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# The global attractor of the damped forced Ostrovsky equation

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**Abstract**

The existence of the global attractor of the damped forced Ostrovsky equation in $\tilde{L}^2(\mathbb{R})$ is proved for the forces in $\tilde{L}^2(\mathbb{R})$. Moreover, the global attractor of the equation in $\tilde{L}^2(\mathbb{R})$ is actually a compact set in $\tilde{H}^3(\mathbb{R})$.


*Keywords:* Global attractor; Fourier restriction norm; Asymptotic smoothing

## 1. Introduction

In the paper, our aim is to study the existence and regularity of the global attractor of the damped forced Ostrovsky equation in $\tilde{L}^2(\mathbb{R})$,

$$u_t - \beta u_{xxx} + \left(u^2\right)_x - \gamma D_x^{-1} u + \lambda u = f, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x) \in \tilde{L}^2(\mathbb{R}), \tag{1.2}$$

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where $\lambda > 0$, $\beta$ and $\gamma$ are real constants and $\beta \cdot \gamma \neq 0$, $D_x^{-1} = \mathcal{F}_x^{-1} \frac{1}{i\xi} \mathcal{F}_x$, $f$ is time independent and belongs to $\tilde{L}^2(\mathbb{R})$; the space $\tilde{L}^2(\mathbb{R})$ is defined as below

$$\tilde{L}^2 = \left\{ f \in L^2(\mathbb{R})\colon\ \mathcal{F}_x^{-1}\left(\frac{\hat{f}(\xi)}{\xi}\right) \in L^2(\mathbb{R}) \right\}$$

with the norm

$$\|f\|_{\tilde{L}^2} = \|f\|_{L^2} + \left\| \mathcal{F}_x^{-1}\left(\frac{\hat{f}(\xi)}{\xi}\right) \right\|_{L^2}.$$

The corresponding Sobolev spaces $\tilde{H}^s$ are defined in a similar way

$$\tilde{H}^s = \left\{ f \in H^s(\mathbb{R})\colon\ \mathcal{F}_x^{-1}\left(\frac{\hat{f}(\xi)}{\xi}\right) \in H^s(\mathbb{R}) \right\}$$

with the norm

$$\|f\|_{\tilde{H}^s} = \|f\|_{H^s} + \left\| \mathcal{F}_x^{-1}\left(\frac{\hat{f}(\xi)}{\xi}\right) \right\|_{H^s}.$$

The Ostrovsky equation [14] governs the propagation of the weakly nonlinear long surface and internal waves of small amplitude in a rotating fluid. The liquid is assumed to be incompressible and inviscid. After the suitable scaling, the equation can be written as above [6]. Parameter $\beta$ determines the type of dispersion. Namely, $\beta = -1$ (negative-dispersion) for surface and internal waves in the ocean and surface waves in a shallow channel with an uneven bottom. Parameter $\beta = 1$ (positive-dispersion) for capillary waves on the surface of liquid or for oblique magneto-acoustic waves in plasma [1,3,6]. Equation (1.1) models the situation when nonlinearity, dispersion, dissipation and rotation are taken into account at the same time.

From the mathematical point of view, the extra term with the factor $\lambda$ accounts for a weak dissipation without regularization or smoothing property. Hence, the well-posedness of the solution of (1.1)–(1.2) and the asymptotic smoothing of the solution operator come essentially from the dispersive regularization property of the equation.

Moreover, if $\gamma = 0$, $\lambda = 0$ and $f = 0$, then Eq. (1.1) reduces to the Korteweg–de Vries equation (KdV)

$$u_t - \beta u_{xxx} + \left(u^2\right)_x = 0.$$

Many works [4,5,7,8,13] consider the existence of global attractors of the weakly damped KdV equation

$$\partial_t u - \beta \partial_x^3 u + u \partial_x u + \lambda u = f. \tag{1.3}$$

For the weakly damped KdV equation (1.3), Goubet and Rosa [8] obtained the existence of global attractor in $L^2(\mathbb{R})$ and its compactness in $H^3(\mathbb{R})$ by the energy equation method together with a splitting of the solution.

If $\lambda = 0$ and $f = 0$, then (1.1) reduces to the Ostrovsky equation

$$\partial_t u - \beta \partial_x^3 u + \partial_x\left(u^2\right) = \gamma D_x^{-1} u. \tag{1.4}$$

Recently Varlamov and Liu [17] introduced the spaces $\tilde{L}^2(\mathbb{R})$ and $\tilde{H}^s(\mathbb{R})$ for considering the solvability of the Cauchy problem of (1.4). They showed that the Cauchy problem is locally well-posed in $\tilde{H}^s(\mathbb{R})$ ($s > \frac{3}{2}$) with $\gamma > 0$. Moreover, the global-in-time solutions were constructed in the space $C(\mathbb{R}, \tilde{L}^2(\mathbb{R}))$ for the small initial data, and the asymptotics of the solution was computed for $x/t = \text{const}$ and $t \to \infty$. Huo and Jia [9] obtained that the Cauchy problem of (1.4) is locally well-posed in $\tilde{H}^s(\mathbb{R})$ ($s \geqslant -\frac{1}{8}$) without the condition $\gamma > 0$ by the so-called Fourier restriction norm (the Bourgain function spaces). The method was first introduced by J. Bourgain [2] to study the KdV and nonlinear Schrödinger equations in the periodic case. It was simplified by Kenig, Ponce and Vega [10,12].

However, it seems that no paper studies the global attractor for the damped forced Ostrovsky equation (1.1). In the paper, we show that global attractor for Eq. (1.1) exists in $\tilde{L}^2(\mathbb{R})$ and is a compact in $\tilde{H}^3(\mathbb{R})$.

First, we consider the well-posedness of the solution of (1.1)–(1.2) by the Bourgain function spaces. We shall only give the outline proof of the well-posedness for simplicity.

Next, we show the long time behavior of solution of (1.1)–(1.2), which is described by global attractor. We obtain the existence of the global attractor in $\tilde{L}^2(\mathbb{R})$ and its boundedness in $\tilde{H}^3(\mathbb{R})$ by the energy equation method together with a splitting of the equation.

For hyperbolic equations, the existence of global attractor is obtained by the asymptotic compactness or the asymptotic smoothing properties of the solution operator, together with the existence of a bounded absorbing set. Those properties are proved by splitting the solutions into a decaying part plus a regular part, or by exploiting suitable energy-type equations, or by both ways [16].

For equations defined on unbounded domains, it is suitable to use the energy equations since it does not depend on compact imbedding of function spaces. It does require, however, the weak continuity of the solution operator in the sense that if the initial data $u_{0n}$ converge to $u_0$ weakly, then the corresponding solutions $u_n(t)$ converge weakly to $u(t)$, at all time $t$. This weak continuity is usually obtained by passing to the limit in the weak formulation of the equation and using the uniqueness of the solutions.

