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Stochastic Processes and their Applications 121 (2011) 24–37
stochastic processes and their applications
www.elsevier.com/locate/spa

Scaling limit for the diffusion exit problem in the Levinson case

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Received 14 June 2010; received in revised form 30 August 2010; accepted 13 September 2010
Available online 19 September 2010

Abstract

The exit problem for small perturbations of a dynamical system in a domain is considered. It is assumed that the unperturbed dynamical system and the domain satisfy the Levinson conditions. We assume that the random perturbation affects the driving vector field and the initial condition, and each of the components of the perturbation follows a scaling limit. We derive the joint scaling limit for the random exit time and exit point. We use this result to study the asymptotics of the exit time for 1D diffusions conditioned on rare events.
Published by Elsevier B.V.

Keywords: Small noise; Exit problem; Levinson case; Rare event

1. Introduction

Small stochastic perturbations of deterministic dynamical systems have been studied for several decades; see, e.g., the set of lectures [4] and references therein. Properties of exit distributions for the resulting diffusions are particularly important. One reason for that is that one can express the solutions of parabolic and elliptic PDE's containing the generator of the diffusion via the exit distributions. Another reason is the possibility of using exit distributions for the analysis of the global behavior of the system. One can cover the state space with several domains and study the process within each of them separately. Using the strong Markov property one can then treat the exit distribution for one of the domains as the starting distribution for the

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0304-4149/$ - see front matter. Published by Elsevier B.V.
doi:10.1016/j.spa.2010.09.002

next one. This approach was vital for the study of noisy heteroclinic networks in the vanishing noise limit; see [1,2].

In this note we study a relatively simple situation called the Levinson case (see [4, Chapter 2]), where the typical exit happens along a trajectory of the deterministic flow. We derive a scaling law for the exit distribution in the limit of vanishing perturbation assuming that the initial random data as well as both deterministic and white noise components of the perturbation follow a scaling limit.

We also show that our main result can be used to study rare events for diffusions. We present a one-dimensional situation where to reach a certain threshold, the diffusion has to evolve against the deterministic flow. By conditioning on this unlikely event, we reduce the analysis to the Levinson case.

The paper is organized as follows. In Section 2 we state the main theorem for the Levinson case, postponing its proof to Section 4. In Section 3 we state the result on the diffusion conditioned on a rare event and derive it from the main theorem and some auxiliary statements proven in Section 5.

2. Main result

We consider a C^2-smooth bounded vector field b in $\mathbb{R}^d$. The unperturbed dynamics is given by the deterministic flow $S = (S^t)_{t\in\mathbb{R}}$ generated by b:

$$\frac{\mathrm{d}}{\mathrm{d}t} S^t x_0 = b(S^t x_0), \qquad S^0 x_0 = x_0.$$

We will introduce three components of perturbations of this deterministic flow. They all depend on a small parameter $\epsilon > 0$.

The first component is white noise perturbation generated by $\epsilon\sigma$, where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a C^2-smooth bounded matrix valued function.

The second one is $\epsilon^{\alpha_1} \Psi_\epsilon$, where Ψ_ϵ is a deterministic Lipschitz vector field on $\mathbb{R}^d$ for each ϵ, converging uniformly to a limiting Lipschitz vector field Ψ_0, and α_1 is a positive scaling exponent. These conditions ensure that the stochastic Itô equation

$$\mathrm{d}X_\epsilon(t) = \big(b(X_\epsilon(t)) + \epsilon^{\alpha_1} \Psi_\epsilon(X_\epsilon(t))\big)\,\mathrm{d}t + \epsilon\sigma(X_\epsilon(t))\mathrm{d}W \tag{1}$$

w.r.t. a standard d-dimensional Wiener process W has a unique strong solution for any $\epsilon > 0$ and all initial conditions (for a general background on stochastic differential equations see, e.g., [6]).

The last component of the perturbation is the initial condition satisfying

$$X_\epsilon(0) = x_0 + \epsilon^{\alpha_2}\xi_\epsilon, \quad \epsilon > 0. \tag{2}$$

Here $\alpha_2 > 0$, and $(\xi_\epsilon)_{\epsilon>0}$ is a family of random variables independent of W, such that for some random variable ξ_0, $\xi_\epsilon \to \xi_0$ as $\epsilon \to 0$ in distribution.

Let M be a smooth C^2-hypersurface in $\mathbb{R}^d$. If

$$\tau_\epsilon = \inf\{t \geq 0 : X_\epsilon(t) \in M\},$$

then on $\{\tau_\epsilon < \infty\}$ we have $X_\epsilon(\tau_\epsilon) \in M$. We are going to study the limiting behavior of τ_ϵ and $X_\epsilon(\tau_\epsilon)$ as $\epsilon \to 0$ under the assumptions above.

Let us describe our assumptions on the joint geometry of the vector field b and the surface M. First we define

$$T = \inf\left\{t > 0 : S^t x_0 \in M\right\},$$

and assume that $0 < T < \infty$. Secondly, we denote $z = S^T x_0 \in M$ and assume that $b(z)$ does not belong to the tangent hyperplane $T_z M$. In other words, we assume that the positive orbit of x_0 intersects M and the crossing is transversal.

In the case of $\xi_\epsilon \equiv 0$ and $\Psi \equiv 0$, Levinson's theorem states (see [7], [5, Chapter 2], and [4, Chapter 2]) that $X_\epsilon(\tau_\epsilon) \to z$ in probability as $\epsilon \to 0$. Levinson worked in the PDE context and showed how to obtain an expansion for the solution of the corresponding elliptic PDE depending on the small parameter ϵ. The main result of this note describes the limiting behavior of the correction $(\tau_\epsilon - T, X_\epsilon(\tau_\epsilon) - z)$ and extends [5, Theorem 2.3] to the situation with generic perturbation parameters ξ_0, Ψ, α_1, and α_2. This extension is essential since, as the analysis in [1] shows, in the sequential study of entrance–exit distributions for multiple domains one has to consider nontrivial scaling laws for the initial conditions; also, considering nontrivial deterministic perturbations will allow us to study rare events; see Section 3.

