Some new oscillation theorems for second order nonlinear elliptic equations with damping

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Abstract

In this paper, some new oscillation criteria are obtained for second order elliptic differential equations with damping of the form
\[ \nabla \cdot (A(x) \nabla y) + B^T(x) \nabla y + q(x)f(y) = 0, \quad x \in \Omega, \]
where \( \Omega \) is an exterior domain in \( \mathbb{R}^N \). These criteria are different from most known ones in the sense that they are based on a class of new functions of the form \( \Phi(r, s, l) \) defined in the sequel.

Keywords: Nonlinear elliptic differential equation; Second order; Generalized partial Riccati transformation; Oscillation; Annulus criteria

1. Introduction

In this paper, we consider the oscillation behavior of solutions to second order elliptic differential equations with damping of the form
\[ \nabla \cdot (A(x) \nabla y) + B^T(x) \nabla y + q(x)f(y) = 0, \quad (1) \]
where \( x \in \Omega \), an exterior domain in \( \mathbb{R}^N \), \( \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N}) \). The following notations will be adopted throughout: \( \mathbb{R} \) and \( \mathbb{R}^+ \) denote the intervals \((-\infty, +\infty)\), \((0, +\infty)\), respectively. The norm of \( x \) is denoted by \(|x| = [\sum_{i=1}^{N} x_i^2]^{1/2}\). For a positive constant \( a > 0 \), let \( S_a = \{ x \in \mathbb{R}^N : |x| = a \} \), \( G(a, \infty) = \{ x \in \mathbb{R}^N : |x| > a \} \), \( G[a, b] = \{ x \in \mathbb{R}^N : a \leq |x| \leq b \} \), \( G(a, b) = \{ x \in \mathbb{R}^N : a < |x| < b \} \). For the exterior domain \( \Omega \) in \( \mathbb{R}^N \), there exists a positive number \( a_0 \) such that \( G(a_0, \infty) \subset \Omega \).

In what follows, we always assume that

\[ (C_1) \ A(x) = (A_{ij}(x))_{N \times N} \text{ is a real symmetric positive definite matrix function (ellipticity condition) with } A_{ij} \in C^{1+\mu}_0(\Omega, \mathbb{R}), \mu \in (0, 1), i, j = 1, \ldots, N, \lambda_{\max}(x) \text{ denotes the largest (necessarily positive) eigenvalue of the matrix } A(x); \text{ there exists a function } \lambda \in C^1(\mathbb{R}^+, \mathbb{R}^+) \text{ such that } \lambda(r) \geq \max_{|x|=r} \lambda_{\max}(x) \text{ for } r > 0; \]

\[ (C_2) \ B^T = (b_i(x))_{1 \times N}, b_i \in C^{1+\mu}_0(\Omega, \mathbb{R}), i = 1, \ldots, N, q \in C^{\mu}_0(\Omega, \mathbb{R}), \mu \in (0, 1) \text{ and } q(x) \neq 0 \text{ for } |x| \geq a_0; \]

\[ (C_3) \ f \in C^1(\mathbb{R}, \mathbb{R}), \ f(y) > 0 \text{ and } f'(y) \geq k > 0 \text{ for all } y \neq 0 \text{ and some constant } k. \]

A function \( y \in C^{2+\mu}_0(\Omega, \mathbb{R}), \mu \in (0, 1), \) is said to be a solution of Eq. (1) in \( \Omega \), if \( y(x) \) satisfies Eq. (1) for all \( x \in \Omega \). For the existence of solutions of Eq. (1), we refer the reader to the monograph [1]. We restrict our attention only to the nontrivial solution \( y(x) \) of Eq. (1), i.e., for any \( a > a_0 \), \( \sup\{|y(x)| : |x| > a\} > 0 \). A nontrivial solution \( y(x) \) of Eq. (1) is called oscillatory if the zero set \( \{ x : y(x) = 0 \} \) of \( y(x) \) is unbounded, otherwise it is called nonoscillatory. Eq. (1) is called oscillatory if all its nontrivial solutions are oscillatory.