Our results are achieved by splitting the solution into two parts. One is regular in $\tilde{H}^3(\mathbb{R})$, the other decays to zero in $\tilde{L}^2(\mathbb{R})$ as time goes to infinity. In this way, the weak continuity of the solution operator is replaced by an asymptotic weak continuity property. We first obtain the boundedness of the regular part of the solution in $\tilde{H}^3(\mathbb{R})$. Then, taking time derivative of Eq. (1.1), we obtain the compactness of the global attractor by applying the energy equation method to the resulting equation.

In the paper, to study well-posedness of the solution of the problem (1.1)–(1.2), we use its integral equivalent formulation

$$u(x,t) = S(t)u_0 - \int_0^t S(t-t')\big(\partial_x\big(u^2\big) + \lambda u - f\big)(t')\,dt', \tag{1.5}$$

where $S(t) = \mathcal{F}_x^{-1} e^{-it(\beta\xi^3 + \frac{\gamma}{\xi})}\mathcal{F}_x$ is the unitary operator associated to the linear equation. For simplicity, denote the phase function by $\phi(\xi) = \beta\xi^3 + \frac{\gamma}{\xi}$.

The major difficulty in this paper comes from the fact: the phase function of semigroup of (1.1) has non-zero points, which makes a difference from that of the linear KdV equation [8]. Therefore, we need to use Fourier restriction operators

$$P^N f = \int_{|\xi| \geqslant N} e^{ix\xi}\hat{f}(\xi)\,d\xi, \quad P_{(\varepsilon,N)} f = \int_{\varepsilon \leqslant |\xi| \leqslant N} e^{ix\xi}\hat{f}(\xi)\,d\xi, \quad \forall N \geqslant \varepsilon > 0, \tag{1.6}$$

to eliminate the singularity of the phase function and to split the solution in Section 6. For simplicity, denote $P_N f = \int_{|\xi| \leqslant N} e^{ix\xi}\hat{f}(\xi)\,d\xi$.

**Definition 1.** For $s, b \in \mathbb{R}$, the space $X_{s,b}$ is the completion of the Schwartz function space on $\mathbb{R}^2$ with respect to the norm

$$\|u\|_{X_{s,b}} = \big\|S(-t)u\big\|_{H_x^s H_t^b} = \big\|\langle\xi\rangle^s\big\langle\tau+\phi(\xi)\big\rangle^b \mathcal{F}u\big\|_{L_\xi^2 L_\tau^2},$$

where $\langle\cdot\rangle = (1+|\cdot|)$.

Similarly to $\tilde{H}^s$, we define the modified Bourgain function space $\tilde{X}_{s,b}$ as below

$$\begin{aligned}\|u\|_{\tilde{X}_{s,b}} &= \big\|\langle\xi\rangle^s\big\langle\tau+\phi(\xi)\big\rangle^b \mathcal{F}u\big\|_{L_\xi^2 L_\tau^2} + \big\|\langle\xi\rangle^s|\xi|^{-1}\big\langle\tau+\phi(\xi)\big\rangle^b \mathcal{F}u\big\|_{L_\xi^2 L_\tau^2} \\ &= \|u\|_{X_{s,b}} + \big\|D_x^{-1}u\big\|_{X_{s,b}}.\end{aligned}$$

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}\psi \subset [-1,1]$. Denote $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for some $\delta \in \mathbb{R}$.

For $T > 0$, we consider the localized Bourgain space $\tilde{X}_{s,b}^T$ endowed with norm

$$\|u\|_{\tilde{X}_{s,b}^T} = \|u\|_{\tilde{X}_{s,b}^{[-T,T]}} = \|\psi_T u\|_{\tilde{X}_{s,b}}.$$

In our arguments, we shall use the trivial embedding relation $\|u\|_{\tilde{X}_{s_1,b_1}} \leqslant \|u\|_{\tilde{X}_{s_2,b_2}}$ whenever $s_1 \leqslant s_2$, $b_1 \leqslant b_2$.

Denote $A \sim B$ by the statement: $A \leqslant C_1 B$ and $B \leqslant C_1 A$ for some constant $C_1 > 0$, and $A \ll B$ by the statement: $A \leqslant \frac{1}{C_2}B$ for some large enough constant $C_2 > 0$.

Denote by $\hat{u}(\tau,\xi) = \mathcal{F}u$ the Fourier transform in $t$ and $x$ of $u$ and by $\mathcal{F}_{(\cdot)}u$ the Fourier transform in the $(\cdot)$ variable.

Let us introduce some variables

$$\sigma = \tau + \beta\xi^3 + \frac{\gamma}{\xi}, \qquad \sigma_1 = \tau_1 + \beta\xi_1^3 + \frac{\gamma}{\xi_1}, \qquad \sigma_2 = \tau_2 + \beta\xi_2^3 + \frac{\gamma}{\xi_2}.$$

By direct calculation, we can obtain

$$\sigma - \sigma_1 - \sigma_2 = 3\beta\xi\xi_1\xi_2\left(1 - \gamma\frac{\xi_1^2+\xi_2^2+\xi_1\xi_2}{3\beta(\xi\xi_1\xi_2)^2}\right). \tag{1.7}$$

Throughout this paper, denote $\int_\star d\delta$ the convolution integral by

$$\int\limits_{\xi=\xi_1+\xi_2;\,\tau=\tau_1+\tau_2} d\tau_1\,d\tau_2\,d\xi_1\,d\xi_2.$$

Now, we give the statement of the main result.

**Theorem 1.1.** *Let $\lambda > 0$, $f \in \tilde{L}^2(\mathbb{R})$. Then the solution operator $\{A(t)\}_{t\in\mathbb{R}}$ associated with Eq.* (1.1) *possesses a connected global attractor $\mathcal{A}$ in $\tilde{L}^2(\mathbb{R})$, which is compact in $\tilde{H}^3(\mathbb{R})$.*

## 2. Preliminary estimates

In this section, several estimates will be deduced. These lemmas can be also found in [9], however for completeness, we give the proofs of them here. First let us use the following notations

$$\|f\|_{L_x^p L_t^q} = \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| f(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

$$\|f\|_{L_t^q L_x^p} = \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| f(x,t) \right|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

$$\|f\|_{L_t^\infty H_x^s} = \left\| \|f\|_{H_x^s} \right\|_{L_t^\infty}, \qquad \mathcal{F}F_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{(1+|\tau+\beta\xi^3+\frac{\gamma}{\xi}|)^\rho},$$

$$a = \max\left(1, \sqrt[4]{\left|\frac{6\gamma}{7\beta}\right|}, \sqrt[2]{\left|\frac{\gamma}{3\beta}\right|}\right),$$

$$D_x^s = \mathcal{F}_x^{-1}|\xi|^s \mathcal{F}_x \quad \text{for fraction } s, \qquad D_x^m = \mathcal{F}_x^{-1}(i\xi)^m \mathcal{F}_x \quad \text{for integer } m.$$

**Lemma 2.1.** *The group* $\{S(t)\}_{-\infty}^{+\infty}$ *satisfies*

$$\left\| D_x P^a S(t) u_0 \right\|_{L_x^\infty L_t^2} \leqslant C \|u_0\|_{L^2}, \tag{2.1}$$

$$\left\| D_x^{-\frac{1}{4}} P^a S(t) u_0 \right\|_{L_x^4 L_t^\infty} \leqslant C \|u_0\|_{L^2}, \tag{2.2}$$

$$\left\| D_x^{\frac{1}{6}} P^a S(t) u_0 \right\|_{L_x^6 L_t^6} \leqslant C \|u_0\|_{L^2}, \tag{2.3}$$

*where the constant* $C$ *depend on* $\gamma$ *and* $\beta$.