We need more notation. Due to the smoothness of b,

$$b(x) = b(y) + Db(y)(x - y) + Q_1(y, x - y), \quad x, y \in \mathbb{R}^d, \tag{3}$$

where

$$|Q_1(u, v)| \le K|v|^2, \tag{4}$$

for some constant $K > 0$ and any $u, v \in \mathbb{R}^d$. We denote by $\Phi_x(t)$ the linearization of S along the orbit of x:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_x(t) = A(t)\Phi_x(t), \qquad \Phi_x(0) = I, \tag{5}$$

where $A(t) = Db(S^t x)$ and I is the identity matrix.

Finally, for any vector $v \in \mathbb{R}^d$, we define $\pi_b v \in \mathbb{R}$ and $\pi_M v \in T_z M$ by

$$v = \pi_b v \cdot b(z) + \pi_M v,$$

i.e., π_b is the (algebraic) projection onto $\mathrm{span}(b(z))$ along $T_z M$ and π_M is the (geometric) projection onto $T_z M$ along $\mathrm{span}(b(z))$.

Theorem 1. *Let* $\alpha = \alpha_1 \wedge \alpha_2 \wedge 1$, *and*

$$\begin{aligned}\phi_0(t) = \mathbf{1}_{\{\alpha_2=\alpha\}}\Phi_{x_0}(t)\xi_0 + \mathbf{1}_{\{\alpha_1=\alpha\}}\Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1}\Psi_0(S^s x)\mathrm{d}s \\ + \mathbf{1}_{\{1=\alpha\}}\Phi_{x_0}(t)\int_0^t \Phi_{x_0}^{-1}(s)\sigma(S^s x_0)\mathrm{d}W(s), \quad t > 0.\end{aligned} \tag{6}$$

Then, in the setting introduced above,

$$\epsilon^{-\alpha}(\tau_\epsilon - T, X_\epsilon(\tau_\epsilon) - z) \to (-\pi_b\phi_0(T), \pi_M\phi_0(T)). \tag{7}$$

in distribution. If additionally we require that $\xi_\epsilon \to \xi_0$ in probability or that $\alpha_2 > \alpha$, then the convergence in (7) is also in probability.

Remark 1. The conditions of Theorem 1 can be relaxed using the standard localization procedure. In fact, one needs to require uniform convergence of $\Psi_\epsilon \to \Psi_0$ and regularity properties of b and σ only in some neighborhood of the set $\{S^t x_0 : 0 \le t \le T(x_0)\}$.

Remark 2. In applications (see [1,2]), the parameters α_1 and α_2 can be chosen such that the r.h.s. of (7) is nondegenerate.

Remark 3. In the case where $d = 1$, the hypersurface M is just a point. Therefore, π_M is identically zero and the only contentful information that Theorem 1 provides is the asymptotics of the exit time.

3. Conditioned diffusions in one dimension

In this section we apply Theorem 1 to the analysis of the exit time of conditioned diffusions in a one-dimensional situation for the large deviation case.

Suppose, for each $\epsilon > 0$, X_ϵ is a weak solution of the following SDE:

$$\mathrm{d}X_\epsilon(t) = b(X_\epsilon(t))\mathrm{d}t + \epsilon\sigma(X_\epsilon(t))\mathrm{d}W(t),$$
$$X_\epsilon(0) = x_0,$$

where b and σ are C^1 functions on $\mathbb{R}$, such that $b(x) < 0$ and $\sigma(x) \neq 0$ for all x in an interval $[a_1, a_2]$ containing x_0. We introduce

$$\tau_\epsilon = \inf\{t \geq 0 : \ X_\epsilon(t) = a_1 \text{ or } a_2\}$$

and $B_\epsilon = \{X_\epsilon(\tau_\epsilon) = a_2\}$. Since $b < 0$, B_ϵ is a rare event: $\lim_{\epsilon\to 0} \mathbf{P}(B_\epsilon) = 0$. More precise estimates on the asymptotic behavior of $\mathbf{P}(B_\epsilon)$ can be obtained in terms of large deviations. However, here we study the diffusion X_ϵ conditioned on the rare event B_ϵ.

Let $T(x_0)$ denote the time that it takes for the solution of $\dot{x} = -b(x)$ starting at x_0 to reach a_2. Given that $b < 0$ on the whole interval $[a_1, a_2]$, a simple calculation shows that

$$T(x_0) = -\int_{x_0}^{a_2} \frac{1}{b(x)}\mathrm{d}x.$$

Theorem 2. *Conditioned on B_ϵ, the distribution of $\epsilon^{-1}(\tau_\epsilon - T(x_0))$ converges weakly to a centered Gaussian distribution with variance*

$$-\int_{x_0}^{a_2} \frac{\sigma^2(y)}{b^3(y)}\mathrm{d}y.$$

To prove this theorem, we will need two auxiliary statements. Their proofs are given in Section 5.

Lemma 3. *Conditioned on B_ϵ, the process X_ϵ is a diffusion with the same diffusion coefficient as the unconditioned process, and with the drift coefficient given by*

$$b_\epsilon(x) = b(x) + \epsilon^2\sigma^2(x)\frac{h_\epsilon(x)}{\int_{a_1}^{x} h_\epsilon(y)\mathrm{d}y},$$

where

$$h_\epsilon(x) = \exp\left\{-\frac{2}{\epsilon^2}\int_{a_1}^{x} \frac{b(y)}{\sigma^2(y)}\mathrm{d}y\right\}. \tag{8}$$

Further analysis requires understanding the limiting behavior of b_ϵ. This is the purpose of the next lemma:

Lemma 4. *There is* $\delta > 0$ *such that*

$$\limsup_{\epsilon \to 0} \epsilon^{-2} \left(\sup_{x \in [x_0 - \delta, a_2 + \delta]} |b_\epsilon(x) + b(x)| \right) < \infty.$$

Remark 4. Although we need the condition that $b(x) < 0$ for all $x \in [a_1, a_2]$ for Theorem 2 to hold, Lemmas 3 and 4 hold independently of the sign properties of b.

Proof of Theorem 2. Let us fix $\beta \in (1, 2)$. Lemmas 3 and 4 imply that X_ϵ conditioned on B_ϵ, up to τ_ϵ, satisfies an SDE of the form

$$\mathrm{d}X_\epsilon(t) = \left(-b(X_\epsilon(t)) + \epsilon^\beta \Psi_{\epsilon,\beta}(X_\epsilon(t))\right) \mathrm{d}t + \epsilon \sigma(X_\epsilon(t)) \mathrm{d}\tilde{W}(t),$$

for some Brownian Motion $\tilde{W}$ and with $\Psi_{\epsilon,\beta} \to 0$ uniformly as $\epsilon \to 0$. We can assume that after time τ_ϵ, this process still follows the same equation at least up to the time it hits $x_0 - \delta$ or $a_1 + \delta$.

