Journal of Pure and Applied Algebra 37 (1985) 167-179
North-Holland

FAMILIES OF SUBGROUPS AND COMPLETION

Stefan JACKOWSKI
Instytut Matematyki, Uniwersytet Warszawski, PKiN IXp, PL-00-901 Warszawa, Poland
Forschungsinstitut für Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland

Communicated by J.D. Stasheff
Received 1 February 1984
Revised 2 October 1984

For a given compact Lie group G and a family of subgroups $\mathscr{F}$ of G, a classifying space $E\mathscr{F}$ is defined. It is a G-space such that every subgroup of G having fixed points on $E\mathscr{F}$ belongs to $\mathscr{F}$, and such that for every $H\in F$ the space $E\mathscr{F}$ is H-contractible i.e. it has an H-equivariant contraction onto a point. We want to compute the equivariant cohomology of classifying spaces for families of subgroups.

Let h_G be an equivariant multiplicative cohomology theory. For any subgroup $H\in\mathscr{F}$, we define an ideal $I(H)=\ker\{h_G(\mathrm{pt})\to h_G(G/H)\}$. The set of ideals $\{I(H_1)\cdot\ldots\cdot I(H_n)\,|\,H_i\in\mathscr{F}\}$ defines a topology on $h_G(\mathrm{pt})$ which is called the $\mathscr{F}$-topology.

The completion conjecture says that for a 'nice' cohomology theory h_G and a 'nice' space X the projection $X\times E\mathscr{F}\to X$ induces an isomorphism $\hat{p}_X(\mathscr{F}):h_G(X)\hat{}\to h_G(X\times E\mathscr{F})$; where $\hat{}$ denotes completion with respect to the $\mathscr{F}$-topology. There is also a more general formulation of the completion conjecture suitable in particular for completing cohomology theories which are modules over the Burnside ring functor with respect to the $\mathscr{F}$-topology on the Burnside ring.

The completion conjecture was formulated by the author [11] and then proved by him [12] for equivariant K-theory when $\mathscr{F}$ is the family of cyclic subgroups of a finite group. The induction method developed in [12] is now extended to give a reduction procedure of the completion conjecture for an arbitrary family of subgroups to the case of the family of all proper subgroups. As a corollary we prove the completion conjecture for equivariant K-theories and arbitrary families of subgroups of compact Lie groups. In this case the completion conjecture was proved independently by J.-P. Haeberly [9] who exploited the method of Atiyah–Segal [4]. Two applications are mentioned. The first one provides a description of the image of the restriction homomorphism $R(G)\to R(H)$ for a normal subgroup H of a compact Lie group G, in terms of group cohomology $H^*(B(G/H);R(H))$ where G/H acts on $R(H)$ by conjugation. The second application is to show how the completion theorem for the family of topologically cyclic subgroups and their subgroups gives a new approach to the description of the prime ideal spectrum $\mathrm{Spec}\,K_G(X)$.

0022-4049/85/\$3.30

I would like to thank Peter Löffler for discussions we had in Oberwolfach in September 1974 during which the completion conjecture was formulated.

1. Families of subgroups and equivariant cohomology

Let G be a compact Lie group. A set $\mathscr{F}$ of closed subgroups is called a *family* if it is closed under conjugation and taking subgroups. A G-space X is called $\mathscr{F}$-free if all isotropy subgroups occuring on X belong to $\mathscr{F}$. It is called $\mathscr{F}$-numerable if it has a numberable open G-covering $\mathscr{U}$ such that for any $U \in \mathscr{U}$ there exists an equivariant map $U \to G/H$ for some $H \in \mathscr{F}$. An $\mathscr{F}$-numerable space is clearly $\mathscr{F}$-free.

The homotopy category of $\mathscr{F}$-numerable G-spaces has a terminal object which will be denoted by $E\mathscr{F}$. Its construction imitates the Milnor construction of a universal free G-space, and it can be found in [8, §7.2]. There is a homotopy-theoretical characterization of classifying spaces for families of subgroups generalizing the classical theorem of A. Dold.

1.1. Theorem. *An $\mathscr{F}$-numerable G-space E is classifying for the family $\mathscr{F}$ if and only if it is H-contractible for every subgroup $H \in \mathscr{F}$.* □

Similarly to the case of free universal spaces for compact groups, one can equip an infinite join of orbits with isotropy subgroups belonging to $\mathscr{F}$ with a compactly generated topology defined by finite joins (cf. [4, footnote on p. 3]). The classifying space for the family $\mathscr{F}$ obtained in such a way is clearly a G-CW-complex in the sense of Matumoto [13].

If $\mathscr{F}$ is a family of subgroups of G and $H \subset G$ is a closed subgroup, let $\mathscr{F} \cap H = \{L \subset H \mid L \in \mathscr{F}\}$ be its restriction to H. We have the following restriction and product formulas for classifying spaces.