**Proof.** First we prove (2.1). It follows that

$$\phi(\xi) = \beta\xi^3 + \frac{\gamma}{\xi}, \qquad \phi'(\xi) = 3\beta\xi^2 - \frac{\gamma}{\xi^2} \quad (\xi \neq 0), \qquad \phi''(\xi) = 6\beta\xi + 2\frac{\gamma}{\xi^3} \quad (\xi \neq 0).$$

If $|\xi| \geqslant a$, $\phi$ is invertible, then

$$\begin{aligned} P^a S(t) u_0 &= \int_{|\xi| \geqslant a} e^{ix\xi} e^{-it\phi(\xi)} \hat{u}_0(\xi)\, d\xi \\ &= \int_{|\phi^{-1}| \geqslant a} e^{ix\phi^{-1}} e^{-it\phi} \hat{u}_0\big(\phi^{-1}\big) \frac{1}{\phi'}\, d\phi \\ &= \mathcal{F}_t\left( e^{ix\phi^{-1}} \chi_{\{|\phi^{-1}| \geqslant a\}} \hat{u}_0\big(\phi^{-1}\big) \frac{1}{\phi'} \right). \end{aligned}$$

Therefore, changing variable $\xi = \phi^{-1}$, we have

$$\begin{aligned} \left\| P^a S(t) u_0 \right\|_{L^2}^2 &= \left\| \mathcal{F}_t\left( e^{ix\phi^{-1}} \chi_{\{|\phi^{-1}| \geqslant a\}} \hat{u}_0\big(\phi^{-1}\big) \frac{1}{\phi'} \right) \right\|_{L^2}^2 \\ &= \int_{|\phi^{-1}| \geqslant a} \left| \hat{u}_0\big(\phi^{-1}\big) \right|^2 \frac{1}{|\phi'|^2}\, d\phi \\ &= \int_{|\xi| \geqslant a} \left| \hat{u}_0(\xi) \right|^2 \frac{1}{|\phi'(\xi)|^2} \phi'(\xi)\, d\xi \end{aligned}$$

$$\leqslant \int\limits_{|\xi|\geqslant a} |\hat{u}_0(\xi)|^2 \frac{1}{|\phi'|}\, d\xi$$
$$\leqslant C\|u_0\|^2_{\dot{H}^{-1}}.$$

This implies the estimate (2.1).

Let us turn to the proof of (2.2) next. The first inequality as below holds with the help of Theorem 2.5 in [11]. It can be shown that

$$\begin{aligned}\|P^a S(t)u_0\|^2_{L^4_x L^\infty_t} &\leqslant C \int\limits_{|\xi|\geqslant a} |\hat{u}_0(\xi)|^2 \left|\frac{\phi'(\xi)}{\phi''(\xi)}\right|^{\frac{1}{2}} d\xi \\ &\leqslant C \int\limits_{|\xi|\geqslant a} |\hat{u}_0(\xi)|^2 |\xi|^{\frac{1}{2}}\, d\xi \\ &\leqslant C\|u_0\|^2_{H^{\frac{1}{4}}}.\end{aligned}$$

Finally, (2.3) follows by interpolation between (2.1) and (2.2). □

**Lemma 2.2.** [17] *The group* $\{S(t)\}_{-\infty}^{+\infty}$ *satisfies*

$$\|S(t)u_0\|_{L^6_x L^6_t} \leqslant C\|u_0\|_{L^2}, \tag{2.4}$$

*where the constant* $C$ *depends on* $\gamma$ *and* $\beta$.

**Remark.** We can use Van der Corput Lemma [15] to obtain the result.

**Lemma 2.3.** *If* $\rho > \frac{3}{8}$, *then*

$$\|D_x^{\frac{1}{8}} P^a F_\rho\|_{L^4_x L^4_t} \leqslant C\|f\|_{L^2_\xi L^2_\tau}, \tag{2.5}$$

*where the constant* $C$ *depends on* $\beta$ *and* $\gamma$.

**Proof.** Changing variable $\tau = \lambda - \phi(\xi)$, we have

$$\begin{aligned}F_\rho(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi+t\tau)} \frac{f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho}\, d\xi\, d\tau \\ &= \int_{-\infty}^{\infty} e^{it\lambda}\left(\int_{-\infty}^{\infty} e^{i(x\xi+t\phi(\xi))} f(\xi,\lambda+\phi(\xi))\, d\xi\right)\frac{d\lambda}{(1+|\lambda|)^\rho}.\end{aligned}$$

Therefore, using (2.3), Minkowski's integral inequality and taking $\rho > \frac{1}{2}$, one is able to show that

$$\|D_x^{\frac{1}{6}} P^a F_\rho\|_{L^6_x L^6_t} \leqslant C\int_{-\infty}^{+\infty} \|f(\xi,\lambda+\phi(\xi))\|_{L^2_\xi} \frac{d\lambda}{(1+|\lambda|)^\rho} \leqslant C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.6}$$

In fact, we have

$$\|F_0\|_{L_x^2 L_t^2} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.7}$$

Then (2.5) follows by interpolation between (2.6) and (2.7). □

**Lemma 2.4.** *If $\rho > \frac{1}{2}\frac{3(q-2)}{2q}$. Then for $2 \leqslant q \leqslant 6$, it holds that*

$$\|F_\rho\|_{L_x^q L_t^q} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}, \tag{2.8}$$

*where the constant $C$ depends on $\beta$ and $\gamma$.*

**Proof.** From the argument (2.6) and the inequality (2.4), we have for $\rho > \frac{1}{2}$,

$$\|F_\rho\|_{L_x^6 L_t^6} \leqslant C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.9}$$