So, having the dynamics from $\dot{x} = -b(x)$ as the underperturbed dynamics, we can apply Theorem 1 (taking into account Remark 1) to see that

$$\epsilon^{-1}(\tau_\epsilon - T(x_0)) \xrightarrow{\mathbf{P}} -\frac{1}{b(a_2)} \Phi_{x_0}(T(x_0)) \int_0^{T(x_0)} \Phi_{x_0}^{-1}(s) \sigma(S^s x_0) \mathrm{d}\tilde{W}(s), \quad \epsilon \to 0, \quad (9)$$

where $S^t x_0$ is the flow generated by the vector field $-b$, the time $T(x_0)$ solves $S^{T(x_0)} x_0 = a_2$, and Φ_{x_0} is the linearization of S near the orbit of x_0. The limit is clearly a centered Gaussian random variable. To compute its variance we must first solve

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_{x_0}(t) = -b'(S^t x_0) \Phi_{x_0}(t), \qquad \Phi_{x_0}(0) = 1.$$

The solution to this linear ODE is

$$\Phi_{x_0}(t) = \exp\left\{ -\int_0^t b'(S^s x_0) \mathrm{d}s \right\},$$

so after the change of variables $u = S^s x_0$ in the integral, we get

$$\Phi_{x_0}(t) = \frac{b(S^t x_0)}{b(x_0)}.$$

Using this expression and Itô isometry for the limiting random variable in (9), we get that the variance of such a random variable is

$$\int_0^{T(x_0)} \frac{\sigma^2(S^t x_0)}{b^2(S^t x_0)} \mathrm{d}t.$$

We can now use the change of variable $y = S^t x_0$ to get the expression in Theorem 2. □

4. Proof of the main result

With high probability, at time T the process X_ϵ is close to z and the hitting time τ_ϵ is close to T. The idea of the proof is that while the diffusion is close to z, the process may be approximated very well by motion with constant velocity $b(z)$.

We start with a lemma and postpone its proof to the end of this section to keep continuity of the text.

Lemma 5. *Let X_ϵ be the solution of the SDE (1) with initial condition (2). Let*

$$\Theta_\epsilon(t) = \epsilon^{\alpha_2-\alpha}\,\Phi_{x_0}(t)\xi_\epsilon + \epsilon^{\alpha_1-\alpha}\,\Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1}\,\Psi_0(S^s x_0)\mathrm{d}s$$
$$+\,\epsilon^{1-\alpha}\,\Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1}\sigma(S^s x_0)\mathrm{d}W(s). \tag{10}$$

Then,

$$X_\epsilon(t) = S^t x_0 + \epsilon^\alpha \phi_\epsilon(t)$$

holds almost surely for every $t > 0$, where $\phi_\epsilon(t) = \Theta_\epsilon(t) + r_\epsilon(t)$, and r_ϵ converges to 0 uniformly over compact time intervals in probability.

If $\xi_\epsilon \to \xi_0$ in distribution, then for any $T > 0$, $\phi_\epsilon \to \phi_0$ in distribution in $C[0, T]$ equipped with uniform norm, where ϕ_0 is the stochastic process defined in (6).

If $\xi_\epsilon \to \xi_0$ in probability or $\alpha_2 > \alpha$, then the uniform convergence for ϕ_ϵ also holds in probability.

Remark 5. This lemma gives the first-order approximation for $X_\epsilon(t)$. Higher-order approximations in the spirit of [3] are also possible. They can be used to refine Theorem 1.

Our task now is to analyze the process $X_\epsilon(t) - z$ for t close to T. Let us first estimate the deviation of the flow S from the motion with constant velocity $b(z)$. Let

$$r_\pm(t, x) = S^{\pm t}x - (x \pm tb(z)), \quad t > 0,\ x \in \mathbb{R}^d. \tag{11}$$

Lemma 6. *There are constants C_1 and C_2 such that for any $t > 0$ and $x \in \mathbb{R}^d$*

$$\sup_{s\le t} |r_\pm(s, x)| \le C_1 \mathrm{e}^{C_2 t}(t|x - z| + t^2).$$

Proof. We prove the result for r_+. The analysis of r_- is similar since $S^{-t}x$ is the solution to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} S^{-t}x = -b(S^{-t}x).$$

Let $L > 0$ be the Lipschitz constant of b. We have

$$\begin{aligned}
|r_+(t, x)| &\le \int_0^t \left|b(S^s x) - b(z)\right| \mathrm{d}s \\
&\le L\int_0^t \left|S^s x - z\right| \mathrm{d}s \\
&\le L\int_0^t |r_+(s, x)|\,\mathrm{d}s + L\int_0^t |x + sb(z) - z|\,\mathrm{d}s \\
&\le L\int_0^t |r_+(s, x)|\,\mathrm{d}s + L\int_0^t |x - z|\mathrm{d}s + L\int_0^t s|b(z)|\mathrm{d}s \\
&\le L\int_0^t |r_+(s, x)|\,\mathrm{d}s + Lt|x - z| + t^2 L|b(z)|/2.
\end{aligned}$$

The result follows as an application of Gronwall's lemma. □

Lemma 7. *Let $\gamma \in (\alpha/2, \alpha)$. Then, there are two a.s.-continuous stochastic processes $\Gamma_{\epsilon,\pm}$ such that*

$$\sup_{t\in[0,\epsilon^\gamma]} |\Gamma_{\epsilon,\pm}(t)| \xrightarrow{\mathbf{P}} 0, \quad \epsilon \to 0,$$

and almost surely for any $t \in [0, \epsilon^\gamma]$

$$X_\epsilon(T-t) = z - tb(z) + \epsilon^\alpha \left(\phi_\epsilon(T-t) + \Gamma_{\epsilon,-}(t)\right) \tag{12}$$

and

$$X_\epsilon(T+t) = z + tb(z) + \epsilon^\alpha \left(\Phi_z(t)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(t)\right). \tag{13}$$

Proof. Due to Lemma 5, the flow property, and (11) we have

$$\begin{aligned} X_\epsilon(T-t) &= S^{T-t}x_0 + \epsilon^\alpha \phi_\epsilon(T-t) \\ &= S^{-t}z + \epsilon^\alpha \phi_\epsilon(T-t) \\ &= z - tb(z) + r_-(t,z) + \epsilon^\alpha \phi_\epsilon(T-t). \end{aligned}$$

The first estimate with $\Gamma_{\epsilon,-}(t) = \epsilon^{-\alpha} r_-(t,z)$ follows from Lemma 6 for $x = z$.