1.2. Proposition. *There is an H-homotopy equivalence*

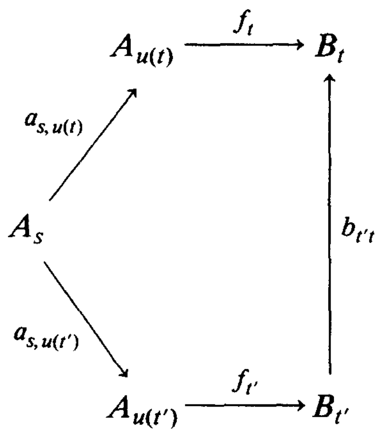
$$E\mathscr{F} \xrightarrow{\simeq} E(\mathscr{F} \cap H). \qquad \square$$

1.3. Proposition. *If $\mathscr{F}_1, \mathscr{F}_2$ are families of subgroups of G, then there is a G-homotopy equivalence*

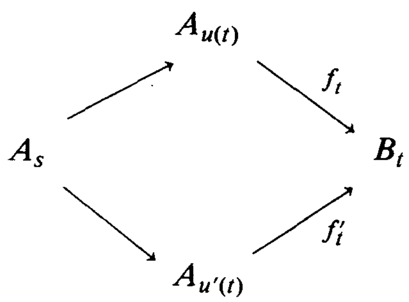
$$E(\mathscr{F}_1 \cap \mathscr{F}_2) \xrightarrow{\simeq} E\mathscr{F}_1 \times E\mathscr{F}_2,$$

where the topology on the cartesian product is the smallest compactly generated topology containing the Tychonoff topology. □

To compute the cohomology of infinite complexes, it is convenient to work in the category of inverse systems of abelian groups or rings. Following [4], we recall the construction of the category Pro($\mathscr{C}$) from a given category $\mathscr{C}$. Objects of the category Pro($\mathscr{C}$) are inverse systems $\{A_s\}_{s \in S}$ of objects of $\mathscr{C}$ indexed by directed sets S. To define a morphism from $\{A_s\}_{s\in S}$ to $\{B_t\}_{t\in T}$ one prescribes the map $u: T\to S$ (not necessarily order-preserving) and morphisms $f_t: A_{u(t)}\to B_t$ of C for each $t\in T$: subject to the condition that if $t<t'$ in T, then for some $s\in S$ such that $s>u(t)$, $s>u(t')$, the diagram



is commutative. But one identifies the morphisms (u,f_t) and (u',f_t') if for each t, there is an $s\in S$ such that



commutes.

It is not difficult to see that if $\mathscr{C}$ is an abelian category, then Pro($\mathscr{C}$) is also abelian. By a pro-cohomology theory will be meant a sequence of functors defined on a topological category with values in the category Pro($\mathscr{A}\mathscr{b}$) and satisfying the Eilenberg-Steenrod axioms. Pro-cohomology theories inherit all formal properties of usual cohomology. In particular, for equivariant pro-cohomology, the comparison theorem remains true.

1.4. Proposition. *Let $T: h_G\to k_G$ be a natural transformation of equivariant pro-cohomology theories. Suppose that for every orbit G/H such that H belongs to a given family $\mathscr{F}$, $T(G/H)$ is an isomorphism. Then for any finite $\mathscr{F}$-free G-CW-complex X, $T(X)$ is an isomorphism.* □

Let h be a cohomology theory defined on the category of compact spaces. Then h can be extended to a pro-cohomology theory on the category of compactly generated spaces by the formula:

$$\mathscr{h}(X)=\{h(K): K\subset X,\ K \text{ compact}\}.$$

Now let h_G be an equivariant cohomology theory defined on the category of compact G-spaces. For any family of subgroups $\mathscr{F}$ of G, one associates with h_G a new pro-cohomology theory $h_G[\mathscr{F}]$ defined on the category of compact G-spaces:

$$h_G[\mathscr{F}](X)=\mathscr{h}_G(X\times E\mathscr{F})=\{h_G(X\times K):K\subset E\mathscr{F},\ K \text{ compact}\}.$$

The projection $X\times E\mathscr{F}\to X$ induces a natural transformation $h_G\to h_G[\mathscr{F}]$. For an arbitrary subgroup $K\subset G$, we define a K-cohomology theory h_K by the induction formula $h_K(X)=h_G(G\times_K X)$. The induction formula holds for various equivariant cohomology theories which are defined geometrically for an arbitrary compact Lie group. The cohomology theory $h_G[\mathscr{F}]$ is characterized by the two universal properties listed in the next theorem.

1.5. Theorem. *The natural transformation* $h_G\to h_G[\mathscr{F}]$ *has the following properties:*

(a) *If X is a $\mathscr{F}$-free compact space, then* $h_G(X)\xrightarrow{\simeq} h_G[\mathscr{F}](X)$ *is an isomorphism.*

(b) *If a G-map $f:X\to Y$ induces an isomorphism $f^*:h_K(Y)\to h_K(X)$ for every $K\in\mathscr{F}$, then $f^*:h_G[\mathscr{F}](Y)\to h_G[\mathscr{F}](X)$ is an isomorphism.* □

The proof is a slight modification of the one given in [12, Theorem 1.5].