Then (2.8) follows by interpolation between (2.9) and (2.7). □

**Lemma 2.5.** *Assume $f$, $f_1$ and $f_2$ belong to Schwartz space on $\mathbb{R}^2$, then*

$$\int_\star \bar{\hat{f}}(\xi,\tau)\hat{f}_1(\xi_1,\tau_1)\hat{f}_2(\xi_2,\tau_2)\,d\delta = \int \bar{f} f_1 f_2(x,t)\,dx\,dt. \tag{2.10}$$

**Proof.** For simplicity, we only discuss the case of one variable.

$$\begin{aligned}
&\int_{\xi=\xi_1+\xi_2} \bar{\hat{f}}(\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\delta \\
&\quad= \int_{\xi=\xi_1+\xi_2} \hat{\bar{f}}(-\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\delta \\
&\quad= \int_{\xi_1}\int_{\xi_2'}\int_{\xi_3'} \hat{\bar{f}}\big(-\xi_3'\big)\hat{f}_1(\xi_1)\hat{f}_2\big(\xi_2'-\xi_1\big)\,d\xi_1\,d\xi_2' \\
&\quad= \hat{\bar{f}} * \hat{f}_1 * \hat{f}_2(0) = \mathcal{F}\bar{f} f_1 f_2(0) \\
&\quad= \int \bar{f} f_1 f_2(x)\,dx. \qquad \square
\end{aligned}$$

## 3. Linear estimates

**Lemma 3.1.** [10,12] *Let $s \in \mathbb{R}$, $\frac{1}{2} < b < 1$. Then for $u_0 \in \tilde{H}^s$, it follows that*

$$\big\|\psi(t)S(t)u_0\big\|_{\tilde{X}_{s,b}} \leqslant C\|u_0\|_{\tilde{H}^s}. \tag{3.1}$$

**Lemma 3.2.** [10,12] *Let $s \in \mathbb{R}$, $\frac{1}{2} < b < 1$ and $0 < \delta \leqslant 1$. Then*

$$\left\|\psi_\delta(t)\int_0^t S(t-t')f(t')\,dt'\right\|_{\tilde{X}_{s,b}} \leqslant C\delta^{b'-b}\|f\|_{\tilde{X}_{s,b-1}}, \tag{3.2}$$

$$\left\|\psi_\delta(t)\int_0^t S(t-t')f(t')\,dt'\right\|_{\tilde{L}^2} \leqslant C\delta^{b'-b}\|f\|_{\tilde{X}_{s,b-1}}, \tag{3.3}$$

$$\left\|\psi_\delta(t)F\right\|_{\tilde{X}_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|F\|_{\tilde{X}_{s,b}}. \tag{3.4}$$

**Lemma 3.3.** [9,10,12] *Let $s\in\mathbb{R}$, $\frac{1}{2}<b<b'<1$ and $0<\delta\leqslant 1$. Then*

$$\left\|\psi_\delta(t)F\right\|_{\tilde{X}_{s,b-1}} \leqslant C\delta^{b'-b}\|F\|_{\tilde{X}_{s,b'-1}}. \tag{3.5}$$

## 4. Bilinear estimate

**Theorem 4.1.** *Let $\frac{1}{2}<b<\frac{9}{16}$, $\frac{1}{2}<b'$. Assume that the Fourier transform $\mathcal{F}u_j=\hat{u}_j(\tau,\xi)$ of $u_j$ is supported in $\{(\xi,\tau)\colon |\xi|\geqslant N\}$, $N>0$, $j=1,2$. Then*

$$\left\|\partial_x(u_1u_2)\right\|_{\tilde{X}_{0,b-1}} \leqslant \frac{C}{N^{\frac{1}{8}}}\|u_1\|_{\tilde{X}_{0,b'}}\|u_1\|_{\tilde{X}_{0,b'}}, \tag{4.1}$$

*where the constant $C$ depends on $\gamma$ and $\beta$, but is independent of $N$.*

**Proof.** First we have

$$\left\|\partial_x(u_1u_2)\right\|_{\tilde{X}_{0,b-1}} = \left\|\partial_x(u_1u_2)\right\|_{X_{0,b-1}} + \left\|D_x^{-1}\partial_x(u_1u_2)\right\|_{X_{0,b-1}}.$$

We only prove bilinear estimate for the first term $\|\partial_x(u_1u_2)\|_{X_{0,b-1}}$ on the right of the equality, as the proof of the second term is easier than that of the first term. That is

$$\left\|\partial_x(u_1u_2)\right\|_{X_{0,b-1}} \leqslant \frac{C}{N^{\frac{1}{8}}}\|u_1\|_{\tilde{X}_{0,b'}}\|u_2\|_{\tilde{X}_{0,b'}}.$$

By duality and the Plancherel identity, it suffices to show that

$$\begin{aligned}
\Upsilon &= \int_\star |\xi|\frac{\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\,d\delta \\
&= \int_\star \frac{|\xi|}{\langle\sigma\rangle^{1-b}\prod_{j=1}^2\langle\sigma_j\rangle^{b'}}\bar{f}(\tau,\xi)f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)\,d\delta \\
&\leqslant \frac{C}{N^{\frac{1}{8}}}\|f\|_{L_2}\prod_{j=1}^2\left(\|f_j\|_{L_2}+\left\||\xi|^{-1}f_j\right\|_{L_2}\right),
\end{aligned}$$

for $\bar{f}\in L_2$, $\bar{f}\geqslant 0$, where $f_j=\langle\sigma_j\rangle^{b'}\hat{u}_j$, $j=1,2$; $\xi=\xi_1+\xi_2$, $\tau=\tau_1+\tau_2$.

One easily obtain that $\|f_j\|_{L^2}+\||\xi|^{-1}f_j\|_{L_2}=\|u_j\|_{\tilde{X}_{s,b'}}$.

Let

$$\mathcal{F}F_\rho^j(\xi,\tau)=\frac{f_j(\xi,\tau)}{(1+|\tau+\beta\xi^3+\frac{\gamma}{\xi}|)^\rho}, \quad j=1,2.$$

In order to bound the integral $\Upsilon$, we split the domain of integration into two pieces. By symmetry it suffices to estimate the integral in the domain

$$|\xi_1|\leqslant|\xi_2|.$$

*Case* 1. Assume: $|\xi| \leqslant 4a$. If $N \leqslant 2a \leqslant |\xi_1| \leqslant |\xi_2|$, then the integral $\Upsilon$ is bounded by

$$\begin{aligned}
&\int_\star \frac{\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\langle\xi_1\rangle^{\frac{1}{8}} \chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{|\xi|^{\frac{1}{8}} \langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\, d\delta \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \int \overline{F_{1-b}} \cdot D_x^{\frac{1}{8}} P^{2a} F^1_{b'} \cdot F^2_{b'}(x,t)\, dx\, dt \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \|F_{1-b}\|_{L^2_x L^2_t} \big\|D_x^{\frac{1}{8}} P^{2a} F^1_{b'}\big\|_{L^4_x L^4_t} \big\|F^2_{b'}\big\|_{L^4_x L^4_t} \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \|f\|_{L^2_\xi L^2_\tau} \|f_1\|_{L^2_\xi L^2_\tau} \|f_2\|_{L^2_\xi L^2_\tau},
\end{aligned}$$

which follows by Lemmas 2.3–2.5.

