A is an $n \times n$ $(0, 1)$ matrix with constant line sum n and thus $A = J_n$ follows. □

8. Miscellaneous

We collect in this last section some observations concerning Hoffman polynomials.

Example 8.1. By Theorem 2.8, a digraph is strongly connected if and only if there is a positive matrix of rank 1 by which the matrix subalgebra spanned is an ideal of the adjacency algebra of Γ, i.e., the matrix algebra spanned by $A(\Gamma)$. Theorem 1.1 tells us that a digraph Γ is strongly connected and regular if and only if the matrix subalgebra spanned by J is an ideal of the adjacency algebra of Γ. Curtin discovers several interesting characterizations by ideal theoretic conditions for some generalization of distance-regular graphs and t-homogeneous graphs [12]. Fiol and Garriga [18] characterize locally pseudo-distance-regular graphs in terms of a vector subspace invariant under the multiplication of the adjacency matrix. It will be good to find some ideal theoretic characterization for digraphs with certain kind of regularity.

In the following, we make some observations on classifying digraphs according to their Hoffman polynomials. Note that in general, it is hard to determine the existence of solutions to a nonnegative integer matrix equation and to give the constructions of as many solutions as possible, not to mention classifying all solutions.

Example 8.2. A *strongly regular digraph* $srd(n; k, t, \ell, c; \lambda, \mu)$ is a k-regular digraph with n vertices such that for $A = A(\Gamma)$

$$AJ = JA = kJ, \quad A^2 + \frac{\mu - \lambda}{c}A + \left(\frac{\ell(\lambda - \mu)}{c} + \mu - t - \ell^2\right) I = \mu J.$$

When $\ell = 0$ and $c = 1$, an $srd(n; k, t, \ell, c; \lambda, \mu)$ coincides with the directed strongly regular graph $dsrg(n; k, t; \lambda, \mu)$ [16,24]; [36, p. 276]. In virtue of Theorem 2.4, if $\Gamma = srd(n; k, t, \ell, c; \lambda, \mu)$ does not have the complete digraph of order n as its subdigraph, then $H_\Gamma(x) = \frac{1}{\mu}\left(x^2 + \frac{\mu-\lambda}{c}x + \frac{\ell(\lambda-\mu)}{c} + \mu - t - \ell^2\right)$. Conversely, let Γ be a k-regular digraph whose adjacency matrix only has ℓ as its diagonal element and 0 and c as its off-diagonal elements. Then we can conclude from $H_\Gamma(x) = a_2x^2 + a_1x + a_0$, $a_2 \neq 0$, that $\Gamma = srd(n; k, t, \ell, c; \lambda, \mu)$, where $\mu = \frac{1}{a_2}$, $\lambda = \frac{1-a_1c}{a_2}$ and $t = \frac{1-a_0-a_1\ell}{a_2} - \ell^2$.

Example 8.3. A *strongly regular symmetric digraph* [32, p. 60] is a symmetric digraph without loops or multiple arcs such that the number of common neighbors of two vertices u and v only depends on whether or not $u = v$, u and v are adjacent, or u and v are nonadjacent. It is not hard to realize that if a quadratic polynomial is a Hoffman polynomial of some symmetric digraph without loops or multiple arcs, then this digraph must be strongly regular. This fact is used in the proof of [32, Theorem 4.11]. Conversely, with the exception of a union of several copies of a complete digraph, each strongly regular symmetric digraph has a quadratic polynomial as its Hoffman polynomial. Indeed, for a strongly regular symmetric digraph $srg(n, k, \lambda, \mu)$ [36, p. 263]; [32, Lemma 4.8], its adjacency matrix A fulfils the equations

$$AJ = kJ, \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

This means that if $\mu \neq 0$, namely the digraph is not a union of complete digraphs, its Hoffman polynomial must be $\frac{1}{\mu}(x^2 + (\mu - \lambda)x + \mu - k)$. We also mention that a symmetric regular digraph without loops and multiple arcs has exactly three distinct eigenvalues if and only if it is strongly regular and not a union of the complete digraphs [36, Problem 31.H]; [32, Theorem 4.11], i.e., if and only if its Hoffman polynomial is quadratic. Note that there do exist symmetric digraphs having exactly three distinct eigenvalues which are non-regular and hence surely not strongly regular [5,6,38].

Example 8.4. Hoffman indicates that a d-regular digraph with Hoffman polynomial $x + x^2$ must be the Kautz digraph $K(d, 2)$ [22]. Gimbert obtains the same result in [19]. Jørgensen extends the assertion of Hoffman by proving that there is a unique d-regular digraph with Hoffman polynomial $\frac{1}{t}x^2 + x$ for any positive integer t [29]. Wu and Li demonstrate that the Kautz digraph $K(d, n)$ is the only d-regular digraph which has $x^{n-1} + x^n$ as its Hoffman polynomial [43].

