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Representing the sporadic Archimedean polyhedra as abstract polytopes[☆]

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ABSTRACT

We present the results of an investigation into the representations of Archimedean polyhedra (those polyhedra containing only one type of vertex figure) as quotients of regular abstract polytopes. Two methods of generating these presentations are discussed, one of which may be applied in a general setting, and another which makes use of a regular polytope with the same automorphism group as the desired quotient. Representations of the 14 sporadic Archimedean polyhedra (including the pseudorhombicuboctahedron) as quotients of regular abstract polyhedra are obtained, and summarised in a table. The information is used to characterize which of these polyhedra have acoptic Petrie schemes (that is, have well-defined Petrie duals).

1. Introduction

Much of the focus in the study of abstract polytopes has been on the regular abstract polytopes. A publication of the first author [6] introduced a method for representing any abstract polytope as a quotient of regular polytopes. In the current work we present the application of this technique to the familiar, but still interesting, Archimedean polyhedra and discuss implications for the general theory of such representations that arose in trying to systematically develop these representations. We discuss the theory and presentations of the thirteen classical (uniform) Archimedean polyhedra as well as the pseudorhombicuboctahedron, which we will refer to as the fourteen sporadic Archimedean polyhedra. In a separate study, we will present and discuss the presentations for the two infinite families of uniform convex polyhedra, the prisms and antiprisms.

1.1. Outline of topics

Section 2 reviews the structure of abstract polytopes and their representation as quotients of regular polytopes and discusses two new results on the structure of the quotient representations of abstract polytopes. Section 3 describes a simple method for developing a quotient presentation for a polyhedron from a description of its faces. In Section 4 we discuss an alternative method of developing a quotient presentation for polytopes that takes advantage of the structure of

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its automorphism group, and in Section 5 we develop this method more fully for the specific polyhedra under study here. Finally, in Section 6 we discuss an example of how these quotient representations may be used to answer questions about their structure computationally and in Section 7 we present some of the open questions inspired by the current work.

2. Abstract polytopes and quotient presentations

To place the current work in the appropriate context we must first review the structure of abstract polytopes and the central results from the first author's [6] for representing any polytope as a quotient of regular abstract polytopes.

An *abstract polytope* $\mathcal{P}$ *of rank* d (or d*-polytope*) is a graded poset with additional constraints chosen so as to generalize combinatorial properties of the face lattice of a convex polytope. Elements of these posets are referred to as *faces*, and a face F is said to be *contained* in a face G if $F < G$ in the poset. One consequence of this historical connection to convex polytopes is that contrary to the usual convention for graded posets, the rank function ρ maps $\mathcal{P}$ to the set $\{-1, 0, 1, 2, \ldots, d\}$ so that the minimal face has rank -1, but otherwise satisfies the usual conditions of a rank function. A face at rank i is an i*-face*. A face F is *incident* to a face G if either $F < G$ or $G < F$. A *proper face* is any face which is not a maximal or minimal face of the poset. A *flag* is any maximal chain in the poset, and the *length* of a chain C we define to be $|C| - 1$. Following [10] we will require that the poset $\mathcal{P}$ also possess the following four properties:

P1 $\mathcal{P}$ contains a least face and a greatest face, denoted F_{-1} and F_d respectively;
P2 Every flag of $\mathcal{P}$ is of length $d + 1$;
P3 $\mathcal{P}$ is strongly connected;
P4 For each $i = 0, 1, \ldots, d - 1$, if F and G are incident faces of $\mathcal{P}$, and the ranks of F and G are $i - 1$ and $i + 1$ respectively, then there exist precisely two i-faces H of $\mathcal{P}$ such that $F < H < G$.

Note that an abstract polytope is *connected* if either $d \leq 1$, or $d \geq 2$ and for any two proper faces F and G of $\mathcal{P}$ there exists a finite sequence of incident proper faces $J_0, J_1, \ldots, J_m$ such that $F = J_0$ and $G = J_m$. A polytope is *strongly connected* if every section of the polytope is connected, where a *section* corresponding to the faces H and K is the set $H/K := \{F \in \mathcal{P} \mid H < F < K\}$. Some texts are more concerned with the notion of *flag connectivity*. Two flags are *adjacent* if they differ by only a single face. A poset is *flag-connected* if for each pair of flags there exists a sequence of adjacent flags connecting them, and a poset is *strongly flag-connected* if this property holds for every section of the poset. It has been shown [10] that for any poset with properties P1 and P2, being strongly connected is equivalent to being strongly flag-connected. A polytope is said to be *regular* if its automorphism group $\mathrm{Aut}(\mathcal{P})$ acts transitively on the set $\mathcal{F}(\mathcal{P})$ of its flags.

To understand what follows, a basic understanding of the structure of string C-groups is necessary, so we will review the essential definitions here. A *C-group* W is a group generated by a set of (distinct) involutions $S = \{s_0, s_1, \ldots, s_{n-1}\}$ such that $\langle s_i | i \in I\rangle \cap \langle s_j | j \in J\rangle = \langle s_i | i \in I \cap J\rangle$ for all I, J (the so-called *intersection property*). Coxeter groups are the most famous examples of C-groups (see [9,10]). A C-group is a *string* C-group if $(s_i s_j)^2 = 1$ for all $|i - j| > 1$. An important result in the theory of abstract polytopes is that the regular polytopes are in one-to-one correspondence with the string C-groups, in particular, that the automorphism group of any regular abstract polytope is a string C-group and that from every string C-group W a unique regular polytope $\mathcal{P}(W)$ may be constructed whose automorphism group is W [10]. Given a C-group W and a polytope $\mathcal{Q}$ (not necessarily related to $\mathcal{P}$), we may attempt to define an action of W on $\mathcal{F}(\mathcal{Q})$ as follows. For any flag Φ of $\mathcal{Q}$, let Φ^{s_i} be the unique flag differing from Φ only by the element at rank i. If this extends to a well-defined action of W on $\mathcal{F}(\mathcal{Q})$, it is called the *flag action* of W on (the flags of) $\mathcal{Q}$. The flag action should not be confused with the natural action of the automorphism group W of a regular polytope $\mathcal{Q}$ on its flags. As noted in [6], it is always possible to find a C-group acting on a given abstract polytope $\mathcal{Q}$ (regular or not) via the flag action.

