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Zeta Measure Associated to $\mathbb{F}_q[T]$

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The object of this paper is to identify the divided power series corresponding to the zeta measure associated to $\mathbb{F}_q[T]$. The first section introduces the zeta function for $\mathbb{F}_q[T]$ and describes some of its interesting properties. In the second section, we describe results on interpolations and measures and state our main result (Theorem VII). The third section summarizes various results about power sums and zeta functions. The last section contains two proofs of the main result. In the appendix, we elaborate on the existence of zeta and beta measures and give alternate descriptions for them.

I. ZETA FUNCTION FOR $\mathbb{F}_q[T]$

Let q be a power of a prime p, $A =: \mathbb{F}_q[T]$, $K =: \mathbb{F}_q(T)$, $K_\infty =: \mathbb{F}_q((1/T))$. Let Ω be the completion of an algebraic closure of K_∞.

We have the well-known analogies $K \leftrightarrow \mathbb{Q}$, $A \leftrightarrow \mathbb{Z}$, $K_\infty \leftrightarrow \mathbb{R}$, $\Omega \leftrightarrow \mathbb{C}$, $A^* = \mathbb{F}_q^* \leftrightarrow \mathbb{Z}^* = \{\pm 1\}$. The cardinalities of A^* and $\mathbb{Z}^*$ (i.e., choices of "signs") are $q-1$ and 2, respectively. Multiples of $q-1$ will be called "even" and other integers "odd". Put $A_+ =: \{$monic polynomials in $\mathbb{F}_q[T]\} \leftrightarrow \mathbb{Z}_+ =: \{$positive integers$\}$. (Note that though both are closed under multiplication, A_+ is not closed under addition in contrast to $\mathbb{Z}_+$.)

Hence, as an analogue to the special values of the Riemann zeta function, Carlitz [Ca 1] introduced the power sums

$$\zeta(k) =: \sum_{n \in A_+} \frac{1}{n^k} \in K_\infty, \qquad k \in \mathbb{Z}_+.$$

Clearly $\zeta(k)$ can be given by Euler product. Observe that, in contrast to the classical case, $\zeta(1)$ (which can now be considered as an analogue of Euler's constant γ, see [Th]) makes sense. Parallel to Euler's theorem that

$$\zeta_{\text{Riemann}}(2m) = \frac{B_{2m}(2\pi)^{2m}}{2(2m)!}, \qquad m \in \mathbb{Z}_+,$$

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Carlitz [Ca 1, Ca 2] proved (see [Go 1] for an exposition and pg. 893 in particular for the outline)

THEOREM I. *We have*

$$\zeta((q-1)m) = \frac{B_{(q-1)m}\tilde{\pi}^{(q-1)m}}{g_{(q-1)m}}, \qquad m \in \mathbb{Z}_+.$$

The notation used in the theorem is explained below.

(1) $\tilde{\pi} \in \Omega$ is an analogue of $2\pi i$, as it is a fundamental period of an appropriate exponential (exponential $e(z)$ corresponding to Carlitz action (see below)). Carlitz' student Wade [Wa] proved that $\tilde{\pi}$ is transcendental over K. $\tilde{\pi}^{q-1} \in K_\infty \leftrightarrow (2\pi i)^2 \in \mathbb{R}$.

(2) $g_k \in A$ is an analogue of factorial function and is defined [Ca 2] as follows. Let $k \in \mathbb{Z}_+ \cup \{0\}$ and $i \in \mathbb{Z}_+$, put

$$[k] =: T^{q^k} - T, \qquad D_0 =: L_0 =: 1, \qquad L_i =: [i] L_{i-1}, \qquad D_i =: [i]\, D_{i-1}^q \tag{1-1}$$

$$g_k =: \prod D_j^{k_j}, \qquad \text{where} \quad k = \sum_{j=0}^{w} k_j q^j \text{ with } 0 \leqslant k_j < q. \tag{1-2}$$

g_k, which is also denoted by $\Pi(k)$, Γ_{k+1}, Γ_k, etc., in various places is an analogue of the usual factorial in many respects [Go 1, Go 6, Th]. Carlitz used F_k for D_k.

(3) $B_{(q-1)m} \in K$ are analogues of Bernoulli numbers. They satisfy analogous "von-Staudt–Clausen" and "Lipschitz–Sylvester" theorems ([Ca 2, Ca 6], see [Go 1] for exposition, but see [Ca 7, pg. 409] for correction, if $q=2$). They are given by the generating function (note $B_m = 0$ if m is "odd")

$$\frac{z}{e(z)} = \sum_{m=0}^{\infty} \frac{B_m}{g_m} z^m.$$

But $B_{(q-1)m}/(g_{(q-1)m}/g_{(q-1)m-1})$, which can be thought of as analogues of $B_{2k}/2k$, do not seem to satisfy congruences of Kummer type.

The exponential function $e(z)$ mentioned above is an entire $\mathbb{F}_q$-linear function given by

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i} = \sum_{i=0}^{\infty} \frac{z^{q^i}}{g_{q^i}}.$$

It has $\tilde{\pi}A$ as kernel, so from standard facts of nonarchimedean analysis,

$$e(z)=z\prod_{\substack{\lambda\in\tilde{\pi}A\\ \lambda\neq 0}}\left(1-\frac{z}{\lambda}\right).$$

It satisfies the functional equation $e(Tz)=Te(z)+e(z)^q$, corresponding to the "Carlitz action" of A on $\mathbb{G}_a$, rather than the usual functional equation $e^{nz}=(e^z)^n$, corresponding to the usual action of $\mathbb{Z}$ on $\mathbb{G}_m$. Values of $e(\tilde{\pi}z)$ at $z\in K$ give "cyclotomic" extensions of K. (See [Ca 3, Ha].)

