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A fractional variational iteration method for solving fractional nonlinear differential equations

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ABSTRACT

Recently, fractional differential equations have been investigated by employing the famous variational iteration method. However, all the previous works avoid the fractional order term and only handle it as a restricted variation. A fractional variational iteration method was first proposed in [G.C. Wu, E.W.M. Lee, Fractional variational iteration method and its application, Phys. Lett. A 374 (2010) 2506–2509] and gave a generalized Lagrange multiplier. In this paper, two fractional differential equations are approximately solved with the fractional variational iteration method.

1. Introduction

Recently the variational iteration method [1] has been widely applied to analytically solve fractional differential equations [2–6], where the term with the fractional derivative was considered as a restricted variation, making the identification of the Lagrange multiplier very inaccurate. To overcome this problem, we investigate the local behavior of fractional differential equations and determine the Lagrange multiplier in a more accurate way with the fractional variation iteration method (FVIM) [7].

2. Properties of local fractional calculus

On the basis of Cantor-like sets, Kolwankar and Gangal [8,9] proposed a concept of a local fractional derivative:

$$D_{x_0}^{\alpha} f(x) = \lim_{x \to x_0} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x_0}^{x} (x-\xi)^{n-\alpha} (f(\xi) - f(x_0)) d\xi, \tag{1}$$

where the derivative on the right-hand side is the Riemann–Liouville fractional derivative.

Chen et al. [10] presented the necessary conditions for

$$D_{x}^{\alpha} f(x) = \lim_{y \to x} \frac{\Gamma(1+\alpha)(f(y) - f(x))}{(y-x)^{\alpha}}, \quad 0 < \alpha \le 1. \tag{2}$$

We now can derive the following useful properties of the Kolwankar–Gangal derivative.

(a) Integration with respect to $(dx)^{\alpha}$:

In the derivation of Eq. (2), for any ϵ, there exists a d, where $|x_{i+1} - x_i| < d$, such that

$$\left| f(x_{i+1}) - f(x_i) - D_{x_i}^{\alpha} f(x_i) \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(1+\alpha)} \right| < \epsilon.$$

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Since $f(x)$ is continuous in the closed interval $[a, b]$, if we consider a finite partition, $[x_0, x_1], \ldots, [x_i, x_{i+1}], \ldots, [x_{n-1}, x_n]$ where $x_0 = a, x_n = b$, and note that

$$f(b) - f(a) = \sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)),$$

we can choose the maximal δ such that

$$\left| f(b) - f(a) - \sum_{i=0}^{n-1} D_{x_i}^{\alpha} f(x_i) \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(1+\alpha)} \right| < \epsilon.$$

As a result, we develop a definition of fractional integration.

If $g(x)$ is continuous in the interval $[a, b]$ and the limit of $\sum_{i=0}^{n-1} g(x_i) \frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(1+\alpha)}$ exists when n tends to infinity, then we say that the function $g(x)$ is α-order integrable in the interval $[a, b]$ denoted by

$$ {}_aI_b^{\alpha} g(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b g(x)(\mathrm{d}x)^{\alpha} = \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i) \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(1+\alpha)}. \tag{3}$$

(b) The fractional Leibniz product law:

If u and v are α-order differentiable functions, we have the generalized Leibniz product law from Eq. (2)

$$D_x^{(\alpha)}(uv) = u^{(\alpha)}v + uv^{(\alpha)}. \tag{4}$$

(c) The fractional Leibniz formulation:

$${}_0I_x^{\alpha} D_x^{\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \le 1. \tag{5}$$

Therefore, integration by parts can be used in the fractional calculus:

$${}_aI_b^{\alpha} u^{(\alpha)} v = (uv)\,|_a^b - {}_aI_b^{\alpha} uv^{(\alpha)}. \tag{6}$$

3. The fractional variational iteration method

In this section, two examples are given to illustrate the effect of the proposed method.

Example 1. As the first example, we consider a time-fractional diffusion equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = c\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial(F(x)u(x,t))}{\partial x}, \quad 0 < \alpha \le 1, \tag{7}$$

where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo derivative, with initial condition $u(x, 0) = f(x)$.

We replace the fractional Caputo derivative with the local fractional derivative in Eq. (7), and assume $c = 1, F(x) = -x$ which leads to

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial(xu(x,t))}{\partial x}, \quad 0 < \alpha \le 1, \tag{8}$$

with the initial condition $u(x, 0) = x^2$.