We consider now the representation of abstract polytopes first presented as Theorem 5.3 of [6].

Theorem 2.1. *Let $\mathcal{Q}$ be an abstract n-polytope, W any string C-group acting on the flags of $\mathcal{Q}$ via the flag action and $\mathcal{P}(W)$ the regular polytope with automorphism group W. If we select any flag Φ as the base flag of $\mathcal{Q}$ and let $N = \{a \in W \mid \Phi^a = \Phi\}$, then $\mathcal{Q}$ is isomorphic to $\mathcal{P}(W)/N$. Moreover, two polytopes are isomorphic if and only if they are quotients $\mathcal{P}(W)/N$ and $\mathcal{P}(W)/N'$ where N and N' are conjugate subgroups of W.*

An interesting fact about these presentations that does not seem to appear explicitly elsewhere in the literature is that there is a strong relationship between the number of transitivity classes of flags under the automorphism group in the polytope and the number of conjugates of the stabilizer subgroup N. This relationship is formalized as follows.

Theorem 2.2. *The number of transitivity classes of flags under the automorphism group in a polytope $\mathcal{Q}$ is equal to the number of conjugates in W of the stabilizer subgroup N for any choice of base flag Φ in its quotient presentation, that is, $|W : \mathrm{Norm}_W(N)|$.*

Proof. Let Φ and Φ' be two flags of a polytope $\mathcal{Q}$, let W be a string C-group acting on $\mathcal{Q}$, and let $\mathcal{P}$ be the regular polytope whose automorphism group is W (so $\mathcal{P} = \mathcal{P}(W)$). Let N be the stabilizer of Φ in W, and let N' be the stabilizer of Φ' in W. Let $\Phi' = \Phi^u$, so that $N' = N^u$. Let ψ be an automorphism of $\mathcal{Q}$ with $\Phi\psi = \Phi'$, and suppose $n \in N$. Observe then that

$$\begin{aligned}(\Phi')^n &= (\Phi\psi)^n && \text{by the definition of } \psi\\ &= (\Phi^n)\psi && \text{by Lemma 4.1 of [6]}\end{aligned}$$

$= (\Phi)\psi$ since $n \in N$, the stabilizer of Φ
$= \Phi'$ by the definition of ψ.

Therefore, $n \in N'$, so $N = N'$.

Conversely, let $N = N'$. Then, a map from $\mathcal{P}/N$ to $\mathcal{P}/N'$ may be constructed as in the proof of Theorem 5.3 of [6] (our Theorem 2.1), which does indeed map Φ to Φ'. □

Theorem 2.1 does not provide much guidance on finding an efficient (i.e. small) presentation for a given polytope. In particular, it is interesting to try to determine what the smallest regular polytope is that may be used as a cover of a given polytope under the flag action of the automorphism group of the regular polytope. Let Core(W, N) be the subgroup of N obtained as $\bigcap_{w\in W} N^w$, in other words, the largest normal subgroup of W in N.

Theorem 2.3. *Let $\mathcal{P}(W/\mathrm{Core}(W, N))$ be a well-defined regular polytope, and $\mathcal{R}$ any other regular cover of $\mathcal{P}(W)/N$ whose automorphism group acts on $\mathcal{P}(W)/N$ via the flag action, and on which W acts likewise. Then $\mathcal{R}$ also covers $\mathcal{P}(W/\mathrm{Core}(W, N))$.*

Proof. Let $\mathcal{R} = \mathcal{P}(W)/K = \mathcal{P}(W/K)$ be a regular cover for $\mathcal{P}(W)/N$. Then the flag action of W/K on $\mathcal{P}(W)/N$ is well defined; that is, for any $w \in W$ and any flag Φ of $\mathcal{P}(W)/N$, we have Φ^{wK} is well defined, because Φ^{wk} independent of the choice of k in K, but depends only on w. It follows that for all $k \in K$, any $w \in W$, and any flag Φ of $\mathcal{P}(W)/N$, we have $(\Phi^{wk})^{w^{-1}} = \Phi$, so, $wkw^{-1} \in N$. Therefore, $k \in N^w$ for all $w \in W$, so $k \in \mathrm{Core}(W, N)$. □

Now, Theorem 3.4 of [7] states that

$$\Gamma(\mathcal{P}(W)/N) \cong W/\mathrm{Core}(W, N),$$

where $\Gamma(\mathcal{P}(W)/N)$ is the image of the homomorphism induced by the flag action from W into $Sym(Flags(\mathcal{P}(W)/N))$. In the case that $\mathcal{P}(W)/N$ is a finite polytope, so that N has finite index in W, it follows that Core(W, N) is a finite index normal subgroup of W. This is because W acts on the finitely many right cosets of N via right multiplication, leading to a homomorphism from W to $\Sigma = Sym(|W : N|)$. The kernel of this homomorphism is Core(W, N), and thus $W/\mathrm{Core}(W, N)$ is isomorphic to a subgroup of the finite group Σ. Hence, a finite polytope always has a finite regular cover if $W/\mathrm{Core}(W, N)$ is a C-group. No proof that $W/\mathrm{Core}(W, N)$ is indeed a C-group has yet been published.

Barry Monson notes [11] that there exist quotients $\mathcal{Q} = \mathcal{P}/N$ of a polytope $\mathcal{P}$, for which the flag action of the automorphism group W of $\mathcal{P}$ on $\mathcal{Q}$ is not well defined. The theory of such exceptional quotients is not well developed. This article therefore concerns itself exclusively with quotients of $\mathcal{P}$ on which the flag action of Aut$(\mathcal{P})$ is well defined.

