A Characterization of Undirected Branching Greedoids

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Undirected branching greedoids are defined by rooted trees of a graph. We give a minor criterion for these greedoids. © 1988 Academic Press, Inc.

INTRODUCTION

One of the most interesting results in algebraic graph theory is Tutte's minor characterization of graphic matroids. In a remarkable series of articles Korte and Lovász [4-11] introduced and studied greedoids. Greedoids may be considered as a generalization of matroids. As in the matroid case, graphs lead to several examples for greedoids. Undirected branching greedoids seem to be a natural counterpart to graphic matroids. Such a greedoid is defined by trees of a graph rooted at a fixed vertex. In this paper we give a minor characterization of these greedoids.

In Section 1 we briefly list some definitions and basic results.

In Sections 2-5 we prepare the proof of the minor criterion. In Section 2 we introduce the notion of a path. A feasible set \( Y \) is said to be a path if there exists \( a \in Y \) such that no proper feasible subset of \( Y \) contains \( a \). Thus a path of a branching greedoid corresponds to a path (without repeated vertices) beginning at the root of an associated graph. We derive some helpful properties of paths.

Since the bases of an undirected branching greedoid define a graphic matroid, these greedoids must have the basis exchange property. We derive this property for greedoids that do not contain the forbidden minors, mainly to work with this property directly, but also to ensure that the basis graph of the greedoid in question is connected.

In Section 4 we construct a rooted graph given a base and prove that the edges of this graph can be labeled with the elements of the greedoids ground set.
In Section 5 we first show that the definition of the graph does not depend on the choice of base B. Finally, the proof of the minor criterion is derived.

1. Definitions and Basic Results

We will assume familiarity with the concept of a graph and of a matroid (cf. Harary [3], Welsh [15]). Greedoids were introduced by Korte and Lovász [4].

A greedoid is a set system \((E, \mathcal{F})\), where \(E\) is a finite set and \(\mathcal{F} \subseteq 2^E\) such that (G1), (G2), and (G3) are satisfied:

(G1) \(\emptyset \in \mathcal{F}\).

(G2) If \(\emptyset \neq X \in \mathcal{F}\) then \(X \setminus \{a\} \in \mathcal{F}\) for some \(a \in X\).

(G3) If \(X, Y\) are members of \(\mathcal{F}\) with \(|X| > |Y|\) there exists \(a \in X \setminus Y\) such that \(Y \cup \{a\} \in \mathcal{F}\).

\((E, \mathcal{F})\) is called an accessible set system if (G1) and (G2) hold.

Sets belonging to \(\mathcal{F}\) are called feasible sets. For \(A \subseteq E\) a maximal feasible subset of \(A\) is called a basis of \(A\). \(\mathcal{B}\) denotes the family of bases of \(E\). A partial alphabet is a union of feasible sets. Let \(\mathcal{A}\) denote the family of partial alphabets.

A greedoid \((E, \mathcal{F})\) is normal if \(E \in \mathcal{A}\), i.e., each element of \(E\) occurs in at least one feasible set. \(x_1, x_2, \ldots, x_k\) is called a feasible ordering of a set \(\{x_1, x_2, ..., x_k\}\) if \(\{x_1, x_2, ..., x_i\} \in \mathcal{F}\) \((1 \leq i \leq k)\) (cf. Korte and Lovász [4] for an alternative definition of greedoids in terms of such strings).

A greedoid \((E, \mathcal{F})\) is said to be an interval greedoid, if \(X \subseteq Y \subseteq Z\), \(X \cup \{a\} \in \mathcal{F}\) and \(Z \cup \{a\} \in \mathcal{F}\) imply \(Y \cup \{a\} \in \mathcal{F}\) (interval property). This condition is equivalent to:

(B) whenever \(X, Y, Z \in \mathcal{F}\) such that \(X, Y \subseteq Z\) then \(X \cup Y \in \mathcal{F}\).

Interval greedoids are a very substantial subclass of greedoids. Korte and Lovász [7] indicate the richness of this structure. An interval greedoid is called a shelling structure if \(E \in \mathcal{F}\). Thus the family of feasible sets of a shelling structure is closed under union. It is immediate that for shelling structures

(UB) for any \(A \subseteq E\), \(A\) has a unique base holds.

**Proposition 1.1.** An accessible set system \((E, \mathcal{F})\) is a shelling structure if and only if \(E \in \mathcal{F}\) and \((E, \mathcal{F})\) satisfies (UB).

**Proof.** It remains to prove that (UB) is sufficient. Korte and Lovász [9] showed that an accessible set system is a shelling structure iff \(E \in \mathcal{F}\).
and $\mathcal{F}$ is closed under union. Now, let $(E, \mathcal{F})$ be an accessible set system such that $\mathcal{F}$ is not closed under union. Choose $X, Y \in \mathcal{F}$ such that $X \cup Y \notin \mathcal{F}$ and $|X \cup Y|$ is minimal. Certainly, $Y - X \neq \emptyset$ and $X - Y \neq \emptyset$. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ be feasible orderings of $X$ and $Y$, respectively, and $j := \max\{k: x_k \notin Y\}$, $i := \max\{k: y_k \notin X\}$. Then we have $\{x_1, x_2, \ldots, x_{j-1}\} \in \mathcal{F}$, $(X \cup Y) - \{x_j\} = \{x_1, x_2, \ldots, x_{j-1}\} \cup Y \in \mathcal{F}$ (by the minimality of $|X \cup Y|$), $\{y_1, y_2, \ldots, y_{i-1}\} \in \mathcal{F}$, and $(X \cup Y) - \{y_i\} \notin \mathcal{F}$.

Now, $(X \cup Y) - \{x_j\}$ and $(X \cup Y) - \{y_i\}$ are two different maximal feasible subsets of $X \cup Y$.

A greedoid $(E, \mathcal{F})$ is called a poset greedoid if $\mathcal{F}$ is closed under union and intersection. An interval greedoid is said to be a local poset greedoid if

\[(A)\] whenever $X, Y, Z \in \mathcal{F}$ such that $X, Y \subseteq Z$ then $X \cap Y \in \mathcal{F}$ holds.

Undirected branching greedoids are special local poset (or special interval) greedoids. Consider an undirected graph $(V, E)$ with specified vertex $P_0$ (root). Let

$$\mathcal{F} := \{X \subseteq E: X \text{ is a tree containing the root}\}.$$ 

Then $(E, \mathcal{F})$ is called an undirected branching greedoid. Directed branching greedoids are defined on arc sets of rooted directed graphs. Let $(V, E)$ be a directed graph with root $P_0$, and

$$\mathcal{F} := \{X \subseteq E: X \text{ is an arborescence rooted at } P_0\}.$$ 

$(E, \mathcal{F})$ is said to be a directed branching greedoid or search greedoid (cf. Schmidt [12] for a characterization of these structures). The bases of $\mathcal{F}$ are the maximal branchings of $(V, E)$ and each feasible ordering of a base of $\mathcal{F}$ corresponds to a search in the graph starting at $P_0$.

Again let $(V, E)$ be a directed graph rooted at $P_0$. Define

$$\mathcal{F} := \{X \subseteq V - \{P_0\}: \text{there is a directed path } Z \subseteq \langle X \cup \{P_0\} \rangle \text{ directed from } P_0 \text{ to } P(\in X)\},$$

where $\langle X \cup \{P_0\} \rangle$ denotes the subgraph of $(V, E)$ induced by the vertices of $X \cup \{P_0\}$. In this case $(V - \{P_0\}, \mathcal{F})$ is called a point search greedoid.

For further examples of greedoids the reader is referred to Björner [1] and Korte and Lovász [7–10].

The rank function of a greedoid is a function $r: 2^E \to \mathbb{Z}$ defined by

$$r(A) := \max\{|X|: X \subseteq A, X \in \mathcal{F}\} \quad (A \subseteq E).$$

$r$ is monotone, subcardinal, and local submodular (i.e., $r(A) \leq r(B)$ $(A \subseteq B \subseteq E)$, $r(A) \leq |A|$ $(A \subseteq E)$, $r(A) = r(A \cup \{x\}) = r(A \cup \{y\})$ implies $r(A) = r(A \cup \{x, y\})$).
A \subseteq E \text{ is rank feasible if}

r(A) = \beta(A) := \max\{|A \cap B| : B \in \mathcal{B}\}.

Let \mathcal{R} denote the family of rank feasible sets. Then \(A \in \mathcal{R}\) iff

\[ r(A \cup C) \leq r(A) + |C| \quad (C \subseteq E - A) \]

and \(r\) is submodular on \(\mathcal{R}: r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \quad (A, B \subseteq E).\)

For interval greedoids, \(\mathcal{A} \subseteq \mathcal{R}\) (Korte and Lovász [5, 8]).

The closure operator \(\sigma\) of a greedoid may be defined as

\[ \sigma(A) := \bigcup \{B \subseteq E : r(A \cup B) = r(A)\} \quad (A \subseteq E). \]

Korte and Lovász [5] proved that \(\sigma\) is subexclusive, idempotent, and satisfies:

\[ (SM) \quad \text{if } X \cup \{a\} \in \mathcal{F} \text{ and } a \in \sigma(X \cup \{b\}) \text{ then } b \in \sigma(X \cup \{a\}). \]

This is a special case of the Steinitz–MacLane exchange property for a matroid closure operator \(\tau:\)

\[ \text{if } A \subseteq E, \ a \notin \tau(A), \text{ and } a \in \tau(A \cup \{b\}) \text{ then } b \in \tau(A \cup \{a\}). \]

We define the kernel closure \(\lambda(A)\) of a set as

\[ \lambda(A) := \bigcup \{X \subseteq \mathcal{F} : r(A \cup X) = r(A)\} \quad (A \subseteq E). \]

\(\lambda\) is said to be the kernel closure operator of \((E, \mathcal{F})\). For \(A \subseteq E\) let

\[ \ker(A) := \bigcup \{X \in \mathcal{F} : X \subseteq A\} \]

be the kernel of \(A\). Korte and Lovász [5] showed that in the case of an interval greedoid \(\ker(\sigma(A)) = \lambda(A) \quad (A \in \mathcal{A}).\)

**Proposition 1.2.** For any greedoid \((E, \mathcal{F})\) with closure operator \(\sigma\) and kernel closure operator \(\lambda\) we have

\[ \lambda(A) = \ker(\sigma(A)) \quad (A \subseteq E). \]

**Proof.** Let \(A \subseteq E\). By definition \(\lambda(A) \subseteq \ker(\sigma(A))\). Consider a feasible subset \(Y\) of \(\sigma(A)\). Then \(A \cup Y \subseteq \sigma(A)\), and hence \(r(A \cup Y) = r(\sigma(A)) = r(A)\). Thus \(Y \subseteq \lambda(A)\) and therefore \(\ker(\sigma(A)) \subseteq \lambda(A)\). \(\square\)

**Proposition 1.3.** \(\lambda\) and \(\sigma\) coincide if and only if \(\{a\} \in \mathcal{F} \quad (a \in E)\).

**Proof.** Let \(\lambda(A) = \sigma(A) \quad (A \subseteq E)\). Then \(\emptyset = \lambda(\emptyset) = \sigma(\emptyset) = \{a \in E : r(\{a\}) = 0\} = \{a \in E : \{a\} \notin \mathcal{F}\}\). Thus \(\{a\} \in \mathcal{F} \quad (a \in E)\). Now suppose
that this is the case. Then each subset \( A \) of \( E \) is a partial alphabet, and so by proposition 2 \( \sigma(A) = \ker(\sigma(A)) = \lambda(A) \) (\( A \subseteq E \)).