Now we describe another important construction which yields a pro-cohomology theory. Let R be a graded ring. Let h be an R-module valued cohomology theory. This means that for any space X the graded group $h(X)$ is a graded R-module, and that induced homomorphisms as well as boundary operators are R-homomorphisms. We will say that the h-cohomology of the space X is R-finite if at least one of the following conditions holds:

(i) $h(X)$ is a finitely generated R-module.

(ii) *If* $R=R^0$, then $h^q(X)$ is a finitely generated R^0-module for every index q.

Let $J=\{I_s\}_{s\in S}$ be a family of ideals in R. The set of all finite products $I_{s_1}\cdot\ldots\cdot I_{s_k}$ of ideals from J (denoted also by J) is the basis of neighbourhoods of $0\in R$ of the J-adic topology on R. Hence the J-adic topology is defined on any R-module.

We define a functor h/J with values in the corresponding pro-category by the formula

$$(h/J)(X):=\{h(X)/I\cdot h(X):I\in J\}.$$

1.6. Proposition. *Let R be a noetherian ring. Then h/J is a pro-cohomology theory on the category of spaces whose h-cohomology is R-finite.*

Proof. The exactness axiom follows from the Artin–Rees lemma as stated in [3, Corollary 10.10]. The homotopy axiom is clearly fulfilled. □

Let $M:\mathscr{O}_G\to\text{Rings}^*$ be a contravariant functor defined on the category of canonical orbits of a group G with values in the category of graded rings. Its value on the orbit G/H we denote M_H for short. For any pair of subgroups $H\subset K$ of G

we define an ideal

$$I_K^M(H) = \ker\{M_K \to M_H\}.$$

In the case $K = G$ we write $I^M(H)$ and if the functor M is fixed, we also omit it and write $I(H)$ for short.

Let $\mathscr{F}$ be a family of subgroups of G. Let $I^M(\mathscr{F}) = \{I^M(H) \mid H \in \mathscr{F}\}$: the $I^M(\mathscr{F})$-adic topology on any M_G-module will be called the $(M, \mathscr{F})$-topology or shortly the $\mathscr{F}$-topology.

Let h_G be any equivariant cohomology theory. Assume that h_G is a module over M, i.e. for every subgroup $K \subset G$ and K-space X, the graded group $h_K(X)$ is a M_K-module in a way such that obvious compactibility conditions are satisfied (cf. [8, §7.4] for the case $M = A$ is the Burnside ring functor).

Hence, any family of subgroups $\mathscr{F}$ defines an $\mathscr{F}$-topology on the theory h_G. The basic property of the $\mathscr{F}$-topology is given by the following proposition:

1.7. Proposition. *If X is a compact $\mathscr{F}$-free G-space, then the $\mathscr{F}$-topology on $h_G(X)$ is discrete.*

Proof. It follows from the existence of tubes around orbits and the Meyer–Vietoris argument. □

1.8. Corollary. *For an arbitrary compact G-space X, the projection $X \times E\mathscr{F} \xrightarrow{p} X$ defines a homomorphism of pro-rings*

$$p_X(\mathscr{F}) : (h_G/I^M(\mathscr{F}))(X) \to h_G[\mathscr{F}](X). \quad \square$$

2. The completion conjecture. Reduction theorems

First we describe a few examples justifying the terminology introduced in the last section.

If h_G is a multiplicative cohomology theory, then h_G is a module over the functor M given by restricting h_G to the orbit category. One can also take $M = h_G^0 \mid \mathscr{O}_G$ and consider $h_G^q(X)$, for every index q, as an h_G^0-module.

For example, equivariant K-theory is a module over the representation ring functor. Equivariant stable cohomotopy theory is a module over the Burnside ring functor which can be identified with $\pi_G^0 \mid \mathscr{O}_G$.

Various equivariant bordism theories are modules over the Burnside ring functor. In fact, every equivariant cohomology theory graded over the representation ring is a module over the Burnside ring functor.

The completion conjecture for the cohomology theory h_G which is a module over the functor M says that the morphism of pro-objects defined in 1.8:

$$p_X(\mathscr{F}) : (h_G/I^M(\mathscr{F}))(X) \to h_G[\mathscr{F}](X)$$

is an isomorphism for spaces X satisfying some finiteness conditions.

For the proofs of our results concerning the completion conjecture, we need to impose certain conditions on cohomology theories and M-module structures on them. The first is the *finiteness condition* (N):

(N) The ring M_G is noetherian and for any subgroup $K \subset G$ the homomorphism $M_G \to M_K$ is finite i.e. M_K becomes a finitely generated M_G-module.