Remark. Due to different normalisations of "Carlitz action", many formulas are different in different papers. For example, $\tilde{\pi}$ and $B_{(q-1)m}$ differ from the corresponding quantities in Carlitz's papers and [Go 1] by the $(q-1)$th root of -1 and by $(-1)^m$, respectively. $e(z)$ here is $\psi(zu)/u$, where u is the $(q-1)$th root of -1 and ψ is the "exponential" in the papers just referred to.

Goss [Go 2] showed how just grouping together the terms of the same degree gives an interpolation of the ζ from its original domain $\mathbb{Z}_+$ to a much bigger domain, containing in particular $\mathbb{Z}$, which is all that we need for our purposes.

THEOREM II (Goss [Go 2]). *For $k\in\mathbb{Z}_+\cup\{0\}$,*

$$\zeta(-k)=:\sum_{i=0}^{\infty}\left(\sum_{\substack{n\in A_+\\ \deg n=i}}n^k\right)\in A. \tag{1-3}$$

If, further, $k\in\mathbb{Z}_+$, then $\zeta(-k)=0$ iff $k\equiv 0 \bmod (q-1)$. Also $\zeta(0)=1$.

This theorem should be compared with the corresponding result for the Riemann zeta function, namely that its values at negative integers are rational (we have integrality, since the sum over i's is finite (see, e.g., Section III)), and zero or not according to whether the argument is even or odd. The proof follows by writing a monic polynomial n of degree i as $Th+b$ with h of degree $i-1$ and $b\in\mathbb{F}_q$ and using the binomial theorem to get the induction formula [Go 2]

$$\zeta(0)=1,\qquad \zeta(-k)=1-\sum_{\substack{f=0\\ (q-1)\mid(k-f)}}^{k-1}\binom{k}{f}T^f\zeta(-f), \tag{1-4}$$

which shows $\zeta(-k)\in A$ since $\zeta(0)=1$. If $(q-1)$ divides k, then induction shows $\zeta(-k)=1-1+0=0$. If $(q-1)$ does not divide k, then there being no term in the summation corresponding to $f=0$, $\zeta(-k)=1-Tp(T)\neq 0$, where $p(T)\in A$.

Goss [Go 2] further observed that, essentially since $\zeta(-k)$ is a finite sum over n's of n^k's and since, for a monic prime $\wp$ of A of degree d and n relatively prime to $\wp$, n^k as a function of k interpolates to a continuous function on $S=:\varprojlim \mathbb{Z}/(q^d-1)p^j\mathbb{Z}$ (for, if $k\equiv 0 \bmod (q^d-1)p^j$, then $n^k\equiv 1 \bmod \wp^{p^j}$), by throwing out the Euler factor at $\wp$ we get a continuous function $\zeta_\wp: S\to A_\wp$ such that $\zeta_\wp(-k)=(1-\wp^k)\zeta(-k)$ if $k\in\mathbb{Z}_+$.

Observe that for $q=2$, $\zeta_\wp\equiv 0$ since $\zeta(-k)=0$, if $k\in\mathbb{Z}_+$. To take care of such problems (among other things, see [G–R, Go 3]), Goss [Go 3] defined modification $\beta(k)\in A$ of $\zeta(-k)$ as follows. For $k\in\mathbb{Z}_+\cup\{0\}$, let

$$\beta(k)=:\zeta(-k) \text{ if } k \text{ is odd}, \qquad \beta(k)=:\sum_{i=0}^{\infty}(-i)\sum_{\substack{n\in A_+\\ \deg n=i}} n^k \text{ if } k \text{ is even}. \tag{1–5}$$

In other words, deformation $Z(x,-k)=:\sum_{i=0}^{\infty}x^{-i}(\sum_{n\in A_+,\deg n=i}n^k)$ of $Z(1,-k)=\zeta(-k)$ has a simple zero at $x=1$ if k is even, and hence one considers $(dZ/dx)(x,-k)|_{x=1}=\beta(k)$ instead. For $k\in\mathbb{Z}_+$, $\beta(k)\neq 0$, and in fact

$$\beta(0)=0, \qquad \beta(k)=1-\sum_{\substack{f=0\\ (q-1)\mid(k-f)}}^{k-1}\binom{k}{f}T^f\beta(f). \tag{1–6}$$

One gets similar Kummer congruences and $\wp$-adic interpolations for $\beta(k)$'s (see [Go 4]).

One has two analogues B_k and $\beta(k)$ of Bernoulli numbers. Since one does not know any "functional equation" for ζ (one has, of course $\zeta(kp^m)=\zeta(k)^{p^m}$), these are not related in any simple way. Nonetheless, both are related to class groups of "cyclotomic" extensions of K in the following way. (See [Ca 3, Ha, G–R] for more on the "cyclotomic" theory.)

Let $\wp$ be a monic prime of A of degree d. Let $\Lambda_\wp=:e(\tilde{\pi}/\wp)\leftrightarrow\zeta_p=:e^{2\pi i/p}$. Then $K(\Lambda_\wp)\leftrightarrow\mathbb{Q}(\zeta_p)$. One also has "maximal totally real" subfield

$$K(\Lambda_\wp)^+=:K\left(\prod_{\theta\in\mathbb{F}_q^*}e\left(\frac{\theta\tilde{\pi}}{\wp}\right)\right)\leftrightarrow\mathbb{Q}(\zeta_p)^+=:\mathbb{Q}\left(\sum_{\theta\in\mathbb{Z}^*}e^{\theta 2\pi i/p}\right).$$

Let C (C^+, $\tilde{C}$, $\tilde{C}^+$, resp.) denote the p-primary component of the class group of $K(\Lambda_\wp)$ ($K(\Lambda_\wp)^+$, ring of integers of $K(\Lambda_\wp)$, $K(\Lambda_\wp)^+$, resp.). Let W be the ring of Witt vectors of $A/\wp$. Then we have decomposition $C\otimes_{\mathbb{Z}_p}W=\bigoplus_{0\leqslant k<q^d-1}C(w^k)$ into isotypical components according to characters of $(A/\wp)^*$, where w is the Teichmuller character. (Similarly for C^+, $\tilde{C}$, $\tilde{C}^+$.)