Crapo [2] considered a closure operator \( - : \mathcal{A} \to \mathcal{A} \) for interval greedoids on partial alphabets which coincides with \( \lambda \) in this special case. In Schmidt [13], we could prove that:

(a) for any partial alphabet \( A \), \( A \subseteq \lambda(A) \),
(b) if \( X \) is a base of \( A \subseteq E \) then \( \lambda(X) = \lambda(A) \),
(c) if \( A \subseteq B \subseteq E \) then \( \lambda(A) \subseteq \lambda(B) \),
(d) for any rank feasible set \( A \), \( x \in E \), \( \lambda(A \cup \{x\}) \neq \lambda(A) \) and \( x \in \lambda(A \cup \{y\}) \) imply \( y \in \lambda(A \cup \{x\}) \),
(e) for interval greedoids \( \lambda \) is monotone.

From (d) we may derive the (SM)-exchange property for a closure operator of a normal matroid. In this case \( \mathcal{R} = 2^E \) and \( \lambda(A \cup \{x\}) \neq \lambda(A) \) is equivalent to \( x \notin \lambda(A) \).

The knowledge of \( r \), \( \sigma \) or \( \lambda \) is sufficient to uniquely determine the greedoid. Hence it is not surprising that there exist axiom systems for a greedoid in terms of each of these concepts. Compare Korte and Lovász [5] for the case of rank function and closure operator and Schmidt [13] for the case of the kernel closure operator.

**Minors** of greedoids may be defined as follows. For \( A \subseteq E \), let \( (A, \mathcal{F}|_A) \) denote the **restriction** of \((E, \mathcal{F})\) to \( A \), where \( \mathcal{F}|_A := \{X \in \mathcal{F} : X \subseteq A\} \). Let \( \mathcal{F} - A := \{X \in \mathcal{F} : X \subseteq E - A\} \). Then \((E - A, \mathcal{F} - A)\) is a greedoid obtained by the **deletion** of \( A \) (or by the restriction of \((E, \mathcal{F})\) to \( E - A \)). For \( A \in \mathcal{R} \) let \( r_A(X) := r(A \cup X) - r(A) \) (\( X \subseteq E - A \)), then \( r_A \) is a rank function of a greedoid \((E - A, \mathcal{F}/A)\), obtained by the **contraction** of \( A \). We have:

\[
\mathcal{F}/A = \{X \subseteq E - A : X \cup Y \in \mathcal{F} \text{ for every base } Y \text{ of } A\}.
\]

In the case of undirected branching greedoids each partial alphabet \( A \in \mathcal{A} \) corresponds to a connected subgraph of the graph \((V, E)\) containing the root, we have \( \mathcal{A} = \mathcal{R} \) and each feasible set \( X \) in \( \mathcal{F}/A \) is a tree containing \( P_0 \) in \((V, E)/A\), the graph obtained from \((V, E)\) by contracting the edges of \( A \) to \( P_0(A) \in \mathcal{A} \). Thus the family of undirected branching greedoids is closed under taking minors.

Minor characterizations have a remarkable tradition in the theory of graphs and matroids. A first result of this kind for greedoids was given by Korte and Lovász [7].

**Theorem 1.4 (Korte and Lovász).** An interval greedoid \((E, \mathcal{F})\) is a local poset greedoid if and only if it does not contain any minor isomorphic to \((\{x, y, z\}, 2^{\{x, y, z\}} - \{z\})\).
UNDIRECTED BRANCHING GREEDOIDS

We want to prove that

**Theorem 1.5.** An interval greedoid is an undirected branching greedoid if and only if it does not contain one of the following greedoids as a minor:

(A) \( E = \{x, y, z\}, \mathcal{F} = 2^E - \{z\} \),

(B) \( E = \{x, y, z\}, \mathcal{F} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}\} \),

(C) \( E = \{x, y, z\}, \mathcal{F} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}\} \),

(D) \( E = \{x, y, z\}; \mathcal{F} = 2^E - E \).

Since these minors are not undirected branching greedoids, the conditions are necessary. The sufficiency will be proved in the rest of the paper.

Korte and Lovász [7] have proved already that for local poset greedoids that do not contain greedoid (C) as a minor, each rank feasible set is a partial alphabet, i.e., \( \mathcal{R} = \mathcal{A} \).

2. Properties of Paths

A feasible set \( Z \) is called a *path*, if there exists \( e \in Z \) such that there exists no proper subset \( Y \subseteq Z, Y \in \mathcal{F} \) such that \( e \in Y \). In this case \( Z \) is called an *e-path* and \( e \) is the *head* of \( Z \).

**Lemma 2.1.** For any local poset greedoid \((E, \mathcal{F})\) the following conditions are equivalent:

(a) \((\text{BR}1)\) \( \sigma(X) \cap \sigma(Y) \subseteq \sigma(X \cup Y) \) \((X, Y \in \mathcal{F})\).

(b) \((E, \mathcal{F})\) does not contain greedoid (B) as a minor.

(c) Each path has a unique feasible ordering.

**Proof.** Compare Schmidt [13]. \( \blacksquare \)

**Remark.** (a) Matroids do satisfy (BR1), since \( \sigma \) is monotone in this case.

(b) Korte and Lovász [11] proved that a shelling structure satisfies (BR1) if and only if it is a point search greedoid.

An interval greedoid is a local poset greedoid (i.e., it does not contain a minor of type (A)) iff for any \( X \in \mathcal{F} \) and \( e \in X \) there is exactly one e-path \( Z \) such that \( Z \subseteq X \) (Schmidt [13]).

Let \( Y \subseteq \mathcal{A}, a, c \in E - \sigma(Y), a \neq c \) and \( \lambda(Y \cup \{c\}) = \lambda(Y \cup \{a\}) \). Then \((Y, a, e)\) is called a d-triple. If \((Y, a, e)\) is a d-triple of a branching greedoid of a graph \((V, E)\), then \( a \) and \( e \) have a common endpoint \( \not\in \langle Y \rangle \) (:= edge subgraph of \((V, E)\) defined by the elements of \( Y \)).
Lemma 2.2. For any normal interval greedoid \((E, \mathcal{F})\) and any \(e \in E\) there exists a \(d\)-triple \((Y, e, a)\) containing \(e\).

Proof. Compare Schmidt [13].

Let for the rest of the paper \((E, \mathcal{F})\) denote a normal interval greedoid without minors (A), (B), (C), and (D).

Proposition 2.3. Let \((Y, e, b)\) be a \(d\)-triple, \(Z_e \subseteq Y \cup \{e\}\) an \(e\)-path and let \(Z \subseteq Y \cup \{b\}\) be a path with head \(b\). Then \(Z_e \cup Z_b \notin \mathcal{F}\).

Proof. Let \(T \subseteq Y \cup \{e, b\}\) be a minimal feasible superset of \(X := (Z_e \cup Z_b) - \{e, b\}\), such that \(e, b \in E - \sigma(T)\) and \(T \cup \{e, b\} \notin \mathcal{F}\). If \(T = X\), then the proposition is true. Suppose \(T - X \neq \emptyset\). Let \(a \in T - X\) such that \(T \setminus \{a\} =: U \in \mathcal{F}\). Then \(U, U \cup \{a\}, U \cup \{b\}, U \cup \{e\}, U \cup \{a, e\}\), and \(U \cup \{a, b\}\) are feasible subsets of \(Y \cup \{e\}\) or \(Y \cup \{b\}\). Since \(T \cup \{e, b\} \notin \mathcal{F}\), \(U \cup \{a, b, e\}\) is not a feasible set. \((E, \mathcal{F})\) does not contain greedoid (D) as a minor, and hence \(U \cup \{b, e\} \notin \mathcal{F}\), contrary to the choice of \(T\).

Proposition 2.4. Let \(Z_a\) and \(Z'_a\) be two different paths with head \(a\) and feasible orderings \(a_1 a_2 \cdots a_m a\) and \(b_1 b_2 \cdots b_n a\), respectively. Let \((Z_a \cup Z'_a) - \{a\} \in \mathcal{F}\). Then we have

(a) \((Z_a \cup Z'_a) - \{x\} \in \mathcal{F}\) iff \(x \in (Z_a - Z'_a) \cup (Z'_a - Z_a) \cup \{a\}\).

(b) \((Z_a \cap Z'_a) - \{a\} \in \mathcal{F}\).

(c) Let \((Z_a \cap Z'_a) - \{a\} = \{a_1, a_2, ..., a_j\}, j > k + 1\), and consider a path \(Z \subseteq Z_a \cup Z'_a\) with head \(a_j\).

Then \(Z = \{b_1, b_2, ..., b_n, a, a_m, a_{m-1}, ..., a_j\}\) or \(Z = \{a_1, a_2, ..., a_j\}\) (Fig. 1).

Proof. (a1) Let \(x \in Z_a \cap Z'_a - \{a\}\), and let \(Z_x \subseteq Z_a\) and \(Z'_x \subseteq Z_a\) be paths with head \(x\). Then \(a \notin Z_x, Z'_x\) and \((Z_x \cup Z'_x) \subseteq (Z_a \cup Z'_a) - \{a\}\). Thus \(Z_x = Z'_x \in \mathcal{F}\) and \(\{x\} = [E - a(Z_x - \{x\})] \cap [Z_a \cup Z'_a]\), and hence \((Z_a \cup Z'_a) - \{x\}\) cannot be a feasible set.

Figure 1
(a2) We have, $|Z_a|, |Z'_a| \geq 2$. $Z_a \cup Z'_a \notin \mathcal{F}$, since $Z_a \cup Z'_a$ contains two different $a$-paths. Let $a_1a_2 \cdots a_ma$ and $b_1b_2 \cdots b_na$ be feasible orderings of $Z_a$ and $Z'_a$, respectively, and let $a_1a_2 \cdots a_k = b_1b_2 \cdots b_k$ be a maximal common beginning section. Hence $\{a_{k+1}, a_{k+2}, \ldots, a_m\} \neq \emptyset$ and $\{b_{k+1}, b_{k+2}, \ldots, b_n\} \neq \emptyset$. Let $t := \max\{i: Z_a \cup \{b_{k+1}, b_{k+2}, \ldots, b_i\} \in \mathcal{F}\}$. Certainly, $t < n$, since $Z_a \cup Z'_a$ is not a feasible set.

Suppose $t < n - 1$. $Y := \{a_1, a_2, \ldots, a_{m-1}\} \cup \{b_1, b_2, \ldots, b_t\}, Y \cup \{a_m\}, Y \cup \{a_m, b_{t+1}\}, Y \cup \{a_m, b_{t+1}, b_{t+2}\}$, and $Y \cup \{a_{t+1}, b_{t+2}\}$ are feasible subsets of $Z_a \cup \{b_{k+1}, b_{k+2}, \ldots, b_t\} \in \mathcal{F}$ and $(Z_a \cup Z'_a) - \{a\} \notin \mathcal{F}$, respectively. From (BR1) and $t < n - 1$, we deduce that $Y \cup \{a\} \notin \mathcal{F}$ and $Y \cup \{b_{t+1}, a\} \notin \mathcal{F}$.

Since $(E, \mathcal{F})$ does not contain any minor of type (C), it follows that $Y \cup \{b_{t+1}\} \in \mathcal{F}$, contrary to the choice of $t$. Thus, $t = n - 1$ and $Z_a \cup Z'_a - \{b_n\} \in \mathcal{F}$.

Let $s := \min\{i: (Z_a \cup Z'_a) - \{b_i\} \in \mathcal{F} (j \geq i)\}$ and suppose $s > k + 1$. By (BR1) we may assume $Z_a \cup \{b_n, b_{n-1}, \ldots, b_s\} \in \mathcal{F}$ ($s + 1 \leq i \leq n$).