There are a great number of papers (see, for example, [2–6,13] and references quoted therein) devoted to the particular cases of Eq. (1), including the following second order ordinary differential equations:

\[ y''(t) + q(t)y(t) = 0, \quad (2) \]

\[ (r(t)y')' + q(t)f(y) = 0, \quad (3) \]

\[ (r(t)y')' + p(t)y' + q(t)f(y) = 0. \quad (4) \]

An important tool in the study of oscillatory behavior of solutions for Eq. (2) is the averaging technique. Here we list some known oscillation criteria for Eq. (2):

\[ \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v) \, dv \, ds = \infty \quad (\text{Wintner [2]}), \quad (5) \]

\[ \liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau) \, d\tau \, ds < \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau) \, d\tau \, ds \leq \infty \quad (\text{Hartman [5]}), \quad (6) \]

\[ \limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t - s)^{m-1} q(s) \, ds = \infty \quad \text{for } m > 2 \quad (\text{Kamenev [6]}). \quad (7) \]

In 1999, Kong [13] proved that
Theorem A. Eq. (2) is oscillatory provided that for each \( l \geq t_0 \), there exists \( \alpha > 1 \) such that

\[
\limsup_{t \to \infty} \frac{1}{t^{\alpha-1}} \int_l^t (s-l)^\alpha q(s) \, ds > \frac{\alpha^2}{4(\alpha-1)}
\]

and

\[
\limsup_{t \to \infty} \frac{1}{t^{\alpha-1}} \int_l^t (t-s)^\alpha q(s) \, ds > \frac{\alpha^2}{4(\alpha-1)}.
\]

Some other oscillatory criteria can be found in [7–9] and references therein. In 2000, Li and Agarwal [15] extended the main results of Kong [13] to Eq. (4) and obtained the following result.

Theorem B. Assume that \( \lim_{t \to \infty} R(t) = \infty \), where \( R(t) = \int_l^t ds/r(s) \) for \( t \geq l \geq t_0 \). Then Eq. (4) is oscillatory provided that for each \( l \geq t_0 \) and for some \( \lambda > 1 \),

\[
\limsup_{t \to \infty} \frac{1}{R(t)^{\lambda-1}} \int_l^t \left\{ \left(q(s) - \frac{p^2(s)}{4kr(s)}\right)[R(t) - R(s)]^\lambda + \frac{p(s)}{2kr(s)} [R(t) - R(s)]^{\lambda-1} \right\} ds > \frac{\lambda^2}{4k(\lambda-1)}
\]

and

\[
\limsup_{t \to \infty} \frac{1}{R(t)^{\lambda-1}} \int_l^t \left\{ \left(q(s) - \frac{p^2(s)}{4kr(s)}\right)[R(s) - R(l)]^\lambda + \frac{p(s)}{2kr(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds > \frac{\lambda^2}{4k(\lambda-1)}.
\]

We see that most oscillation results involve the function class \( X \). We say that a function \( H = H(t, s) \) belongs to the function class \( X \), denoted by \( H \in X \), if \( H \in C(D, \mathbb{R}^+) \), where \( D = \{(t, s): t_0 \leq s \leq t < \infty \} \), which satisfies \( H(t, t) = 0 \), \( H(t, s) > 0 \) for \( t > s \), and has partial derivatives \( \partial H / \partial t \) and \( \partial H / \partial s \) on \( D \) such that

\[
\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)},
\]

where \( h_1, h_2 \in L_{loc}(D, \mathbb{R}) \).

Recently, Sun [16] establish some new Kamenev-type oscillation criteria for Eq. (4) by defining a new function class \( Y \): Let \( E = \{(t, s, l): a_0 \leq l \leq s \leq t < \infty \} \), a function \( \Phi = \Phi(t, s, l) \) is said to belong to \( Y \), if \( \Phi \in C(E, \mathbb{R}) \) and satisfies \( \Phi(t, t, l) = 0, \Phi(t, l, l) = 0, \Phi(t, s, l) \neq 0 \) for \( l < s < \infty \), and has the partial derivative \( \partial \Phi / \partial s \) on \( D \) such that \( \partial \Phi / \partial s \) is locally integrable with respect to \( s \) in \( E \). One of the main results in [16] is as follows.