*Case* 2. Assume: $|\xi| \geqslant 4a$. If $N \leqslant 2a \leqslant |\xi_1| \leqslant |\xi_2|$, from the identities (1.7), it follows that, if $|\xi| \geqslant a$, $|\xi_1| \geqslant a$ and $|\xi_2| \geqslant a$, then

$$\max\big(|\sigma|, |\sigma_1|, |\sigma_2|\big) \geqslant C|\xi\xi_1\xi_2|.$$

This implies that one of the following cases always occurs:

(a) $|\sigma| \geqslant C|\xi\xi_1\xi_2|$,
(b) $|\sigma_1| \geqslant C|\xi\xi_1\xi_2|$,
(c) $|\sigma_2| \geqslant C|\xi\xi_1\xi_2|$.

In this domain, the integral $\Upsilon$ is bounded by

$$\int_\star \frac{|\xi|\chi_{|\xi|\geqslant 4a} \bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\, d\delta.$$

We consider the three cases (a)–(c) separately. Without loss of generality, we can assume $|\xi| \sim |\xi_2| \geqslant |\xi_1|$.

If (a) holds, for $1 \leqslant \frac{1}{8} + 2(1-b)$, then the integral $\Upsilon$ is bounded by

$$\begin{aligned}
&\int_\star \frac{|\xi|\chi_{|\xi|\geqslant 4a} \bar{f}(\tau,\xi)}{(|\xi_1||\xi_2||\xi|)^{1-b}} \frac{\langle\xi_1\rangle^{\frac{1}{8}} \chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{1}{8}} \langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\, d\delta \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \int_\star \bar{f}(\tau,\xi) \frac{|\xi_1|^{\frac{1}{8}} \chi_{|\xi_1|\geqslant 2a} f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{|\xi_2|^{\frac{1}{8}} \chi_{|\xi_2|\geqslant 2a} f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\, d\delta \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \int \overline{F_0} \cdot D_x^{\frac{1}{8}} P^{2a} F^1_{b'} \cdot D_x^{\frac{1}{8}} P^{2a} F^2_{b'}(x,t)\, dx\, dt \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \|F_0\|_{L^2_x L^2_t} \big\|D_x^{\frac{1}{8}} P^{2a} F^1_{b'}\big\|_{L^4_x L^4_t} \big\|D_x^{\frac{1}{8}} P^{2a} F^2_{b'}\big\|_{L^4_x L^4_t} \\
&\quad\leqslant \frac{C}{N^{\frac{1}{8}}} \|f\|_{L^2_\xi L^2_\tau} \|f_1\|_{L^2_\xi L^2_\tau} \|f_2\|_{L^2_\xi L^2_\tau},
\end{aligned}$$

which follows by Lemmas 2.3 and 2.5.

If (b) holds, then the integral $\Upsilon$ is bounded by

$$\begin{aligned}
&\int_\star \frac{|\xi|\chi_{|\xi|\geqslant 4a}\bar{f}(\tau,\xi)}{\langle\sigma\rangle 1-b}\frac{\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)}{(|\xi_1||\xi_2||\xi|)^{b'}}\frac{|\xi_2|^{\frac18}\chi_{|\xi_2|\geqslant 2a}f_2(\tau_2,\xi_2)}{|\xi_2|^{\frac18}\langle\sigma_2\rangle^{b'}}\,d\delta\\
&\quad\leqslant \frac{C}{N^{\frac18}}\int_\star \frac{\chi_{|\xi|\geqslant 4a}\bar{f}(\tau,\xi)}{\langle\sigma_1\rangle^{1-b}}\chi_{|\xi_1|\geqslant 2a}f_1(\tau_1,\xi_1)\frac{|\xi_2|^{\frac18}\chi_{|\xi_2|\geqslant 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\,d\delta\\
&\quad\leqslant \frac{C}{N^{\frac18}}\int \overline{F_{1-b}}\cdot F_0^1\cdot D_x^{\frac18}P^{2a}F_{b'}^2(x,t)\,dx\,dt\\
&\quad\leqslant \frac{C}{N^{\frac18}}\|F_{1-b}\|_{L_x^4L_t^4}\|F_0^1\|_{L_x^2L_t^2}\|D_x^{\frac18}P^{2a}F_{b'}^2\|_{L_x^4L_t^4}\\
&\quad\leqslant \frac{C}{N^{\frac18}}\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2},
\end{aligned}$$

which follows by Lemmas 2.3–2.5.

If (c) holds, the argument is similar to case (b). This completes the proof of Theorem 4.1. □

**Theorem 4.2.** [9] *Let $\frac12 < b < \frac{9}{16}$. For $\frac12 < b'$ and $s \geqslant -\frac18$, it follows that*

$$\|\partial_x(u_1u_2)\|_{\tilde{X}_{s,b-1}} \leqslant C\|u_1\|_{\tilde{X}_{s,b'}}\|u_2\|_{\tilde{X}_{s,b'}}, \tag{4.2}$$

*where the constant $C$ depends on $\beta$ and $\gamma$.*

**Remark.** The proof is similar to that of Theorem 4.1. Here the condition $s \geqslant -\frac18$ is required only because of the mathematical point of view.

## 5. Global well-posedness in $\tilde{L}^2$ and absorbing sets

In this section, we use the approaches in [9] to obtain the local well-posedness of the problem (1.1)–(1.2). By the $\tilde{L}^2$ energy equation for the solution, we can obtain the global well-posedness and the existence of the bounded absorbing sets in $\tilde{L}^2$.