Hence $T := Z_a \cup \{b_n, b_{n-1}, \ldots, b_s\} \cup \{b_1, b_2, \ldots, b_{s-2}\}, T \cup \{b_{s+1}\}, T \cup \{b_{s+1}, b_{s-1}\}$, and $T \cup \{b_{s+1}, b_{s-1}\}$ are feasible subsets of $(Z_a \cup Z'_a) - \{b_s\}$ and $(Z_a \cup Z'_a) - \{b_{s+1}\}$, respectively. If $T \cup \{b_s\} \in \mathcal{F}$, then $\{b_1, b_2, \ldots, b_k, b_s\} = (T \cup \{b_s\}) \cap (\{b_1, b_2, \ldots, b_k, b_s\}) \in \mathcal{F}$, in contrary to $s > k + 1$. Furthermore $T \cup \{b_{s-1}, b_{s+1}\} = (Z_a \cup Z'_a) \notin \mathcal{F}$, and hence $T \cup \{b_{s-1}, b_{s+1}\} = (Z_a \cup Z'_a) - \{b_{s-1}\} \in \mathcal{F}$ (consider minor (C)), contrary to the choice of $s$.

Thus, $s = k + 1$ and $Z_a \cup Z'_a - \{b_k\} \in \mathcal{F}$ ($k + 1 \leq i \leq n$). This completes the proof of (a). Especially, $\{b_{k+1}, b_{k+2}, \ldots, b_n\} \subseteq Z_a - Z'_a$, by part (a1), and hence part (b) is proved, too.

(c) Let $m > k + 1$ and let $a_j \in \{a_1, a_2, \ldots, a_m\}$. First we show that $Z'_j = \{b_1, b_2, \ldots, b_n, a, a_m, a_{m-1}, \ldots, a_j\}$ is an $a_j$-path.

Since $(Z_a \cup Z'_a) - \{a_{k+1}\} \in \mathcal{F}$, there is an $a_j$-path, contained in $(Z_a \cup Z'_a) - \{a_{k+1}\}$, not containing $a_{k+1}$. This path must contain $Z'_a$.

Let $i \in \{k+2, k+3, \ldots, m\}$ and $a_j \in E - \sigma(Z'_a)$. Suppose $i < m$. Then $Z'_a \cup \{a_i\}$ and $\{a_1, a_2, \ldots, a_i\}$ are feasible subsets of $(Z_a \cup Z'_a) - \{a_m\} \in \mathcal{F}$, and hence $(Z_a \cup Z'_a) - \{a_i\} \in \mathcal{F}$ follows as above. Thus $m = k + 1 = j$, contrary to our assumption, and so $Z'_a \cup \{a_m\}$ is a path with head $a_m$.

Hence we may assume that an $a_j$-path has a feasible ordering $b_1b_2 \cdots b_na_{k+1}a_{k+1}a_{k+1}a_j$, with $j > k$. Suppose $j \neq l - 1$. Then $Z'_j = \{b_1, b_2, \ldots, b_n, a, a_m, a_{m-1}, \ldots, a_{l-1}, a_j\}$ and $\{a_1, a_2, \ldots, a_{l-1}\}$ are feasible subsets of $(Z_a \cup Z'_a) - \{a_{l-1}\} \in \mathcal{F}$, and so $\{a_j\} = Z'_j \cap \{a_1, a_2, \ldots, a_{l-2}\}$.
would be feasible, a contradiction. Thus \( l - 1 = j \) and \( Z'_j \) is a \( a_j \)-path. Each basis \( B \) of \( Z_a \cup Z'_a \) such that \( a_j \in B \) contains \( Z'_j \) or \( \{a_1, a_2, \ldots, a_j\} \), thus (c) follows.

**Proposition 2.5.** Let \( Z, Z' \) be paths with head \( a \) and feasible orderings

\[ a_1 a_2 \cdots a_m \quad \text{and} \quad b_1 b_2 \cdots b_n, \]

respectively, such that \( a_1 \neq b_1 \) \((Z \cup Z') - \{a_m\} \notin \mathscr{F}\) and \( r(Z \cup Z') = |Z \cup Z'| - 1 \).

Let

\[
I := \min \{t : \{E - \sigma(\{b_1, b_2, \ldots, b_t\})\} \cap \{a_2, a_3, \ldots, a_m\} \neq \emptyset\},
\]

\[
k := \min \{t \geq 2 : a_t \in E - \sigma(\{b_1, b_2, \ldots, b_t\})\},
\]

\[
k' := \min \{t : \{E - \sigma(\{a_1, a_2, \ldots, a_t\})\} \cap \{b_2, b_3, \ldots, b_n\} \neq \emptyset\},
\]

\[
l := \min \{t \geq 2 : b_t \in E - \sigma(\{a_1, a_2, \ldots, a_{k'}\})\}.
\]

Then

(a) \( l = l \) if \( Z \cup \{b_1\} \in \mathscr{F} \).

(b) If \( Z \cup \{b_1\} \notin \mathscr{F} \) then \( l = 1 \) and \( l = 2 \).

(c) \((Z \cup Z') \{x\} \in \mathscr{F} \) iff \( x \in \{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_t\} \).

(d) \( Z \cap Z' = \{a_{k+1}, a_{k+2}, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_t\} \).

(e) Let \( j \in \{2, 3, \ldots, k\} \) and let \( Z_j \) be an \( a_j \)-path, \( Z_j \subseteq Z \cup Z' \). Then

\( Z_j = \{a_1, a_2, \ldots, a_j\} \) or \( Z_j = \{b_1, b_2, \ldots, b_t, a_k, a_{k-1}, \ldots, a_j\} \).

(f) Let \( j \in \{k+1, k+2, \ldots, m\} \) and let \( Z_j \) be an \( a_j \)-path, \( Z_j \subseteq Z \cup Z' \). Then

\( Z_j = \{a_1, a_2, \ldots, a_j\} \) or \( Z_j = \{b_1, b_2, \ldots, b_t, a_{k+1}, a_{k+2}, \ldots, a_j\} \) (Fig. 2).

**Proof.** Let \( Z, Z' \) as prescribed as above. Since \( a_m = a = b_n \) is an element of \( Z \cap Z' \), \( l, I, k \), and \( I \) do exist. Let \( a_i = b_j \in Z \cap Z' \). Then, \( a_i \in E - \sigma(\{b_1, b_2, \ldots, b_{j-1}\}) \), and so \( j - 1 \geq l \), and \( b_j \in E - \sigma(\{a_1, a_2, \ldots, a_{l-1}\}) \), and

![Figure 2](image-url)
so \(i - 1 \geq k\). Thus \(\{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\}\) does not contain any element of \(Z \cap Z'\).

(a) Let \(Z \cup \{b_1\}, \ Z' \cup \{a_1\} \in \mathcal{F}\). For the case \(Z \cup \{b_1\} \in \mathcal{F}\), \(Z' \cup \{a_1\} \notin \mathcal{F}\) see (b).

Then \(l > 1\), for if \(a_s \in E - \sigma(\{b_1\})\) for some \(s \in \{2, 3, ..., m\}\) then \(Z \cup \{b_1\}\) would contain two different \(a_s\)-paths \(\{b_1, a_s\}\) and \(\{a_1, a_2, ..., a_s\}\). Also \(k > 1\).

We show \(l \leq l\) and \(k \leq k\). By symmetry, \(l = l\) and \(k = k\) then follows.

(1) Suppose \(b_i \notin E - \sigma(\{a_1, a_2, ..., a_l\})\) \((2 \leq i \leq l, 1 \leq t \leq k)\) \((i.e., k > k)\).

We have \(a_k \in E - \sigma(\{b_1, b_2, ..., b_l\})\) and \(\{b_1, b_2, ..., b_l, a_k\}\) is an \(a_k\)-path, by the choice of \(l\) and \(k\). Thus \(R := \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\}\) contains two different \(a_k\)-paths. From \(R \in \mathcal{A} \subseteq \mathcal{R}\) and \(r(Z \cup Z') = |Z \cup Z'| - 1\) we conclude \(r(R) = |R| - 1\).

(2) Suppose \(R - \{b_s\} \in \mathcal{F}\) for some \(s \in \{1, 2, ..., l - 1\}\).

Then \(R - \{b_s\} = \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_{s-1}\} \cup \{b_{s+1}, b_{s+2}, ..., b_l\}\). Hence \(b_j \in E - \sigma(\{a_1, a_2, ..., a_k\})\) for some \(j \in \{s + 1, s + 2, ..., l\}\), by (BR1), contrary to (1). Therefore \(R - \{b_s\} \notin \mathcal{F}\).

(3) Suppose \(R - \{a_k\} \in \mathcal{F}\).

Let \(T := \{a_1, a_2, ..., a_{k-1}\} \cup \{b_1, b_2, ..., b_{l-2}\}\).

Then \(T, T \cup \{a_k\}, T \cup \{b_{l-1}\}, T \cup \{a_k, b_{l-1}\}, T \cup \{b_{l-1}, b_{l-2}\}\) are feasible subsets of \(R - \{a_k\}\) or \(R - \{b_s\}\). We have \(b_i \notin [E - \sigma(\{b_1, b_2, ..., b_{l-1}\})] \cup [E - \sigma(\{a_1, a_2, ..., a_{k-1}\})] \cup [E - \sigma(\{a_1, a_2, ..., a_k\})]\), and hence \(T \cup \{b_i\} \notin \mathcal{F}\) and \(T \cup \{a_k, b_i\} \notin \mathcal{F}\), by (BR1). But since \(T \cup \{a_k, b_{l-1}, b_{l-2}\} = R \notin \mathcal{F}\), \((E, \mathcal{F})\) contains the greedoid \((C)\) as a minor, a contradiction.

Thus \(R - \{a_k\} \notin \mathcal{F}\), and so \(R - \{a_i\} \in \mathcal{F}\) for some \(i < k\). \(R - \{a_i\} = \{a_1, a_2, ..., a_{i-1}\} \cup \{b_1, b_2, ..., b_l\} \cup \{a_{i+1}, a_{i+2}, ..., a_k\}\), and from (BR1) it follows that \(a_j \in E - \sigma(\{b_1, b_2, ..., b_l\})\) for some \(j \in \{i + 1, i + 2, ..., k\}\).

(4) Suppose that \(\{b_1, b_2, ..., b_l, a_j\}\) is not an \(a_j\)-path.

Then \(\{b_1, b_2, ..., b_{l-1}, a_j\} \in \mathcal{F}\). But now \(\{a_1, a_2, ..., a_j\} \cup \{b_1, b_2, ..., b_{l-1}\}\) is a feasible subset of \(R - \{b_j\}\) that contains two different \(a_j\)-paths.

Thus \(\{b_1, b_2, ..., b_l, a_j\}\) is an \(a_j\)-path.

Let \(t_0 := \max\{t: R - \{a_i\} \in \mathcal{F}\}\), \(t_1 := \min\{t \in \{2, 3, ..., k\}: a_i \in E - \sigma(\{b_1, b_2, ..., b_i\})\}\).

\(a_{t_1} \notin E - \sigma(\{b_1, b_2, ..., b_i\})\) \((1 \leq t < l - 1)\) is already proved.

(5) Suppose \(t_1 < t_0\).
Then \( \{a_1, a_2, ..., a_t\} \cup \{b_1, b_2, ..., b_l\} \) would be a feasible subset of \( R - \{a_{t_0}\} \) containing two different \( a_{t_i} \)-paths.

(6) Suppose \( t_1 > t_0 + 1 \).