Theorem C. Eq. (4) with \( r(t) \equiv 1 \) is oscillatory provided that for each \( l \geq t_0 \), there exists a constant \( \alpha > 1/2 \) such that
\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha+1}} \int_{l}^{t} (t-s)^{2\alpha} (s-l)^2 \left[ 4kq(s) - p^2(s) + 4\frac{t - (1 + \alpha)s + \alpha l}{(t-s)(s-l)} p(s) \right] ds
\]
\[
> \frac{4\alpha}{(2\alpha - 1)(2\alpha + 1)}.
\]

For the semilinear elliptic equation
\[
\nabla \cdot (A(x) \nabla y) + q(x) f(y) = 0,
\]
Noussair and Swanson [10] first extended the Wintner theorem by using the following partial Riccati type transformation:
\[
W(x) = -\frac{\alpha(|x|)}{f(y(x))} (A \nabla y)(x),
\]
where \(\alpha \in C^2\) is an arbitrary positive function. Swanson [11] summarized the oscillation results for Eq. (13) up to 1979. For recent contributions, we refer the reader to Xu et al. [12] and references therein.

This paper is motivated by [10,12–16], where the generalized Riccati transformation and integral average technique are used to establish oscillation criteria for Eqs. (2), (3) and (4), respectively. In Sections 2 and 3, we establish some new oscillation criteria for Eq. (1) in terms of the above definitions. Our results are extensions of the results of Kong [13], Li and Agarwal [14,15] and Sun [16], etc. Some of them are different from most known ones in the sense that they are based on the information only on a sequence of annuluses of \(\Omega \subset \mathbb{R}^N\), rather than on the whole exterior domain \(\Omega\).

2. Kamenev-type oscillation criteria

For convenience, we let
\[
Q(r) = \int_{S_{r}} \left[ q(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \nabla \cdot B \right] d\sigma,
\]
\[
g(r) = \frac{\omega}{k} \lambda(r) r^{N-1},
\]
where \(S_{r} = \{ x \in \mathbb{R}^N : |x| = r \}, r > 0, B^T = (b_1(x), \ldots, b_N(x)), d\sigma\) denotes the spherical integral element in \(\mathbb{R}^N\), \(\omega\) is the area of unit sphere in \(\mathbb{R}^N\) and \(k\) is defined in (C3).

We will use the operator \(T[\cdot ; l, r]\) introduced by Sun [16]. Let \(\Phi \in Y\) and \(g \in C[a_0, \infty)\), define
\[
T[g; l, r] = \int_{l}^{r} \Phi^2(r, s, l) g(s) ds \quad \text{for} \ r \geq s \geq l \geq a_0,
\]
and the function \(\phi = \phi(r, s, l)\) by
\[
\frac{\partial \Phi(r, s, l)}{\partial s} = \phi(r, s, l) \Phi(r, s, l).
\]

It is easy to verify that \(T[\cdot ; l, r]\) is a linear operator and satisfies
\[
T[g'; l, r] = -2T[g\phi; l, r] \quad \text{for} \ g \in C^1[a_0, \infty).
\]
Theorem 2.1. Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exists a function \( \Phi \in Y \) such that
\[
\limsup_{r \to \infty} T\left[ Q - g\Phi^2; l, r \right] > 0. \tag{18}
\]

Proof. Suppose to the contrary that there exists a solution \( y(x) \) of Eq. (1) such that \( y(x) > 0 \) for \( |x| \geq a_1 \geq a_0 \). Define
\[
W(x) = \frac{1}{f(y)} (A \nabla y)(x) + \frac{1}{2k} B, \quad x \in G[a_1, \infty), \tag{19}
\]
and
\[
V(r) = \int_{S_r} W(x) \cdot \gamma(x) \, d\sigma, \quad x \in G[a_1, \infty), \tag{20}
\]
where \( \nabla y \) denotes the gradient of \( y(x) \), \( \gamma(x) = x/|x|, |x| \neq 0 \) is the outward unit normal to \( S_r \).

