Note

The connectivity of acyclic orientation graphs

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Abstract

The *acyclic orientation graph*, AO(G), of an undirected graph, G, is the graph whose vertices are the acyclic orientations of G and whose edges are the pairs of orientations differing only by the reversal of one edge. Edelman (1984) has observed that it follows from results on polytopes that when G is simple, the connectivity of AO(G) is at least $n - c$, where n is the number of vertices and c is the number of components of G. In this paper we give a simple graph-theoretic proof of this fact. Our proof uses a result of independent interest. We establish that if H is a triangle-free graph with minimum degree at least k, and the graph obtained by contracting the edges of a matching in H is k-connected, then H is k-connected.

The connectivity bound on AO(G) is tight for various graphs including K_n, $K_{p,q}$, and trees. Applications and extensions are discussed, as well as the connection with polytopes. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

For an undirected graph G and an acyclic orientation D of G, an edge e of D is called *dependent* if reversing the direction of e creates a directed cycle in D; otherwise, e is *independent* [18]. Two orientations D and D' of G are called *adjacent* if they differ only in the orientation of a single (necessarily independent) edge. The *acyclic orientation graph* of G, AO(G), is the graph whose vertices are the acyclic orientations of G and whose edges are the pairs of adjacent orientations.

As observed in [15], when G is a tree on n vertices, AO(G) is isomorphic to the $(n - 1)$-cube; when G is K_n, AO(G) is isomorphic to the Cayley graph of the symmetric group S_n, of permutations of $1 \ldots n$, generated by the adjacent transpositions

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$\{(1\ 2),\ (2\ 3),\dots,(n\ -\ 1\ n)\}$. Although these acyclic orientation graphs are Hamiltonian, many are not. For example, when G is a cycle of even length, $AO(G)$ is not Hamiltonian. Several other counterexamples appear in [15].

In this note we investigate the connectivity of $AO(G)$. It is known that $AO(G)$ is connected if and only if G is simple (for example, [15]). We will use the following result of Edelman, proved by West in a graph-theoretic setting [18], which shows that the minimum degree of $AO(G)$ is at least $n - c$, where n is the number of vertices and c is the number of components of G.

Lemma 1.1. *Every acyclic orientation of an n-vertex simple graph G has at least $n - c$ independent edges, where c is the number of components of G.*

Section 2 contains a simple graph-theoretic proof that the connectivity of $AO(G)$ is at least $n - c$. Since the connectivity cannot exceed the minimum degree, this lower bound is tight when $G = K_n$, since $AO(K_n)$ has minimum degree $n - 1$. Also, since West has shown in [18] that the minimum degree of $AO(K_{p,q})$ is $n - 1$, where $p + q = n$, the connectivity of $AO(K_{p,q})$ is exactly $n - 1$. We prove and use a result about the connectivity of graphs obtained by contracting matchings. We establish that if H is a triangle-free graph with minimum degree at least k, and the graph obtained by contracting the edges of a matching in H is k-connected, then H is k-connected. In Section 3 we discuss some possible extensions.

It was communicated to us by Edelman [4] that this result on the connectivity of the $AO(G)$ follows from earlier work on polytopes. In the appendix, we present an outline of this approach, both for completeness and because this fact does not appear to be recorded elsewhere in the literature.

2. The connectivity of acyclic orientation graphs

For a simple graph G and an edge $e = xy$ of G, let G/e denote the underlying simple graph obtained by contracting the edge e. That is, if w_{xy} is the vertex obtained by identifying the ends of e, then

$$V(G/e) = V(G) - \{x, y\} \cup \{w_{xy}\}$$

and

$$E(G/e) = E(G - e) \cup \{uw_{xy}:\ ux \in E(G)\ \text{or}\ uy \in E(G)\}.$$

The following two lemmas are straightforward.

Lemma 2.1. *If e_1 and e_2 are edges of G, then $(G/e_1)/e_2 = (G/e_2)/e_1$.*

Lemma 2.2. *If M is a matching in G and $e \in M$ then $M - e$ is a matching in G/e.*

The two lemmas above imply that the edges in M can be contracted in arbitrary order to produce a unique graph which we denote by G/M.