Due to the strong Markov property and Lemma 5 the process $\tilde{X}_\epsilon(t) = X_\epsilon(t+T)$ is a solution of the initial value problem

$$\begin{aligned} &\mathrm{d}\tilde{X}_\epsilon(t) = (b(\tilde{X}_\epsilon(t)) + \epsilon^{\alpha_1} \Psi_\epsilon(\tilde{X}_\epsilon(t)))\mathrm{d}t + \epsilon\sigma(\tilde{X}_\epsilon(t))\mathrm{d}\tilde{W}, \\ &\tilde{X}_\epsilon(0) = X_\epsilon(T) = z + \epsilon^\alpha \phi_\epsilon(T), \end{aligned}$$

with respect to the Brownian Motion $\tilde{W}(t) = W(t+T) - W(T)$. So, again, applying Lemma 5 to this shifted equation, we obtain $\tilde{X}_\epsilon(t) = S^t z + \epsilon^\alpha \hat{\phi}_\epsilon(t)$, where, for $t > 0$,

$$\hat{\phi}_\epsilon(t) = \Phi_z(t)\phi_\epsilon(T) + \theta_\epsilon(t),$$

and

$$\begin{aligned} \theta_\epsilon(t) = {} & \epsilon^{1-\alpha} \Phi_z(t) \int_0^t \Phi_z(s)^{-1}\sigma(S^s z)\mathrm{d}\tilde{W}(s) \\ & + \epsilon^{\alpha_1-\alpha} \Phi_z(t) \int_0^t \Phi_z(s)^{-1} \Psi_0(S^s z)\mathrm{d}s + \tilde{r}_\epsilon(t), \end{aligned}$$

where $\tilde{r}_\epsilon$ converges to 0 uniformly over compact time intervals in probability. Then due to (11),

$$\begin{aligned} \tilde{X}_\epsilon(t) &= S^t z + \epsilon^\alpha (\Phi_z(t)\phi_\epsilon(t) + \theta_\epsilon(t)) \\ &= z + tb(z) + r_+(t,z) + \epsilon^\alpha (\Phi_z(t)\phi_\epsilon(t) + \theta_\epsilon(t)). \end{aligned}$$

Hence, with $\Gamma_{\epsilon,+}(t) = \theta_\epsilon(t) + \epsilon^{-\alpha} r_+(t,z)$ the result is a consequence of Lemma 6. □

Let us now parametrize, locally around z, the hypersurface M as a graph of a C^2-function F over $T_z M$, i.e., $y \mapsto z + y + F(y) \cdot b(z)$ gives a C^2-parametrization of a neighborhood of z in M by a neighborhood of 0 in $T_z M$. Moreover, $DF(0) = 0$ so $|F(y)| = O(|y|^2)$, $y \to 0$. With this definition, it is clear that, for $w \in \mathbb{R}^d$ with $w - z$ small enough, $w \in M$ if and only if $\pi_b(w - z) = F(\pi_M(w - z))$.

Let us define

$$\Omega_{1,\epsilon} = \left\{\tau_\epsilon = \inf\{t \geq 0 : \pi_b\left(X_\epsilon(t) - z\right) = F\left(\pi_M\left(X_\epsilon(t) - z\right)\right)\}\right\},$$
$$\Omega_{2,\epsilon} = \left\{|\tau_\epsilon - T| \leq \epsilon^\gamma\right\},$$
$$\Omega_\epsilon = \Omega_{1,\epsilon} \cap \Omega_{2,\epsilon}.$$

Lemma 8. $\mathbf{P}(\Omega_\epsilon) \to 1$ *as* $\epsilon \to 0$.

Proof. The definition of F and Lemma 5 imply that as $\epsilon \to 0$, $\mathbf{P}(\Omega_{1,\epsilon}) \to 1$.

We use (13) to conclude that

$$\pi_b\left(X_\epsilon(T + \epsilon^\gamma) - z\right) = \epsilon^\gamma\left(1 + \epsilon^{\alpha-\gamma}\pi_b\left(\Phi_z(\epsilon^\gamma)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\epsilon^\gamma)\right)\right),$$

and

$$F\left(\pi_M\left(X_\epsilon(T + \epsilon^\gamma) - z\right)\right) = F\left(\epsilon^\alpha\pi_M\left(\Phi_z(\epsilon^\gamma)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\epsilon^\gamma)\right)\right).$$

Since $|F(x)| = O(|x|^2)$, these estimates imply that

$$\begin{aligned}\limsup_{\epsilon\to 0}\mathbf{P}\left(\left\{\tau_\epsilon > T + \epsilon^\gamma\right\} \cap \Omega_{1,\epsilon}\right) &\leq \limsup_{\epsilon\to 0}\mathbf{P}\left\{\pi_b\left(X_\epsilon(T + \epsilon^\gamma) - z\right)\right.\\ &\leq \left. F\left(\pi_M\left(X_\epsilon(T + \epsilon^\gamma) - z\right)\right)\right\} = 0.\end{aligned}$$

It remains to prove

$$\lim_{\epsilon\to 0}\mathbf{P}\left\{\tau_\epsilon < T - \epsilon^\gamma\right\} = 0. \tag{14}$$

Let us denote the Hausdorff distance between sets by $d(\cdot,\cdot)$. Then an obvious estimate

$$d(\{S^t x_0 : 0 \leq t \leq T - \delta\}, M) \geq c\delta$$

holds true for some $c > 0$ and all sufficiently small $\delta > 0$. Now (14) follows from Lemma 5, and the proof is complete. □

Lemma 9. *Define* $\tau'_\epsilon = \tau_\epsilon - T$. *Then,*

$$\epsilon^{-\alpha}\tau'_\epsilon + \pi_b\phi_\epsilon(T) \xrightarrow{\mathbf{P}} 0, \quad \epsilon \to 0.$$