3. An example in detail

From Theorem 2.1 we learn that any given polytope $\mathcal{Q}$ admits a presentation as the quotient of a regular polytope. To find such a presentation we must first identify a string C-group W acting on the flags of $\mathcal{Q}$ via the flag action, and then having selected a base flag $\Phi \in \mathcal{Q}$, we must identify the stabilizer of Φ in W. To illustrate the mechanics of this process we will consider here the case of the cuboctahedron. As in [4] we will associate to each uniform or Archimedean polyhedron a symbol of type $p_1.p_2...p_k$, which specifies an oriented cyclic sequence of the number of sides of the faces surrounding each vertex. For example, 3.4.3.4 designates the cuboctahedron, which is an isogonal polyhedron with a triangle, a square, a triangle and a square about each vertex in that cyclic order. Fig. 1 shows the corresponding graph of the one-skeleton of the cuboctahedron.

First we select as our C-group the group $W = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^{12} = (bc)^4 = e\rangle$, where e is the identity. For ease of notation we write a, b, c instead of s_0, s_1, s_2, respectively. In general, one possible choice of the string C-group acting on a 3-polytope is the group $W = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^j = (bc)^k = e\rangle$, where j is the least common multiple of the number of sides of polygons in $\mathcal{Q}$ and k is the least common multiple of the degrees of the vertices of $\mathcal{Q}$. Here the action of the generators a, b or c on a flag F of $\mathcal{Q}$ yields the adjacent flag differing from F only by the vertex, edge or face, respectively.

For our choice of base flag in this example we select a flag Φ on a square face (in our diagram this corresponds to the outside face), and we mark it with a solid black flag. Construction of the stabilizer subgroup N of Φ in W is a bit more involved. For each of the faces of $\mathcal{Q}$ we may construct a sequence of consecutively adjacent flags starting at the base flag, going out to the face, forming a circuit of the edges and vertices of the face, and returning to the base flag. Each of these flags may be obtained from Φ via the flag action of W on Φ; for example, the flag marked with a ① is obtained from Φ via the action of the generator c of W. Starting at flag ⑨, a complete circuit of the face N is obtained from flag ⑨ by application of the element $(ab)^4 \in W$. Thus the group element corresponding to starting at the base flag and traversing the face marked N and returning is $((ab)^4)^{cbacbacbc}$.

Let N be the group in W generated by

$$\big\{(ab)^4, ((ab)^3)^c, ((ab)^4)^{cbabc}, ((ab)^3)^{cba}, ((ab)^4)^{cbcabab}, ((ab)^3)^{cbab}, ((ab)^4)^{cbacb}, ((ab)^3)^{cb}, ((ab)^4)^{cbc}, ((ab)^3)^{cbcabc}, ((ab)^3)^{cbcabcba}, ((ab)^3)^{cbcabcabab}, ((ab)^3)^{cbabacbc}, ((ab)^4)^{cbacbacbc}\big\}. \quad (1)$$

The generators in (1) correspond to faces A through N in Fig. 1 in that order.

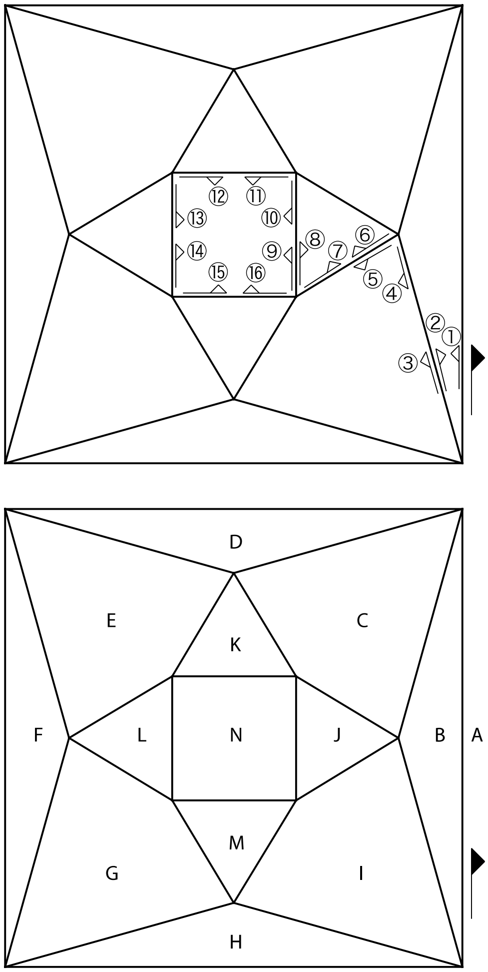

Fig. 1. At the top is pictured the cuboctahedron with a sequence of labeled flags used in the construction of the stabilizer subgroup of the base flag indicated in black. On the bottom is the same figure with labels indicated for each of the faces of the cuboctahedron.

Note that in general, finding elements of W that, as above, traverse each face of $\mathcal{Q}$ may only suffice to generate a proper subgroup of N. Inspection of $\mathcal{Q}$ should then reveal other elements of W that stabilize Φ —these can then be added to the generating set for N. In the example here, however, the elements listed do indeed generate the whole of the base flag stabilizer N. Then by Theorem 2.1 the cuboctahedron $\mathcal{Q}$ is isomorphic to $\mathcal{P}(W)/N$.

4. Representation via isomorphism

In the context of the current work, an important observation is that the automorphism group of a polyhedron is often shared with a better understood regular polytope. For example, the automorphism group of the cuboctahedron is that of the cube. It turns out that the quotient presentation can be characterized with the help of the symmetry group of the associated regular polytope. Again, we let $\mathcal{P}$ be a regular n-polytope, with automorphism group W. Let $\mathcal{Q}$ be a quotient $\mathcal{P}/N$ of $\mathcal{P}$ (not necessarily regular) admitting the flag action by W with Ψ a base flag for $\mathcal{Q}$ chosen so that N is the stabilizer for Ψ, and let $\mathcal{R}$ be a regular d-polytope whose automorphism group is isomorphic to $\mathrm{Aut}(Q)$. Note that we do not assume that $d = n$. Let $\mathrm{Aut}(\mathcal{R}) = \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \rangle$. Let ϕ be an isomorphism from $\mathrm{Aut}(\mathcal{R})$ to $\mathrm{Aut}(\mathcal{P}/N)$.