From Eqs. (1) and (19), it follows that
\[
\nabla \cdot W(x) = -\frac{f''(y)}{f^2(y)} (\nabla y)^T A \nabla y - \frac{1}{f(y)} \left[ q(x) f(y) + B^T \nabla y \right] + \frac{1}{2k} \nabla \cdot B
\leq -k \left[ W - \frac{1}{2k} B \right]^T A^{-1} \left[ W - \frac{1}{2k} B \right] - q(x)
- B^T A^{-1} \left[ W - \frac{1}{2k} B \right] + \frac{1}{2k} \nabla \cdot B
= -k W^T A^{-1} W - q(x) + \frac{1}{4k} B^T A^{-1} B + \frac{1}{2k} \nabla \cdot B. \tag{21}
\]

Using Green’s formula in (20), we get
\[
V'(r) = \int_{S_r} \nabla \cdot W(x) \, d\sigma
\leq -k \int_{S_r} W^T A^{-1} W \, d\sigma - \int_{S_r} \left[ q(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \nabla \cdot B \right] \, d\sigma. \tag{22}
\]

In view of \((C_1)\), we have \((W^T A^{-1} W)(x) \geq \lambda_{\max}^{-1}(x)|W(x)|^2\). Then, by Cauchy–Schwartz inequality, we obtain
\[
\int_{S_r} \left| W(x) \right|^2 \, d\sigma \geq \frac{r_{1-N}^{1-N}}{\omega} \left[ \int_{S_r} W(x) \cdot \gamma(x) \, d\sigma \right]^2.
\]
Moreover, by (20) and (22), we get
\[
V'(r) \leq -\frac{k r_{1-N}^{1-N}}{\omega \lambda(r)} \left[ \int_{S_r} W(x) \cdot \gamma(x) \, d\sigma \right]^2 - \int_{S_r} \left[ q(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \nabla \cdot B \right] \, d\sigma
= -Q(r) - \frac{1}{g(r)} V^2(r). \tag{23}
\]
Applying $T[\cdot; r_1, r]$ ($r > r_1$) to (23), we have
\[ T[V'; r_1, r] \leq -T\left[ \frac{1}{g} V^2; r_1, r \right] - T[Q; r_1, r]. \tag{24} \]
By (17) and (24), we have for $r \geq r_1$,
\[ T[Q; r_1, r] \leq 2T[V\phi; r_1, r] - T[\mathcal{Q}; r_1, r]. \tag{25} \]
Completing square of $V$ in (25) and using (16), we have
\[ T[Q-g\phi^2; r_1, r] \leq 0, \quad r > r_1. \]
Taking the super-limit in the above inequality, we have
\[ \limsup_{r \to \infty} T[Q-g\phi^2; r_1, r] \leq 0, \]
which contradicts the assumption (18). This completes the proof of Theorem 2.1. \qed

If we choose $\Phi(r, s, l) = \rho(s)(r-s)^{\alpha}(s-l)^{\beta}$ for $\alpha, \beta > 1/2$ and $\rho(r) \in C^1([a_0, \infty), \mathbb{R}^+)$, then
\[ \phi(r, s, l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta r - (\alpha + \beta)s + \alpha l}{(r-s)(s-l)}. \]
By Theorem 2.1, we have the following oscillation result.