Proof. Let us define $A_\epsilon = \left\{0 \leq \tau'_\epsilon \leq \epsilon^\gamma\right\} \cap \Omega_{1,\epsilon}$ and $B_\epsilon = \left\{-\epsilon^\gamma \leq \tau'_\epsilon < 0\right\} \cap \Omega_{1,\epsilon}$, so $\Omega_\epsilon = A_\epsilon \cup B_\epsilon$. We can use (13) and the definition of $\Omega_{1,\epsilon}$ to get

$$\mathbf{1}_{A_\epsilon}\tau'_\epsilon + \mathbf{1}_{A_\epsilon}\epsilon^\alpha\pi_b\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right) = \mathbf{1}_{A_\epsilon}F\left(\epsilon^\alpha\pi_M\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right)\right).$$

This implies

$$\begin{aligned}\mathbf{1}_{A_\epsilon}\epsilon^{-\alpha}\tau'_\epsilon &= \epsilon^{-\alpha}\mathbf{1}_{A_\epsilon}F\left(\epsilon^\alpha\pi_M\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right)\right)\\ &\quad - \mathbf{1}_{A_\epsilon}\pi_b\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right)\\ &= -\mathbf{1}_{A_\epsilon}\pi_b\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T)\right) + r_{\epsilon,1}\\ &= -\mathbf{1}_{A_\epsilon}\pi_b\phi_\epsilon(T) + \mathbf{1}_{A_\epsilon}\pi_b\left((I - \Phi_z(\tau'_\epsilon))\phi_\epsilon(T)\right) + r_{\epsilon,1},\end{aligned} \tag{15}$$

where $r_{\epsilon,1}$ is a random variable that converges to 0 in probability as $\epsilon \to 0$.

Likewise, since $\tau_\epsilon = T - (-\tau'_\epsilon)$ and $\mathbf{1}_{B_\epsilon}\tau'_\epsilon \le 0$, we can use (12) and the definition of $\Omega_{1,\epsilon}$ to see that

$$\mathbf{1}_{B_\epsilon}\tau'_\epsilon + \mathbf{1}_{B_\epsilon}\epsilon^\alpha \pi_b \left(\phi_\epsilon(T+\tau'_\epsilon) + \Gamma_{\epsilon,-}(-\tau'_\epsilon)\right) = \mathbf{1}_{B_\epsilon} F\left(\epsilon^\alpha \left(\phi_\epsilon(T+\tau'_\epsilon) + \Gamma_{\epsilon,-}(-\tau'_\epsilon)\right)\right).$$

Hence, proceeding as before, we see that

$$\begin{aligned}\mathbf{1}_{B_\epsilon}\epsilon^{-\alpha}\tau'_\epsilon &= -\mathbf{1}_{B_\epsilon}\pi_b\phi_\epsilon(T+\tau'_\epsilon) + r_{\epsilon,2}\\ &= -\mathbf{1}_{B_\epsilon}\pi_b\phi_\epsilon(T) + \mathbf{1}_{B_\epsilon}\pi_b\left(\phi_\epsilon(T) - \phi_\epsilon(T+\tau'_\epsilon)\right) + r_{\epsilon,2}\end{aligned}$$

for some random variable $r_{\epsilon,2}$ such that $r_{\epsilon,2} \to 0$ in probability as $\epsilon \to 0$. Adding this identity and (15), we see that on Ω_ϵ

$$\begin{aligned}\epsilon^{-\alpha}\tau'_\epsilon = &-\pi_b\phi_\epsilon(T) + \mathbf{1}_{A_\epsilon}\pi_b\left((I - \Phi_z(\tau'_\epsilon))\phi_\epsilon(T)\right)\\ &+ \mathbf{1}_{B_\epsilon}\pi_b\left(\phi_\epsilon(T) - \phi_\epsilon(T+\tau'_\epsilon)\right) + r_{\epsilon,1} + r_{\epsilon,2}.\end{aligned}$$

Due to Lemma 8, to finish the proof it is sufficient to notice that as $\epsilon \to 0$

$$\sup_{0\le t\le \epsilon^\gamma} |(I - \Phi_z(t))\phi_\epsilon(T)| \xrightarrow{\mathbf{P}} 0, \tag{16}$$

and

$$\sup_{0\le t\le \epsilon^\gamma} |\phi_\epsilon(T) - \phi_\epsilon(T+t)| \xrightarrow{\mathbf{P}} 0. \quad \square \tag{17}$$

Lemma 9 takes care of the time component in Theorem 1. We shall consider the spatial component now.

Let A_ϵ and B_ϵ be as in the proof of Lemma 9. Then, (13) implies

$$\begin{aligned}\mathbf{1}_{A_\epsilon}&(X_\epsilon(\tau_\epsilon) - z)\,\epsilon^{-\alpha} = \mathbf{1}_{A_\epsilon}\epsilon^{-\alpha}\tau'_\epsilon b(z) + \mathbf{1}_{A_\epsilon}\left(\Phi_z(\tau'_\epsilon)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right)\\ &= \mathbf{1}_{A_\epsilon}\left(\epsilon^{-\alpha}\tau'_\epsilon b(z) + \phi_\epsilon(T)\right) + \mathbf{1}_{A_\epsilon}\left[(\Phi_z(\tau'_\epsilon) - I)\phi_\epsilon(T) + \Gamma_{\epsilon,+}(\tau'_\epsilon)\right].\end{aligned} \tag{18}$$

Likewise, from (12) we get that

$$\begin{aligned}\mathbf{1}_{B_\epsilon}&(X_\epsilon(\tau_\epsilon) - z)\,\epsilon^{-\alpha} = \mathbf{1}_{B_\epsilon}\epsilon^{-\alpha}\tau'_\epsilon b(z) + \mathbf{1}_{B_\epsilon}\left(\phi_\epsilon(T+\tau'_\epsilon) + \Gamma_{\epsilon,-}(-\tau'_\epsilon)\right)\\ &= \mathbf{1}_{B_\epsilon}\left(\epsilon^{-\alpha}\tau'_\epsilon b(z) + \phi_\epsilon(T)\right) + \mathbf{1}_{B_\epsilon}\left[(\phi_\epsilon(T+\tau'_\epsilon) - \phi_\epsilon(T)) + \Gamma_{\epsilon,-}(-\tau'_\epsilon)\right].\end{aligned} \tag{19}$$