Let $\mathscr{F}$ be a fixed family of subgroups of G. The second assumption is the following *restriction $\mathscr{F}$-topology property* ($R_{\mathscr{F}}$):

($R_{\mathscr{F}}$) For any subgroup $H \subset G$, the $\mathscr{F}$-topology defined on M_H by restriction $M_G \to M_H$ coincides with the $\mathscr{F} \cap H$-topology on M_H; i.e. the topology defined by the ideals $\{I_H(K) \mid K \in \mathscr{F} \cap H\}$.

We will discuss certain reduction theorems for the completion conjecture.

2.1. Theorem. *Let $\mathscr{F}$ be a family of subgroups of G. Assume that the cohomology theory h_G is an M-module satisfying conditions* (N) *and* ($R_{\mathscr{F}}$). *Let X be a compact G-space of M_K-finite cohomology $h_K(X)$, for every subgroup $K \subset G$. Assume also that:*

(i) *for the family $\mathscr{P}_G$ of all proper subgroups of G, the morphism $p_X(\mathscr{P}_G)$ is an isomorphism,*

(ii) *for any proper subgroup $K \subset G$ and the cohomology theory h_K the morphism $p_X(\mathscr{F} \cap K)$ is an isomorphism.*

Then the morphism $p_X(\mathscr{F})$ is an isomorphism.

We will give a proof of Theorem 2.1 at the end of this section. An easy inductive argument gives the following corollary:

2.2. Corollary. *Let h_G be a cohomology theory which is an M-module satisfying conditions* (N) *and* ($R_{\mathscr{F}}$) *for any family of subgroups $\mathscr{F}$ of G. Let X be a G-space as in Theorem* 2.1. *Suppose that for any subgroup $H \subset G$ the morphism $p_X(\mathscr{P}_H)$ is an isomorphism of corresponding H-equivariant pro-cohomology theories. Then for any family of subgroups $\mathscr{F}$ of G, the morphism $p_X(\mathscr{F})$ is an isomorphism.* □

We formulate another reduction principle which may be useful to prove the completion theorem for some cohomology theories. Assume that topologies defined by families of subgroups $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the following intersection property:

(I) The $\mathscr{F}_1 \cap \mathscr{F}_2$-topology on M_G coincides with the topology defined by the ideals $\{I_1 + I_2 : I_1 \in I(\mathscr{F}_1), I_2 \in I(\mathscr{F}_2)\}$.

2.3. Theorem. *Let h_G be a module over a functor M satisfying assumption* (I). *Assume that for a given G-space X such as in Theorem* 2.1 *and every subgroup $H\in\mathscr{F}_1$, $p_X(\mathscr{F}_1)$ and $p_X(\mathscr{F}_2\cap H)$ are isomorphisms. Then $p_X(\mathscr{F}_1\cap\mathscr{F}_2)$ is also an isomorphism.*

Our reduction Theorem 2.2 is analogous to the easy part of Carlsson's proof of the Segal conjecture as presented by Adams [1] – i.e. the reduction of the general Segal conjecture for p-groups to the computation of p-adically completed stable cohomotopy of the unit sphere of an infinite-dimensional representation. The sphere of an infinite-dimensional representation $\bigoplus^\infty V$ such that $V^H\neq 0$ for any subgroup $H\subsetneq G$ and $V^G=0$ is actually a classifying space for the family of all proper subgroups (cf. [7] and Section 3). At least for finite groups, the stable cohomotopy theory considered as a module over the Burnside ring functor satisfies conditions (N) and ($\mathrm{R}_{\mathscr{F}}$) for any family of subgroups $\mathscr{F}$. Hence to prove the completion conjecture in stable cohomotopy for an arbitrary finite G-CW-complex, it remains to compute the pro-group $\pi_G^*(S(\bigoplus^\infty V))$ for any finite group G.[1]

In the next section we prove the completion conjecture for families of all proper subgroups only for cohomology theories satisfying certain orientability conditions.

Proof of Theorem 2.1. The theorem is trivially true for the family of all subgroups of G; in that case, the classifying space is G-contractible and the F-topology is discrete. Hence assume that $\mathscr{F}\subset\mathscr{P}$. Then there is a G-homotopy equivalence $E\mathscr{F}=E\mathscr{P}\times E\mathscr{F}$ (cf. Proposition 1.3) and the pro-cohomology theory $h_G[\mathscr{F}]$ can be written in the following form

$$h_G[\mathscr{F}](X)=\{h_G(X\times A\times B): A\subset E\mathscr{P}, B\subset E\mathscr{F}, A, B \text{ compact}\}.$$

The $\mathscr{F}$-topology on the ring M_G coincides with the topology defined by the ideals $\{I_1+I_2: I_1\in I(\mathscr{P}), I_2\in I(\mathscr{F})\}$. Hence we have the following decomposition of the theory $h_G/I(\mathscr{F})$. We write $h_G=h_G(X)$ for short.