**Theorem 2.2.** Eq. (1) is oscillatory provided that for each $l \geq a_0$, there exist a function $\rho(r) \in C^1([a_0, \infty), \mathbb{R}^+)$ and two constants $\alpha, \beta > 1/2$ such that
\[ \limsup_{r \to \infty} \int_{l}^{r} \rho^2(s)(r-s)^{2\alpha}(s-l)^{2\beta} \times \left[ Q(s) - g(s)\left( \frac{\rho'(s)}{\rho(s)} + \frac{\beta r - (\alpha + \beta)s + \alpha l}{(r-s)(s-l)} \right)^2 \right] ds > 0. \tag{26} \]
Define
\[ R(r) = \int_{a_0}^{r} \frac{1}{g(s)} ds, \quad r \geq a_0. \]
Let $\Phi(r, s, l) = \rho(s)\left[ R(r) - R(s)\right]^{\alpha}\left[ R(s) - R(l)\right]^{\beta}$ for $\alpha, \beta > 1/2$ and $\rho(r) \in C^1([a_0, \infty), \mathbb{R}^+)$, then
\[ \phi(r, s, l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta R(r) - (\alpha + \beta)R(s) + \alpha R(l)}{g(s)\left[ R(r) - R(s)\right]\left[ R(s) - R(l)\right]}. \]
By Theorem 2.1, we have the following oscillation result.

**Theorem 2.3.** Eq. (1) is oscillatory provided that for each $l \geq a_0$, there exist a function $\rho(r) \in C^1([a_0, \infty), \mathbb{R}^+)$ and two constants $\alpha, \beta > 1/2$ such that
\[
\limsup_{r \to \infty} \int_{l}^{r} \rho^2(s) [R(r) - R(s)]^{2\alpha} [R(s) - R(l)]^{2\beta} \\
\times \left[ Q(s) - g(s) \left( \frac{\rho'(s)}{\rho(s)} + \frac{\beta R(r) - (\alpha + \beta)R(s) + \alpha R(l)}{g(s)[R(r) - R(s)][R(s) - R(l)]} \right)^2 \right] ds > 0. \tag{27}
\]

Taking \( \beta = 1 \) and \( \rho \equiv 1 \), we have the following theorem.

**Theorem 2.4.** Assume that \( \lim_{r \to \infty} R(r) = \infty \). Then Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exists a constant \( \alpha > 1/2 \) such that

\[
\limsup_{r \to \infty} \frac{1}{R^{2\alpha+1}(r)} \int_{l}^{r} [R(r) - R(s)]^{2\alpha} [R(s) - R(l)]^{2} Q(s) ds > \frac{\alpha}{(2\alpha - 1)(2\alpha + 1)}. \tag{28}
\]

**Proof.** Note that

\[
\int_{l}^{r} [R(r) - R(s)]^{2\alpha-2} [R(r) - (\alpha + 1)R(s) + \alpha R(l)]^{2} \frac{1}{g(s)} ds \\
= \int_{l}^{r} [R(r) - R(s)]^{2\alpha-2} [(R(r) - R(s)) - \alpha(R(s) - R(l))]^{2} \frac{1}{g(s)} ds \\
= \int_{l}^{r} [R(r) - R(s)]^{2\alpha-1} \frac{1}{g(s)} ds + \alpha^2 \int_{l}^{r} [R(r) - R(s)]^{2\alpha-2} [R(s) - R(l)]^{2} \frac{1}{g(s)} ds \\
- 2\alpha \int_{l}^{r} [R(r) - R(s)]^{2\alpha-1} [R(s) - R(l)] \frac{1}{g(s)} ds \\
= \frac{\alpha}{(2\alpha - 1)(2\alpha + 1)} [R(r) - R(l)]^{2\alpha+1}. \tag{29}
\]

From (28) and (29), we have

\[
\limsup_{r \to \infty} \frac{1}{R^{2\alpha+1}(r)} \int_{l}^{r} [R(r) - R(s)]^{2\alpha} [R(s) - R(l)]^{2} \\
\times \left[ Q(s) - g(s) \left( \frac{R(r) - (\alpha + 1)R(s) + \alpha R(l)}{g(s)[R(r) - R(s)][R(s) - R(l)]} \right)^2 \right] ds \\
= \limsup_{r \to \infty} \frac{1}{R^{2\alpha+1}(r)} \int_{l}^{r} [R(r) - R(s)]^{2\alpha} [R(s) - R(l)]^{2} Q(s) ds \\
- \limsup_{r \to \infty} \frac{1}{R^{2\alpha+1}(r)} \int_{l}^{r} [R(r) - R(s)]^{2\alpha-2} [R(r) - (\alpha + 1)R(s) + \alpha R(l)]^{2} \frac{1}{g(s)} ds
\]
\[ \frac{1}{R^{2\alpha+1}(r)} \int_l^r \left[ R(r) - R(s) \right]^{2\alpha} \left[ R(s) - R(l) \right]^2 Q(s) \, ds = \limsup_{r \to \infty} \frac{\alpha \left[ R(r) - R(l) \right]^{2\alpha+1}}{(2\alpha - 1)(2\alpha + 1) R^{2\alpha+1}(r)}. \]  