Theorem 2.3. *Let H be a triangle-free graph with minimum degree at least t and let M be a matching in H. If the graph H/M is t-connected, then H is also t-connected.*

Proof. We use the vertex-split operation. A *vertex-split* in a simple graph G replaces a vertex v by two adjacent vertices x and y whose neighborhoods have union $N_G(v) \cup \{x, y\}$. Call the resulting graph G'. We first prove that if G is t-connected and the degrees of x and y in G' are at least t then G' is also t-connected.

Suppose S is a vertex cut of G' of size $t' < t$. If both x and y are in S, then $S - \{x, y\} \cup \{v\}$ is a vertex cut of size $t' - 1 < t$ of G. If neither x nor y are in S, then if $z \in V(G')$ is not adjacent to v in $G' - S$, z is not adjacent to x or y in $G - S$, so S is a vertex cut of size $t' < t$ in G. Finally, if $x \in S$, but $y \notin S$, then every vertex adjacent to y in G' is either in S or in the component, C, of $G' - S$ containing y. Since the degree of y in G' is at least t, some neighbor of y is in C and therefore $S - \{x\} \cup \{v\}$ is a vertex cut of size $t' < t$ in G, again a contradiction.

Let $M = \{e_1, \ldots, e_m\}$ where $e_i = x_i y_i$. Let $H_0 = H$ and for $1 \leqslant i \leqslant m$, let $H_i = H_{i-1}/\{e_i\}$, where e_i is contracted to the vertex v_i. Then H can be recovered from $H/M = H_m$ by a sequence of vertex splits of $v_m, v_{m-1}, \ldots, v_1$, in that order. For $i \geqslant 1$, in splitting v_i in H_i, the degree of x_i (and y_i) in H_{i-1} must be at least t. For, x_i must have degree at least t in $H_0 = H$ and yet since M is a matching, the degree of x_i could be increased by a later split of some v_j, $j < i$, if and only if x_i is adjacent to both x_j and y_j. This is impossible since then x_i, x_j, y_j would form a triangle in H. $\quad\square$

Fix an acyclic orientation, D^*, of G. Any orientation D of G can be represented by a function

$$\sigma \colon E(G) \to \{0, 1\}$$

defined with respect to D^*, by $\sigma(e) = 1$ if e has the same orientation in D and D^*, and otherwise, $\sigma(e) = 0$. Let $\sigma - e$ denote the orientation of $G - e$ in which σ is restricted to $E(G - e)$.

For $e \in E(G)$, let M_e be the matching in $\mathrm{AO}(G)$ defined by

$$M_e = \{\sigma\tau \in E(\mathrm{AO}(G)) \colon \sigma - e = \tau - e, \text{ but } \sigma(e) \neq \tau(e)\}.$$

Lemma 2.4. *For a simple graph G, $AO(G)/M_e \cong \mathrm{AO}(G - e)$.*

Proof. If $\sigma \in \mathrm{AO}(G - e)$, let u and v be the ends of e and let D be the digraph corresponding to σ. Since D is acyclic, D cannot have both a directed path from u to v and a directed path from v to u. If there are two distinct extensions, σ', σ'' of σ to acyclic orientations of G, then $\sigma'\sigma'' \in M_e$, so σ corresponds to a single vertex of $\mathrm{AO}(G)/M_e$. Otherwise, there is a unique extension σ' of σ to an acyclic orientation

of G and σ' is an uncontracted vertex of $AO(G)/M_e$. For the reverse mapping, any $\sigma \in AO(G)/M_e$ either satisfies $\sigma \in AO(G)$, so that σ corresponds to $\sigma - e \in AO(G - e)$ or σ was formed by contracting an edge $\alpha\beta \in M_e$, in which case σ corresponds to $\alpha - e = \beta - e$ in $AO(G - e)$.

Finally, $\alpha\beta$ is an edge of $AO(G)/M$ if and only if α and β differ on some edge $f \neq e$ in $E(G)$ and possibly on e itself. But this occurs if and only if $\alpha - e$ and $\beta - e$ are adjacent in $AO(G - e)$. $\square$

For the proof of the main theorem we need the following well-known result, which is easy to prove by induction.