$$\begin{aligned}(h_G/I(\mathscr{F}))(X)&=\{h_G/(I_1+I_2)h_G: I_1\in I(\mathscr{P}), I_2\in I(\mathscr{F})\}\\&=\{h_G\otimes_{M_G}(M_G/I_1)\otimes_{M_G}(M_G/I_2): I_1\in I(\mathscr{P}), I_2\in I(\mathscr{F})\}.\end{aligned}$$

Consequently, the map $p_X(\mathscr{F}):(h_G/I(\mathscr{F}))(X)\to h_G[\mathscr{F}](X)$ can be decomposed

$$\begin{array}{c}\{h_G/I_1h_G\otimes_{M_G}(M_G/I_2): I_1\in I(\mathscr{P}), I_2\in I(\mathscr{F})\}\\ \Big\downarrow{\scriptstyle p_X(\mathscr{P})\otimes\mathrm{id}}\\ \{h_G(X\times A)\otimes_{M_G}(M_G/I_2): A\subset E\mathscr{P}, I_2\in I(\mathscr{F})\}\\ \Big\downarrow{\scriptstyle \{p_{X\times A}(\mathscr{F})\}}\\ \{h_G(X\times A\times B): A\subset E\mathscr{P}, B\subset E\mathscr{F}, A, B \text{ compact}\}.\end{array}\qquad p_X(\mathscr{F})$$

[1] *Added in September 1984*: A proof of the completion conjecture for equivariant stable cohomotopy will appear in a joint paper by J.F. Adams, J.-P. Haeberly, S. Jackowski and J.P. May.

We prove that both factors are isomorphisms. The assumption (i) implies that for each fixed $I_2 \in I(\mathscr{F})$, the map $p_X(\mathscr{P})\otimes \mathrm{id}_{M_G/I_2}$ is an isomorphism. Therefore, from general properties of pro-categories (cf. [4, Lemma 3.2]), the map $p_X(\mathscr{P})\otimes \mathrm{id}$ is an isomorphism. Now fix a compact subcomplex $A\subset E\mathscr{P}$. We prove that

$$p_{X\times A}(\mathscr{F}): h_G(X\times A)\otimes_{M_G} M_G/I_2 \to \{h_G(X\times A\times B): B\subset E\mathscr{F}\}$$

is an isomorphism. Indeed, for any proper subgroup $K\subset G$, the assumption (ii) together with condition $(\mathrm{R}_{\mathscr{F}})$ implies that $p_{X\times G/K}(\mathscr{F})$ can be decomposed into three isomorphisms:

$$\begin{array}{ccc} \{h_G(X\times G/K)\otimes_{M_G} M_G/I_2 : I_2\in I(\mathscr{F})\} & \to & \{h_K(X)/J_2h_K(X): J_2\in I(\mathscr{F}\cap K)\} \\ {\scriptstyle p_{X\times G/K}(\mathscr{F})}\downarrow & & \downarrow{\scriptstyle p_X(\mathscr{F}\cap K)} \\ \{h_G(X\times (G/K)\times B): B\subset E\mathscr{F}\} & \longleftarrow & \{h_K(X\times B): B\subset E(\mathscr{F}\cap H)\}\,. \end{array}$$

Propositions 1.4 and 1.6 imply that $p_{X\times A}(\mathscr{F})$ is an isomorphism for every finite G-CW-complex A such that $A^G=\emptyset$. Hence, as before the map of inverse systems $\{p_{X\times A}(\mathscr{F})\}$ is an isomorphism. □

The proof of Theorem 2.3 is analogous to the last one. It uses Proposition 1.3 and its algebraic analogue (I).

3. The completion conjecture for families of subgroups defined by linear representations

We recall here the results of [12, §3]; stating them in a slightly generalized form.

Let $\mathscr{V}$ be a set of real representations of a group G. It determines a family of subgroups $\mathscr{F}_{\mathscr{V}} := \bigcup_{V\in\mathscr{V}}\{H\subset G: V^H\neq 0\}$. Let h_G be a multiplicative cohomology theory and assume that h_G has suspension isomorphisms for all representations $V\in\mathscr{V}$. Then their Euler classes $e(V)\in h_G(\mathrm{pt})$ are defined (cf. [8, §7.1]). We consider h_G as a module over its restriction to the orbit category.