Let \( S := \{b_1, b_2, ..., b_l\} \cup \{a_1, a_2, ..., a_{t_0}, a_{t_0+1}, ..., a_t\} \in \mathcal{A} \). \( S \notin \mathcal{F} \), by the choice of \( t_0 \), \( r(S) = |S| - 1 \). \( S' := \{b_1, b_2, ..., b_l\} \cup \{a_1, a_2, ..., a_{t_0+1}\} \) is a feasible subset of \( R - \{a_{t_0}\} \). \( S' \cup \{a_{t_0}\} \notin \mathcal{F} \), by the choice of \( t_0 \), and \( S' \cup \{a_{t_0}, a_{t_0+1}\} \notin \mathcal{F} \), by (BR1) and \( t_1 > t_0 + 1 \). Hence \( |S'| = r(S' \cup \{a_{t_0}, a_{t_0+1}\}) < |(S' \cup \{a_{t_0}, a_{t_0+1}\}) \cap (R - \{a_{t_0}\})| \leq \beta(S' \cup \{a_{t_0}, a_{t_0+1}\}) \), contrary to \( S' \cup \{a_{t_0}, a_{t_0+1}\} \in \mathcal{A} \).

(7) Suppose \( t_1 = t_0 + 1 \).

\( W := \{a_1, a_2, ..., a_{t_0-1}\} \cup \{b_1, b_2, ..., b_{l-1}\} \), \( W \cup \{a_{t_0-1}\} \), \( W \cup \{b_l\}, \) \( W \cup \{a_{t_0-1}, b_l\} \) and \( W \cup \{a_{t_0-1}, b_l\} \) are feasible subsets of \( R - \{a_{t_0}\} \) or \( R - \{b_l\} \). From \( a_{t_0} \notin [E - \sigma(\{a_1, a_2, ..., a_{t_0-2}\})] \cup [E - \sigma(\{b_1, b_2, ..., b_{l-1}\})] \) and (BR1) we conclude \( W \cup \{a_{t_0}\} \notin \mathcal{F} \). Also \( W \cup \{a_{t_0}, b_l\} \notin \mathcal{F} \), by the choice of \( t_1 = t_0 + 1 \) and (BR1). Further \( W \cup \{a_{t_0-1}, a_{t_0}, b_l\} \notin \mathcal{F} \), by the choice of \( t_0 \), and so \((E, \mathcal{F})\) contains a minor of type (C), a contradiction. Thus, \( t_0 = t_1 \).

\( A := \{b_1, b_2, ..., b_{l-1}\} \cup \{a_1, a_2, ..., a_{t_0-1}\}, \) \( A \cup \{b_{l-1}\}, \) \( A \cup \{a_{t_0}\}, \) \( A \cup \{a_{t_0}, b_{l-1}\}, \) and \( A \cup \{a_{t_0-1}, b_l\} \) are feasible subsets of \( R - \{a_{t_0}\} \) and \( R - \{b_l\} \), respectively. We have \( b_l \notin \left[ E - \sigma(\{b_1, b_2, ..., b_{l-2}\}) \right] \cup \left[ E - \sigma(\{a_1, a_2, ..., a_{t_0-1}\}) \right] \cup \left[ E - \sigma(\{a_1, a_2, ..., a_{t_0-1}\}) \right] \cup \left[ E - \sigma(\{a_1, a_2, ..., a_{t_0-1}\}) \right] \), by assumption and (1). Thus, \( A \cup \{b_l\} \notin \mathcal{F} \) and \( A \cup \{a_{t_0}, b_l\} \notin \mathcal{F} \). Since \((E, \mathcal{F})\) does not contain greedoid (C) as a minor, \( A \cup \{b_{l-1}, b_l, a_{t_0}\} \in \mathcal{F} \) follows, contrary to the choice of \( t_0 \). This contradicts (1), and hence there exists \( i \in \{2, 3, ..., l\} \) and \( t \in \{1, 2, ..., k\} \) such that \( b_i \in E - \sigma(\{a_1, a_2, ..., a_k\}) \). This proves \( i \leq l \) and \( k \leq k \).

(b) Let \( Z \cup \{b_1\} \notin \mathcal{F} \) and \( j := \min \{t: \{a_1, a_2, ..., a_t\} \cup \{b_1\} \notin \mathcal{F} \} \).

We consider two cases.

(1) \( j = 1 \).

Then, \( \{a_1, b_1\} \notin \mathcal{F} \), and so \( \{a_1, b_2\} \) and \( \{b_1, a_2\} \) are bases of \( \{a_1, b_1, b_2\} \) and \( \{a_1, a_2, a_3\} \), respectively. Hence, \( l = 2, l = 1, k = 1, k = 2 \).

(2) \( j > 1 \).

\( T := \{a_1, a_2, ..., a_{j-1}\}, \) \( T \cup \{a_{j-1}\}, \) \( T \cup \{b_1\}, \) \( T \cup \{a_{j-1}, a_j\} \) and \( T \cup \{a_{j-1}, b_j\} \) are feasible sets, and \( T \cup \{a_j\} \notin \mathcal{F}, \) \( T \cup \{a_{j-1}, a_j, b_j\} \notin \mathcal{F}. \)

It follows that \( T \cup \{a_j, b_1\} \in \mathcal{F} \). Hence, \( a_j \in E - \sigma(\{b_1\}) \), by (BR1), and so \( l = 1 \). Let \( S \) be a base of \( \{a_1, a_2, ..., a_j, b_1, b_2\} \), \( \{b_1, b_2\} \in S \). We have \( |S| = j + 1 \). Now augment \( \{a_1, a_2, ..., a_j\} \) from \( S \). It follows \( \{a_1, a_2, ..., a_j, b_2\} \in \mathcal{F} \), and hence \( l = 2 \) and \( k = k = j \). [In the case \( Z \cup \{b_1\} \notin \mathcal{F} \) and \( Z' \cup \{a_j\} \in \mathcal{F} \) of part (a) we have \( \min \{t: \{a_1, a_2, ..., a_t\} \cup \{b_1\} \notin \mathcal{F} \} > 1 \), and so \( k = k \) follows from part (2) above.]

(c) Let \( Z \cup \{b_1\} \in \mathcal{F} \), i.e., \( l = l \geq 2 \).
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Let \( X := \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\} \) contains two different \( b_i \)-paths \( \{a_1, a_2, ..., a_k, b_i\} \) and \( \{b_1, b_2, ..., b_l\} \), and so \( r(X) = |X| - 1 \). Let \( i + 1 := \max\{t : X - \{b_i\} \in \mathcal{F}\} \), and \( j + 1 := \max\{t : X - \{a_1\} \in \mathcal{F}\} \). Then \( Y := \{b_1, b_2, ..., b_{i-1}\} \cup \{a_1, a_2, ..., a_j\} \in \mathcal{F} \). \( Y \cup \{b_i\} \), \( Y \cup \{a_{j+1}\} \), \( Y \cup \{b_i, a_{j+1}\} \) and \( Y \cup \{b_i, b_{i+1}\} \) are feasible subsets of \( X - \{b_{i+1}\} \) and \( X - \{a_{j+1}\} \), respectively. \( Y \cup \{b_{i+1}\} \) and \( Y \cup \{b_i, b_{i+1}, a_{j+1}\} \) are no feasible sets. Since \( (E, \mathcal{F}) \) does not contain a minor of type \( (C) \) it follows \( y^2 = (b_i, a_{j+1} \notin \mathcal{F} \), and hence \( i + 1 = E - \sigma(\{a_1, a_2, ..., a_{j+1}\}) \), by \( (BR1) \). Thus \( i = i + 1 \). Therefore \( X - \{b_i\} = \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\} \) is a feasible set. From \( \{a_1, a_2, ..., a_k\} \cap \{b_1, b_2, ..., b_l\} = \emptyset \) and Proposition 2.4, we conclude that \( X - \{x\} \in \mathcal{F} \) \( \forall x \in X \). Since \( r(Z' \cup \bar{Z}) = |Z \cup \bar{Z}'| - 1 \), this means that \( (Z \cup \bar{Z}') - \{x\} \in \mathcal{F} \) iff \( x \in X \). If \( Z \cup \{b_1\} \notin \mathcal{F} \), \( Z' \cup \{a_{j+1}\} \notin \mathcal{F} \) then \( i = 1 = k \), and the proposition is trivially true.

(d) Let \( k \geq 2 \) and \( j \in \{2, 3, ..., k\} \).

First we show that \( Z_j := \{b_1, b_2, ..., b_j, a_0, a_{j-1}, ..., a_k\} \) is an \( a_j \)-path. For \( k = j \) this is already proved. Since \( (Z \cup Z') - \{a_1\} \) is a base of \( Z \cup Z' \) there is at least one \( a_j \)-path not containing \( a_1 \). \( \{b_1, b_2, ..., b_l\} \) is a feasible subset of this path, by the definition of \( l \). Let \( Z_j \) be this path and suppose that \( \bar{Z}_j - \{a_j\} \) is an \( a_j \)-path. W.l.o.g. we may assume that \( b_1b_2\cdots b_a b_k a_{k-1} \cdots a_{i-j+i-1} a_i \) is a feasible ordering of \( \bar{Z}_j - \{a_j\} \) and \( i \geq j \). Suppose \( i - 1 \neq j \). Then \( \{a_1, a_2, ..., a_j\} \) and \( \{a_1, a_2, ..., a_{i-2}\} \) are feasible subsets of \( \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\} \) \( \{a_{i-1}\} \), and so \( \{a_j\} = \{a_1, a_2, ..., a_j\} \cap \{a_1, a_2, ..., a_{i-2}\} \in \mathcal{F} \). Hence, \( j = 1 \), a contradiction. Thus \( i = j + 1 \) and \( \bar{Z}_j = Z'_j \).

Each \( a_j \)-path \( Z \cup Z' \) is contained in \( \{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_l\} \). If \( B \) is an arbitrary base of \( Z \cup Z' \) and \( a_j \in B \) then \( Z_j \) or \( \{a_1, a_2, ..., a_j\} \subseteq B \). Hence the proposition follows.

(e) We have already proved that \( (Z \cap Z') \cap \{a_1, a_2, ..., a_k, b_1, b_2, ..., b_l\} = \emptyset \).

Let \( X := \{a_1, a_2, ..., a_k, b_1, b_2, ..., b_l\} \), then \( \{a_{k+1}, a_{k+2}, ..., a_m, b_{l+1}, b_{l+2}, ..., b_n\} \in \mathcal{F} \). Thus both \( a \)-paths \( \{a_{k+1}, a_{k+2}, ..., a_m, b_{l+1}, b_{l+2}, ..., b_n\} \) with respect to \( \mathcal{F} \) must be identical, which means that \( Z \cap Z' = \{a_{k+1}, a_{k+2}, ..., a_m\} = \{b_{l+1}, b_{l+2}, ..., b_n\} \).

The proof of (f) is similar to the proof of (e) and is omitted.

3. The Basis Exchange Property

The family of bases of an undirected branching greedoid defines a matroid, hence this family satisfies the basis exchange property.

Now we prove that the bases of \( (E, \mathcal{F}) \) also have this property.

Remark 3.1. Let \( A \) be a feasible set, \( a \in A, b \in E - A, A \cup \{b\} \notin \mathcal{F} \),
\( A - a \cup b \in \mathcal{F} \) and let \( X_a := \ker(A - \{a\}) \). Then \( X_a \cup \{a\}, X_a \cup \{b\} \in \mathcal{F} \), and \( a \in Z \) for each \( x \)-path \( Z \subseteq A \) \( (x \in A - X_a) \).