Let Φ be a flag of $\mathcal{R}$, and Ψ a base flag of $\mathcal{Q} = \mathcal{P}/N$, stabilized by N under the flag action. For each ρ_i, let ν_i be an element of W that maps Ψ to $\Psi(\rho_i\phi)$ under the flag action, that is, $\Psi^{\nu_i} = \Psi(\rho_i\phi)$. Let V be the subgroup of W generated by the ν_i. Finally, define a map ψ taking words w in the generators of $\mathrm{Aut}(\mathcal{R})$ to the group W, via $w\psi = (\rho_{i_1}\rho_{i_2}\ldots\rho_{i_k})\psi = \nu_{i_k}\ldots\nu_{i_2}\nu_{i_1}$. Note that the action of ψ reverses the order of the generators.

The following result goes a long way towards characterizing N in terms of $\mathrm{Aut}(\mathcal{R})$.

Theorem 4.1. *The set $N \cap V$ is the set of all images $w\psi$ of words w in the ρ_i such that $w = 1$ as an element of* $\mathrm{Aut}(\mathcal{R})$.

Proof. Note that $\rho_{i_1}\ldots\rho_{i_k} = 1$ in $\mathrm{Aut}(\mathcal{R})$ if and only if $\Psi((\rho_{i_1}\ldots\rho_{i_k})\phi) = \Psi$. This will be so if and only if $\Psi(\rho_{i_1}\phi)\ldots(\rho_{i_k}\phi) = \Psi$. Since the flag action commutes with the action of the automorphism group (Lemma 4.1 of [6]), we have

$$\begin{aligned}\left(\Psi(\rho_{i_j}\phi)\dots(\rho_{i_k}\phi)\right)^{\nu_{i_{j-1}}\cdots\nu_{i_1}} &= \left(\Psi^{\nu_{i_j}}(\rho_{i_{j+1}}\phi)\dots(\rho_{i_k}\phi)\right)^{\nu_{i_{j-1}}\cdots\nu_{i_1}} \\ &= \left(\Psi(\rho_{i_{j+1}}\phi)\dots(\rho_{i_k}\phi)\right)^{\nu_{i_j}\cdots\nu_{i_1}}.\end{aligned}$$

Thus, $\Psi(\rho_{i_1}\phi)\dots(\rho_{i_k}\phi) = \Psi$ if and only if $\Psi^{\nu_{i_k}\cdots\nu_{i_1}} = \Psi$, that is, if and only if $\nu_{i_k}\dots\nu_{i_1} = w\psi \in N$. This completes the proof. □

So the elements of $N \cap V$ have been characterized. To characterize the whole of N, it is sufficient to characterize elements of $N \cap V\mu$, for arbitrary cosets $V\mu$ of V in W. This is not as difficult as it may seem. Note that if $\mu \in N$, then $N \cap V\mu = (N \cap V)\mu$.

Theorem 4.2. *Let T be a right transversal of V in W, such that for all $\mu \in T$, if $N \cap V\mu \neq \emptyset$, then $\mu \in N$. Then*

$$N = \bigcup_{\mu \in N\cap T} \{(w\psi)\mu : w = 1 \text{ in } \mathrm{Aut}(\mathcal{R})\}.$$

Proof. For any right transversal T of V in W,

$$N = N \cap W = N \cap \left(\bigcup_{\mu\in T} V\mu\right) = \bigcup_{\mu\in T} (N \cap V\mu).$$

For the transversal chosen here, $N \cap V\mu$ is empty unless $\mu \in N$, whence also $N \cap V\mu = (N \cap V)\mu$. It follows that

$$N = \bigcup_{\mu\in N\cap T} ((N \cap V)\mu),$$

which by Theorem 4.1 is

$$N = \bigcup_{\mu\in N\cap T} \{(w\psi)\mu : w = 1 \text{ in } \mathrm{Aut}(\mathcal{R})\}$$

as desired. □

This gives a characterization of the elements of N, in terms of the elements of $\mathrm{Aut}(\mathcal{R})$, the map ϕ, and the transversal T.

Theorems 4.1 and 4.2 are particularly useful for the purposes of this article, since every uniform sporadic Archimedean solid has an automorphism group that is also the automorphism group of a regular polytope $\mathcal{R}$. In most cases, the choice of $\mathcal{R}$ is obvious — it will be the underlying platonic solid. The snub cube and snub dodecahedron have as automorphism group the *rotation* group of the cube and dodecahedron respectively, not the full automorphism groups. However, these rotation groups are isomorphic (respectively) to the automorphism groups of the hemi-cube $\{4, 3\}_3$ and the hemi-dodecahedron $\{5, 3\}_5$, so these theorems may still be applied.

In the following sections, Theorems 4.1 and 4.2 are used to construct each of the Archimedean solids as a quotient $\mathcal{P}/N$ of some regular polytope $\mathcal{P}$ by a subgroup N of its automorphism group. The steps in construction are as follows.

1. Find a polytope $\mathcal{P}$ that is known to cover the desired Archimedean solid.
2. Identify, using Theorems 4.1 and 4.2, a subset S of N.
3. Prove, or computationally verify, that S generates a subgroup of $\mathrm{Aut}(\mathcal{P})$ whose index is the same as the (known) index of N.
4. Finally, use Theorem 2.3 to find a minimal regular cover $\mathcal{P}/\mathrm{Core}(\mathrm{Aut}(\mathcal{P}), N)$ for the Archimedean solid $\mathcal{P}/N$.