THEOREM III (Goss–Sinnot [Go 3, G–S]). $C(w^k) \neq 0$ *iff* $\wp \mid \beta(q^d-1-k)$, $0<k<q^d-1$.

THEOREM IV (Goss, Okada [Go 5, Ok]). $\tilde{C}(w^k) \neq 0$ *implies* $\wp \mid B_k$, $0<k<q^d-1$, $k \equiv 0 \bmod (q-1)$.

Conjecture (Gekeler [Ge 3]). If $\wp \mid B_{k'}$ for all k' such that $k' \equiv p^m k \bmod (q^d-1)$ with $0<k$, $k'<q^d-1$, $k \equiv 0 \bmod (q-1)$, then $\tilde{C}(w^k) \neq 0$.

Remark. Let $0<k<q^d-1$. It is easy to see that

(a) if k is "odd" then $C(w^k) \cong \tilde{C}(w^k)$ and $C^+(w^k)$ and $\tilde{C}^+(w^k)$ are trivial;

(b) if k is "even" then $C(w^k) \cong C^+(w^k)$ and $\tilde{C}(w^k) \cong \tilde{C}^+(w^k)$.

(See [Go 5, Propositions 2–3], where "even" and "odd" are reversed by mistake.)

For some formulas for B_i's see [Ca 2] and [Ge 2]; more results on $\zeta(k)$, $\zeta_v(k)$ when $k \in \mathbb{Z}_+$ are contained in [A–T], [Yu]. For generalizations of many results in this section to a much broader context, see [Go 5] and references therein.

II. MEASURES AND INTERPOLATION

For a quick and nice introduction to p-adic measures see [Ka].

For topological spaces X, Y, let Conti(X, Y) denote the set of continuous functions from X to Y. A $\mathbb{Z}_p$-valued p-adic measure on $\mathbb{Z}_p$ is just a $\mathbb{Z}_p$-linear map μ: Conti$(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{Z}_p$. One writes $\mu(f)$ symbolically as $\int_{\mathbb{Z}_p} f(x)\, d\mu(x)$.

Binomial coefficients $\binom{x}{k} =: x(x-1)\cdots(x-(k-1))/k!$ take $\mathbb{Z}$-values on $\mathbb{Z}$ and hence by continuity assume $\mathbb{Z}_p$-values on $\mathbb{Z}_p$. In fact, they form a basis of the $\mathbb{Z}$-module of polynomials (in x) over $\mathbb{Q}$ which map $\mathbb{Z}$ into $\mathbb{Z}$ (and also of the $\mathbb{Z}_p$-module of polynomials over $\mathbb{Q}_p$ that map $\mathbb{Z}_p$ into $\mathbb{Z}_p$).

THEOREM V (Mahler [Ma]). *Any $f \in$ Conti$(\mathbb{Z}_p, \mathbb{Z}_p)$ can be uniquely written as*

$$f(x) = \sum_{k \geq 0} a_k \binom{x}{k}, \qquad a_k \in \mathbb{Z}_p, \quad a_k \to 0.$$

The a_k may be recovered as the higher differences of f:

$$a_k = (\Delta^k f)(0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(i).$$

Conversely, any series $\sum_{k\geqslant 0} a_k\binom{x}{k}$, $a_k \in \mathbb{Z}_p$, $a_k \to 0$ *converges to an element of* $\mathrm{Conti}(\mathbb{Z}_p, \mathbb{Z}_p)$.

COROLLARY. *A* $\mathbb{Z}_p$*-valued measure* μ *on* $\mathbb{Z}_p$ *is uniquely determined by the sequence* $\mu_k =: \int_{\mathbb{Z}_p} \binom{x}{k}\, d\mu(x)$ *of elements of* $\mathbb{Z}_p$. *Any sequence* μ_k *determines a* $\mathbb{Z}_p$*-valued measure* μ *by the formula* $\int_{\mathbb{Z}_p} f\, d\mu =: \sum_{k\geqslant 0} a_k\mu_k = \sum_{k\geqslant 0} (\Delta^k f)(0)\mu_k$.

Convolution $*$ of measures μ, μ' is defined by $\int f(x)\, d(\mu * \mu')(x) =: \iint f(x+t)\, d\mu(x)\, d\mu'(t)$.

By comparing coefficients in $(1+m)^{x+t} = (1+m)^x\,(1+m)^t$ one sees

$$\binom{x+t}{k} = \sum \binom{x}{i}\binom{t}{k-i}.$$

Hence

$$\sum a_k(\mu * \mu')_k = \iint \sum_{k\geqslant 0} a_k \sum_{i=0}^{k} \binom{x}{i}\binom{t}{k-i}\, d\mu(x)\, d\mu'(t) = \sum a_k\left(\sum_{i=0}^{k} \mu_i\mu'_{k-i}\right).$$

In other words, if one identifies measure μ with $\sum \mu_k X^k$ then convolution $*$ on measures is just multiplication on the corresponding power series.

Carlitz [Ca 5] (see [Wa 1, Wa 2] also) constructed an analogue $G_k(x)/g_k \in K[x]$ for $\binom{x}{k} \in \mathbb{Q}[x]$ as follows. (We use the suggestive notation $\left\{ {x \atop k} \right\}$ for $G_k(x)/g_k$.)