Assume that $\lambda \in \mathbb{R}$ and $f = f(x,t) \in \tilde{X}^T_{0,b-1}$, for some $T > 0$. For $u_0 \in \tilde{L}^2$, we define the operator

$$\Phi(u) = \psi_1(t)S(t)u_0 + \psi_1(t)\int_0^t S(t-t')\psi_T(t')\bigl(\partial_x\bigl(u^2\bigr) + \lambda u - f\bigr)(t')\,dt',$$

and the set

$$\mathcal{B} = \bigl\{u \in \tilde{X}_{0,b}\colon \|u\|_{\tilde{X}_{0,b}} \leqslant 4C\|u_0\|_{\tilde{L}^2}\bigr\}.$$

In order to show that $\Phi$ is a contraction mapping on $\mathcal{B}$, we first prove

$$\Phi(\mathcal{B}) \subset \mathcal{B}.$$

Using Theorem 4.2 and Lemmas 3.1–3.3, for $\frac12 < b < b' < 1$, we have the next chain of inequalities

$$\|\Phi(u)\|_{\tilde{X}_{0,b}} \leqslant C\|u_0\|_{\tilde{L}^2} + CT^{b'-b}\bigl(\|u\|^2_{\tilde{X}_{0,b}} + \|u\|_{\tilde{X}_{0,b}} + \|f\|_{\tilde{X}_{0,b-1}}\bigr).$$

Therefore, if $CT^{b'-b}\|u_0\|_{\tilde{L}^2} \leqslant \frac{1}{2}$, $CT^{b'-b} \leqslant \frac{1}{2}$ and $CT^{b'-b}\|f\|_{\tilde{X}^T_{0,b-1}} \leqslant C\|u_0\|_{\tilde{L}^2}$, then

$$\Phi(\mathcal{B}) \subset \mathcal{B}.$$

For $u, v \in \mathcal{B}$, in an analogous way to above, we obtain

$$\big\|\Phi(u) - \Phi(v)\big\|_{\tilde{X}_{0,b}} \leqslant CT^{b'-b}\big(\|u\|_{\tilde{X}_{0,b}} + \|v\|_{\tilde{X}_{0,b}} + 1\big)\|u - v\|_{\tilde{X}_{0,b}} \leqslant \frac{1}{2}\|u - v\|_{\tilde{X}_{0,b}}.$$

Therefore, $\Phi$ is a contraction mapping on $\mathcal{B}$. There exists a unique fixed point which solves the Cauchy problem (1.1)–(1.2) for $T$.

If $f$ is time independent and belongs to $\tilde{L}^2$, by multiplying (1.1) by $u$ and $D_x^{-2}u$; integrating it over $\mathbb{R}$ respectively, we can obtain the following energy-type equation,

$$\frac{d}{dt}\big\|u(t)\big\|^2_{\tilde{L}^2} + 2\lambda\big\|u(t)\big\|^2_{\tilde{L}^2} = 2\big(f, u(t)\big)_{\tilde{L}^2}. \tag{5.1}$$

Therefore, we have the following result:

**Theorem 5.1.** *Let $\lambda \in \mathbb{R}$, $f \in \tilde{L}^2(\mathbb{R})$ and $u_0 \in \tilde{L}^2(\mathbb{R})$. Then problem* (1.1)–(1.2) *admits a unique global solution $u(x,t) \in C(\mathbb{R}; \tilde{L}^2)$, which belongs to $\tilde{X}^T_{0,b}$ for all $T > 0$ and $b$ close to $\frac{1}{2}$. Moreover, the map which associates the data $(\lambda, f, u_0)$ to the corresponding unique solution $u$ is continuous from $\mathbb{R} \times \tilde{L}^2(\mathbb{R}) \times \tilde{L}^2(\mathbb{R})$ into $C([-T,T]; \tilde{L}^2(\mathbb{R})) \cap \tilde{X}^T_{0,b}$ for all $T > 0$, with, in particular,*

$$\|u\|_{\tilde{X}^T_{0,b}} \leqslant C\big(\lambda, \|u_0\|_{\tilde{L}^2(\mathbb{R})}, \|f\|_{\tilde{L}^2(\mathbb{R})}, T\big).$$

Thanks to Theorem 5.1 we can define a group associated with Eq. (1.1):

**Definition 2.** For $\lambda \in \mathbb{R}$, $f \in \tilde{L}^2(\mathbb{R})$ fixed, we denote by $\{A(t)\}_{t\in\mathbb{R}}$ the group in $\tilde{L}^2(\mathbb{R})$ defined by $A(t)u_0 = u(t)$, where $u = u(t)$ is the unique solution of Eq. (1.1) which belongs to $\tilde{X}^T_{0,b}$ for all $T > 0$.

From now on, we are interested in the long time behavior of Eq. (1.1) taking the dissipation into account. Therefore, we assume that $\lambda > 0$. We also assume that the forcing term $f$ belongs to $\tilde{L}^2(\mathbb{R})$. We want to obtain the existence of bounded absorbing sets for the solution operator $\{A(t)\}_{t\in\mathbb{R}}$. This is achieved with the help of the energy-type equation proved in above.

By applying Cauchy–Schwartz's and Young's inequalities to the term on the right-hand side of (5.1), it follows that

$$\frac{d}{dt}\big\|u(t)\big\|^2_{\tilde{L}^2} + \lambda\big\|u(t)\big\|^2_{\tilde{L}^2} = \frac{1}{\lambda}\|f\|^2_{\tilde{L}^2}. \tag{5.2}$$

Therefore, upon integrating in time,

$$\big\|u(t)\big\|^2_{\tilde{L}^2} \leqslant \|u_0\|^2_{\tilde{L}^2}e^{-\lambda t} + \frac{1}{\lambda^2}\|f\|^2_{\tilde{L}^2}\big(1 - e^{-\lambda t}\big). \tag{5.3}$$

Whence we deduce that

$$\limsup_{t\to\infty}\big\|u(t)\big\|_{\tilde{L}^2} \leqslant \rho_0 = \frac{1}{\lambda}\|f\|_{\tilde{L}^2} \tag{5.4}$$

uniformly for $u_0$ bounded in $\tilde{L}^2(\mathbb{R})$. Thus, we have proved the following result:

**Theorem 5.2.** *Let $\lambda > 0$, $f \in \tilde{L}^2(\mathbb{R})$. Then the solution operator associated with Eq.* (1.1) *possesses a bounded absorbing set in $\tilde{L}^2(\mathbb{R})$, with the radius of absorbing ball given according to* (5.4).

## 6. Splitting of the solutions

The splitting used is obtained by writing $u = v + w$ and splitting the nonlinear term as

$$uu_x = vv_x + P_N\big((vw)_x + ww_x\big) + P^N\big((vw)_x + ww_x\big). \tag{6.1}$$

Then, we obtain

$$v_t - \beta v_{xxx} - \gamma D_x^{-1} v + vv_x + \lambda v = f - P_N\big((vw)_x + ww_x\big), \tag{6.2}$$

and

$$w_t - \beta w_{xxx} - \gamma D_x^{-1} w + P^N(ww_x) + \lambda w = -P^N\big((vw)_x\big), \tag{6.3}$$

with the initial data

$$v(t=0) = P_N u_0, \qquad w(t=0) = P^N u_0. \tag{6.4}$$

First, we use that $v = u - w$ to write the equation for $w$ without explicit use of $v$. Whence, we deduce the global existence of $w$ and the decay in time of $w(t)$ in $\tilde{L}^2(\mathbb{R})$. Then, the global existence of $v$ follows, and we prove the regularity of $v$ in $\tilde{H}^3(\mathbb{R})$ and the $\tilde{L}^2(\mathbb{R})$ energy equation for $v$.