Adding (18) and (19) and proceeding as in the proof of Lemma 9 we see that

$$(X_\epsilon(\tau_\epsilon) - z)\,\epsilon^{-\alpha} - \pi_M\phi_\epsilon(T) = \left(\epsilon^{-\alpha}\tau'_\epsilon + \pi_b\phi_\epsilon(T)\right) b(z) + \rho_\epsilon,$$

where, due to (16), (17) and Lemma 7, $\rho_\epsilon \to 0$ in probability as $\epsilon \to 0$. From this expression and Lemma 9 we get that

$$(X_\epsilon(\tau_\epsilon) - z)\,\epsilon^{-\alpha} - \pi_M\phi_\epsilon(T) \xrightarrow{\mathbf{P}} 0, \quad \epsilon \to 0.$$

Then, using this and the convergence in Lemma 9

$$\epsilon^{-\alpha}(\tau_\epsilon - T, X_\epsilon(\tau_\epsilon) - z) = R_\epsilon + G(\phi_\epsilon(T)),$$

where R_ϵ is a random variable such that $R_\epsilon \to 0$ in probability as $\epsilon \to 0$. G is the continuous function $x \mapsto (-\pi_b x, \pi_M x)$. Hence, Theorem 1 follows from the convergence in Lemma 5.

It remains to prove Lemma 5, the core of the proof of the main result.

Proof of Lemma 5. Let $\Delta_\epsilon^t = X_\epsilon(t) - S^t x_0$ and note that it satisfies the equation

$$\mathrm{d}\Delta_\epsilon^t = \left(\left(b(X_\epsilon(t)) - b(S^t x_0)\right) + \epsilon^{\alpha_1} \Psi_\epsilon(X_\epsilon(t))\right) \mathrm{d}t + \epsilon\sigma(X_\epsilon(t))\mathrm{d}W(t),$$

with initial condition $\Delta_\epsilon^0 = \epsilon^{\alpha_2}\xi_\epsilon$. We want to study the properties of this equation. We start with the difference in b. Since b is a C^2 vector field, we may write

$$b(X_\epsilon(t)) - b(S^t x_0) = Db(S^t x_0)\Delta_\epsilon^t + Q_1(S^t x_0, \Delta_\epsilon^t). \tag{20}$$

Also, we can write

$$\Psi_\epsilon(X_\epsilon(t)) = \Psi_0(S^t x_0) + Q_2(S^t x_0, \Delta_\epsilon^t) + R_\epsilon(S^t x_0), \tag{21}$$

and

$$\sigma(X_\epsilon(t)) = \sigma(S^t x_0) + Q_3(S^t x_0, \Delta_\epsilon^t), \tag{22}$$

where

$$R_\epsilon(x) = \Psi_\epsilon(x) - \Psi_0(x) = o(1), \quad \epsilon \to 0,$$

uniformly in x; $Q_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d,\ i = 2, 3$, satisfies

$$|Q_i(u, v)| \le K|v|, \quad u, v \in \mathbb{R}^d. \tag{23}$$

We can assume that the constant $K > 0$ in (4) and (23) is the same for simplicity of notation.

Let $Q = Q_1 + \epsilon^{\alpha_1} Q_2 + \epsilon^{\alpha_1} R_\epsilon$. Combine (20)–(22) to get

$$\begin{aligned} \mathrm{d}\Delta_\epsilon^t &= \left(A(t)\Delta_\epsilon^t + \epsilon^{\alpha_1}\Psi_0(S^t x_0) + Q(S^t x_0, \Delta_\epsilon^t)\right)\mathrm{d}t \\ &\quad + \epsilon\left(\sigma(S^t x_0) + Q_3(S^t x_0, \Delta_\epsilon^t)\right)\mathrm{d}W(t), \end{aligned} \tag{24}$$

$$\Delta_\epsilon^0 = \epsilon^{\alpha_2}\xi_\epsilon. \tag{25}$$

Hence, applying Duhamel's principle to (24) and using (10), we get

$$\begin{aligned} \Delta_\epsilon^t &= \epsilon^\alpha \Theta_\epsilon(t) + \Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1} Q(S^s x_0, \Delta_\epsilon^s)\mathrm{d}s \\ &\quad + \epsilon\Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1} Q_3(S^s x_0, \Delta_\epsilon^s)\mathrm{d}W(s) \\ &= \epsilon^\alpha \Theta_\epsilon(t) + \Theta'_\epsilon(t), \end{aligned} \tag{26}$$

where Θ'_ϵ is defined by (26). A simple inspection of (10) shows that $(\Theta_\epsilon)_{\epsilon>0}$ converges in distribution in $C(0, T)$ to the process $\phi_0(t)$. This convergence is in probability if $\alpha_2 > \alpha$ or $\xi_\epsilon \to \xi_0$ in probability. Therefore, the lemma will follow with $\phi_\epsilon = \Theta_\epsilon + \epsilon^{-\alpha}\Theta'_\epsilon$ if we show that

$$\epsilon^{-\alpha}\sup_{t\le T}|\Theta'_\epsilon(t)| \xrightarrow{\mathbf{P}} 0, \quad \epsilon \to 0. \tag{27}$$

For any $\delta \in (1/2, 1)$, we introduce the stopping time

$$l_\epsilon(\delta) = \inf\left\{t > 0 : |\Delta_\epsilon^t| \ge \epsilon^{\alpha\delta}\right\}.$$

Now, $\Theta'_\epsilon = \Theta'_{\epsilon,1} + \epsilon\Theta'_{\epsilon,2}$, where

$$\Theta'_{\epsilon,1}(t) = \Phi_{x_0}(t)\int_0^t \Phi_{x_0}(s)^{-1} Q(S^s x_0, \Delta_\epsilon^s)\mathrm{d}s,$$

and

$$\Theta'_{\epsilon,2}(t) = \epsilon \Phi_{x_0}(t) \int_0^t \Phi_{x_0}(s)^{-1} Q_3(S^s x_0, \Delta_\epsilon^s) \mathrm{d}W(s).$$

Bounds (4) and (23) imply

$$\sup_{t \le T \wedge l_\epsilon(\delta)} |\Theta'_{\epsilon,1}(t)| = O(\epsilon^{2\alpha\delta} + \epsilon^{\alpha_1 + \alpha\delta}) + o(\epsilon^{\alpha_1}) = o(\epsilon^\alpha). \tag{28}$$