**Proof.** Let \( Z_a \) be an \( a \)-path, \( Z_a \subseteq A \), then \( Z_a - \{a\} \subseteq X_a \), and hence \( X_a \cup \{a\} = X_a \cup Z_a \in \mathcal{F} \). Consider \( x \in A - X_a \) and a path \( Z_x \subseteq A \) with head \( x \). If \( a \notin Z_x \), then \( x \in Z_x \subseteq X_a \), contrary to the choice of \( x \). Hence, \( a \in Z_x \). We have \( A - a \cup b \in \mathcal{F} \), and \( X_a \) cannot be augmented by any element \( x \in A - (X_a \cup \{a\}) \) hence \( X_a \cup \{b\} \) is a feasible set. 

**Proposition 3.2.** Let \( A \in \mathcal{F} \), \( a \in A \), \( b \in E - A \), \( X_a := \ker(A - \{a\}) \) and let \( X_a \cup \{b\} \in \mathcal{F} \), \( A \cup \{b\} \notin \mathcal{F} \). Then \( A - a \cup b \in \mathcal{F} \).

**Proof.** For \( |A| = 1 \) the proposition is certainly true. Let \( |A| \geq 2 \) and suppose that the proposition is true for all feasible sets containing less than \( n \) elements. Let \( |A| = n \). If \( X_a \cup \{a\} = A \) then \( A - a \cup b = X_a \cup \{b\} \in \mathcal{F} \).

Hence we may assume that \( A - (X_a \cup \{a\}) \neq \emptyset \).

Let \( x \in A - (X_a \cup \{a\}) \) such that \( A - \{x\} \) is feasible. If \( A - x \cup b \notin \mathcal{F} \) then \( A - \{x\} \) is a base of \( A - x \cup b \). Since \( X_a = \ker(A - \{a\}) = \ker(A - \{x, a\}) = (A - \{x\}) - a \cup b \in \mathcal{F} \) follows by induction hypothesis. Now augment this set from \( A \). We get \( A - a \cup b \in \mathcal{F} \). Thus we may also assume that \( A - x \cup b \in \mathcal{F} \) \((x \in A - (X_a \cup \{a\}) \) s.t. \( A - \{x\} \in \mathcal{F} \).

Let \( x, y \) be two different elements from \( A - (X_a \cup \{a\}) \) such that \( A - \{x\}, A - \{y\} \in \mathcal{F} \). Then \( A - x \cup b, A - y \cup b, A - \{x, y\} \in \mathcal{F} \). \( X_a \cup \{b\} \) and \( A - \{x, y\} \) are feasible subsets of \( A - x \cup b \), and hence \( (A - \{x, y\}) \cup \{b\} = (X_a \cup \{b\}) \cup (A - \{x, y\}) \in \mathcal{F} \). Thus \( \emptyset, \{x\}, \{y\}, \{b\}, \{x, b\}, \{x, y\}, \{y, b\} \subseteq \mathcal{F} \cap (A - \{x, y\}) \). Since \( (E, \mathcal{F}) \) does not contain greedoid (D) as a minor, \( \{x, y, b\} \in \mathcal{F} \). Further \( A \cup \{b\} \in \mathcal{F} \), contrary to our assumption.

Thus, \( A - \{x\} \in \mathcal{F} \) for exactly one \( x \in A - (X_a \cup \{a\}) \).

Let \( x_m \) be this element and let \( Z_m \subseteq A \) be a path with head \( x_m \). We show \( A = X_a \cup Z_m \).

Suppose that this is not the case. Let \( y \in A - (X_a \cup Z_m) \). Then a \( y \)-path \( Z_y \subseteq A \) is contained in a maximal path \( Z \subseteq A \). Let \( c \) be the head of \( Z \). We have \( c \neq x_m \) and so \( A - \{x_m\} \neq A - \{c\} \notin \mathcal{F} \), in contrary to our assumption. Thus \( A = X_a \cup Z_m \).

\( Z_m - X_a \) and \( \{b\} \) are paths in \((E - X_a)/X_a\). Further \((A - x_m \cup b) - X_a \in \mathcal{F} / X_a \).

Let \( c \in (A \setminus \{x_m\}) \cap (A - X_a) \) such that \( W := A - \{x_m, c\} \in \mathcal{F} \). Hence \( W, W \cup \{b\}, W \cup \{c\}, W \cup \{b, c\}, \) and \( W \cup \{x_m, c\} \) are feasible subsets of \((A - x_m \cup b) \) and \( A \), respectively. \( W \cup \{x_m\} = \emptyset \notin \mathcal{F} \), \( W \cup \{x_m, c, b\} = A \cup \{b\} \notin \mathcal{F} \), and hence \( W \cup \{x_m, b\} \in \mathcal{F} \), since \((E, \mathcal{F})\) does not contain any minor of type (C). We have \( x_m \notin E - \sigma(W) \), and hence \( x_m \in E - \sigma(X_a \cup \{b\}) \), by (BR1). Thus \( X_a \cup \{b, x_m\} \) is a feasible set and \( \{b, x_m\} \) is a
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path with head $x_m$ in $\mathcal{F}/X_a$. Now $U := \{b, x_m\}$ and $V := Z_m - X_a$ are two different $x_m$-paths and $(U \cup V) - \{x_m\} \in \mathcal{F}/X_a$, and hence $(U \cup V) - \{a\} \in \mathcal{F}/X_a$, by Proposition 2.4. Thus $A - a \cup b = X_a \cup [(U \cup V) - \{a\}] \in \mathcal{F}$.

**LEMMA 3.3.** If $B, B' \in \mathcal{B}$ and $a \in B - B'$ there exists $b \in B' - B$ such that $B - a \cup b \in \mathcal{F}$.

**Proof.** Let $a \in B - B'$ and $X_a := \ker(B - \{a\})$. Then $r(X_a \cup \{c\}) = r(X_a)$ ($c \in B - (X_a \cup \{a\})$), by 3.1, and so $r(X_a \cup (B - (X_a \cup \{a\}))) = r(X_a)$. Suppose $r(X_a \cup \{b\}) = r(X_a)$ ($b \in B' - B$). It follows that $r(X_a \cup (B' - B)) = r(X_a)$, and hence $r(B) > r(X_a) = r(X_a \cup (B' - B) \cup (B - (X_a \cup \{a\}))) = r((B - \{a\}) \cup (B' - B))$, since $a \notin B'$, $B' \subseteq (B - \{a\}) \cup (B' \cup B)$. Thus, by the monotonicity of the rank function, we have a contradiction, and hence there exists a $b \in B' - B$ such that $X_a \cup \{b\} \in \mathcal{F}$. Now apply 3.2 and the proposition follows.

4. CONSTRUCTION OF A GRAPH

Let $B = \{a_1, a_2, \ldots, a_q\}$ be a base of $E$ with feasible ordering $a_1, a_2, \ldots, a_q$. Now we associate with $B$ a rooted tree such that a branching greedoid of this tree is identical with $(B, \mathcal{F} \mid_B)$.

Let $Z_j \subseteq B$ be an $a_j$-path ($1 \leq j \leq q$). We define (Fig. 3)

$$
P_0 := \{a_j \in B: \{a_j\} \in \mathcal{F}\}
$$

$$
P_i := \{a_i\} \cup \{a_j \in B: Z_j - \{a_j\} = Z_i\} \quad (1 \leq i \leq q).
$$

Let $G(B, B)$ denote the intersection graph of the family of sets $P_0, P_1, \ldots, P_q$. Obviously, each $a_j \in B$ is in exactly two sets $P_{j+1}$ and $P_{j+2}$ and $P_{j+1} \cap P_{j+2} = \{a_j\}$. That is, the edges of the graph can be identified with the elements of $B$.

**Figure 3**
Now we associate with \((E, \mathcal{F})\) a rooted graph such that the rooted tree constructed above is a spanning tree of this graph:

\[
P_0 := \{ x \in E : \{ x \} \in \mathcal{F} \}
\]

\[
P_i := \{ a_i \} \cup \{ x \in E : Z_i \cup \{ x \} \text{ is an } x\text{-path} \}
\]

\[
\cup \{ x \in E : \text{there exists } Y \subseteq B - \{ a_i \} \text{ s.t.} \ 
(Y, a_i, x) \text{ is a } d\text{-triple} \} \quad (1 \leq i \leq q).
\]

Let \(G = G(E, B)\) denote the intersection graph of \(\{P_0, P_1, \ldots, P_q\}\). If \(P_i \cap P_j \neq \emptyset\) then \(P_i\) and \(P_j\) are connected by \(|P_i \cap P_j|\) edges \((i \neq j)\). In the remaining part of this section we prove that the edges of \(G\) can be identified with the elements of \(E\).

**Remark 4.1.** Let \(a \in B \in \mathcal{B}, X_a = \ker(B - \{a\})\), and \(e \in E - B\) such that \(X_a \cup \{ e \} \in \mathcal{F}\) and \(X_a \cup \{ a, e \} \notin \mathcal{F}\). Then \(X_a\) is a maximal feasible subset of \(B\) augmentable by \(e\).

**Proof.** We have \(a \in Y \ (X_a \subseteq Y \subseteq B, Y \in \mathcal{F})\). If \(Y \cup \{ e \} \in \mathcal{F}\) then \((X_a \cup \{ a \}) \cup (X_a \cup \{ e \}) = X_a \cup \{ a, e \} \in \mathcal{F}\).

**Remark 4.2.** Let \(X \cup \{ a \} \subseteq B \in \mathcal{B}, e \in E - B\), and let \((X, a, e)\) be a \(d\)-triple. Then \((Y, e, a)\) is a \(d\)-triple, too, for every feasible set \(Y\) such that \(X \subseteq Y \subseteq \ker(B - \{a\})\).

**Proof.** Let \(X_a := \ker(B - \{a\})\). We augment \(X \cup \{ e \}\) from \(B\) to a base \(B'\) of \(E\). Then \(a \notin B'\). Now we augment \(X_a\) from \(B'\). We have \((E - \sigma(X_a)) \cap B' = \{ e \}\), by the choice of \(X_a\). Since \(X \cup \{ a, e \} \notin \mathcal{F}\) it follows that \(X_a \cup \{ a, e \} \notin \mathcal{F}\). Consider \(Y \in \mathcal{F}\) such that \(X \subseteq Y \subseteq X_a\). Obviously, \(Y \cup \{ e \}, Y \cup \{ a \} \in \mathcal{F}, \text{ and } Y \cup \{ a, e \} \notin \mathcal{F}\). Thus \((Y, a, e)\) is a \(d\)-triple.

**Remark 4.3.** For any \(e \in E - B\) there exists an \(e\)-path \(Z\) such that \(Z - \{ e \} \subseteq B\), a feasible set \(X \subseteq B\), and an element \(a \in B - Z\) such that \((X, a, e)\) is a \(d\)-triple.

**Proof.** From \(r(B \cup \{ e \}) = r(B)\) we have \(B \cup \{ e \} \in \mathcal{R} = \mathcal{A}\). Hence there exists a minimal feasible set \(Y \subseteq B\) such that \(Y \cup \{ e \} \in \mathcal{F}\). By \((BR1)\), \(Y\) is a path. Let \(X\) be a maximal feasible subset of \(B\) such that \(Y \subseteq X\) and \(X \cup \{ e \} \in \mathcal{F}\). Then \(X \notin B\), and so \(X \cup \{ a \} \in \mathcal{F}\) for some \(a \in B - X\). Thus \(X \cup \{ a, e \} \notin \mathcal{F}\), by the maximality of \(X\), and hence \(\lambda(X \cup \{ a \}) = \lambda(X \cup \{ e \})\).