A correction functional for Eq. (8) can be constructed as follows [1,5]:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t,\tau) \left\{ \frac{\partial^{\alpha} u_n(x,\tau)}{\partial \tau^{\alpha}} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} - \frac{\partial(x\tilde{u}_n(x,\tau))}{\partial x} \right\} (\mathrm{d}\tau)^{\alpha} \tag{9}$$

with the property, from Eqs. (4)–(6), that $\lambda(t, \tau)$ must satisfy

$$\frac{\partial^{\alpha}\lambda(t,\tau)}{\partial \tau^{\alpha}} = 0, \quad \text{and} \quad 1 + \lambda(t,\tau)|_{\tau=t} = 0. \tag{10}$$

Therefore, $\lambda(t, \tau)$ can be identified as $\lambda(t, \tau) = -1$.

With the fractional Taylor series [11], we can determine the trial function or the initial value $u_0(x, t) = u_0(x, 0) = f(x) = x^2$ in the iteration formulation as follows:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^{\alpha} u_n(x,\tau)}{\partial \tau^{\alpha}} - \frac{\partial^2 u_n(x,t)}{\partial x^2} - \frac{\partial(xu_n(x,t))}{\partial x} \right\} (\mathrm{d}\tau)^{\alpha}.$$

We can derive

$$u_1(x,t) = x^2 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left\{\frac{\partial^\alpha u_0(x,\tau)}{\partial\tau^\alpha} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2} - \frac{\partial(xu_0(x,\tau))}{\partial x}\right\}(\mathrm{d}\tau)^\alpha$$
$$= x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)},$$
$$u_2(x,t) = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$u_3(x,t) = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(26+27x^2)t^{3\alpha}}{\Gamma(1+3\alpha)}.$$

As a result, the exact solution can be given in a compact form:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{k^i t^{i\alpha}}{\Gamma(1+i\alpha)} = E_\alpha(kt^\alpha), \quad (11)$$

where $k^i = x^2 + (1+x^2)(3^i - 1)$ and $E_\alpha(kt^\alpha)$ is the Mittag-Leffler function. We note that Eq. (12) is also the exact solution of the fractional diffusion equation [5] if we take $\alpha = \frac{1}{2}$.

Example 2. In order to illustrate the FVIM for higher fractional order equations, we only consider an initial value problem given in [12]:

$$y^{(2\alpha)} = y^2 + 1, \quad 0 < \alpha \le 1,\ 0 \le x \le 1, \quad (12)$$

with $y(0) = 0$ and $y^{(\alpha)}(0) = 1$, where $y^{(2\alpha)} = D_x^\alpha D_x^\alpha y$.

Construct the following functional:

$$y_{n+1}(x,t) = y_n(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \lambda\{y_n^{(2\alpha)} - \tilde{y}_n^2(\xi) - 1\}(\mathrm{d}\xi)^\alpha.$$

We have

$$\delta y_{n+1} = \delta y_n + \frac{1}{\Gamma(1+\alpha)}\delta\int_0^x \lambda(y_n^{(2\alpha)} - y_n^2(\xi) - 1)(\mathrm{d}\xi)^\alpha$$
$$= \delta y_n + \lambda\delta y_n^{(\alpha)}|_{\xi=x} - \lambda^{(\alpha)}(\tau)\delta y_n(\tau)|_{\xi=x} + \frac{1}{\Gamma(1+\alpha)}\int_0^x \lambda^{(2\alpha)}\delta y_n(\mathrm{d}\xi)^\alpha.$$

Similarly, we can set the coefficients of $\delta u_n(\tau)$ and $\delta y_n^{(\alpha)}$ to zero:

$$1 - \lambda^{(\alpha)}|_{\xi=x} = 0, \qquad \lambda^{(2\alpha)} = 0 \quad \text{and} \quad \lambda|_{\xi=x} = 0.$$