(30)

From (28) and (30), we obtain

\[ \limsup_{r \to \infty} \frac{1}{R^{2\alpha+1}(r)} \int_l^r \left[ R(r) - R(s) \right]^{2\alpha} \left[ R(s) - R(l) \right]^2 Q(s) \, ds - \frac{\alpha}{(2\alpha - 1)(2\alpha + 1)} > 0. \]

Hence, Eq. (1) is oscillatory by Theorem 2.3. This completes the proof of Theorem 2.4.

Taking \( \alpha = 1 \) and \( \rho \equiv 1 \), similar to the proof of Theorem 2.4, we have the following theorem.

**Theorem 2.5.** Assume that \( \lim_{r \to \infty} R(r) = \infty \). Then Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exists a constant \( \beta > 1/2 \) such that

\[ \limsup_{r \to \infty} \frac{1}{R^{2\beta+1}(r)} \int_l^r \left[ R(r) - R(s) \right]^{2\beta} \left[ R(s) - R(l) \right]^2 \frac{Q(s)}{Q(s)} ds > \frac{\beta}{(2\beta - 1)(2\beta + 1)}. \]  

(31)

Taking \( \Phi(r,s,l) = \sqrt{H_1(s,l)H_2(r,s)} \), where \( H_1, H_2 \in X \), we have the following result by Theorem 2.1.

**Theorem 2.6.** Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exist two functions \( H_1, H_2 \in X \) such that

\[ \lim_{r \to \infty} \int_l^r H_1(s,l)H_2(r,s) \left[ Q(s) - \frac{g(s)}{4} \left( \frac{\sqrt{H_1(s,l)}}{\sqrt{H_1(s,l)}} - \frac{\sqrt{H_2(r,s)}}{\sqrt{H_2(r,s)}} \right)^2 \right] ds > 0, \]  

(32)

where \( h_1 \) and \( h_2 \) are defined as follows:

\[ \frac{\partial H_1(s,l)}{\partial s} = h_1(s,l)\sqrt{H_1(s,l)}, \quad \frac{\partial H_2(r,s)}{\partial s} = -h_2(r,s)\sqrt{H_1(r,s)}. \]

**Example 1.** Consider the second order nonlinear elliptic differential equation

\[ \frac{\partial}{\partial x_1} \left[ \alpha_1 \frac{\partial y}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ \alpha_2 \frac{\partial y}{\partial x_2} \right] + \frac{\alpha_1}{r^2} \frac{\partial y}{\partial x_1} + \frac{\alpha_2}{r^2} \frac{\partial y}{\partial x_2} + \frac{(\alpha_1 + \alpha_2 + 4m)r - 4(\alpha_1 x_1 + \alpha_2 x_2)}{4r^4} (y + y^3) = 0, \]  