For $i \in \mathbb{Z}_+ \cup \{0\}$ put

$$e_i(x) =: \psi_i(x) =: \prod_{\substack{a\in A \\ \deg a < i}} (x-a) \in A[x] \tag{2-1}$$

(where $\deg 0$ is $-\infty$ by convention, consistent with $0 = \mathbb{N}0 = q^{\deg 0}$). For example, $e_i(x^i) = D_i$. For m, $i \in \mathbb{Z}_+ \cup \{0\}$, put

$$\begin{bmatrix} m \\ i \end{bmatrix} =: \frac{D_m}{D_i L_{m-i}^{q^i}} \text{ if } m \geqslant i, \qquad =: 0 \text{ otherwise.} \tag{2-2}$$

Then one has [Ca 1]

$$e_m(x) =: \sum_{i\geqslant 0} (-1)^{m-i} \begin{bmatrix} m \\ i \end{bmatrix} x^{q^i}. \tag{2-3}$$

For

$$k \in \mathbb{Z}_+ \cup \{0\}, \qquad k = \sum k_i q^i, \qquad 0 \leqslant k_i < q \tag{2-4}$$

let

$$G_k(x) =: \prod e_i(x)^{k_i} \in A[x] \tag{2–5}$$

$$G'_k(x) =: \prod G'_{k_i q^i}(x)$$

with (2–6)

$$G'_{k_i q^i}(x) =: \begin{cases} e_i(x)^{k_i} & \text{if } 0 \leqslant k_i < q-1 \\ e_i(x)^{q-1} - D_i^{q-1} & \text{if } k_i = q-1. \end{cases}$$

Why is $\left\{ {x \atop k} \right\} = G_k(x)/g_k$ a good analogue of $\binom{x}{k}$? There are several analogies.

First, we present a curious mix of analogies. Definition by multiplication of "basic" objects (which correspond to q^i) using digit multiplication is prevalent throughout this subject (e.g., Definition (1–2)). Hence we concentrate on the analogy $\left\{ {x \atop q^i} \right\} = e_i(x)/D_i \leftrightarrow \binom{x}{q^i} = x(x-1) \cdots (x-(q^i-1))/q^i!$. Here $\{0, 1, ..., q^i - 1\}$ is a natural full residue class system for $\mathbb{Z}/q^i\mathbb{Z}$, whereas $\{n \in A, \deg n < i\}$ is a natural full residue class system for A/aA with a such that $\mathbb{N}a = q^i$. Hence by (2–1) the numerators are analogous. We have seen that the denominator $D_i = g_{q^i}$ is an analogue of $q^i!$ (One should note now that $q^i! = q^i(q^i - 1) \cdots 1$, but D_i is a product of monic elements of A degree i [Ca 1], which is why we called this analogy a curious mixture.)

Second, the binomial coefficient $\binom{x}{k}$ is the coefficient of t^k in the expansion of $(1+t)^x = e^{x \log(1+t)}$, whereas $\left\{ {x \atop q^i} \right\}$ is the coefficient of t^{q^i} in $e(x \log t)$ (where log is the inverse function of e of section I and the coefficients of t^m, $m \neq q^i$ are zero). On translation from "analytic" to "algebraic" language, multiplication by $x \in \mathbb{Z}$ for $\mathbb{G}_m$ is represented by $(1+t)^x - 1 = \sum \binom{x}{k} t^k$, whereas "Carlitz action" of $x \in A$ on $\mathbb{G}_a$ is represented by $\sum \left\{ {x \atop q^i} \right\} t^{q^i}$. (See [Ca 3, (1–6)] and [Ha, (1–2)], but note that because of different normalizations of Carlitz action, there is a sign involved in each term in [Ca 3, (1–6).]

Third, Carlitz showed that $\left\{ {x \atop k} \right\}$ (and also $G'_k(x)/g_k$) takes A-values on A [Ca 5] and in fact forms a basis of the A-module of polynomials in x over K which map A into A. Carlitz' student Wagner showed

THEOREM VI (Wagner [Wa 1, Wa 2]; see also [Go 7]). *Any $f \in \mathrm{Conti}(A_{\wp}, A_{\wp})$ can be uniquely written as*

$$f(x) = \sum_{k \geqslant 0} a_k \left\{ {x \atop k} \right\}, \qquad a_k \in A_{\wp}, \quad a_k \to 0.$$

The a_k may be recovered as

$$a_k = (-1)^m \sum_{\deg a < m} \frac{G'_{q^m-1-k}(a)}{g_{q^m-1-k}} f(a), \qquad \textit{where} \quad q^m > k.$$

Conversely, any series $\sum_{k \geq 0} a_k \{{x \atop k}\}$, $a_k \in A_\wp$, $a_k \to 0$ converges to an element of $\mathrm{Conti}(A_\wp, A_\wp)$.

Remark. The sum in the formula for a_k can also be written as $\sum_{0 \leq \mathbb{N}a \leq k}$. Also, if none of the q-digits for $q^m - 1 - k$ is $(q-1)$, then $G'_{q^m-1-k}(a)/g_{q^m-1-k} = \{{a \atop q^m-1-k}\}$, a "binomial coefficient." For the analogy with "higher differences", see [Ca 1] and [Wa 5, p. 386].