The index of N in $\mathrm{Aut}(\mathcal{P})$ is known, from Theorem 2.5 of [7], to be just the number of flags of the quotient $\mathcal{P}/N$, which is easy to compute. Indeed, the Archimedean solid with symbol $p_1.p_2\dots p_k$ has exactly $2k$ flags at every vertex.

5. Isomorphism for geometric operations

From a combinatorial – but not geometric – standpoint, each of the uniform sporadic Archimedean polyhedra may be constructed from a Platonic solid by (possibly repeated) application of either truncation, full truncation, rhombification or snubbing. By Theorem 4.1, we may construct quotient presentations for these polyhedra by determining the appropriate choices for the $\nu_i \in V$ that correspond to these operations. Let us now carefully define what each of these operations does. Geometrically, *truncation* (t) cuts off each of the vertices of the polyhedron, replacing them with the corresponding vertex figure as a facet. *Full truncation* (ft) performs essentially the same operation, but the cut is taken deeper so that new facets share a vertex if the corresponding vertices shared an edge, and all of the original edges are replaced with single vertices. *Rhombification* (r) is a little more complicated geometrically, but from a combinatorial standpoint is equivalent to applying full truncation twice (the difficulty is in getting the new facets to be geometrically regular). Finally, to construct the *snub* of a polyhedron requires first constructing the rhombification, and then triangulating the squares generated by the second full truncation in such a way as to preserve the rotational symmetries of the figure (in Fig. 2 the triangulation step is indicated

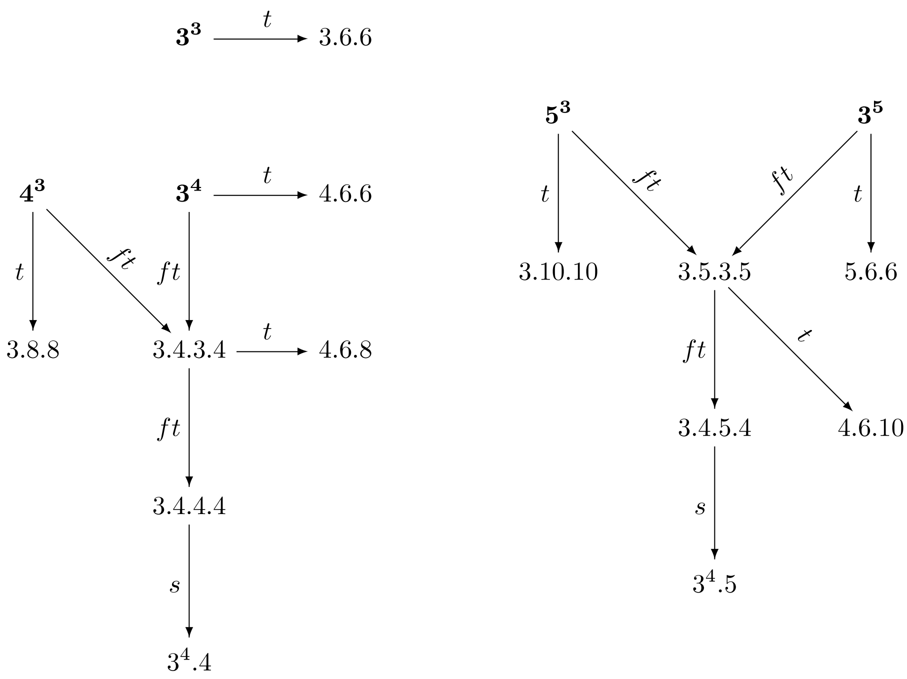

Fig. 2. The construction of the sporadic uniform Archimedean polyhedra from the Platonic solids.

by *s*). The ways in which each of the sporadic uniform Archimedean polyhedra may be obtained (hierarchically) from the Platonic solids via these operations is given in Fig. 2. Note for instance that 4^3 abbreviates the symbol 4.4.4 for the cube. More information on these, and other, operations on the maps associated with polyhedra is available in [12].

5.1. Generators of V

For the convenience of the reader, we present here the morphisms ψ from the words in the generators of the symmetry groups of the regular polyhedra $\mathcal{R}$ into the symmetry groups of the regular covers $\mathcal{P}$ of the quotient polytopes $\mathcal{Q}$ that provide the generators for the subgroup V of Theorems 4.1 and 4.2. It is also important to note that different morphisms (and corresponding sets of generators) arise if one makes different choices for the base flag in the quotient polytope than those made here, and that ν_0 and ν_2 may be interchanged by using the dual choice for the polytope $\mathcal{R}$ (where possible and appropriate). The map ψ in each case is determined by its action on the generators of $\mathrm{Aut}(\mathcal{R})$, denoted ρ_0, ρ_1 and ρ_2, in terms of the generators of $W = \mathcal{P} = \langle a, b, c\rangle$ in the usual way.

5.1.1. Truncation

There are five Archimedean polyhedra obtained by truncation of each of the Platonic solids, namely, the truncated tetrahedron, cube, octahedron, icosahedron and dodecahedron. In each instance the vertex star contains either two hexagons, two octagons or two decagons. Here we choose as a base flag Ψ on one of those hexagons, octagons or decagons whose edge is shared with another polygon of the same type. Thus

$$\rho_0\psi = \nu_0 = a,$$
$$\rho_1\psi = \nu_1 = bab$$
$$\rho_2\psi = \nu_2 = c,$$

so $V = \langle a, bab, c\rangle$.

5.1.2. Full truncation

Full truncation provides derivations for two of the Archimedean polyhedra, the cuboctahedron and the icosidodecahedron. We have chosen to perform full truncation to the cube and the dodecahedron, respectively, and our base flags on square or pentagonal faces respectively. Thus

$$\rho_0\psi = \nu_0 = b,$$
$$\rho_1\psi = \nu_1 = a,$$
$$\rho_2\psi = \nu_2 = cbc,$$

so $V = \langle b, a, cbc\rangle$.