Lemma 2.5. *The n-cube, Q_n, is n-connected for $n \geqslant 1$.*

Theorem 2.6. *For a simple graph G with n vertices and c components, the acyclic orientation graph $AO(G)$ is $(n - c)$-connected.*

Proof. We use induction on $m - (n - c)$, where m is the number of edges in G. If $m - (n - c) = 0$, then G is a forest with c components and $n - c$ edges. Each of the 2^{n-c} orientations of G is acyclic and in any orientation, every edge is independent. Thus, G is the $(n - c)$-cube which has connectivity $n - c$, by Lemma 2.5.

If $m - (n - c) > 0$, then G contains an edge e that lies on a cycle of G. Then $G - e$ still has n vertices and c components, so the induction hypothesis implies that $AO(G - e)$ is $(n - c)$-connected.

By Lemma 2.4, $AO(G - e) \cong AO(G)/M_e$, so that $AO(G)/M_e$ is also $(n - c)$-connected. In addition, $AO(G)$ is bipartite and, by Lemma 1.1, has minimum degree at least $n - c$. Thus, Theorem 2.3 applies and therefore establishes that $AO(G)$ is $(n - c)$-connected. $\square$

Although $AO(G)$ is not always Hamiltonian, Fleischner [5,6,11] has shown that the square of any 2-connected graph with at least three vertices is Hamiltonian and therefore $AO^2(G)$ is Hamiltonian when $n - c \geqslant 2$. An explicit construction of a Hamiltonian cycle in $AO^2(G)$ is given in [16].

An *eulerian spanning subgraph* of a graph G is a closed trail of G that passes through every vertex. We note that the high connectivity of $AO(G)$ in fact guarantees the existence of two eulerian spanning subgraphs which together contain all of the edges of $AO(G)$ when $n - c \geqslant 4$. This follows from the lemmas below since then the edge connectivity of $AO(G)$ will be at least 4.

Lemma 2.7 (Nash-Williams [10] and Tutte [17], or Kundu [9]). *Every 4-edge-connected graph contains two edge-disjoint spanning trees.*

Lemma 2.8 (Jaeger [8] or Catlin [2]). *If G is a graph containing two edge-disjoint spanning trees, then G contains two spanning eulerian subgraphs F_1, F_2 such that $E(F_1) \cup E(F_2) = E(G)$.*

3. Further studies

Since not every AO(G) has a Hamiltonian cycle, one could at least consider:

(1) If AO(G) has a bipartition into partite sets of equal size, does AO(G) then have a Hamiltonian cycle?

Note that a Hamiltonian cycle is a spanning eulerian subgraph with maximum degree 2.

(2) Does every AO(G) have a spanning eulerian subgraph with maximum degree at most 4?

For a simple undirected graph G and a subset R of $E(G)$, fix an acyclic orientation, (R, σ) of the edges of R. Let AO$_{(R,\sigma)}(G)$ be the subgraph of AO(G) induced by the acyclic orientations of G which agree with (R, σ) on R. If $R = \emptyset$, then AO$_{(R(G),\sigma)}(G) = $ AO(G). It is known that AO$^2_{(R,\sigma)}(G)$ is not necessarily Hamiltonian. Counterexamples appear in [16,14].

(3) Does every AO$_{(R,\sigma)}(G)$ have a spanning eulerian subgraph of maximum degree at most 4?

Another interesting case arises when G is K_n and the binary relation on $V(G)$ implied by (R, σ) is a partially ordered set. In this case, the vertices of AO$_{(R,\sigma)}(G)$ are the linear extensions of the partially ordered set $(V(G), (R, \sigma))$. It is shown in [13] that in this case, AO$^2_{(R,\sigma)}(G)$ is Hamiltonian.

Finally, we can consider the connectivity of AO$_{(R,\sigma)}(G)$.

(4) What is the connectivity of AO$_{(R,\sigma)}(G)$?