COROLLARY. *An $A_\wp$-valued measure μ on $A_\wp$ (called just measure μ for short) is uniquely determined by the sequence $\mu_k =: \int_{A_\wp} \{{x \atop k}\}\, d\mu$ of elements of $A_\wp$. Any sequence μ_k determines measure μ by $\int_{A_\wp} f\, d\mu =: \sum_{k \geq 0} a_k \mu_k$.*

Carlitz [Ca 5, Wa 6] showed further that

$$\left\{ {x+t \atop k} \right\} = \sum \binom{k}{i} \left\{ {x \atop i} \right\} \left\{ {t \atop k-i} \right\}.$$

As Anderson observed (see [Go 7]), this means that if we identify measure μ with the divided power series $\sum \mu_k (x^k/k!)$ (recall the formal nature of $x^k/k!$ in the context of divided power series) then by a proof similar to that in the classical case above, convolution $*$ on measures is multiplication on corresponding divided power series.

In light of results of Leopaldt, Mazur, Iwasawa (see [Ka]), and the results of Goss stated in Section I, Anderson and Goss then asked what the divided power series corresponding to measure μ (called "zeta measure", see the appendix) whose moments $\int_{A_\wp} x^k\, d\mu$ are $\zeta(-k)$ (0 resp.) for $k > 0$ ($k = 0$ resp.) would be?

Our main result, proved in Section IV, is

THEOREM VII. *If μ is the zeta measure, then the corresponding power series $\sum \mu_k (x^k/k!)$ is given by*

$$\mu_k = \begin{cases} (-1)^m & \textit{if} \quad k = cq^m + (q^m - 1),\ 0 < c < q-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Remark. (1) This result can be viewed as giving many interesting identities for $\zeta(-k)$'s.

(2) This result is trivially true for $q = 2$, since the zeta measure is then identically zero by Theorem II.

(3) We will use shorthand $k=k(c,m)$ for $k=cq^m+(q^m-1)$ with $0<c<q-1$. We will see in Section IV that $k=k(c,m)$ iff $\deg \zeta(-k)=\deg g_k$ for $k>0$.

For comparison, which is not fully understood, we state the corresponding classical result (see [Ka]). (First recall one difference noted in Section One: $\zeta(-k)$ satisfy stronger integrality than $\zeta_{\text{Riemann}}(-k)$. Hence one needs some twisting factor in the classical case.) If μ is the measure whose moments are $\int_{\mathbb{Z}_p} x^k\,d\mu=(1-a^{k+1})\,\zeta_{\text{Riemann}}(-k)$, where $a\geqslant 2$ and $(a,p)=1$, then $\sum \mu_k x^k=(1+x)/(1-(1+x))-a(1+x)^a/(1-(1+x)^a)$.

III. Power Sums

We state several results on power sums. But note that the proof of the main result needs from this section only part (3) of Theorems VIII and X (or part (3) of Theorem VIII and the remark following theorem VIII), for the first proof; and only part (4) of Theorem VIII (or part (4) of Theorem VIII and part (1) of Theorem X) for the second proof.

Let $k>0$, unless stated otherwise; though many statements below are valid (and trivial) for $k=0$.

Let $S_i(k)=:\sum_{n\in A_+,\deg n=i} n^k$. If $i>k$, $S_i(k)=\sum_{a_0,\dots,a_{i-1}\in\mathbb{F}_q}(T^i+a_{i-1}T^{i-1}+\cdots+a_0)^k=\sum_{a_0,\dots,a_{i-1}\in\mathbb{F}_q}$ (terms containing not all of $a_0,\dots,a_{i-1}$) $=0$ since $\#\mathbb{F}_q=q=0$ in characteristic p. Hence one sees Goss' result in Section I that $\zeta(-k)=\sum_{i=0}^{\infty} S_i(k)$ is a finite sum and hence $\zeta(-k)\in A$.

One has much better results on vanishing of $S_i(k)$ and closed expressions for $S_i(k)$, which we now recall.

Let $l(k)(v(k)$ resp.) be the sum of digits of k base q (base p resp.).

Theorem VIII. (1) (Paley [Pa, Lemma 2]) *If $q=p^t$ and $i>v(k)/t(p-1)$ then $S_i(k)=0$.*

(2) (Lee [Le, Theorem 3–2]) *If $q=p$, then $S_i(k)=0$ iff $i>l(k)/(q-1)$.*

(3) (Carlitz [Ca 8, Theorem 4–2]) *Let $k\geqslant 1$. $S_i(k)\neq 0$ iff $k=m_0+m_1+\cdots+m_i$, where $q-1\mid m_j$, $m_j>0$ for $1\leqslant j\leqslant i$, and the sum is such that there is no carry over of digits base p.*

(4) (Carlitz [Ca 1, Theorem 9–5; Ca 5, p. 497]) *If $k<q^i-1$, $S_i(k)=0$.*

(5) (Lee [Le, Lemma 7–1; see also Ge 1, 2–12]) *If $i>l(k)/(q-1)$, then $S_i(k)=0$.*

Remark. Part (3) implies the other parts easily. Part (3) itself follows by expanding $S_i(k)$ using the multinomial theorem and analysing when the multinomial coefficients vanish mod p. In fact, the argument [Ca 8] shows that degree of $S_i(k)$ is the maximum value of the expression $im_0+(i-1)m_1+\cdots+m_{i-1}$, over all choices of m_j's satisfying the conditions in the statement.

There are many variations of such results in these references. For example,

THEOREM IX (Lee [Le, Lemma 7–1]). *If* $i>l(k)/q-1$, *then* $\sum_{a\in A,\,\deg a<i} a^k=0$.

Now if $q-1\mid k$, then $\sum_{a\in A,\,\deg a<i} a^k=(q-1)\,\zeta(-k)$ if $i\gg 0$, so $\zeta(-k)=0$, which is Goss's result in Section I. (It also follows from [Ca 8, Theorem 4–1].)