3.1. Proposition. *The topology defined on $h_G(\mathrm{pt})$ by the family of principal ideals $\{e(V)h_G(\mathrm{pt}): V\in\mathscr{V}\}$ coincides with the $\mathscr{F}_{\mathscr{V}}$-topology.* □

3.2. Corollary. *For any G-space X, there is an isomorphism of pro-rings*

$$(h_G/I(\mathscr{F}_{\mathscr{V}}))(X) = \{h_G(X)/e(V)h_G(X): V\in\mathscr{V}^+\},$$

where $\mathscr{V}^+$ denotes the closure of $\mathscr{V}$ with respect to direct sums, ordered by G-linear inclusions. □

A classifying space for the family $\mathscr{F}_{\mathscr{V}}$ can be constructed as a sphere in an infinite dimensional representation. More precisely, we can assume that $\mathscr{V} = \{V_i\}_{i=1}^{\infty}$ is countable, and then set

$$E\mathscr{F}_{\mathscr{V}} = \lim_n S(\underbrace{V_1 \oplus \cdots \oplus V_1}_{n} \oplus \cdots \oplus \underbrace{V_n \oplus \cdots \oplus V_n}_{n}).$$

To compute $h_G[\mathscr{F}_{\mathscr{V}}](X)$, we apply the Gysin sequence and use the last lemma, proceeding as in the proof of Lemma 3.1 in [4].

3.3. Theorem. *Let X be a compact G-space such that $h_G(X)$ is a noetherian $h_G(\mathrm{pt})$-module. Then*

$$p_X(\mathscr{F}_{\mathscr{V}}) : (h_G/I(\mathscr{F}_{\mathscr{V}}))(X) \to h_G[\mathscr{F}_{\mathscr{V}}](X)$$

is an isomorphism of pro-rings. □

3.4. Corollary. *If the assumption of the last theorem is fulfilled and h_G is an additive cohomology theory, then $p_X(\mathscr{F}_{\mathscr{V}})$ defines an isomorphism*

$$h_G(X)^{\wedge} \to h_G(X \times E\mathscr{F}_{\mathscr{V}}),$$

where $^{\wedge}$ *denotes completion with respect to the $\mathscr{F}_{\mathscr{V}}$-topology.* □

The above corollary can be regarded as a statement dual to tom Dieck's result [7, Satz 5] describing in homotopy terms the localization of the homology theory with respect to Euler classes.

We say that the theory h_G has sufficiently many suspensions if every orbit G/H can be imbedded into a representation V for which the suspension isomorphism is defined in h_G. For example, complex, orthogonal and real K_G-theories have sufficiently many suspensions.

Clearly not every family of subgroups is defined by representations: for example if G does not have free representations, then the family consisting of the identity subgroup is not. However if h_G has sufficiently many suspensions, then the family of all proper subgroups $\mathscr{P}$ is defined by the set of all representations without the trivial summand for which suspension isomorphisms are defined in h_G. As we have shown in the previous section, the computation of the theory $h_G[\mathscr{P}]$ is an important ingredient of the proof of the completion theorem for an arbitrary family of subgroups.

4. $\mathscr{F}$-topology in the representation ring

We prove that conditions $(\mathrm{R}_{\mathscr{F}})$ and (I) of the last section are fulfilled for equivariant K-theory. We restrict ourselves to the case of the complex representation ring. However the corresponding results remain true for coefficient ring of KO_G-theory.

All the results of this section are based on the description of the prime ideals in the representation ring of a compact Lie group given by Segal [15].

4.1. Proposition. *In the noetherian ring A, the topology defined by the ideals $\{I\}$ coincides with the topology defined by the set of their radicals $\{r(I)\}$.* □

4.2. Corollary. *If M_G is a noetherian ring, then its $\mathscr{F}$-topology coincides with the topology defined by the set of ideals $I'(\mathscr{F})=\{I(H_1)\cap\cdots\cap I(H_k): H_i\in\mathscr{F}\}$.*

Proof. Clearly $r(I(H_1)\cdot\ldots\cdot I(H_k))=r(I(H_1)\cap\cdots\cap I(H_k)$. □

The representation ring of a compact Lie group is noetherian and the restriction homomorphism to a subgroup is finite. Hence condition (N) is fulfilled. Now we prove $(\mathrm{R}_{\mathscr{F}})$.

4.3. Proposition. *For any family of subgroups $\mathscr{F}$ of a compact Lie group G and any subgroup $H\subset G$, the $\mathscr{F}$-topology on the representation ring $R(H)$ coincides with the $\mathscr{F}\cap H$-topology.*

Proof. Denote by $i_*:\operatorname{Spec} R(H)\to\operatorname{Spec} R(G)$, the map induced by the restriction homomorphism on prime ideal spectra. If a prime ideal $\mathfrak{p}\in\operatorname{Spec} R(H)$ has support $S_{\mathfrak{p}}$, the subgroup $S_{\mathfrak{p}}$ is also the support of $i_*(\mathfrak{p})$. It is enough to prove that for any $I\in I(\mathscr{F})$ there exists $J\in I(\mathscr{F}\cap H)$ such that $J\subset r(I)$. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be minimal prime ideals containing $I(K_1)\cdot\ldots\cdot I(K_n)R(H)$ and let $S_1,\ldots,S_n$ be their supports in H. Now $J=I_H(S_1)\cdot\ldots\cdot I_H(S_n)$. □

Let us observe that the functor R also fulfills condition (I). However (I) is not used for the proof of the completion theorem in equivariant K-theory.