Likewise, (23) for Q_3 and BDG inequality imply that for any $\kappa > 0$ there is a constant K_κ such that

$$\mathbf{P}\left\{ \sup_{t \le T \wedge l_\epsilon(\delta)} |\Theta'_{\epsilon,2}(t)| > K_\kappa \epsilon^{1+\alpha\delta} \right\} < \kappa \tag{29}$$

for all $\epsilon > 0$ small enough. Then, this, together with (28), implies that

$$\epsilon^{-\alpha\delta} \sup_{t \le T \wedge l_\epsilon(\delta)} |\Theta'_\epsilon(t)| \xrightarrow{\mathbf{P}} 0, \quad \epsilon \to 0. \tag{30}$$

Then, if $l_\epsilon(\delta) < T$ we use (26) to get

$$\begin{aligned} 1 &= \epsilon^{-\alpha\delta} \sup_{t \le T \wedge l_\epsilon(\delta)} |\Delta_\epsilon^t| \\ &\le \epsilon^{\alpha(1-\delta)} \sup_{t \le T \wedge l_\epsilon(\delta)} |\Theta_\epsilon(t)| + \epsilon^{-\alpha\delta} \sup_{t \le T \wedge l_\epsilon(\delta)} |\Theta'_\epsilon(t)|. \end{aligned}$$

The r.h.s. converges to 0 in probability due to (30) and the tightness of distributions of Θ_ϵ. Hence, $\mathbf{P}\{l_\epsilon(\delta) < T\} \to 0$ as $\epsilon \to 0$. Using T instead of $T \wedge l_\epsilon(\delta)$ in (28) and (29), we see that with the choice of $\delta > 1/2$, (27) follows and the proof is finished. □

5. Proof of Lemmas 3 and 4

Proof of Lemma 3. Let us find the generator of the conditioned diffusion. To that end we denote the generator of the original diffusion by L_ϵ:

$$L_\epsilon f(x) = b(x) f'(x) + \frac{\epsilon^2}{2} \sigma^2(x) f''(x) = \lim_{t \to 0} \frac{\mathbf{E}_x f(X_\epsilon) - f(x)}{t}, \tag{31}$$

where f is any bounded C^2-function with bounded first two derivatives and $\mathbf{E}_x$ denotes expectation with respect to the measure $\mathbf{P}_x$, the element of the Markov family describing the Markov process emitted from point x.

Let us define $u_\epsilon(x) = \mathbf{P}_x(B_\epsilon)$. This function solves the following boundary-value problem for the backward Kolmogorov equation:

$$L_\epsilon u_\epsilon(x) = 0, \qquad u_\epsilon(a_1) = 0, \qquad u_\epsilon(a_2) = 1.$$

Using (31), it is easy to check that a unique solution is given by

$$u_\epsilon(x) = \frac{\int_{a_1}^x h_\epsilon(y) \mathrm{d}y}{\int_{a_1}^{a_2} h_\epsilon(y) \mathrm{d}y},$$

where h_ϵ is defined in (8).

Now we can compute the generator $\bar{L}_\epsilon$ of the conditioned flow. For any smooth and bounded function $f \in C^2$ with bounded first two derivatives, we can write

$$\begin{aligned}
\mathbf{E}_x[f(X_\epsilon)|B_\epsilon] &= u_\epsilon^{-1}(x)\mathbf{E}_x f(X_\epsilon(t))\mathbf{1}_{B_\epsilon} \\
&= u_\epsilon^{-1}(x)\mathbf{E}_x f(X_\epsilon(t))\mathbf{1}_{B_\epsilon}\mathbf{1}_{\{\tau_\epsilon \geq t\}} + R_\epsilon \\
&= u_\epsilon^{-1}(x)\mathbf{E}_x\mathbf{E}_x[f(X_\epsilon(t))\mathbf{1}_{B_\epsilon}\mathbf{1}_{\{\tau_\epsilon \geq t\}}|\mathcal{F}_t] + R_\epsilon \\
&= u_\epsilon^{-1}(x)\mathbf{E}_x f(X_\epsilon(t))\mathbf{P}_{X_\epsilon(t)}(B_\epsilon) + R_\epsilon \\
&= u_\epsilon^{-1}(x)\mathbf{E}_x f(X_\epsilon(t))u_\epsilon(X_\epsilon(t)) + R_\epsilon,
\end{aligned}$$

where

$$|R_\epsilon| = u_\epsilon^{-1}(x)|\mathbf{E}_x f(X_\epsilon)\mathbf{1}_{B_\epsilon}\mathbf{1}_{\{\tau_\epsilon < t\}}| \leq C(x)\mathbf{P}\{\tau_\epsilon < t\} = o(t)$$

for some $C(x) > 0$. Therefore, we obtain

$$\begin{aligned}
\bar{L}_\epsilon f(x) &= \lim_{t\to 0} \frac{\mathbf{E}_x[f(X_\epsilon(t))|B_\epsilon] - f(x)}{t} \\
&= \lim_{t\to 0} \frac{u_\epsilon^{-1}(x)\mathbf{E}_x f(X_\epsilon(t))u_\epsilon(X_\epsilon(t)) - f(x)}{t} \\
&= \frac{1}{u_\epsilon(x)} \lim_{t\to 0} \frac{\mathbf{E}_x f(X_\epsilon(t))u_\epsilon(X_\epsilon(t)) - f(x)u_\epsilon(x)}{t} \\
&= \frac{1}{u_\epsilon(x)} L_\epsilon(f u_\epsilon)(x) \\
&= \left(b(x) + \epsilon^2\sigma^2(x)\frac{u_\epsilon'(x)}{u_\epsilon(x)}\right) f'(x) + \epsilon^2 \frac{\sigma^2(x)}{2} f''(x) \\
&= \left(b(x) + \epsilon^2\sigma^2(x)\frac{h_\epsilon(x)}{\int_{a_1}^x h_\epsilon(y)\mathrm{d}y}\right) f'(x) + \epsilon^2 \frac{\sigma^2(x)}{2} f''(x),
\end{aligned}$$

completing the proof. □

Proof of Lemma 4. The proof is a variation of Laplace's method. Let

$$\Phi(x) = 2\int_{a_1}^x \frac{b(y)}{\sigma^2(y)}\mathrm{d}y, \quad x \geq a_1, \tag{32}$$

so we have $h_\epsilon(x) = \mathrm{e}^{-\Phi(x)/\epsilon^2}$. We take any $\beta \in (1, 2)$ and break the integral of h_ϵ into two parts:

$$\int_{a_1}^x \mathrm{e}^{-\Phi(y)/\epsilon^2}\mathrm{d}y = I_{\epsilon,1}(x) + I_{\epsilon,2}(x),$$

where

$$I_{\epsilon,1}(x) = \int_{a_1}^{x-\epsilon^\beta} \mathrm{e}^{-\Phi(y)/\epsilon^2}\mathrm{d}y, \tag{33}$$

and

$$I_{\epsilon,2}(x) = \int_{x-\epsilon^\beta}^{x} \mathrm{e}^{-\Phi(y)/\epsilon^2}\mathrm{d}y. \tag{34}$$

The idea is to prove that $I_{\epsilon,1}$ is exponentially smaller than $I_{\epsilon,2}$ and then estimate $I_{\epsilon,2}$.