THEOREM X. (1) (Carlitz [Ca 4, p. 941; Ge 1, Theorem 4–1]) $S_i(q^{k+i}-1)=(-1)^i D_{i+k}/L_i D_k^{q^i}$.

(2) (Lee [Le, Theorem 4–1]) *If* $k+1=q^{l_1}+\cdots+q^{l_s}$, $s<q$, *then* $S_i(k)=(-1)^i [l_1]_i\cdots[l_s]_i/L_i$, *where* $[h]_m=: D_h/D_{h-m}^{q^m}$ *if* $m\leqslant h$ *and* 0 *otherwise.*

(3) (Gekeler [Ge 1, Theorem 3–13]) *Let* $k<q^{i+1}-1$ *have base* q *expansion* $k=\sum k_j q^j$. *Then* $S_i(k)=(-1)^r\cdot M\cdot\prod_{j\leqslant i} L_{i-j}^{q^j(k_j-(q-1))}g_k$, *where* $r=: i+\sum_{j<i}(i-j+1)k_j$ *and* M *is the multinomial coefficient* $\binom{k_i}{k_0',\ldots,k_i'}$. $k_j'=: q-1-k_j\,(j<i)$ *and* $k_i'=: l(k)-i(q-1)$.

Many more variations (e.g., estimates on $\deg S_i(k)$ [Ge 1, Th 1], generating function for $S_i(k)$ [Ca 1, Ge 1]) are included in these references.

Consider now $k=cq^m+(q^m-1)$, $0\leqslant c<q-1$. Making extensive computations [Go 4, Go 6] Goss empirically observed that $S_m(k)=\pm g_k$. For $c=0$, it follows immediately from Carlitz' result (1) of Theorem X. For general c, if we choose $s=c+1$, $l_j=m$, $i=m$ in (2) of Theorem X, one sees $S_m(k)=(-1)^m D_m^c(D_0\cdots D_{m-1})^{q-1}=(-1)^m g_k$. See [Ge 1, Section 6] for a different proof. Results for $S_i(k)$, $i>m$ or $i<m$ stated there also follow from Lee's result. Hence one sees as in [Ge 1] that $c\neq 0$ implies $\deg\zeta(-k)=\deg g_k$.

THEOREM XI (Carlitz [Ca 1, Theorem 9–4]). *If* $m>0$, *then* $S_k(-m)=(-1)^k A_{m-1}^{(k)}/L_k$, *where*

$$\left(1-\sum_{i=0}^{k}\frac{(-1)^{k-i}t^{q^i}}{D_i L_{k-i}^{q^i}}\right)^{-1}=\sum_{m=0}^{\infty} A_m^{(k)} t^m.$$

(There are some sign errors in the statement and proof in [Ca 1].)

THEOREM XII (Thomas [Th 1] for case $q=p$). *If* $l(k)<q-1$, *then* $\beta(k)=\zeta(-k)=1$.

This follows from Lee's (5) of Theorem VIII: $S_i(k)=0$ if $i\geqslant 1$ and $S_0(k)=1$.

IV. PROOF OF THE MAIN RESULT

For this section, let μ be the zeta measure of Section II, so that for $k>0$, $\int_{A_\wp} x^k\, d\mu=\zeta(-k)$. Recall that $\mu_k=\int \{{x \atop k}\}\, d\mu$. Let $k>0$.

First we give a quick proof of a weaker result even though we will not need it:

PROPOSITION. $\mu_k=0$ *in either of the following situations*:

(1) $k>0$ *and* $(q-1)$ *divides* k *or*

(2) $k\geqslant q$ *and* $l(k)<q-1$.

Proof. By (2–3), $e_i(x)/D_i$ is a K-linear combination of powers x^{q^j} $(0\leqslant j\leqslant i)$ and so expanding $\{{x \atop k}\}=\prod (e_i(x)/D_i)^{k_i}$ in powers of x, one sees that in case (1), powers x^m occurring in the expansion have $m\equiv\sum k_i\equiv k\equiv 0 \bmod (q-1)$ and hence $\int x^m\, d\mu=\zeta(-m)=0$ by Theorem II; whereas in case (2) $l(m)<q-1$ and hence by Theorem XII, $\zeta(-m)=1$, so μ_k is obtained by putting $x=1$ in $G_k(x)/g_k$, but $e_j(1)=0$ if $j>0$ and $e_0(1)=1$. Hence $\mu_k=0$.

Remark. I had shown earlier that $\mu_k=(-1)^m$ for $k=k(c,m)$. This proposition showed that (infinitely) many other μ_k's are zero. Computations for small k's suggested that all other μ_k's are zero. Later, putting the computation on computer, Goss [Go 8] verified it for a larger range of k's.

First Proof of the Main Result

LEMMA 1. $\mu_k\in A$.

Since $G_k(x)/g_k\in K[x]$ and $\zeta(-i)\in A$, one sees that $\mu_k\in K$. But $\mu_k\in A_\wp$ for every prime $\wp$ of A, so $\mu_k\in A$.

LEMMA 2. (1) *If* $k=cq^m+(q^m-1)$, $0<c<q-1$, *then* $S_m(k)=(-1)^m g_k$, $S_{m+w}(k)=0$, $\deg S_{m-w}(k)<\deg g_k$ *if* $w>0$.

(2) *If* $k=q^m-1$, *then* $S_m(k)=(-1)^m g_k$, $S_{m+w}(k)=0$, $S_{m-1}(k)=$

$(-1)^{m-1}\, T^{\deg g_k}$ + *polynomial of lower degree and* $\deg S_{m-1-w}(k) < \deg g_k$ *if* $w > 0$.

(3) *If* $k \neq cq^m + (q^m - 1)$, $0 \leqslant c < q-1$, *then* $\deg S_i(k) < \deg g_k$ *for* $i \geqslant 0$.