4.4. Proposition. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be families of subgroups of a group G. Then the $\mathscr{F}_1\cap\mathscr{F}_2$-topology on $R(G)$coincides with the topology defined by the set of ideals $\{I_1+I_2: I_k\in I(\mathscr{F}_k)\}$.*

Proof. Clearly the $(I(\mathscr{F}_1)+I(\mathscr{F}_2))$-topology coincides with the $(I'(\mathscr{F}_1)+I'(\mathscr{F}_2))$-topology. It is enough to prove that for every ideal $I_1+I_2\in I'(\mathscr{F}_1)+I'(\mathscr{F}_1)+I'(\mathscr{F}_2)$ there is some $I\in I(\mathscr{F}_1\cap\mathscr{F}_2)$ such that $I\subset r(I_1+I_2)$. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_k$ be minimal prime ideals containing I_1+I_2 and let $S_1,\ldots,S_k$ be their supports. These subgroups clearly belong to $\mathscr{F}_1\cap\mathscr{F}_2$. Now $I=I(S_1)\cdot\ldots\cdot I(S_k)$. □

If a family $\mathscr{F}$ contains all cyclic subgroups, and hence all Cartan subgroups, then the $\mathscr{F}$-topology is clearly discrete. We close this section by observing that this is actually the only case when the $\mathscr{F}$-topology is complete.

4.5. Proposition. *For a family of subgroups $\mathscr{F}$ of a compact Lie group G, the following conditions are equivalent*

(1) *$\mathscr{F}$ contains all cyclic subgroups of G,*
(2) *$\mathscr{F}$-topology on $R(G)$ is discrete,*
(3) *$\mathscr{F}$-topology on $R(G)$ is complete and Hausdorff,*
(4) *$\mathscr{F}$-topology on $R(G)$ is complete.*

Proof. Implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)$ are obvious. We prove $(4)\Rightarrow(1)$. Let S be a cyclic subgroup of G. Since the restriction $R(G)\rightarrow R(S)$ is a finite homomorphism, Proposition 4.3 implies that $R(S)$ is complete in the $\mathscr{F}\cap S$-topology. Assume $S\notin\mathscr{F}$. Then $\mathscr{F}\cap S\subset\mathscr{P}_S$, and hence $R(S)$ must be also complete in the $\mathscr{P}_S$-topology. It is now easy to see that the representation ring of a cyclic group is not $\mathscr{P}$-complete. □

5. Equivariant K-theory and spectral sequences

The results of the previous sections imply a completion theorem generalizing results of Atiyah–Segal [4] obtained independently by J.-P. Haeberly [9].

5.1. Theorem. *Let X be a compact G-space such that $K_H^*(X)$ is finite over $R(H)$ for any subgroup $H\subset G$. Then for any family of subgroups F, the map*

$$p_X(\mathscr{F}):(K_G^*/I(\mathscr{F}))(X)\rightarrow K_G^*[\mathscr{F}](X)$$

is an isomorphism of pro-rings.

Proof. The theorem follows from 4.3, 2.2, 3.3, and the remarks following 3.4. □

In the case of the family consisting only of the identity subgroup, 5.1 provides a new proof of the Atiyah–Segal theorem [4]. Note that the theorem corresponding to 5.1 is true for KO_G-theory and the proof carries over verbatim.

5.2. Corollary. *Let X be as in the last theorem. Then $p_X(\mathscr{F})$ defines an isomorphism*

$$K_G^*(X)\hat{}\rightarrow K_G^*(X\times E\mathscr{F})$$

where $\hat{}$ denotes completion with respect to the $\mathscr{F}$-topology. □

5.3. Corollary. *Let X, Y be G-spaces satisfying assumption of Theorem 5.1. Let $f:Y\rightarrow X$ be a G-map inducing an isomorphism $f^*:K_H^*(X)\rightarrow K_H^*(Y)$ for any $H\in\mathscr{F}$. Then $f^*:K_G^*(X)\rightarrow K_G^*(Y)$ induces an isomorphism of the $I(\mathscr{F})$-completions.*

Proof. The corollary follows from 1.5 and 5.2. □

This last corollary is completed by the following generalization of an observation due to Petrie [14].

5.4. Proposition. *The induced homomorphism* $f^*:K_G^*(X)\to K_G^*(Y)$ *induces an isomorphism of the* $I(\mathscr{F})$*-completions if and only if for every ideal* $\mathfrak{p}\in\operatorname{Spec}R(G)$ *such that its support* S *belongs to* $\mathscr{F}$*, the localized induced homomorphism* $f^*:K_G^*(X^{(S)})_\mathfrak{p}\to K_G^*(Y^{(S)})_\mathfrak{p}$ *is an isomorphism.* □

For a G-space X satisfying assumption of the completion theorem 5.1 we have $\lim^1\{K_G(X\times A):A\subset E\mathscr{F}\}=0$. Hence the equivariant Atiyah–Hirzebruch spectral sequence (cf. [10])

$$E_2^*=H_G^*(X\times E\mathscr{F};R(\cdot))\Rightarrow K_G^*(X)\hat{}$$

is convergent (cf. [6, §3]). The edge homomorphism of this sequence

$$\alpha:K_G^*(X)\hat{}\to H_G^0(X\times E\mathscr{F};R(\cdot))$$

has its values in the equivariant singular 0-cohomology group (in the sense of Illman [10]) which is canonically imbedded into the product $\prod_{x\in X}\prod_{H\in\mathscr{F}\cap G_x}R(H)$ (cf. [6, Proposition 3.3]).