Example 8.5. A digraph of out-degree at most k and diameter at most d is a *Moore digraph* if the number of its vertices attains the Moore bound, i.e., $1 + k + k^2 + \cdots + k^d$. In other words, a Moore digraph is a digraph whose adjacency matrix A satisfies $I + A + \cdots + A^d = J$. It is known that the only Moore digraphs are the complete digraphs of $k + 1$ vertices or cycles of $d + 1$ vertices, corresponding to the case $d = 1$ and $k = 1$, respectively [7,39]. Clearly, a regular digraph is a Moore digraph if and only if it has a Hoffman polynomial of the form $x^{n-1} + \cdots + x + 1$.

A symmetric digraph of degree at most k and diameter at most d is called a (k, d) *Moore symmetric digraph* if the number of its vertices achieves the (undirected) Moore bound, namely $1 + k + k(k - 1) + \cdots + k(k - 1)^{d-1}$ [1]. The Moore symmetric digraphs of diameter 1 are just complete digraphs. A Moore symmetric digraph of diameter 2 can only have valence 2, 3, 7 or 57 [1]. It is known that the only Moore symmetric digraphs with parameters $(2, 2)$, $(3, 2)$ and $(7, 2)$ are the pentagon, the Peterson graph and the Hoffman–Singleton graph, respectively [3, Chapter 23]. However, the existence of a $(57, 2)$ Moore symmetric digraph remains an enigma.

The Moore symmetric digraphs of diameter $d \geqslant 3$ can only be the symmetric cycle of length $2d + 1$.

Example 8.6. A digraph is a $(k, 1)$ Moore symmetric digraph if and only if it is a loopless symmetric digraph with Hoffman polynomial $x + 1$. A digraph is a $(k, 2)$ Moore symmetric digraph if and only if it is a loopless regular symmetric digraph with Hoffman polynomial $x^2 + x - (k - 1)$.

Problem 8.7. Is it possible to characterize (k, d) Moore symmetric digraphs for $d \geqslant 3$ using their Hoffman polynomials together with a bit of some other parameters? Note that the Hoffman polynomial of the symmetric cycle of length $n = 2d + 1$ is $\prod_{i=1}^{d}(x - 2\cos(2\pi i/n))$. We also mention that the symmetric cycle of length n is characterized by its spectrum $\{2\cos(2\pi i/n) : i \in [n]\}$ [13, p. 72].

We now consider the Hoffman polynomials of vertex transitive digraphs.

Example 8.8. A *circulant digraph* is a Cayley digraph on a cyclic group. A classical result is that all vertex transitive digraphs of prime order are circulant digraphs [32, Theorem 7.7]. In addition, Lazarus [34, p. 115, Corollary] proves that the minimal polynomial of a circulant digraph of prime order either splits into linear factors or is a linear factor times one irreducible factor. This then tells us that the Hoffman polynomials of all vertex transitive digraphs of prime order either splits into linear factors or is irreducible.

Let us introduce two more results of Hoffman himself about Hoffman polynomials.

Example 8.9. Let Q_m be the m-dimensional cube. Hoffman proves that for $m = 2$ or 3, Q_m is the only symmetric digraph with 2^m vertices and Hoffman polynomial equal to $H_{Q_m}(x)$ [25, Theorem 2]. Hoffman also finds that there are only one symmetric digraph of 16 vertices besides Q_4 which has $H_{Q_4}(x)$ as its Hoffman polynomial.

Example 8.10. For any positive integer t, let Γ_t be the digraph with $V(\Gamma_t) = \{(i, j) : i, j \in \mathbb{Z}_t\}$ and arcs going from (i, j) to $(i, j + 1)$ and $(i + 1, j)$ for all $i, j \in \mathbb{Z}_t$. Hoffman proves that any digraph of order t^2 with Hoffman polynomial $H_{\Gamma_t}(x)$ for $t = 2, 4$ or an odd prime must be isomorphic to Γ_t [26, Theorem 2].

Our final example is due to Feng and Kwak.

Example 8.11. Ref. [17, Lemma 2] asserts that a symmetric digraph of order $2(k + 1)$ with Hoffman polynomial $(x^3 + kx^2 - x - k)/(k^2 - k)$ must be the complement of the $2 \times (k + 1)$-grid, i.e., the complete bipartite symmetric digraph $K_{k+1,k+1}$ with one perfect symmetric matching deleted.

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