5.1.3. Rhombification

There are two Archimedean polyhedra obtained by rhombification, the small rhombicuboctahedron and the small rhombicosidodecahedron. Here we begin with the cube and dodecahedron, respectively, and our base flag lies on an edge of a square or pentagonal face shared with the square face introduced by the second full truncation. Thus

$$\begin{aligned}\rho_0\psi &= \nu_0 = a,\\ \rho_1\psi &= \nu_1 = b,\\ \rho_2\psi &= \nu_2 = cbabc,\end{aligned}$$

so $V = \langle a, b, cbabc\rangle$. While it is true that the octahedron may be obtained by full truncation from the tetrahedron (and so the cuboctahedron may be obtained by rhombification of the tetrahedron), the maps given do not provide an isomorphism since the symmetry group of the octahedron, and hence the cuboctahedron, is larger than that of the tetrahedron.

5.1.4. Truncation of full truncation

There are two Archimedean polyhedra obtained in this way, the great rhombicuboctahedron and the great rhombicosidodecahedron. Here we begin with a cube and a dodecahedron, respectively, and our base flag lies on either an octagonal or decagonal face with an edge shared with a square. Thus

$$\begin{aligned}\rho_0\psi &= \nu_0 = a,\\ \rho_1\psi &= \nu_1 = bab,\\ \rho_2\psi &= \nu_2 = cbabc\end{aligned}$$

and so $V = \langle a, bab, cbabc\rangle$.

5.1.5. Snubbing

There are two Archimedean polyhedra obtained by the snubbing operation, the snub cube and the snub dodecahedron. For the presentation given below for V, we have chosen to start with the hemi-cube and the hemi-dodecahedron, respectively. These regular polyhedra are non-orientable, so the group of $\mathcal{R}$ is coincides with its rotation subgroup, and we need only consider the generators of this group in determining V. In each case the base flag lies on either a square or pentagonal face.

$$\begin{aligned}\rho_1\rho_0\psi &= \nu_0\nu_1 = ab,\\ \rho_2\rho_1\psi &= \nu_1\nu_2 = bcbabcbc\end{aligned}$$

so $V = \langle ab, bcbabcbc\rangle$.

5.2. The cuboctahedron

To better understand how this works in practice, let us return to the example of the cuboctahedron, conceived as the full truncation of the cube. In this case $\mathrm{Aut}(\mathcal{R}) = \langle s, t, u \mid s^2 = t^2 = u^2 = (su)^2 = (st)^4 = (tu)^3\rangle$, and $V = \langle b, a, cbc\rangle < W$ (this W was defined in Section 3). By Theorem 4.1 (and Theorem 4.2 if necessary), if we can find a set of words in the generators s, t, u of $\mathrm{Aut}(\mathcal{R})$ that are equivalent to the identity in $\mathrm{Aut}(\mathcal{R})$ and whose images generate a group of the appropriate index (in this case 96) in W, then we will have found the necessary subgroup of W for use in the quotient presentation of the cuboctahedron. Recall that if ϕ is the isomorphism from $\mathrm{Aut}(\mathcal{R})$ to $\mathrm{Aut}(\mathcal{P}/N)$, and ψ the associated map from $\mathrm{Aut}(\mathcal{R})$ to W, then $s\psi = b$, $t\psi = a$ and $u\psi = cbc$; using this map we generate the list of words given below in Eq. (2), which satisfies the conditions of Theorem 4.1:

$$\begin{aligned}&\big\{(st)^4, (ut)^3, ((st)^4)^{utu}, ((ut)^3)^{st}, ((st)^4)^{utsts}, ((ut)^3)^{sts}, ((st)^4)^{uts}, ((ut)^3)^{s}, ((st)^4)^{u}, ((ut)^3)^{stu},\\ &\quad ((ut)^3)^{stus}, ((ut)^3)^{stutsts}, ((ut)^3)^{ststu}, ((st)^4)^{utstu}\big\}.\end{aligned} \tag{2}$$

Each of the terms in Eq. (2) corresponds to either a circuit of one of the square faces of the cube, or to a traversal of one of the vertex stars of the cube (starting at, and returning to a chosen base flag), and so clearly is equivalent to 1 in $\mathrm{Aut}(\mathcal{R})$. By Theorem 4.1, if we apply ψ to each of these terms we obtain an element of the subgroup N required to construct a quotient representation under the flag action of W. Conveniently, in this example each of the terms in Eq. (2) corresponds to one of the generators given in Eq. (1) and are listed in the same order. To see this, consider for example the sixth item on the list, $((ut)^3)^{sts}$. When we apply the map ψ, we see that

$$\begin{aligned}((ut)^3)^{sts}\psi &= (stsutututsts)\psi\\ &= babacbcacbcacbcbab \quad \text{(by definition of } \psi)\\ &= bab\mathbf{ca}b\mathbf{ac}cb\mathbf{ac}cbcbab \quad \text{(by commutativity of } a \text{ and } c \text{ in } W)\\ &= babcababab cbab = ((ab)^3)^{cbab} \quad (\text{since } c^2 = 1)\end{aligned}$$

as was desired.

Table 1
This summarizes the representations of the Archimedean solids as quotients of abstract regular polytopes $\mathcal{P} = \mathcal{P}(W)$. These $\mathcal{P}$ are the minimal regular polytopes whose automorphism groups act on the Archimedean solids via the flag action.