**Proposition 4.4.** Let \(a, b, c \in B\), and let \(Z_a, Z_b, Z_c \subseteq B\) be paths with head \(a, b\) and \(c\), respectively. Let \(e \in E - B\). If \(Z_a \cup \{ e \}, Z_b \cup \{ e \}, \text{ and } Z_c \cup \{ e \}\) are \(e\)-paths, then \(|\{ a, b, c \}| \leq 2\).
Proof. Let \( a, b, c, e, Z_a, Z_b, Z_e \) as prescribed as above. Suppose \( a \neq b \neq c \neq a \).

We have, \( e \in \sigma(Z_a \cup Z_c) \), otherwise \( Z_a \cup Z_c \cup \{ e \} \) would contain two different \( e \)-paths. Hence, \( Z_a \cup Z_b \cup Z_c \cup \{ e \} \notin \mathcal{F} \). Consider two feasible sets \( X \subseteq Z_b, Y \subseteq Z_c \) such that \( |X| + |Y| \) is maximal and \( Z_a \cup X \cup Y \cup \{ e \} \in \mathcal{F} \). Suppose \( X \neq Z_b \) and \( Y \neq Z_c \). Let \( x \in Z_b - X \) and \( y \in Z_c - Y \) such that \( X \cup \{ x \}, Y \cup \{ y \} \in \mathcal{F} \). Hence \( Z_a \cup X \cup Y \cup \{ e \} =: W \) and \( V := Z_a \cup X \cup Y \cup \{ x, y \} \) are feasible sets, but \( W \) is not augmentable from \( V \), by the choice of \( X \) and \( Y \). W.l.o.g. we may assume \( Y = Z_c \). Now, \( Z_a \cup Z_c \cup \{ e \} \in \mathcal{F} \), contrary to \( e \in \sigma(Z_a \cup Z_c) \).

\[ \text{PROPOSITION 4.5.} \] Let \( a, b, c \in B \), \( e \in E - B \), and \( X, Y, Z \subseteq B \) such that \( (X, a, e), (Y, b, e), \) and \( (Z, c, e) \) are \( d \)-triples. Then \(| \{ a, b, c \} | \leq 2 \).

Proof. W.l.o.g. we may assume \( X = \ker(B - \{ a \}), Y = \ker(B - \{ b \}), Z = \ker(B - \{ c \}) \).

(1) Let \( a \notin Y \). Then \( a \in E - \sigma(X \cup Y) \), and hence \( X \cup Y \subseteq X \), by the choice of \( X \). Since \( Y \cup \{ e, b \} \notin \mathcal{F} \) it follows that \( X \cup \{ e, b \} \notin \mathcal{F} \). Thus \( b \notin X \), and hence \( X \cup \{ b \} \in \mathcal{F} \). So \( X \subseteq Y \), by the choice of \( Y \). Therefore \( X = Y \). \( X \cup \{ a \} \) and \( X \cup \{ a, b \} \) are feasible subsets of \( B \). Thus, \( a = b \), by the choice of \( X \) and \( Y \). Analogously, we consider the cases \( c \notin Y \), \( b \notin X \), \( b \notin Z \), \( a \notin Z \), and \( c \notin X \).

(2) Let \( a, c \in Y, b, c \in X, \) and \( a, b \in Z \). Suppose \( a \neq b \neq c \neq a \). Let \( Z_i \subseteq U_i \) be a path such that \( Z_i \cup \{ e \} \) is a path with head \( e \) \((i = 1, 2, 3; U_1 = X, U_2 = Y, U_3 = Z) \). It follows from 4.4 that at least two of these paths are identical. Let \( Z_1 = Z_2 \). We have \( a \in Y - X \) and \( b \in X - Y \). Augment \( T := X \cap Y \) from \( X \) and \( Y \). We get two feasible sets \( T \cup \{ x \}, T \cup \{ y \} \) with \( x \in X - Y \) and \( y \in Y - X \). \( T \cup \{ e \} = (X \cap Y) \cup (Z_1 \cup \{ e \}), T \cup \{ x, y \}, T \cup \{ x, e \}, \) and \( T \cup \{ y, e \} \) are feasible sets, too, and hence \( T \cup \{ x, y, e \} \in \mathcal{F}, \) \( X \cup \{ e, y \}, Y \cup \{ e, x \} \notin \mathcal{F} \), by the choice of \( X \) and \( Y \)(cf. 4.1), but \( X \cup \{ e, y \} = (X \cup \{ e \}) \cup (T \cup \{ y \}) \in \mathcal{A}, \) \( Y \cup \{ e, x \} = (Y \cup \{ e \}) \cup (T \cup \{ x \}) \in \mathcal{A} \), \( X \cup \{ y \} \in \mathcal{F} \) and \( Y \cup \{ x \} \in \mathcal{F} \).

We have: \( r(X \cup Y \cup \{ e, x, y \}) = r(X \cup Y \cup \{ e \}) - r(X \cup Y) - |X \cup Y|, \)
\[ r((X \cup \{ e, y \}) \cap (Y \cup \{ e, x \})) = r(T \cup \{ e, x, y \}) = |X \cap Y| + 3, \]
and hence \( r((X \cup \{ e, y \}) \cup (Y \cup \{ e, x \})) + r((X \cup \{ e, y \}) \cap (Y \cup \{ e, x \})) > r(X \cup \{ e, y \}) + r(Y \cup \{ e, y \}) = |X \cap Y| + |X \cap Y| + 2, \) contrary to the fact that \( r \) is submodular on \( \mathcal{A} \).

Thus at least two elements of \( a, b, c \) must be identical. \[ \square \]

\[ \text{PROPOSITION 4.6.} \] Let \( X, Y \) be feasible subsets of \( B \), \( a, b \in B, a \neq e \) and let
If \((X, a, e)\) and \((Y, b, e)\) are \(d\)-triples then there exists an \(a\)-path \(Z_a \subseteq B\) and a \(b\)-path \(Z_b \subseteq X\) such that \(Z_a \cup \{e\}\) and \(Z_b \cup \{e\}\) are paths with head \(e\).

**Proof.** Let \(X, Y, a, b, e\) as prescribed as above and let \((X, a, e)\) and \((Y, b, e)\) be two \(d\)-triples. W.l.o.g. we may assume \(X = \ker(B - \{a\})\) and \(Y = \ker(B - \{b\})\).

Suppose \(a \notin Y\).

Then \(X \cup Y \neq X \cup Y \cup \{a\} \in \mathcal{F}\), and hence \(Y\) is a subset of \(X\), by the choice of \(X\). If \(b \notin X\), then \(X \cup \{b\} \in \mathcal{F}\) and \(X = Y\), by the choice of \(Y\). But then \(X \cup \{e\}\) would be a feasible set which cannot be augmented from \(X \cup \{a, b\}\). Thus \(b \in X\). Now, \(Y \cup \{b\}\) and \(Y \cup \{e\}\) are feasible subsets of \(X \cup \{a\}\). Since \(Y \cup \{b, e\} \notin \mathcal{F}\), this is a contradiction. Thus \(a \in Y\) and \(b \in X\).

Let \(Z_a \subseteq Y\) be an \(a\)-path and let \(Z_b \subseteq X\) be a path with head \(b\). \(Z_a - \{a\}\) and \(Z_b - \{b\}\) are feasible subsets of \(X \cap Y\), and hence \(X \cap Y\), \((X \cap Y) \cup \{a\}\), \((X \cap Y) \cup \{b\}\), and \((X \cap Y) \cup \{a, b\}\) are feasible. Let \(Z \subseteq X \cup \{e\}\) be an \(e\)-path, and \(Z := Z - \{e\}\).

Suppose that \(b \notin Z\). Then \(Y \cup \{b\} \cup Z \in \mathcal{F}\) and hence \(Z \subseteq Y\), by the choice of \(Y\). Therefore \(Z \subseteq X \cap Y\). Thus \((X \cap Y) \cup \{e\}\), \((X \cap Y) \cup \{a, b\}\), and \((X \cap Y) \cup \{a, e\}\) are paths with head \(b\) # \(a\), and \((X \cap Y) \cup \{e, b\}\) follows. Now we have: 
\[
\begin{align*}
r(X \cup \{e, a\}) &= |X| + 1, \\
r(Y \cup \{e, b\}) &= |Y| + 1, \\
r((X \cup Y) \cup \{a, b\}) &= r(X \cap Y) = |X| + |Y| - |X \cap Y| \quad \text{(cf. 4.1)}, \\
r((X \cup \{e, a\}) \cap (Y \cup \{e, b\})) &= r((X \cap Y) \cup \{a, b\}) = |X \cap Y| + 3. 
\end{align*}
\]
From this we conclude 
\[
r((X \cup \{e, a\}) \cup (Y \cup \{e, b\})) + r((X \cup \{e, a\}) \cap (Y \cup \{e, b\})) < r((X \cup \{e, a\}) \cup (Y \cup \{e, b\})) + r((X \cup \{e, a\}) \cap (Y \cup \{e, b\}))
\]
otherwise \(Y \cup \{e\}\) would contain two different \(e\)-paths \(\bar{Z}\) and \(\bar{Z}'\). We may assume \(Z - \{x\} \subseteq Y \cup \{b\}\). Let \(S := (Z' - \{e\}) \cup (Z - \{x\})\). Then \((S, x, e)\) is \(d\)-triple, contrary to \(x \neq a, x \neq b\), and 4.5.

Thus \(Z - \{e\} = Z_b\) and \(Z' - \{e\} = Z_a\). 

**Remark 4.7.** If \(Z, Z' \subseteq B \cup \{e\}\) are paths with head \(e\) s.t. \(Z - \{e\}\) is an \(a\)-path and \(Z' - \{e\}\) is path with head \(b \neq a\), then \(((Z \cup Z') - \{a, b, e\}, a, e)\) and \(((Z \cup Z') - \{a, b, e\}, b, e)\) are \(d\)-triples.

**Proof.** Compare Proposition 2.4.

Now, from 4.1.–4.7 we deduce the main result of this section:

**Lemma 4.7.** \(\{|i : i \in \{0, 1, \ldots, q\}, a \in P,\} = 2\ a \in E\).
Thus the edges of $G(E, B)$ may be labeled by the elements of $E$.

Remark 4.9. If $E' \subseteq E$, $B' \subseteq B$ and if $B'$ is a base of $E'$, then $G(E', B')$ is a subgraph of $G(E, B)$.

5. Proof of the Theorem

In 5.1–5.5 we first show that the definition of $G(E, B)$ does not depend on the choice of $B$.

Lemma 5.1. If $B$ and $B' = B - a \cup \{e\}$ are bases of $(E, \mathcal{P})$ then $G(B \cup \{e\}, B)$ and $G(B \cup \{e\}, B')$ are isomorphic.

Proof. Let $a \neq e, R, R - a \cup \{e\} \subseteq B, X'_a := \ker(R - \{a\}), X'_e := \ker(B' - \{e\})$. Then $X'_a = X'_e$ and $X'_a \cup \{a\}, X'_a \cup \{e\} \in \mathcal{P}$. Let $a_1 a_2 \ldots a_{j+1} \ldots a_q$ be a feasible ordering of $B, a = a_{j+1}, X_a = \{a_1, a_2, \ldots, a_j\}$. Let $Z_i \subseteq B$ be an $a_i$-path ($1 \leq i \leq q$), $Z'_e$ be an $e$-path, $Z'_e \subseteq B'$ and let $Z'_i \subseteq B'$ be an $a_i$-path ($1 \leq i \leq q, t \neq j + 1$).

We show that the linegraphs of $G_1 := G(B \cup \{e\}, B)$ and $G_2 := G(B \cup \{e\}, B')$ are isomorphic.

We use the following notation: $x \perp_B y$ ($x$ is incident with $y$ relative to $B$) if there is a vertex $P_i$ in a graph defined relative to $B$ such that $x, y \in P_i$.