Let us consider this spectral sequence for three interesting families of subgroups.

5.5. Example. Let H be a normal subgroup of G. Denote by $\mathscr{F}_H$ the family of all subgroups contained in H. Then it is easy to see that $H_G^*(E\mathscr{F}_H;R(\cdot))=H^*(B(G/H);R(H))$ where G/H acts on $R(H)$ by conjugation. Hence for $X=\mathrm{pt}$, the spectral sequence has the form

$$E_2^*=H^*(B(G/H);R(H))\to R(G)\hat{}$$

where $\hat{}$ denotes completion with respect to the ideal $\ker\{R(G)\to R(H)\}$. The image of the edge homomorphism $R(G)\hat{}\to H^0(B(G/H);R(H))=R(H)^{G/H}$ coincides with the image of the restriction homomorphism $R(G)\to R(H)^{G/H}$. Hence we obtain an implicit description of the characters of H which extend to G in terms of group cohomology. This spectral sequence – similar to the Hochschild–Serre Spectral sequence – is a stronger version of Atiyah's spectral sequence (cf. Atiyah [2, Theorem 7.8]). Indeed, Atiyah's sequence can be obtained from ours by completing it with respect to the augmentation ideal in $R(G)$.

5.6. Example. Let $\mathscr{F}=\mathscr{C}$ be a family consisting of subgroups contained in topologically cyclic subgroups of G. Then the $\mathscr{C}$-topology is discrete. Character theory and Proposition 1.7 imply that the kernel of the edge homomorphism

$$\alpha:K_G^*(X)\to H_G^0(X\times E\mathscr{C};R(\cdot))$$

is nilpotent. One can also prove that α induces a bijection of prime ideal spectra.

For more details cf. [6]. A decomposition theorem for Spec $K_G^*(X)$ – proved using different arguments – is contained in [5].

5.7. Example. Let G be a finite group and let $\mathscr{E}$ be the family of its elementary subgroups. Then Dress' theory implies that $H_G^q(E\mathscr{E}; R(\cdot))=0$ for $q>0$. Therefore our spectral sequence for $X=\mathrm{pt}$ reduces to the Brauer theorem in character theory.

References

[1] J.F. Adams, An exposition of Carlsson's proof, Lecture to the Newfoundland Conference, August 1983.
[2] M.F. Atiyah, Characters and cohomology of finite groups, Publ. Math. IHES 9 (1960) 23–64.
[3] M.F. Atiyah and I. Macdonald, Introduction to Commutative Algebra (Addison-Wesley, Reading, MA, 1969).
[4] M.F. Atiyah and G.B. Segal, Equivariant K-theory and completion, J. Diff. Géom. 3 (1969) 1–18.
[5] A. Bojanowska, The spectrum of equivariant K-theory. Math. Z. 183 (1983) 1–19.
[6] A. Bojanowska and S. Jackowski, A spectral sequence convergent to equivariant K-theory, in: Topology Symposium Siegen 1979, Lecture Notes in Math. 788 (Springer, Berlin, 1980).
[7] T. tom Dieck, Orbittypen und äquivariante Homologie I, Arch. Math. 23 (1972) 307–317.
[8] T. tom Dieck, Transformation Groups and Representation Theory, Lecture Notes in Math. 766 (Springer, Berlin, 1979).
[9] J.-P. Haeberly, Completions in equivariant K-theory, Thesis, University of Chicago (1983).
[10] S. Illman, Equivariant singular homology and cohomology I, Memoirs AMS 156 (1975).
[11] S. Jackowski, Families of subgroups and completion, Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht 37/1975, 6 (1975).
[12] S. Jackowski, Equivariant K-theory and cyclic subgroups, in: Transformation Groups, London Math. Soc. Lecture Note Series 26 (Cambridge Univ. Press, Cambridge, 1977).
[13] T. Matumoto, On G-CW-complexes and a theorem of J.H.C. Whitehead, J. Fac. Sci. Univ. Tokyo 18 (1971) 363–374.
[14] T. Petrie, A setting for smooth S^1-actions with applications to real algebraic actions on $P(C^{4n})$, Topology 13 (1974) 363–374.
[15] G.B. Segal, The representation ring for compact Lie group, Publ. Math. IHES 34 (1968) 113–128.