When G is complete and $(V(G), (R, \sigma))$ is a partially ordered set, we conjecture that the connectivity of AO$_{(R,\sigma)}(G)$ is at least one less than the width of the partially ordered set. Pruesse and Ruskey [12] have shown this to be true for partially ordered sets of width 3.

Acknowledgements

Appendix

Balinski in [1] considered convex polyhedral sets in $\mathbf{R}^n$, defined by a system of linear inequalities. These polyhedra can be viewed as graphs in which a vertex (edge) of the polyhedron corresponds to a vertex (edge) of the graph. He showed that if a polyhedron so defined had dimension n (i.e., its convex hull does not lie in any $n - 1$-

dimensional hyperplane), then the polyhedron, regarded as a graph, is n-connected. Edelman [4] observed that this can be used to give a lower bound on the connectivity of AO(G). We sketch his argument below.

Let $G = (V, E)$ be a simple undirected graph with $V = \{v_1, \ldots, v_n\}$. In [7], Greene and Zaslavsky described a bijection between the vertices of AO(G) and the regions of a certain arrangement in $\mathbf{R}^n$. For each edge $v_i v_j \in E$ with $i < j$, define the hyperplane H_{ij} by

$$H_{ij} = \{(x_i, \ldots, x_n) \in \mathbf{R}^n \colon x_i - x_j = 0\}.$$

The arrangement $\mathscr{H}(\mathscr{G})$ is the collection of hyperplanes $\mathscr{H}(\mathscr{G}) = \{H_{ij} \colon v_i v_j \in E, \ i < j\}$. The regions of $\mathscr{H}(\mathscr{G})$ are the connected subsets of $\mathbf{R}^n \setminus \bigcup_{H \in \mathscr{H}(\mathscr{G})} H$.

For an orientation α of G and an edge $v_i v_j$ of E with $i < j$, define the region $R_\alpha(i, j)$ by

$$R_\alpha(i, j) = \begin{cases} \{(x_1, \ldots, x_n) \colon x_i < x_j\} & \text{if } \alpha \text{ orients } v_i v_j \text{ as } (v_i, v_j), \\ \{(x_1, \ldots, x_n) \colon x_i > x_j\} & \text{otherwise.} \end{cases}$$

It was shown in [7] that the mapping $\alpha \to \bigcap_{v_i v_j \in E, i < j} R_\alpha(i, j)$ is a one-to-one correspondence between the acyclic orientations of G and the regions of $\mathscr{H}(\mathscr{G})$. Furthermore, vertices adjacent in AO(G) correspond to adjacent regions in the arrangement: two acyclic orientations α and β differ only by the reversal of edge $v_i v_j$ with $i < j$, if and only if R_α and R_β have (a subset of) H_{ij} as a common boundary. Define the *region graph* of $\mathscr{H}(\mathscr{G})$ to be the graph whose vertices are the regions of $\mathscr{H}(\mathscr{G})$, with two regions adjacent if their boundaries intersect at more than a point. Then the result of Greene and Zaslavsky showed that AO(G) is isomorphic to the region graph of the arrangement $\mathscr{H}(\mathscr{G})$.

For each $v_i v_j \in E$ with $i < j$, define the vector w_{ij} in R^n by

$$w_{ij}(k) = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then w_{ij} is normal to H_{ij}. Let $W = \{w_{ij} \colon v_i v_j \in E, \ i < j\}$. Since $(1, 1, \ldots, 1)$ is normal to all elements in W, the vectors in W span a subspace of dimension at most $n - 1$. In fact it can be shown that the dimension of this subspace is $n - c$, where c is the number of components of G [7]. Edelman showed in [3] that $Z = \{x \colon x = \sum_{w \in W} \lambda_w w$ and $-1 \leqslant \lambda_w \leqslant 1\}$ (called a zonotope) is a convex polyhedron of dimension $n - c$ which has as its associated graph (in fact its one-skeleton) the region graph of $\mathscr{H}(\mathscr{G})$, which is isomorphic to AO(G). It then follows from the result of Balinski that AO(G) has connectivity at least $n - c$.

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