As a result, the Lagrange multiplier can be identified as

$$\lambda = \frac{(\xi - x)^\alpha}{\Gamma(1+\alpha)}. \quad (13)$$

By a similar manipulation, we can derive a more generalized Lagrange multiplier:

$$\lambda = (-1)^{(m)}\frac{(\xi - x)^{(m-1)\alpha}}{\Gamma(1+(m-1)\alpha)},$$

for higher fractional nonlinear ordinary differential equations:

$$y^{(m\alpha)} = N(y), \quad 0 < \alpha \le 1.$$

For Eq. (13), we can check when $\alpha = 1$ and $\lambda = \xi - x$ is the multiplier for the differential equation of integer order

$$y^{('')} = y^2 + 1. \quad (14)$$

The iteration formulation for Eq. (12) can be rewritten as

$$y_{n+1}(x) = y_n(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(\xi - x)^\alpha}{\Gamma(1+\alpha)}\{y_n^{(2\alpha)} - y_n^2 - 1\}(\mathrm{d}\xi)^\alpha.$$

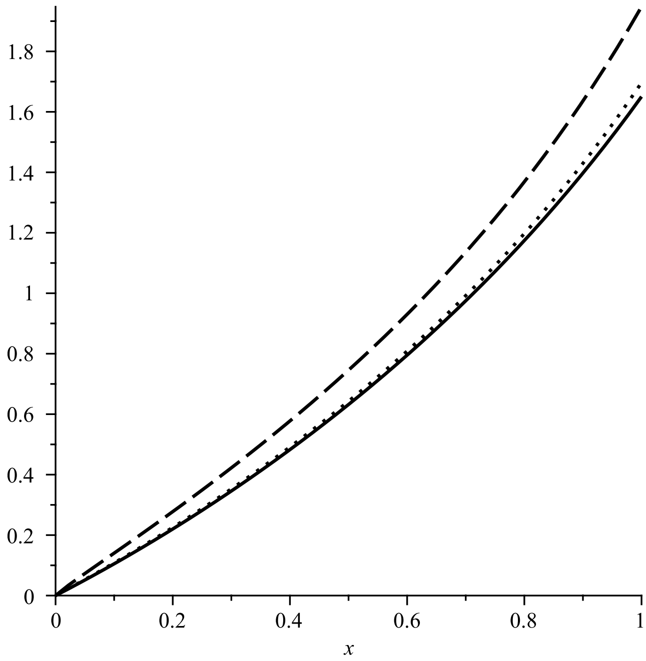

Fig. 1. The second-order approximate solution vs. the exact solution when $\alpha = 1$. The discontinuous line (--) is the approximate solution when $\alpha = 0.9$ and the dotted line (...) is that when $\alpha = 0.99$. The continuous line is the exact solution when $\alpha = 1$. However, the figure only shows the trend of the approximate solutions on a large scale, which cannot illustrate the local behavior of fractional differential equations.

Taking the initial value

$$y_0 = y(0) + \frac{y^{(\alpha)}(0)x^\alpha}{\Gamma(1+\alpha)} = \frac{x^\alpha}{\Gamma(1+\alpha)},$$

we can obtain

$$\begin{aligned}
y_1(x) &= y_0(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\{y_0^{(2\alpha)} - y_0^2 - 1\}(\mathrm{d}\xi)^\alpha \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+2\alpha)x^{4\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)}, \\
y_2(x) &= y_1(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\{y_1^{(2\alpha)} - y_1^2 - 1\}(\mathrm{d}\xi)^\alpha \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+2\alpha)x^{4\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)} + \frac{2\Gamma(1+3\alpha)x^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+5\alpha)} \\
&\quad + \frac{\Gamma(1+4\alpha)x^{6\alpha}}{\Gamma^2(1+2\alpha)\Gamma(1+6\alpha)} + \frac{2\Gamma(1+5\alpha)\Gamma(1+2\alpha)x^{7\alpha}}{\Gamma^3(1+\alpha)\Gamma(1+4\alpha)\Gamma(1+7\alpha)} + \frac{2\Gamma(1+6\alpha)x^{8\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)\Gamma(1+8\alpha)} \\
&\quad + \frac{\Gamma(1+8\alpha)\Gamma^2(1+2\alpha)x^{10\alpha}}{\Gamma^4(1+\alpha)\Gamma^2(1+4\alpha)\Gamma(1+10\alpha)}.
\end{aligned}$$

If the second-order approximation, $y_2(x)$, is sufficient, we can compare the solution with the exact solutions for the case $\alpha = 1$ as in Fig. 1.

4. Conclusions

In this paper, a fractional variational iteration method is proposed, and proved to be an efficient tool for solving fractional differential equations because the Lagrange multiplier can be identified in a more accurate way using the fractional variational theory. Some other recent work in calculation of variation can be found in Refs. [13–15].

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