We show: $x \perp_B y$ iff $x \perp_B y$ ($x, y \in B \cup \{e\}$). For this it suffices to show $x \perp_B y \Rightarrow x \perp_B y$ ($x, y \in B \cup \{e\}$).

Let $x \perp_B y$ We consider different cases.

(0) If $\{x\}, \{y\} \in \mathcal{P}$ then certainly $x \perp_B y$.

(1) Let $x = a_i \in X_a$ and $\perp_B y$.

(1a) $y = a_s \in B$.

Then $s \leq j + 1$. (1a) If $Z_i - \{a_i\}$ is an $a_s$-path then we have $Z'_i = Z_s, Z_i = Z'_i$, and hence $a_i \perp_B a_s$.

(1ab) If $Z_s = Z_i \cup \{a_s\}$ and $s < j + 1$ then $Z_i = Z'_i$ and $Z_s = Z'_s$. Thus $a_i \perp_B a_s$. 

Figure 4
(1a) If \( Z_i \cup \{a_i\} = Z_s \) and \( s = j + 1 \) then \( Z_i = Z'_s \) is a path in \( B' \) and \( Z'_s \cup \{a\} \) is an \( a \)-path. Therefore \( a \perp B a_s = a \).

(1b) Let \( y = e \).

If \( Z_i \cup \{e\} \) is a path, then \( a \perp B e \), since \( Z'_i = Z_i \subseteq B' \). There is no \( d \)-triple containing \( a \) and \( e \), by 2.3 and \( \{a_i, e\} \subseteq X_a \cup \{e\} \in \mathcal{F} \).

(2) Let \( x = e \perp_B a_s \) (cf. Fig. 4).

The case \( s < j \) has already been considered. Let \( s > j \). \( Z_s \cup \{e\} \) is not an \( e \)-path, since \( a_s \notin X_a \cup \{e\} \in \mathcal{F} \). Hence there exists a feasible set \( Y \subseteq B \) such that \( (Y, e, a_s) \) is a \( d \)-triple. W.l.o.g. we may assume that \( Y = \ker (B - \{a_s\}) \). If \( a_s = a \) then \( Y = X_a \subseteq B' \), and hence \( e \perp_B a_s \). Let \( s > j + 1 \). We show that \( Z'_s - \{a_s\} \) is an \( e \)-path in \( B' \). Since \( Y \cup \{e\} \in \mathcal{F} \) we have \( B - a_s \cup e \in \mathcal{F} \) (cf. 3.2 and 3.3). Certainly \( a \in Y \), otherwise \( Y \cup \{a_s, e\} \) would be a feasible subset of \( B' = B - a \cup e \). \( X_a \) and \( Y \cup \{e\} \) are feasible subsets of \( B - a_s \cup e \), and hence \( X_a \cup Y \cup \{e\} \in \mathcal{F} \). Thus \( Z_s \subseteq Y \), by the choice of \( Y \). Since \( a_s \notin X_a \), \( a \in Z_{j+1} \subseteq Z_s \) follows. Therefore \( e \in Z_s \) and so \( Z'_s \subseteq Z_s \). Suppose that \( Z'_s - \{a_s\} \) is a \( z \)-path, and \( z \neq e \). \( Z_s \cup Z'_s - \{a_s\} \) is a feasible subset of \( B - a_s \cup e \) and hence \( z \notin Z_s \), for otherwise \( (Z_s \cup Z'_s) - \{a_s\} \) would contain two different \( z \)-paths (one of these paths contains \( e \), the other one does not). Now we apply Proposition 2.4, and \( (Z_s \cup Z'_s) - \{z\} \in \mathcal{F} \) follows. But then \( Z_s \cup Z'_s \) would be a feasible subset of this set, contradicting Proposition 2.3. Thus \( Z'_s - \{a_s\} = Z'_s \) and \( e \perp_B a_s \).

(3) Let \( x = a \).

The cases \( y \in X_a \cup \{e\} \) have already been considered in (1) and (2). Let \( y = a_s \in \{a_{j+2}, a_{j+3}, \ldots, a_g\} \). Then \( Z_s - \{a_s\} = Z_{j+1} \). There exists a feasible set \( Y \subseteq B - a \cup e \) and \( z \in B' \) such that \( (Y, a, z) \) is a \( d \)-triple. We may assume \( Y = \ker (B' - \{z\}) \). Hence \( B' - z \cup a = B - z \cup e \) is also a base of \( E \), by 3.2 and 3.3. Let \( Z \subseteq B \) be a \( z \)-path. We can show as in part (2) that \( Z - \{z\} \) is an \( a \)-path: \( Z - \{z\} = Z_{j+1} \), i.e., \( Z_s = Z - z \cup a_s \). In particular, \( z \notin X_a \). If \( z = a_s \), then \( a \perp_B a_s \). Let \( z \neq a_s \) and let \( Z' \subseteq B' \) be a path with head \( z \). Then \( e \in Z' \) and \( Z \subseteq Z_s \cup Z' \) (Fig. 5).
This means that \( Z_s \cup Z' \notin \mathcal{F} \) and \( (Z_s \cup Z') - \{a_s\} \notin \mathcal{F} \), since \( Z, Z' \subseteq (Z_s \cup Z') - \{a_s\} \). \( R := Z_s \cup (Z' - \{z\}) \) is a feasible subset of \( B - z \cup e \).

Let \( a_{i_1}a_{i_2} \cdots e \cdots a_{i_p} \) be a feasible ordering of \( Z' \) and let \( a_{j_1}a_{j_2} \cdots a_{j_q} \) be a feasible ordering of \( Z_s \). \( R - \{a_s\} = (Z \cup Z') - \{z\} \) is a feasible set. Since \( a \notin Z' \) it follows that \( (Z \cup Z') - \{a\} = (Z_{j+1} \cup Z') - \{a\} \in \mathcal{F} \), by Proposition 2.4. We augment \( \{a_{j_1}, a_{j_2}, \ldots, a_{j_q}\} \cup \{a_{i_1}, a_{i_2}, \ldots, a_{i_p}, z\} \) from \( R \).

We have \( Z \cup Z' \notin \mathcal{F} \), and hence \( ((Z \cup Z') - \{a\}) \cup \{a_s\} \in \mathcal{F} \). Therefore \( a_s \in E - \sigma(Z') \), by (BR1) and \( a_s \notin E - \sigma(Z_{j+1} - \{a\}) \).

Suppose \( a_s \in E - \sigma(Z' - \{z\}) \). Then \( z \in Z_s' \) and \( Z_s' \cup Z_s' \) is a feasible subset of \( B - z \cup e \). On the other hand, \( a \in Z_s \) and \( e \in Z_s' \), and so \( Z_s \) and \( Z_s' \) are two different \( a_s \)-paths contained in \( B - z \cup e \). This is a contradiction. Thus \( Z_s' = Z' \cup \{a_s\} \) and \( a_s \perp_{B'} z \).

We have \( e \in Z_{j+1}' = Z' - \{z\} \), thus \( a_{i_p} \in B' - X_a \). In particular, \( a_{i_p} \notin Z - \{z\} = Z_{j+1} \), and hence \( (Z \cup Z') - \{a_{i_p}\} \in \mathcal{F} \), by 2.4.

(i) Let \( e \neq a_{i-p} \) and suppose that \( a \perp_{B'} a_{i_p} \).

If \( U := \ker(B' - \{a_{i_p}\}) \) and if \( (U, a_{i_p}, a) \) is a \( d \)-triple, then \( z \notin U \) and so \( U \cup \{a, a_{i_p}\} \) would be a feasible subset of \( B - z \cup e \), contradicting \( \lambda(U \cup \{a\}) = \lambda(U \cup \{a_{i_p}\}) \). Thus \( Z_{i_p}' \cup \{a\} \) or \( Z_{i_p}' - a_{i_p} \cup a \) is an \( a \)-path. But in both cases \( B - z \cup e \) would contain two different \( a \)-paths, since \( e \in Z_{i_p}' - \{a_{i_p}\} \) and \( e \notin Z_{j+1} \). Thus we have a \( L_{B'} a_{i_p} \), furthermore \( z \perp_{B'} a_{i_p} \), \( a_s \perp_{B'} a_{i_p} \), \( z \perp_{B'} a_s \), and \( z \perp_{B'} a_s \). It follows that \( a \perp_{B'} a_s \), as required (Fig. 6).

(ii) Let \( a_{i_p} = e \).

Then \( Z'_{i_p} = Z_{j+1} - a \cup e \), \( a \perp_{B'} e \), \( a_s \perp_{B'} e \), \( z \perp_{B'} e \), \( z \perp_{B'} a_s \), \( z \perp_{B'} a_s \), and \( a \perp_{B'} a_s \) follows (Fig. 7).
(4) Let \( x = a_i \in B - (X_a \cup \{a\}) \).
We may assume that \( y = a_s \in B - (X_a \cup \{a\}) \).

(4a) Let \( Z_i - \{a_i\} \) be an \( a_s \)-path.
Then \( Z_i, Z'_i \subseteq B \cup \{e\} \) and so \( r(Z_i \cup Z'_i) = |Z_i \cup Z'_i| - 1 \).

(4aa) Let \( (Z_i \cup Z'_i) - \{a_i\} \in \mathcal{F} \).
Hence \( (Z_i \cup Z'_i) - \{z\} \in \mathcal{F} \) (\( z \in (Z_i \cup Z'_i) - (Z_i \cap Z'_i) \)), by Proposition 2.4 (Fig. 8). Suppose \( a_s \in Z'_i \). This means that \( (Z_i \cap Z'_i) - \{a_i\} = Z_i - \{a_i\} = Z'_i - \{a_i\} \) (cf. 2.4), contrary to \( a \in Z_i, a \notin Z'_i \). Thus \( a_s \notin Z_i \cap Z'_i \) and so \( Z_i \cup \{a_s\} \) is an \( a_s \)-path, again by 2.4(c). Thus \( a_i \uparrow a_s \).

(4ab) Let \( (Z_i \cup Z'_i) - \{a_i\} \notin \mathcal{F} \) and let \( a_{i_1}a_{i_2} \cdots a_{i_l}a_i \) and \( a_{j_1}a_{j_2} \cdots a_{j_l}a_i \) be feasible orderings of \( Z_i \) and \( Z'_i \), respectively. We have \( r(Z_i \cup Z'_i) = |Z_i \cup Z'_i| - 1 \). We may assume that \( a_{i_l} \neq a_{j_l} \) (otherwise contract a common feasible beginning section). \( Z_i \) and \( Z'_i \) satisfy the conditions of 2.5. If \( a_s \in Z'_i \) then \( Z'_i - \{a_i\} \) is an \( a_s \)-path, and hence \( a_s \uparrow_{Z'_i} a_i \). If \( a_s \notin Z'_i \) and \( a_s \neq a_{i_l} \) then \( Z'_i - a_i \cup a_s = Z'_i \). We have \( Z'_i - \{a_s\} = Z'_i - \{a_i\} \) (Fig. 9).
Then $a_i \perp_B a_j, a_{j_1} \perp_B a_i, a_{j_2} \perp_B a_{j_1}, \ldots, a_i$. It follows $a_i \perp_B a_i$. The case $a_i = a_i$ is similar and is omitted.

(4b) If $Z_i \cup \{a_s\} = Z_s$ interchange $s$ and $i$ and proceed as in (4a).

Lemma 3.3 shows that $\mathcal{M} := \{X \subseteq E : \beta(X) = |X|\}$ is the family of feasible sets of a matroid. $C \subseteq E$ is called a circuit of $(E, \mathcal{F})$ if $C$ is a minimal non feasible set of $(E, \mathcal{M})$. If $B \in \mathcal{B}$ and $e \in E - B$ then $B \cup \{e\}$ contains exactly one circuit $C = \text{cir}(B, e)$.