Polytope	Vertex figure	Schläfli type of $\mathcal{P}(W)$	$\|W\|$	$\|N\|$
Trunc. tetrahedron	3.6.6	{6, 3}	144	2
Trunc. octahedron	4.6.6	{8, 3}	6912	48
Cuboctahedron	3.4.3.4	{12, 4}	2304	24
Trunc. cube	3.8.8	{24, 3}	82944	576
Icosadodecahedron	3.5.3.5	{15, 4}	14400	120
Trunc. icosahedron	5.6.6	{30, 3}	2592000	7200
Sm. rhombicuboctahedron	3.4.4.4	{12, 4}	1327104	6912
Pseudorhombicuboctahedron	3.4.4.4	{12, 4}	$2^{35}3^55^27 \cdot 11$	$2^{29}3^45^27 \cdot 11$
Snub cube	3.3.3.3.4	{12, 5}	$2^{32}3^{11}5^1$	$2^{28}3^{10}$
Sm. rhombicosidodecahedron	3.4.5.4	{60, 4}	207360000	432000
Gt. rhombicosidodecahedron	4.6.10	{60, 3}	559872000000	777600000
Snub dodecahedron	3.3.3.3.5	{15, 5}	$2^{23}3^{11}5^{11}$	$2^{20}3^{10}5^9$
Trunc. dodecahedron	3.10.10	{30, 3}	2592000	7200
Gt. rhombicuboctahedron	4.6.4.8	{24, 4}	5308416	18432

We conclude this discussion with the results of constructing such presentations for each of the sporadic uniform Archimedean solids.

Theorem 5.1. *Each of the sporadic uniform Archimedean solids has a finite regular cover whose automorphism group acts on the Archimedean solid via the flag action. Moreover, the regular covers are minimal in this sense, as detailed in Table 1.*

The minimal cover of the truncated tetrahedron is in fact $\{6, 3\}_{(2,2)}$. That the latter covers the truncated tetrahedron was noted in [8], but it was not shown to be a minimal cover.

6. Analysis of presentations

Having obtained a quotient presentation, there are a variety of questions that one may now ask about the structure of the presentation, both algebraically and combinatorially, that may be approached by algebraic methods.

6.1. Acoptic Petrie schemes

One such question is the determination of whether or not the given polytope has acoptic Petrie schemes[1], a question related to understanding under what conditions a polyhedron will have Petrie polygons that form simple closed curves. First, we require some definitions; we will follow the second author's [13]. A *Petrie polygon* of a polyhedron is a sequence of edges of the polyhedron where any two consecutive elements of the sequence have a vertex and face in common, but no three consecutive edges share a common face. For the regular polyhedra, the Petrie polygons form the equatorial skew polygons. The definition of a Petrie polygon may be extended to polytopes of rank $n > 3$ as well. An *exchange map* ϱ_i is a map on the flags of the (abstract or geometric) polytope sending each flag Φ to the unique flag that differs from it only by the element at rank i (this corresponds to earlier discussion of flag action for a suitable Coxeter group). A *Petrie map* σ of a polytope $\mathcal{Q}$ of rank d is any composition of the exchange maps $\{\varrho_0, \varrho_1, \ldots, \varrho_{d-1}\}$ on the flags of $\mathcal{Q}$ in which each of these maps appears exactly once. For example, the map $\sigma = \varrho_{d-1}\varrho_{d-2}\ldots\varrho_2\varrho_1\varrho_0$ is a Petrie map. In particular, suppose $\mathcal{Q} \simeq \mathcal{P}(W)/N$ admits a flag action by the string C-group W. Then the flag action of a Coxeter element in W, such as $s_n \ldots s_1 s_0$, on a given flag in $\mathcal{Q}$ is a Petrie map.

Definition 6.1. A *Petrie sequence* of an abstract polytope is an infinite sequence of flags which may be written in the form $(\ldots, \Phi\sigma^{-1}, \Phi, \Phi\sigma, \Phi\sigma^2, \ldots)$, where σ is a fixed Petrie map and Φ is a flag of the polytope.

Definition 6.2. A *Petrie scheme* is the shortest possible listing of the elements of a Petrie sequence. If a Petrie sequence of an abstract polytope contains repeating cycles of elements, then the Petrie scheme is the shortest possible cycle presentation of that sequence. Otherwise, the Petrie scheme is the Petrie sequence.

For example, there is no finite presentation for a Petrie scheme of the regular tiling of the plane by squares, but while any Petrie sequence of a tetrahedron is infinitely long, any of its Petrie schemes has only four elements (and we consider cyclic permutations of a Petrie scheme to be equivalent).

A polytope possesses *acoptic Petrie schemes* if each proper face appears at most once in each Petrie scheme. We borrow this terminology from Branko Grünbaum who coined the term acoptic (from the Greek $\kappa o\pi\tau\omega$, to cut) to describe polyhedral surfaces with no self-intersections (cf. [1–3,13]). Let $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ be the collection of *distinct* Coxeter elements in W

[1] Such polytopes are referred to as *Petrial* polytopes in [13].

Table 2
The ranks at which the Archimedean polyhedra have acoptic Petrie schemes.

Polyhedron	Acoptic Ranks
Cuboctahedron	{0, 1, 2}
Great rhombicosidodecahedron	{0, 1, 2}
Great rhombicuboctahedron	{0, 1, 2}
Icosadodecahedron	{0, 1, 2}
Small rhombicosidodecahedron	{0, 1, 2}
Small rhombicuboctahedron	{0, 1, 2}
Pseudorhombicuboctahedron	∅
Snub cube	∅
Snub dodecahedron	∅
Truncated cube	{0, 1}
Truncated dodecahedron	{0, 1}
Truncated icosahedron	{0, 1, 2}
Truncated octahedron	{0, 1, 2}
Truncated tetrahedron	{0, 1}