(33)
where \( r = \sqrt{x_1^2 + x_2^2}, \ r \geq 1, \ N = 2, \ \alpha_1 \geq \alpha_2 > 0, \ m > 0. \) It is easy to see that
\[
\lambda(r) = \frac{\alpha_1}{r}, \quad q(x) = \frac{(\alpha_1 + \alpha_2 + 4m)r - 4(\alpha_1 x_1 + \alpha_2 x_2)}{4r^4}, \quad f(y) = y + y^3.
\]
Hence
\[
f'(y) = 1 + 3y^2 \geq 1 = k, \quad \omega = 2\pi, \quad g(r) = 2\pi \alpha_1,
\]
\[
Q(r) = \frac{2\pi m}{r^2}, \quad R(r) = \frac{r - 1}{2\pi \alpha_1}.
\]
Then, for any \( b \geq 1, \)
\[
\lim_{r \to \infty} \frac{1}{R^2 + 1}(r) \int_b^r [R(r) - R(s)]^2 [R(s) - R(l)]^2 \beta Q(s) \ ds = \lim_{r \to \infty} \frac{f(r)^2}{(2\beta + 1)R^2} \int_b^r [R(r) - R(s)]^2 [R(s) - R(l)]^2 \beta Q(s) \ ds
\]
\[
= \lim_{r \to \infty} \frac{2 \int_b^r [R(r) - R(s)]^2 [R(s) - R(l)]^2 \beta Q(s) \ ds}{(2\beta + 1)R^2} - \lim_{r \to \infty} \frac{2 \int_b^r R(s)[R(s) - R(l)]^2 \beta Q(s) \ ds}{(2\beta + 1)R^2}\]
\[
= \lim_{r \to \infty} \frac{2 \int_b^r [R(r) - R(s)]^2 [R(s) - R(l)]^2 \beta Q(s) \ ds}{(2\beta + 1)(2\beta - 1)R^2} - \lim_{r \to \infty} \frac{2 \int_b^r R(s)[R(r) - R(l)]^2 \beta Q(s) \ ds}{(2\beta + 1)(2\beta - 1)R^2}
\]
\[
= \frac{2m}{(2\beta + 1)(2\beta - 1)\alpha_1} - \frac{m}{(2\beta + 1)\beta \alpha_1} = \frac{m}{(2\beta + 1)(2\beta - 1)\beta \alpha_1}.
\]
(34)

We can choose an appropriate constant \( \beta > 1/2 \) such that \( m/\alpha_1 > \beta^2 \), and hence
\[
\frac{m}{(2\beta + 1)(2\beta - 1)\beta \alpha_1} > \frac{\beta}{(2\beta - 1)(2\beta + 1)}, \quad \text{(35)}
\]
which implies that (31) holds. By Theorem 2.5, Eq. (33) is oscillatory when \( m/\alpha_1 > \beta^2 \) and \( \beta > 1/2 \).

3. Annulus criteria

We see that Theorems 2.1–2.6 and other results in Xu et al. [12] involve the integral of the coefficients \( A_{ij}, b_i, \) and \( q \), and hence require the information of the coefficients on the entire exterior domain \( \Omega \). It is difficult to apply them to the cases when Eq. (1) is “bad” on a big part of \( \Omega \). However, from the Sturm Separation Theorem, we see that oscillation for Eq. (2) is only an interval property, i.e., if there exists a sequence of subintervals \([a_i, b_i]\) of \([t_0, \infty)\), \( a_i \to \infty \), such that for each \( i \) there exists a solution of Eq. (2) that has at least two zeros in \([a_i, b_i]\), then every solution of Eq. (2) is oscillatory, no matter how “bad” Eq. (2) (or \( q \)) is on the remaining parts of \([t_0, \infty)\). This should motivate further study of the annulus property for Eq. (1).

In the following, we will establish some annulus criteria for Eq. (1). The results are different from most known ones in the sense that they are based on the information only on a sequence of annuluses of \( \Omega \subset \mathbb{R}^N \), rather than on the whole exterior domain \( \Omega \).
Theorem 3.1. Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exist a function \( \Phi \in Y \) and two constants \( b > a \geq l \) such that
\[
T \left[ Q - g\phi^2; a, b \right] > 0,
\]
where the operator \( T \) is defined by (15) and the function \( \phi = \phi(b, s, a) \) is defined by (16).

Proof. Following the proof of Theorem 2.1 with \( r \) and \( l \) replaced by \( b \) and \( a \), respectively, we can easily see that every solution of Eq. (1) has at least one zero in annulus domain \( \Omega(a, b) \), i.e., every solution of Eq. (1) has arbitrarily large zero on \( \Omega(a_0) \). This completes the proof of Theorem 3.1. \( \square \)

Similar to the discussions in Section 2, we have the following corollaries.