**Proposition 5.2.** If $B$, $B' = B - a \cup e$ are bases of $(E, \mathcal{F})$ and $x \in E - (B \cup \{e\})$ then either $B - a \cup x \in \mathcal{B}$ or there exists $y \in \text{cir}(B, x)$ such that $B - y \cup x \in \mathcal{B}$ and $(B - y \cup x) - a \cup e \in \mathcal{B}$.

**Proof.** Let $B$, $B' := B - a \cup e \in \mathcal{B}$, $a \in B$, $e \in E - B$, $x \in E - (B \cup \{e\})$, $C_1 = \text{cir}(B, x)$ and let $C_2 := \text{cir}(B, e)$. Then $a \in C_2$. If $a \in C_1 \cap C_2$ then $B - a \cup x \in \mathcal{B}$. Otherwise choose $y \in B \cap C_1 \neq \emptyset$ ($(E, \mathcal{F})$ is normal!!). If $y \in C_2$ then there exists a circuit $C$ such that $a \in C \subseteq (C_1 \cup C_2) - \{y\}$, and hence $C \subseteq (B - y \cup x) \cup \{e\}$. Thus $a \in C = \text{cir}(B - y \cup x, e)$ and so $(B - y \cup x) - a \cup e \in \mathcal{B}$. If $y \notin C_2$ then $C_2 \subseteq (B - y \cup x) \cup \{e\}$. Therefore $a \in C_2 = \text{cir}(B - y \cup x, e)$ and hence $(B - y \cup x) - a \cup e \in \mathcal{B}$.

**Proposition 5.3.** Let $\{x, y\}$ be a circuit of $(E, \mathcal{F})$ and let $Z$ be a path with head $y$. Then $Z - y \cup x$ is a $x$-path. In particular for any feasible set $A$ containing $y$, $A - y \cup x$ is feasible, too.

**Proof.** If $\{x, y\}$ is a circuit of $(E, \mathcal{F})$ and $y \in B' \in \mathcal{B}$ then also $B' - y \cup x \in \mathcal{B}$. Let $y \in B \in \mathcal{B}$, $X := \ker(B - \{y\})$. Then $X \cup \{x\} \in \mathcal{F}$, by 3.1 and $\mathcal{A} = \mathcal{B}$. Let $Z \subseteq B$ be a path with head $y$. Suppose that $Z - y \cup x \notin \mathcal{F}$. Then we choose a minimal feasible set $A$ such that $Z - \{y\} \subseteq A \subseteq X$ and $A \cup \{x\} \in \mathcal{F}$. Hence $A \neq Z \cup \{y\}$ and $A \cup \{a\} \notin \mathcal{F}$ for some $a \in A - (Z - \{y\})$. It follows that $\{a\}, \{y\}, \{a, y\}, \{a, x\} \in \mathcal{F}/(A - \{a\})$, furthermore $\{x\}, \{x, y\}, \{x, y, a\} \notin \mathcal{F}/(A - \{a\})$, by assumption and by the choice of $A$. But this leads a contradiction, since $(E, \mathcal{F})$ does not contain any minor of type (C). Thus $Z - y \cup x$ is a feasible set and also a $x$-path. If $A$ is an arbitrary feasible set, $y \in A$ and $Z \subseteq A$ a path with head $y$, then $Z - y \cup x \in \mathcal{F}$. Augmenting this set from $A$ we get $A - y \cup x \in \mathcal{F}$.

**Lemma 5.4.** $G(E, B) = G(E, B')$ $(B, B' \in \mathcal{B}, |B \cap B'| \geq |B| - 1)$.

**Proof.** Let $B$, $B' := B - a \cup e \in \mathcal{F}$, $x \in E - (B \cup \{e\})$. We show $x \perp_B c \Rightarrow x \perp_B c$ $(c \in B \cup \{e\})$.

We consider two cases.

(1) Let $B'' = B - a \cup x \in \mathcal{B}$.

We define $G_1 := G(B \cup \{x\}, B)$, $G_2 := G(B \cup \{e\}, B)$, $G_3 := G(B -$
\{a\} \cup \{x, e\}, B''\). Then \(G_1 = G(B \cup \{x\}, B'')\), \(G_2 = G(B \cup \{e\}, B')\), \(G_3 = G(B - \{a\} \cup \{x, e\}, B')\), by Lemma 5.1. \(G_1\) and \(G_2\) are subgraphs of \(G := G(B \cup \{x, e\}, B)\). \(\langle B - a \cup x \rangle_{G_3} = \langle B - a \cup x \rangle_{G_1} (= \text{a subgraph of } G_1 \text{ defined by the edges of } B - a \cup x)\) and \(\langle B - a \cup e \rangle_{G_1} = \langle B - a \cup e \rangle_{G_2}\) are subgraphs of \(G_1\) and \(G_2\), respectively. Hence \(\langle (B - a \cup x) \cup (B - a \cup e) \rangle_{G_3} = \langle B - \{a\} \cup \{x, e\} \rangle_{G_3} = G_3\) is a subgraph of \(G\), and so \(G = G_2 \cup G_3 = G(B \cup \{e, x\}, B')\).

(2) Let \(B - a \cup x \notin \mathcal{B}\).

There exists \(y \in \text{cir}(B, x)\) such that \(B'' := B - y \cup x\) and \(B' := B' - y \cup x = (B - y \cup x) - a \cup e\) are bases of \(E\). Let \(G := G(B \cup \{e, x\}, B), G' := (B \cup \{e, x\}, B'), G_1 := G(B \cup \{e\}, B), G_2 := G(B \cup \{x\}, B), G_3 := G(B - \{y\} \cup \{e, x\}, B'')\) and let \(G_4 := G(B - \{a\} \cup \{e, x\}, B'')\).

Then \(G_1 = G(B \cup \{e\}, B'), G_2 = G(B \cup \{x\}, B''), G_3 = G(B - \{y\} \cup \{e, x\}, B''),\) and \(G_4 = G(B - \{a\} \cup \{x, e\}, B')\), by 5.1. Let \(\{a_1, a_2, ..., a_p\} = B \cap B' \cap B'' \cap B'''\). Let \(c \in \{a_1, a_2, ..., a_p\}\). Then \(x \perp_B c\) follows by applying 5.1, repeatedly.

(2b) Let \(c = y\). If (2ba) in this case \(\{x, y\}\) is a circuit of \((E, \mathcal{F})\) then \(x \perp_B c\) follows from 5.3.

(2bb) Let \(\text{cir}(B, x) \neq \{y, x\}\) and \(a \notin \text{cir}(B, x)\), by assumption, and hence \(a \neq a_i \in B' \cap B'' \cap B'''\). Thus \(B^{***} := B - a \cup x\) and \(B^{**} := (B - a \cup x) - a \cup e\) are bases of \(E\) (cf. 5.2 + proof). Now \(x \perp_B y\) follows from part (2a), since \(y \in B' \cap B^{**} \cap B^{***} \cap B\) (Fig. 10).

(2c) Let \(c = a\), and let \(\{P_0, P_1, ..., P_q\}, \{P'_0, P'_1, ..., P'_q\}\) be the vertex-sets of \(G\) and \(G'\), respectively. \(\langle B \rangle_G = \langle B \rangle_{G_1}\) and \(G_1\) are subgraphs of \(G'\) and \(\langle B \rangle_{G} = \langle B \rangle_{G'}\). We have shown already that \(P_i \cap (B - a \cup x) = P'_{\pi(i)} \cap (B - a \cup x)\) (1 \(\leq i \leq q\)) for some permutation \(\pi\). That is, \(\langle B - a \cup x \rangle_G\) is a subgraph of \(G'\), too (if \(\langle B - a \cup x \rangle_G = K_3\langle B - a \cup x \rangle_G - K_{1,3}\) is impossible) and \(\langle B - a \cup x \rangle_{G'} = K_{1,3}\) then 0 or 2 of the edges of \(B - a \cup x\) (Fig. 11) would be incident with \(P_0\) and hence 1 or 3 of these edges would be incident with \(P'_0\). Since \(P_0 = P'_0\), this is impossible.) It follows that also \(\langle B \cup (B - a \cup x) \rangle_G = \langle B \cup \{x\} \rangle_G\) is a subgraph of \(G'\) and hence \(x \perp_B a\).

(2d) Let \(c = e\). Since \(\langle B \cup \{x\} \rangle_G\) and \(\langle B \cup \{e\} \rangle_G\) are subgraph of \(G'\), \(\langle B \cup \{x, e\} \rangle_G\) is a subgraph of \(G'\), too. Thus \(x \perp_B e\).
Corollary 5.5. \( G(E, B) = G(E, B') \) (\( B, B' \in \mathcal{B} \)).

Proof. Compare Lemma 3.3 and Lemma 5.4.

Proof of 1.5. Let \( \{P_0, P_1, ..., P_q\} \) be a vertex set of \( G := G(E, \mathcal{F}) := G(E, B) \) and let \((E, \mathcal{N})\) be an undirected branching greedoid on \( G \). We claim \( \mathcal{N} = \mathcal{N} \). \( \mathcal{N} \subseteq \mathcal{N} \) is clear, by Corollary 5.5. Conversely, let \( X \in \mathcal{N} \cap \mathcal{G} \) and \( X \cup \{a\} \in \mathcal{G} \) for some \( a \in E - X \). We may assume that each arborescence in \( G(E - \{b\}, \mathcal{F} - \{b\}) \) is a feasible set in \((E - \{b\}, \mathcal{F} - \{b\}) \) \( (b \in E) \). We consider two cases.

1. Let \( a \in \mathcal{B} \) (\( B \in \mathcal{B} \)).

Then \( Y, Y \cup \{a\} \in \mathcal{F} \) for some superset \( Y \subseteq E - \{a\} \) of \( X \). Let \( Z' \subseteq Y \cup \{a\} \), \( Z' \in \mathcal{F} \) be an \( a \)-path. Let \( P_i, P_j \) be the endpoints of \( a \) in \( G \), and assume that no edge of \( Y \) is incident with \( P_j \). Then \( P_j \) is incident with some edge \( e \) of \( X \in \mathcal{F} \subseteq \mathcal{N} \). Let \( Z \subseteq X \) be an \( e \)-path, \( Z \in \mathcal{F} \subseteq \mathcal{G} \). \( Z \) and \( Z' - \{a\} \) are two paths in \( G(E, \mathcal{F}) \) with the same endpoint and subsets of \( Y \cup \{a\} \in \mathcal{F} \subseteq \mathcal{G} \). Therefore \( Z = Z' - \{a\} \) and so \( X \cup Z' = X \cup \{a\} \) is a feasible subset of \( Y \).

2. Let \( X \subseteq B \in \mathcal{B} \) and \( a \notin \mathcal{B} \). Then \( X \cup \{e\} \in \mathcal{F} \) for some \( e \in B - X \). Each arborescence in \( G(E - \{e\}, \mathcal{F} - \{e\}) \) is a feasible set of \( \mathcal{F} - \{e\} \subseteq \mathcal{F} \). We have \( X \cup \{a\} \in \mathcal{G} \) and \( e \notin X \cup \{a\} \). Thus \( X \cup \{a\} \) is an arborescence in \( G(E - \{e\}, \mathcal{F} - \{e\}) \) and \( X \cup \{a\} \in \mathcal{F} - \{e\} \subseteq \mathcal{F} \), i.e., \( X \cup \{a\} \in \mathcal{F} \).
REFERENCES