(we assume here that W is finite), and choose $\{u_1 = 1, u_2, u_3, \ldots, u_{|W:N|}\}$ such that $\{\Phi^{u_1} = \Phi, \Phi^{u_2}, \ldots, \Phi^{u_{|W:N|}}\} = \mathcal{F}(\mathcal{P}(W)/N)$. Note that all Coxeter elements in W are conjugates since the covering Coxeter group has a string diagram. Following [6], we denote by H_i the parabolic subgroups of W of the form $\langle s_j : j \neq i\rangle$. Since faces of the polytope are in one-to-one correspondence with double cosets of the form Nu_jH_i, and the flag action of an element $v \in W$ sends a face Nu_jH_i in flag Φ^{u_j} to the face Nu_jvH_i (see [7]), it suffices to consider the conditions under which $Nu_j(\sigma_l)^kH_i = Nu_jH_i$. In this instance, $u_j(\sigma_l)^k \in Nu_jH_i$, so there exist $n \in N$, $h \in H_i$ such that $nu_jh = u_j(\sigma_l)^k$. In other words, $u_j^{-1}nu_j = (\sigma_l)^kh^{-1}$, which is equivalent to $(\sigma_l)^kH_i \cap N^{u_j} \neq \emptyset$. Note that this intersection condition depends not on our choice of u_j, but only on the conjugates of N. In other words, by Theorem 2.2, we may restrict our attention only to a subcollection of the Φ^{u_j}, one taken from each automorphism class. Therefore, a Petrie scheme fails to be acoptic precisely when $(\sigma_l)^kH_i \cap N^{u_j} \neq \emptyset$ and k is less than the size of the orbit of Φ^{u_j} under the action of σ_l. We have thus shown the following theorem.

Theorem 6.3. *Let $\{u_1 = 1, u_2, u_3, \ldots, u_r\}$ be chosen such that $\{\Phi^{u_j} : 1 \le j \le r\}$ are representatives of each of the r transitivity classes of flags under the automorphism group of the polytope $\mathcal{P}(W)/N$. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ be the collection of distinct Coxeter elements in W and let $m_{j,l} = |\{\Phi^{u_j}\alpha : \alpha \in \langle\sigma_l\rangle\}|$. Then $\mathcal{P}(W)/N$ has acoptic Petrie schemes if $(\sigma_l)^kH_i \cap N^{u_j} = \emptyset$ for all $1 \le k < m_{j,l}$.*

The results of applying such a test to the sporadic Archimedean solids are given in Table 2. This expands the list of known polytopes with acoptic Petrie schemes given in [13] to include eight of the sporadic Archimedean polyhedra. We say that a polytope has *acoptic Petrie schemes at rank i* if each face of rank i appears at most once in each Petrie scheme, so a polyhedron has acoptic Petrie schemes if it has acoptic Petrie schemes at ranks 0, 1 and 2.

As a practical matter, one need not check all of the distinct Coxeter elements, but instead only half of them, since the inverse of a Coxeter element is itself a Coxeter element, and inverse pairs generate the same sequences of flags, only in reverse order. Thus for polyhedra, one need only check $\sigma_1 = s_0s_1s_2$ and $\sigma_2 = s_0s_2s_1$.

Let $|\sigma_l|$ denote the order of σ_l. It is worth noting that it is easy to construct examples of polytopes for which $m_{j,l} < |\sigma_l|$ for all j and l, even when the covering regular polytope is finite and all of the schemes are acoptic. One such is obtained by taking the quotient of the universal square tessellation $\{4, 4\}$, whose automorphism group W is the Coxeter group $[4, 4]$. Now let $N = \langle(\nu_1\nu_2)^3, (\nu_1\nu_2^{-1})^5\rangle$ where $\nu_1 = s_0s_1s_2s_1$ and $\nu_2 = s_1s_0s_1s_2$. Then $\mathcal{P}(W)/N = [4, 4]/N$ is a toroidal polyhedron. In this case, $m_{j,l}$ is either 6 or 10, but $|\sigma_l| = 30$ in $W/Core(W, N)$. For a further discussion of Petrie polygons and polytopes with acoptic Petrie schemes see [13].

6.2. Size of presentations

The pseudorhombicuboctahedron (also known as the elongated square gyrobicupola, or Johnson solid J_{37})[2] provides an interesting case for discussion, because while it has the same local structure as the small rhombicuboctahedron (vertex stars of type 3.4.4.4), it has significantly less symmetry. Theorem 2.2 provides a computationally very fast method of determining that there are in fact twelve equivalence classes of flags (a fact otherwise tedious to determine), while Theorem 6.3 provides a rapid method of verifying that the Petrie schemes of J_{37} are not all acoptic at any rank. Perhaps more surprising to the reader might be the comparison of the sizes of the group presentation with the small rhombicuboctahedron. While the minimal cover of the small rhombicuboctahedron is of order 1 327 104 the cover for the pseudorhombicuboctahedron is more than ten orders of magnitude larger at 16 072 626 615 091 200.

2 The pseudorhombicuboctahedron has been "discovered" independently on numerous occasions and has proved to be an excellent example of the difficulties mathematicians have in constructing definitions about intuitively understood objects that are sufficiently rigorous so as to specify precisely the objects they wish to study without accidentally assuming unstated constraints (such as symmetry). The interested reader is encouraged to review Grünbaum's excellent discussion of the history in [5].

7. Some open questions

We include here some questions motivated by the current work. Theorem 2.3 provides a minimal presentation for a polytope as a quotient of a regular polytope, but only in the instance that $\mathcal{P}(W/\mathrm{Core}(W, N))$ is a well-defined polytope. Does there exist an example of a (finite) polytope for which $\mathcal{P}(W/\mathrm{Core}(W, N))$ is not polytopal? Also, in the examples studied to date, finite polytopes have all yielded representations as the quotients of finite regular polytopes. Is there an example of a finite polytope which does not admit a presentation as the quotient of a finite regular polytope? Both of these questions would be answered in the negative if the following conjecture – and thus its corollary by Theorem 2.2 – are true (for definitions and a more detailed discussion of the role semisparse subgroups play in the theory of quotient representations, see [8]).

Conjecture 7.1. *If N is semisparse in W then* $\mathrm{Core}(W, N)$ *is also semisparse.*

Corollary 7.2. *Assuming Conjecture 7.1, every finite abstract polytope admits a presentation as the quotient of a finite regular abstract polytope.*

A computer survey of the symmetry groups of abstract regular polytopes found no counterexamples to Conjecture 7.1 for groups W of order less than 639.

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