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Yetter–Drinfeld modules and projections of weak Hopf algebras

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Abstract

In this paper we prove that if $g: B \to H$ is a morphism of weak Hopf algebras which is split as an algebra–coalgebra morphism, then the subalgebra of coinvariants B_H of B is a Hopf algebra in the category of Yetter–Drinfeld modules associated to H.

Keywords: Yetter–Drinfeld module; Weak Hopf algebra

Introduction

Weak Hopf algebras or quantum groupoids have been proposed by Böhm, Nill and Szlachányi in [4,5], as a new generalization of Hopf algebras. Roughly speaking, a weak Hopf algebra H, in a strict symmetric monoidal category with split idempotents $\mathcal{C}$, is an object that has both algebra and coalgebra structures with some relations between them and that possesses an antipode λ_H which does not necessarily verify $\lambda_H \wedge id_H = id_H \wedge \lambda_H = \varepsilon_H \otimes \eta_H$ where ε_H, η_H

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are the counity and unity morphisms respectively and $\wedge$ denotes the convolution product. The main difference with other Hopf algebraic constructions such as Hopf algebras or quasi-Hopf algebras is the following: weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity η_H or, equivalently, the counity is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras.

Let H be a Hopf algebra over a field K and let A be a K-algebra. A well-known result of Radford [15] gives equivalent conditions for an object $A \otimes H$ equipped with smash product algebra and coalgebra to be a Hopf algebra and characterizes such objects via bialgebra projections. Majid in [11] interpreted this result in the modern context of Yetter–Drinfeld modules and stated that there is a correspondence between Hopf algebras in this category, denoted by ${}^H_H\mathcal{YD}$, and Hopf algebras B with morphisms of Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$. Later, Bespalov proved the same result for braided categories with split idempotents in [3]. The key point in Radford–Majid–Bespalov's theorem is to define an object B_H, called the algebra of coinvariants, as the equalizer of $(B \otimes g) \circ \delta_B$ and $B \otimes \eta_H$. This object is a Hopf algebra in the category ${}^H_H\mathcal{YD}$ and there exists a Hopf algebra isomorphism between B and $B_H \bowtie H$ (the smash (co)product of B_H and H). It is important to point out that in the construction of $B_H \bowtie H$ they use that B_H is the image of the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (f \circ \lambda_H \circ g)) \circ \delta_B$.

In [8], Bulacu and Nauwelaerts generalize Radford's theorem about Hopf algebras with projection to the quasi-Hopf algebra setting. Namely, if H and B are quasi-Hopf algebras with bijective antipode and with morphisms of quasi-Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$, then they define a subalgebra B^i (the generalization of B_H to this setting) and with some additional structures B^i becomes, a Hopf algebra in the category of left–left Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$ defined by Majid in [12]. Moreover, as the main result in [8], Bulacu and Nauwelaerts state that $B^i \times H$ is isomorphic to B as quasi-Hopf algebras where the algebra structure of $B^i \times H$ is the smash product defined in [7] and the quasi-coalgebra structure is the one introduced in [8].

The basic motivation of [1] is to explain in detail how the above ideas can be generalized to weak Hopf algebras in a strict symmetric monoidal category with split idempotents. In [1], the authors construct the algebra of coinvariants B_H, associated to a weak Hopf algebra projection (i.e., a pair of morphisms of weak Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$) and, using the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (f \circ \lambda_H \circ g)) \circ \delta_B : B \to B$ (factorized as $q_H^B = i_H^B \circ p_H^B$), they prove that B_H is also a coalgebra in $\mathcal{C}$. In this setting it is also possible to define morphisms $\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B) : H \otimes B_H \to B_H$ and $\varrho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B : B_H \to H \otimes B_H$ such that (B_H, φ_{B_H}) is a left H-module, (B_H, ϱ_{B_H}) is a left H-comodule and to prove that B_H is an object in the category of weak Yetter–Drinfeld modules defined in [1] and denoted by ${}^H_H\mathcal{WYD}$. The algebra–coalgebra B_H satisfies similar conditions to the ones included in the definition of weak Hopf algebra but changing the natural symmetry isomorphism of $\mathcal{C}$ by $t_{B_H,B_H} = (\varphi_{B_H} \otimes B_H) \circ (H \otimes c_{B_H,B_H}) \circ (\varrho_{B_H} \otimes B_H) : B_H \otimes B_H \to B_H \otimes B_H$. Finally, in Theorem 4.1 of [2] we prove that B is isomorphic to the image, denoted by $B_H \times H$, of an idempotent morphism $\nabla_{B_H \otimes H} : B_H \otimes H \to B_H \otimes H$ as weak Hopf algebras, being the (co)algebra structure in $B_H \times H$ the smash (co)product.