Proof. (There is more than one way to prove this and more by combining information in Section III) We have trivial estimate $\deg S_i(k) \leqslant ik$. Let $k = \sum_{j=0}^{w} k_j q^j$ be expansion base q.

Claim (A). If $S_i(k) \neq 0$ and $ik \geqslant \deg g_k$ then if k is odd then $i = w$ and if k is even $i = w$ or $w+1$. By (1–1) and (1–2), it is easy to see that $\deg g_k = \sum jk_j q^j$, so the hypothesis implies that $i \geqslant w$. By (3) of Theorem VIII, $k = m_0 + \cdots + m_i$ with $q-1 \mid m_j > 0$ for $1 \leqslant j \leqslant i$, and there is no carry over base p (and hence base q) in the sum. So $l(k) \geqslant i(q-1) + h$, where h is the least nonnegative residue of k modulo $q-1$. But $l(k) \leqslant w(q-1) + h$ if k is odd and $l(k) \leqslant (w+1)(q-1)$ in general, giving the required inequalities between i and w in the other direction. Hence (A) is proved.

Claim (B). If $k < q^{i+1} - 1$ and k as in case (3), then $\deg S_i(k) < \deg g_k$. This follows from (3) of Theorem X, since not all k_j except the first are $q-1$. (Alternate proof, using the estimate on $\deg S_i(k)$ stated in the remark following Theorem VIII, instead of (3) of Theorem X, is an easy exercise for the intersted reader.)

Vanishing and inequalities on degrees of the power sums claimed in the lemma follow easily by combining (A), (B), and (4) of Theorem VIII. Other parts of the lemma follow from remarks after Theorem X and the fact that $\zeta(-k) = 0$ in case (2).

COROLLARY 1. *Consider cases* (1), (2), (3) *of the lemma.*

(1) $\zeta(-k) = (-1)^m\, T^{\deg g_k}$ + *polynomial of lower degree*

(2) $\zeta(-k) = 0$

(3) $\deg \zeta(-k) < \deg g_k$.

COROLLARY 2. *Consider cases* (1), (2), (3) *of the lemma.*

(1) $\beta(k) = (-1)^m\, T^{\deg g_k}$ + *polynomial of lower degree*

(2) $\beta(k) = (-1)^{m-1}\, T^{\deg g_k}$ + *polynomial of lower degree*

(3) $\deg \beta(k) < \deg g_k$.

LEMMA 3. *Write* $\prod e_j(x)^{k_j} = \sum_{i=0}^{k} a_i x^i$; *then*

(a) $\deg a_i + \deg \zeta(-i) \leqslant \deg g_k$ *for all* i

(b) *Equality holds in* (a) *iff* $k = k(c, m)$ *and* $i = k$.

Proof of Lemma 3. For the top degree $i=k$, $a_k=1$ as e_j are monic. Hence in this case, (a) and (b) follow from Corollary 1 of Lemma 2. For general u, if $a_u \neq 0$, the term $a_u x^u$ occurs in the expansion of $\prod e_j(x)^{k_j}$ in powers of x, if some $x^{q^{a_u}}$ (with appropriate coefficient) are picked up from each e_j (see (2–3)). Hence $u=q^{a_1}+\cdots+q^{q_b}$ with $b=\sum k_j$. Suppose that the exponents u and v are formed in such a way that all except one of $x^{q^{a_u}}$ and $x^{q^{a_v}}$ are obtained in exactly the same fashion and that the exception is that for v, x^{q^c} is picked up from one e_j whereas for u, $x^{q^{c-d}}$ $(d>0)$ is chosen from the e_j, so that $v=u+q^c-q^{c-d}>u$.

We will prove (1) $\deg a_u+\deg g_u \leqslant \deg a_v+\deg g_v$. This follows from (2) $\deg g_{r+q^c}-\deg g_{r+q^{c-d}} \geqslant \deg\genfrac{[}{]}{0pt}{}{c}{c-d}$ (take $r=u-q^{c-d}$) and (3) $\deg\genfrac{[}{]}{0pt}{}{j}{c-d}-\deg\genfrac{[}{]}{0pt}{}{j}{c}=\deg\genfrac{[}{]}{0pt}{}{c}{c-d}$. Now (3) is straightforward computation using (1–1) and (2–2) and (2) follows by minimizing with respect to "carying over" possibilities in addition of numbers written in base q. (Hint: $\deg\genfrac{[}{]}{0pt}{}{c}{c-d}=\deg(D_{c-1}\cdots D_{c-d})^{q-1}$.)

Hence $\deg a_u+\deg\zeta(-u)\leqslant \deg a_u+\deg g_u \leqslant \deg a_v+\deg g_v \leqslant \deg g_k$ by inductive process. This proves (a) for general i; (b) follows by straightforward tracing of when the equality holds in (2), if we note that $\deg \zeta(-u)<\deg g_u$ if u is not of the form $k(a,b)$. This completes the proof.

Proof of the Theorem. Clearly $\mu_k=\sum_{i=0}^k a_i\zeta(-i)/g_k \in A$ by Lemma 1. But if $k\neq k(c,m)$, $\deg \mu_k<0$ by Lemma 3, so $\mu_k=0$. On the other hand, when $k=k(c,m)$, Lemma 3 and Corollary 1 to Lemma 2 imply $\mu_k=(-1)^m$, finishing the proof.

Straightforward modification, using Corollary 2 of Lemma 2 in place of Corollary 1 of Lemma 2, one gets a variation.