## 7. Well-posedness and decay of the $w$ part of the solution

We use $v = u - w$ to write Eq. (6.3) for $w$ without explicit use of $v$:

$$w_t - \beta w_{xxx} - \gamma D_x^{-1} w - P^N(ww_x) + \lambda w = -P^N\big((uw)_x\big), \tag{7.1}$$

with the initial data

$$w(t=0) = P^N u_0 = w_0 \in \tilde{L}^2(\mathbb{R}). \tag{7.2}$$

Next, we mainly use the arguments in Section 5 to obtain that for $T$ sufficiently small, there exists unique local solution of (7.1)–(7.2), and the bound

$$\|w\|_{\tilde{X}^T_{0,b}} \leqslant C\|w_0\|_{\tilde{L}^2(\mathbb{R})}, \tag{7.3}$$

for some $b > \frac{1}{2}$.

The decay of $w$ can be obtained with the help of bilinear estimate (4.1) and the energy equation method. By taking the inner product of Eq. (7.1) with $2w$ in $\tilde{L}^2(\mathbb{R})$ and integrating the equation, we obtain

$$\big\|w(t)\big\|^2_{\tilde{L}^2} \leqslant \|w_0\|^2_{\tilde{L}^2} e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\big[\big(u_x(s), w^2(s)\big)_{\tilde{L}^2} - \lambda\big\|w(s)\big\|_{\tilde{L}^2}\big]\,ds. \tag{7.4}$$

Next, we estimate the term $\int_0^t e^{-\lambda(t-s)}(u_x(s), w^2(s))_{\tilde{L}^2}\,ds$ by the bilinear estimate (4.1). Let $\frac{1}{2} < b$, using integration by parts and duality, we obtain

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\big(u_x(s), w^2(s)\big)_{\tilde{L}^2}\,ds \\
&\quad = \int_{-\infty}^{\infty} e^{-\lambda(t-s)}\big(\psi_{[0,t]}u, \partial_x(\psi_{[0,t]}w)^2\big)_{\tilde{L}^2}(s)\,ds \\
&\quad \leqslant \|\psi_{[0,t]}u\|_{\tilde{X}_{0,b-1}}\big\|\partial_x(\psi_{[0,t]}w)^2\big\|_{\tilde{X}_{0,b-1}}. \qquad (7.5)
\end{aligned}$$

By applying the bilinear estimate (4.1), it follows that

$$\big\|\partial_x(\psi_{[0,t]}w)^2\big\|_{\tilde{X}_{0,b-1}} \leqslant \frac{C}{N^{\frac{1}{8}}}\|w\|^2_{\tilde{X}^{[0,t]}_{0,b}}. \qquad (7.6)$$

On the other hand, we easily have

$$\|\psi_{[0,t]}u\|_{\tilde{X}_{0,b-1}} \leqslant \|u\|_{\tilde{X}^{[0,t]}_{0,b}}. \qquad (7.7)$$

Hence, we have by (7.6) and (7.7)

$$\int_0^t e^{-\lambda(t-s)}\big(u_x(s), w^2(s)\big)_{\tilde{L}^2}\,ds \leqslant \frac{C}{N^{\frac{1}{8}}}\|w\|^2_{\tilde{X}^{[0,t]}_{0,b}}\|u\|_{\tilde{X}^{[0,t]}_{0,b}}. \qquad (7.8)$$

Then, using (7.3), we have

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\big(u_x(s), w^2(s)\big)_{\tilde{L}^2}\,ds \\
&\quad \leqslant \frac{C}{N^{\frac{1}{8}}}\|u\|_{\tilde{X}^{[0,t]}_{0,b}}\frac{1}{|t|}\int_0^t \|w\|^2_{\tilde{X}^{[0,t]}_{0,b}}\,ds \leqslant \frac{C}{N^{\frac{1}{8}}}\|u\|_{\tilde{X}^{[0,t]}_{0,b}}\frac{1}{|t|}\int_0^t \big\|w(s)\big\|^2_{\tilde{L}^2}\,ds. \qquad (7.9)
\end{aligned}$$

Thus, we have

$$\big\|w(t)\big\|^2_{\tilde{L}^2} \leqslant \|w_0\|^2_{\tilde{L}^2}e^{-\lambda t} + \int_0^t \left[\frac{C}{|t|N^{\frac{1}{8}}}\|u\|_{\tilde{X}^{[0,t]}_{0,b}} - e^{-\lambda(t-s)}\lambda\right]\big\|w(s)\big\|^2_{\tilde{L}^2}\,ds. \qquad (7.10)$$

For $N$ large enough, the second term in the right-hand side above is negative, hence

$$\big\|w(t)\big\|^2_{\tilde{L}^2} \leqslant \|w_0\|^2_{\tilde{L}^2}e^{-\lambda t}. \qquad (7.11)$$

This now can be iterated and shown to hold for all $t \geqslant 0$ in the interval of definition of $w$. This shows that $w(t)$ decays exponentially to zero in $\tilde{L}^2(\mathbb{R})$.

## 8. Regularity of the $v$ part of the solution

Since $u$ and $w$ are defined globally in time, we mainly use the argument in Section 5 and energy equation to prove an $\tilde{H}^3(\mathbb{R})$ bound for $v = P_N v + P^N v$. For details, we can refer to [8].

Using (5.4), we first observe that

$$\limsup_{t\to\infty}\big\|P_N v(t)\big\|_{\tilde{H}^3} \leqslant \limsup_{t\to\infty} N^3\big\|v(t)\big\|_{\tilde{L}^2} \leqslant \limsup_{t\to\infty} N^3\big\|u(t)\big\|_{\tilde{L}^2} \leqslant \rho_0 N^3. \qquad (8.1)$$

Hence, we focus on an $\tilde{H}^3(\mathbb{R})$ estimate for $P^N v = Z$, which is solution to

$$Z_t - \beta Z_{xxx} - D_x^{-1} Z + P^N\big((P_N v + Z)(P_N v + Z)_x\big) + \lambda Z = P^N f, \tag{8.2}$$

$$Z(0) = 0 \tag{8.3}$$

which follow from (6.2), (6.4) and $P^N(P_N) = 0$.