Corollary 3.1. Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exist a function \( \rho(r) \in C^1([a_0, \infty), \mathbb{R}^+) \) and constants \( \alpha, \beta > 1/2, b > a \geq l \) such that
\[
\int_a^b \rho^2(s) \left[ R(b) - R(s) \right]^{2\alpha} \left[ R(s) - R(a) \right]^{2\beta} \left[ Q(s) - g(s) \left( \frac{\rho'(s)}{\rho(s)} + \frac{\beta R(b) - (\alpha + \beta) R(s) + \alpha R(a)}{g(s)[R(b) - R(s)][R(s) - R(a)]} \right)^2 \right] ds > 0.
\]

Corollary 3.2. Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exist a function \( \rho(r) \in C^1([a_0, \infty), \mathbb{R}^+) \) and constants \( \alpha, \beta > 1/2, b > a \geq l \) such that
\[
\int_a^b \rho^2(s) \left[ R(b) - R(s) \right]^{2\alpha} \left[ R(s) - R(a) \right]^{2\beta} \left[ Q(s) - g(s) \left( \frac{\rho'(s)}{\rho(s)} + \frac{\beta R(b) - (\alpha + \beta) R(s) + \alpha R(a)}{g(s)[R(b) - R(s)][R(s) - R(a)]} \right)^2 \right] ds > 0.
\]

Corollary 3.3. Eq. (1) is oscillatory provided that for each \( l \geq a_0 \), there exist two functions \( H_1, H_2 \in X \) and two constants \( b > a \geq l \) such that
\[
\int_a^b H_1(s, a) H_2(b, s) \left[ Q(s) - g(s) \left( \frac{h_1(s, a)}{\sqrt{H_1(s, a)}} - \frac{h_2(b, s)}{\sqrt{H_2(b, s)}} \right)^2 \right] ds > 0.
\]

Example 2. Consider the second order nonlinear elliptic differential equation
\[
\frac{\partial}{\partial x_1} \left[ \frac{\alpha}{r} \frac{\partial y}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ \frac{\alpha}{r} \frac{\partial y}{\partial x_2} \right] + \cos r \frac{\partial y}{\partial x_1} + \cos r \frac{\partial y}{\partial x_2} + \frac{(r - x_1 - x_2) \sin r + \alpha r^2 \cos^2 r}{2r} (y + y^3) = 0,
\]
where \( r = \sqrt{x_1^2 + x_2^2}, r \geq 1, N = 2, \alpha > 0 \). It is easy to see that
\[
\lambda(r) = \frac{\alpha}{r}, \quad q(x) = \frac{(r - x_1 - x_2) \sin r + \alpha r^2 \cos^2 r}{2r}, \quad f(y) = y + y^3.
\]
Hence

\[ f'(y) = 1 + 3y^2 \geq 1 = k, \quad \omega = 2\pi, \quad g(r) = 2\pi\alpha, \quad Q(r) = 2\pi \sin r. \]

Then, for any \( l \geq 1 \), there exists \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) such that \( 2n\pi \geq l \). Letting \( a = 2n\pi \), \( b = (2n + 1)\pi \), and choosing \( \Phi(r, s, l) = \sqrt{|\sin(r-s)\sin(s-l)|} \), we have \( \Phi(b, s, a) = \sin s \), \( \Phi^2(b, s, a)\Phi^2(b, s, a) = \cos s \), and

\[
T\left[Q - g\Phi^2; a, b\right] = \int_{a}^{b} \Phi^2(b, s, a)\left[Q - g\Phi^2\right] ds
\]

\[
= \int_{0}^{\pi} \left[\sin^2 s 2\pi \sin s - 2\pi\alpha \cos^2 s\right] ds
\]

\[
= 2\pi \left(\frac{4}{3} - \alpha\right).
\]

From Theorem 3.1, we see that Eq. (40) is oscillatory for \( \alpha < 4/3 \).

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References