We start with some preliminaries for the function Φ. Since both b and σ are C^1 and $\sigma \neq 0$ in $[a_1, a_2]$ we conclude that Φ is a C^2 function, so we can find a function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a number $\delta_0 > 0$ such that for every $x, y \in [a_1, a_2 + \delta_0]$, we have the expansion

$$\Phi(y) = \Phi(x) + \Phi'(x)(y - x) + R(x, y - x), \tag{35}$$

and

$$|R(x, v)| \leq K_1 |v|^2, \quad x \in [a_1, a_2 + \delta_0],\ v \in \mathbb{R}, \tag{36}$$

for some $K_1 > 0$.

To estimate $I_{\epsilon,1}$, we introduce

$$J_{\epsilon,1}(x) = \frac{e^{\Phi(x)/\epsilon^2}}{\epsilon^2 \sigma^2(x)} I_{\epsilon,1}(x), \quad x \in [a_1, a_2 + \delta_0].$$

Since Φ is decreasing, we have that for some constant $K_2 > 0$ independent of $x \in [a_1, a_2 + \delta_0]$,

$$J_{\epsilon,1}(x) \leq \frac{K_2}{\epsilon^2} e^{(\Phi(x) - \Phi(x - \epsilon^\beta))/\epsilon^2}. \tag{37}$$

Since $\beta < 2$ and Φ' is negative and bounded away from zero, we conclude that there is $\alpha(\epsilon)$ such that $\alpha(\epsilon) = o(\epsilon^2)$ as $\epsilon \to 0$ and

$$\sup_{x \in [a_1, a_2 + \delta_0]} J_{\epsilon,1}(x) \leq \alpha(\epsilon). \tag{38}$$

We now estimate $I_{\epsilon,2}$. Using expansion (35) and the change of variables $u = -\Phi(x)(y - x)/\epsilon^2$, we get

$$\begin{aligned} I_{\epsilon,2}(x) &= e^{-\Phi(x)/\epsilon^2} \int_{x-\epsilon^\beta}^{x} e^{-\Phi'(x)(y-x)/\epsilon^2 - R(x,y-x)/\epsilon^2} dy \\ &= -\frac{\epsilon^2}{\Phi'(x)} e^{-\Phi(x)/\epsilon^2} \int_{\Phi'(x)/\epsilon^{2-\beta}}^{0} e^{u - R(x, -\epsilon^2 u/\Phi'(x))/\epsilon^2} du \\ &= -\frac{\epsilon^2 \sigma^2(x)}{2b(x)} e^{-\Phi(x)/\epsilon^2} J_{\epsilon,2}(x), \end{aligned} \tag{39}$$

where we use (32) to compute the derivative of Φ, and we define $J_{\epsilon,2}$ by (39). Hence, combining (37) with the definition of b_ϵ and (39), we get

$$b_\epsilon(x) = b(x) + \frac{1}{J_{\epsilon,1}(x) - \frac{1}{2b(x)} J_{\epsilon,2}(x)}.$$

Due to (38), the proof will be complete once we prove that for sufficiently small $\delta > 0$,

$$\limsup_{\epsilon \to 0} \epsilon^{-2} \left(\sup_{x \in [x_0 - \delta, a_2 + \delta]} |J_{\epsilon,2}(x) - 1| \right) < \infty.$$

Note that for any $\delta \in (0, x_0 - a_1)$, some constant $K_3 = K_3(\delta) > 0$ and all $x \in [x_0 - \delta, a_2 + \delta]$,

$$\begin{aligned} |J_{\epsilon,2}(x) - 1| &= \left| \int_{\Phi'(x)/\epsilon^{2-\beta}}^{0} e^u (1 - e^{-R(x, -\epsilon^2 u/\Phi'(x))/\epsilon^2}) du + \int_{-\infty}^{\Phi'(x)/\epsilon^{2-\beta}} e^u du \right| \\ &\leq \int_{\Phi'(x)/\epsilon^{2-\beta}}^{0} e^u |1 - e^{-R(x, -\epsilon^2 u/\Phi'(x))/\epsilon^2}| du + e^{-K_3/\epsilon^{2-\beta}}. \end{aligned} \tag{40}$$

Using (36) we see that for some constant $K_4 > 0$ independent of $x \in [x_0 - \delta, a_2 + \delta]$ and $u \in \mathbb{R}$,

$$|R(x, -\epsilon^2 u/\Phi'(x))|/\epsilon^2 \leq K_4 \epsilon^2 u^2.$$

In particular,

$$\sup_{x \in [x_0 - \delta, a_2 + \delta]} \sup_{u \in [\Phi'(x)/\epsilon^{2-\beta}, 0]} |R(x, -\epsilon^2 u/\Phi'(x))|/\epsilon^2 \leq K_4 \epsilon^{2(\beta-1)}.$$

Since $\beta > 1$, the r.h.s. converges to 0 and we can apply a basic Taylor estimate which implies that for all $\epsilon > 0$ small enough,

$$\sup_{x \in [x_0 - \delta, a_2 + \delta]} \sup_{u \in [\Phi'(x)/\epsilon^{2-\beta}, 0]} |1 - \mathrm{e}^{-R(x, -\epsilon^2 u/\Phi'(x))/\epsilon^2}| \leq K_5 \epsilon^2 u^2,$$

for some $K_5 > 0$. Using this fact in the integral of (40), we can find a constant $K_6 = K_6(\delta) > 0$ such that

$$\sup_{x \in [x_0 - \delta, a_2 + \delta]} |J_{\epsilon,2}(x) - 1| \leq K_6 \epsilon^2 + \mathrm{e}^{-K_3/\epsilon^{2-\beta}},$$

which finishes the proof. □

Acknowledgement

Yuri Bakhtin is grateful to NSF for partial support via a CAREER grant DMS-0742424.

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