We can find to obtain the $\tilde{H}^3(\mathbb{R})$ bound for $Z$ is equivalent to prove an $\tilde{L}^2(\mathbb{R})$ estimate on $Z' = Z_t$, which solves

$$\begin{aligned} &Z'_t - \beta Z'_{xxx} - D_x^{-1} Z' + \frac{1}{2} P^N \partial_x\big((P_N v + Z) Z'\big) + \lambda Z' \\ &\quad = -\frac{1}{2} P^N \partial_x\big((P_N v + Z)\partial_t(P_N v)\big), \end{aligned} \tag{8.4}$$

$$Z'(0) = P^N f - P^N\big(P_N v(0)\partial_x P_N v(0)\big). \tag{8.5}$$

Similar with arguments in Sections 5 and 7, using the energy equation, bilinear estimate (4.1), the local well-posedness of (8.4)–(8.5) in $\tilde{L}^2(\mathbb{R})$, we have

$$\big\|Z'(t)\big\|_{\tilde{L}^2(\mathbb{R})} \leqslant C\big(\lambda, \|u_0\|_{\tilde{L}^2(\mathbb{R})}, \|f\|_{\tilde{L}^2(\mathbb{R})}, N\big), \quad \forall t \geqslant 0. \tag{8.6}$$

Therefore, we have

$$\big\|v(t)\big\|_{\tilde{H}^3(\mathbb{R})} \leqslant C\big(\lambda, \|u_0\|_{\tilde{L}^2(\mathbb{R})}, \|f\|_{\tilde{L}^2(\mathbb{R})}, N\big), \quad \forall t \geqslant 0. \tag{8.7}$$

Therefore, there exists the following energy equation for $v$,

$$\begin{aligned} \big\|v(t)\big\|^2_{\tilde{L}^2(\mathbb{R})} \leqslant \big\|v(0)\big\|^2_{\tilde{L}^2(\mathbb{R})} + \int_0^t e^{-2\lambda t}\big[&2\big(f, v(s)\big)_{\tilde{L}^2(\mathbb{R})} \\ &+ \big(P_N\big(2v(s)w(s) + w^2(s)\big), v_x(s)\big)_{\tilde{L}^2(\mathbb{R})}\big]\,ds. \end{aligned} \tag{8.8}$$

## 9. Asymptotic smoothing and the global attractor

In this section, we use the results in Sections 7, 8 to prove the existence of global attractor in $\tilde{L}^2(\mathbb{R})$ and its compactness in $\tilde{H}^3(\mathbb{R})$. For details, we can refer to [8]. First, we prove the asymptotic compactness of the group in $\tilde{L}^2(\mathbb{R})$. Hence we prove that a bounded sequence of initial conditions $\{u_{0n}\}_n$ in $\tilde{L}^2(\mathbb{R})$ and a sequence of positive numbers $t_n \to \infty$, the solution $u_n(t_n) = v_n(t_n) + w_n(t_n)$ are precompact in $\tilde{L}^2(\mathbb{R})$, with $w_n(t_n)$ decaying to zero in $\tilde{L}^2(\mathbb{R})$ and $v_n(t_n)$ being precompact in $\tilde{L}^2(\mathbb{R})$ and weakly precompact in $\tilde{H}^3(\mathbb{R})$. This will give us the existence of the global attractor $\mathcal{A}$ in $\tilde{L}^2(\mathbb{R})$, and, at the same time, the boundedness of $\mathcal{A}$ in $\tilde{H}^3(\mathbb{R})$.

In fact, from Section 8, we have

$$\big\{v_n(t_n + \cdot)\big\}_n \quad \text{is bounded in } C\big([-T, T]; \tilde{H}^3(\mathbb{R})\big), \tag{9.1}$$

and, for the time-derivative,

$$\big\{\partial_t v_n(t_n + \cdot)\big\}_n \quad \text{is bounded in } C\big([-T, T]; \tilde{L}^2(\mathbb{R})\big), \tag{9.2}$$

for each $T > 0$ (and starting with $n$ sufficiently large so that $t_n - T \geqslant 0$). By Arzela–Ascoli Theorem, we can find that a subsequence of $v_n(t_n)$ such that

$$\begin{aligned} v_{n_j}(t_{n_j} + \cdot) \rightharpoonup \bar{u}(\cdot) \quad &\text{strongly in } C\big([-T, T]; H^s_{\mathrm{loc}}(\mathbb{R})\big),\ \forall s \in [0, 3), \\ &\text{weakly star in } L^\infty\big([-T, T]; \tilde{H}^3(\mathbb{R})\big),\ \forall T > 0. \end{aligned} \tag{9.3}$$

Moreover,

$$v_{n_j}(t_{n_j}+t) \rightharpoonup \bar{u}(t) \quad \text{weakly in } \tilde{H}^3(\mathbb{R}), \text{ for every } t \in \mathbb{R}. \tag{9.4}$$

From (7.11), we have

$$\left\| w_n(t_n+t) \right\|_{\tilde{L}^2(\mathbb{R})} \to 0, \quad \text{uniformly for } t \geqslant -T,\ \forall T > 0. \tag{9.5}$$

With (9.3) and (9.5), one can pass to the limit in the weak formulation of the equation for $v_{n_j}$ to find that $\bar{u}$ is a solution of (1.1), and satisfy the energy equality (5.1).

We use energy equality (8.8) for $v_n$ with $t = t_n$ and $0 = t_n - T$, uniform for $v$ in $\tilde{H}^3(\mathbb{R})$, the decay (9.5) of $w_n$ and weak-star limit of $v_{n_j}$ in (9.3) to obtain

$$\limsup_{j\to\infty} \left\| v_{n_j}(t_{n_j}) \right\|_{\tilde{L}^2}^2 \leqslant \left\| \bar{u}(0) \right\|_{\tilde{L}^2}^2. \tag{9.6}$$

This, together with (9.4) and (9.5), implies that

$$u_{n_j}(t_{n_j}) = v_{n_j}(t_{n_j}) + w_{n_j}(t_{n_j}) \to \bar{u}(0) \quad \begin{array}{l} \text{strongly in } \tilde{L}^2(\mathbb{R}), \\ \text{weakly in } \tilde{H}^3(\mathbb{R}). \end{array} \tag{9.7}$$

This shows the solution operator is asymptotically compact in $\tilde{L}^2(\mathbb{R})$. Hence there exists a global attractor $\mathcal{A}$ in $\tilde{L}^2(\mathbb{R})$, and, at the same time, $\mathcal{A}$ is bounded set in $\tilde{H}^3(\mathbb{R})$.

Then, we work with the equations for $u_n' = \frac{du_n}{dt}$ and show, using the energy equation method applied to $u_n'$, that with the initial conditions $\{u_{0n}\}_n$ belonging to $\mathcal{A}$ (and, hence, bounded in $\tilde{H}^3(\mathbb{R})$), the sequence $u_n'(t_n)$ is precompact in $\tilde{L}^2(\mathbb{R})$. This implies, from the equation for $u$, that $u_n(t_n)$ is precompact in $\tilde{H}^3(\mathbb{R})$. This shows that the flow restricted to the global attractor is asymptotically compact in $\tilde{H}^3(\mathbb{R})$ and, hence, that the global attractor is compact in $\tilde{H}^3(\mathbb{R})$. The proof of asymptotically compact in $\tilde{H}^3(\mathbb{R})$ is similar with one of asymptotically compact in $\tilde{L}^2(\mathbb{R})$, here we do not need to split the solutions $u_n(t_n)$. Hence we omit the details.

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