THEOREM XIII. *If $\bar\mu$ denotes the "beta measure" (see the appendix), i.e., $\int x^k\, d\bar\mu=\beta(k)$, then the corresponding divided power series $\sum \bar\mu_k(x^k/k!)$ is given by*

$$\bar\mu_k=\begin{cases}(-1)^m & \textit{if} \quad k=cq^m+(q^m-1)\ 0<c\leqslant(q-1)\\ 0 & \textit{otherwise.}\end{cases}$$

This also gives the two variable versions as stated in [Go 8].

Second Proof of the Main Result

Let $k=\sum_{j=0}^w k_j q^j$, $0\leqslant k_j<q$, $k_w>0$, and $G_k(x)=\sum a_i x^i$, as before. We want to compute $\mu_k=(\sum a_i\zeta(-i))/g_k$.

First consider $k\neq q^{w+1}-1$, so that $q^{w+1}-1>k>q^w-1$. Hence, by part (4) of Theorem VIII,

$$\mu_k=\frac{1}{g_k}\sum a_i \sum_{n\ \text{monic},\ \deg n\leqslant w} n^i=\frac{1}{g_k}\sum_{n\ \text{monic},\ \deg n\leqslant w} G_k(n).$$

(In this expression of μ_k as a sum of "binomial coefficients," w can of course be replaced by a quantity greater than w; in particular, one can take the limit as w tends to infinity. This remark is not necessary for the proof.)

Now, if $\deg n < w$, $e_w(n) = 0$ and hence $G_k(n) = 0$. On the other hand, if $\deg n = w$, then $e_w(n) = D_w$, hence

$$\mu_k = \frac{D_w^{k_w}}{g_k} \sum_{n \text{ monic}, \deg n = w} G_{k - k_w q^w}(n).$$

One has the following "Mahler inversion" type result.

THEOREM XIV (Carlitz [Ca 5, p. 494]). *If* $p(t) = \sum_{i=0}^{d} A_i' G_i'(t)$ *and* $q^m > d$ *then*

$$(-1)^m \frac{D_m}{L_m} A_i' = \sum_{n \text{ monic}, \deg n = m} G_{q^m - 1 - i}(n)\, p(n).$$

Take $p(t) = 1$, $m = w$, so that $A_0' = 1$, $A_i' = 0$ if $i \neq 0$ and hence $\mu_k = 0$ if $k - k_w q^w \neq q^w - 1$ and $\mu_k = D_w^{k_w}/g_k(-1)^w D_w/L_w = (-1)^w$ if $k = k(k_w, w)$. In the remaining case, $k = q^{w+1} - 1$, the same proof (where one has now to consider the sum over degrees w and $w+1$) gives

$$\mu_k = \frac{1}{g_k}((-1)^{w+1} D_{w+1}/L_{w+1} + D_w^{q-1}(-1)^w D_w/L_w) = 0.$$

Hence the result is proved.

Remark. We used a very special case, $p(t) = 1$, of Theorem XIV, for which one can give a direct proof as follows. As $G_{q^m-1-i}(n)$ is a polynomial in n of degree $q^m - 1 - i$, the sum $\sum G_{q^m-1-i}(n)$ is zero if $i > 0$, by part (4) of Theorem VIII. On the other hand, if $i = 0$, the sum is $(-1)^m D_m/L_m$, by part (1) of Theorem X (in fact, by its special case, $k = 0$, which was already in [Ca 1, Theorem 9.5]).

A similar proof can be given for Theorem XIII. One only has to note that if k is even (odd resp.), then $a_i \neq 0$ only for i even (odd resp.) and hence there is no "mixing" of moments for even and odd k's.

We finish by commenting that if one tries to generalize analogies in Section II to "general A" (see [Go 2] for notation) in naïve fashion, one may lose the integrality of binomial coefficients if one uses the first analogy (note that there are no "natural" residue class representatives), whereas one has integrality and "the addition formula for binomial coefficients" which implied the divided power series correspondence if one uses the second analogy, but then the integral basis property fails.

Appendix: Existence of Zeta and Beta Measures

Integrality of t_i and $(\sum a_i t_i)/g_k$ (where as before $G_k(x)/g_k = (\sum a_i x^i)/g_k$) immediately implies existence of a measure whose ith moment is t_i. Hence the second proof of the main result and of Theorem XIII implies the existence of zeta and beta measures.

In fact, there is an alternate approach to measures, where one defined measure μ by first defining "measures" $\mu_{f,m}$ of the basic compact opens $f + \wp^m A_\wp$ and then defining the "integrals" $\int_{A_\wp} f\, d\mu$ by taking limits of "Riemann sums" in the usual fashion.

We assume $\delta =: \deg f < m\, d$, where, as before, d is degree of the monic prime $\wp$. Goss [Go 2, Go 7] defined measures μ and μ' by defining $\mu_{f,m} = 2$, $\mu'_{f,m} = -\delta - m\, d$ when f is monic and $\mu_{f,m} = 1$, $\mu'_{f,m} = -m\, d$ otherwise. He showed that when k is odd (even, resp.), the kth moment of μ (μ', resp.) is $\zeta(-k)$ ($\beta(k)$, resp.). By computing moments, it is easy to see that μ is, in fact, the zeta measure and that the beta measure $\bar{\mu}$ can be given by setting $\bar{\mu}_{f,m}$ to be $2 + \delta - m\, d$ or $1 + \delta - m\, d$ according as whether f is monic or not.

As noted in [Go 2], μ is obtained, analogously to the classical case, by reducing the Stickelberger elements for the full cyclotomic extensions to characteristic p. Following the same procedure (rather than taking "derivatives" [Go 2] which leads to μ') for the maximal totally real subfields of the full cyclotomic extensions, one gets a measure whose moments are $-\beta(i)$ (0, resp.) for i even (odd, resp.). "Subtraction" of this measure from the zeta measure gives the beta measure.

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