The aim of the present paper is to improve and to complete the results related in the previous paragraph. Firstly, when the antipode of H is an isomorphism, we find a condition relating the category ${}^H_H\mathcal{WYD}$ to the category of Yetter–Drinfeld modules defined by Böhm in [6] and denoted by ${}^H_H\mathcal{YD}$. This category is a subcategory of ${}^H_H\mathcal{WYD}$ and it is braided monoidal but not strict because the tensor product $M \times N$ for two objects M and N in ${}_H^H\mathcal{YD}$ is defined as the image of an idempotent morphism $\nabla_{M\otimes N} : M \otimes N \to M \otimes N$. Secondly, we prove the main result of this paper, this is, for a weak Hopf algebra projection the object B_H is a Hopf algebra in ${}_H^H\mathcal{YD}$. Finally, using the weak smash product and the weak smash coproduct defined in [2], we give a good weak Hopf algebra interpretation of well-known theorems proved by Radford [15], Majid [11] and others (see for example [3]), in the Hopf algebra setting, that provides a correspondence between Hopf algebra projections and Hopf algebras in the category of Yetter–Drinfeld modules.

1. Weak Hopf algebras in monoidal categories

In this section, we review the basics of weak Hopf algebras. We denote a braided monoidal category $\mathcal{C}$ as $(\mathcal{C}, \otimes, K, a, l, r, c)$ where $\mathcal{C}$ is a category and $\otimes$ provides $\mathcal{C}$ with a monoidal structure with unit object K whose associator is denoted by a and whose left and right unit constraints are given by l and r. The braiding is denoted by c. If the braiding is a symmetry, the category $\mathcal{C}$ is a symmetric monoidal category and if the associator and the unit constraints are the identity morphisms, the category $\mathcal{C}$ will be named strict. It is well know that, given a monoidal category, we can construct a strict monoidal category $\mathcal{C}^{\mathrm{st}}$ which is tensor equivalent to $\mathcal{C}$ (see [10] for the details).

We denote the class of objects of a category $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $id_M : M \to M$. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Assumption 1.1. From now on $\mathcal{C}$ denotes a strict symmetric monoidal category that admits split idempotents, i.e. for every morphism $q : Y \to Y$ such that $q = q \circ q$ there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B is a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

By weak Hopf algebras we understand the objects introduced in [4], as a generalization of ordinary Hopf algebras. Here we recall the definition of these objects.

Definition 1.2. A weak Hopf algebra H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H).$

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H).$

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) verifying:
 (a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
 (a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
 (a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Note that, in this definition, the conditions (a2), (a3) weaken the conditions of multiplicativity of the counit, and comultiplicativity of the unit that we can find in the Hopf algebra definition. On the other hand, axioms (a4-1), (a4-2) and (a4-3) weaken the properties of the antipode in a Hopf algebra. Therefore, a weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

1.3. If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}; \qquad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H; \tag{1}$$

$$\lambda_H \circ \eta_H = \eta_H; \qquad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2}$$

If we define the morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \circ \big((\delta_H \circ \eta_H) \otimes H\big);$$
$$\Pi_H^R = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ (c_{H,H} \otimes H) \circ \big(H \otimes (\delta_H \circ \eta_H)\big);$$
$$\overline{\Pi}_H^L = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ \big((\delta_H \circ \eta_H) \otimes H\big);$$
$$\overline{\Pi}_H^R = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ \big(H \otimes (\delta_H \circ \eta_H)\big)$$

it is straightforward to show (see [4]) that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H; \qquad \Pi_H^R = \lambda_H \wedge id_H. \tag{3}$$

Moreover, we have that

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L; \qquad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R; \qquad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L; \qquad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R; \tag{4}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L; \qquad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R; \qquad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L; \qquad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{5}$$

Also it is easy to show the formulas

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L; \qquad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R; \tag{6}$$

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R; \qquad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L. \tag{7}$$

If λ_H is an isomorphism (for example, when H is finite), we have the equalities:

$$\overline{\Pi}_H^L = \mu_H \circ \big(H \otimes \lambda_H^{-1}\big) \circ c_{H,H} \circ \delta_H; \qquad \overline{\Pi}_H^R = \mu_H \circ \big(\lambda_H^{-1} \otimes H\big) \circ c_{H,H} \circ \delta_H. \tag{8}$$

A morphism between weak Hopf algebras H and B is a morphism $f : H \to B$ which is both algebra and coalgebra morphism. If $f : H \to B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see 1.4 of [1]).

1.4. Let H be a weak Hopf algebra. We say that (M, φ_M) is a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_H \otimes M) = id_M$, $\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M)$. Given two left H-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-modules if $\varphi_N \circ (H \otimes f) = f \circ \varphi_M$. We denote the category of right H-modules by ${}_H\mathcal{C}$. In an analogous way we define the category of right H-modules and we denote it by $\mathcal{C}_H$.

If (M, φ_M) and (N, φ_N) are left H-modules we denote by $\varphi_{M\otimes N}$ the morphism $\varphi_{M\otimes N} : H \otimes M \otimes N \to M \otimes N$ defined by

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

We say that (M, ϱ_M) is a left H-comodule if M is an object in $\mathcal{C}$ and $\varrho_M : M \to H \otimes M$ is a morphism in $\mathcal{C}$ satisfying $(\varepsilon_H \otimes M) \circ \varrho_M = id_M$, $(H \otimes \varrho_M) \circ \varrho_M = (\delta_H \otimes M) \circ \varrho_M$. Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if $\varrho_N \circ f = (H \otimes f) \circ \varrho_M$. We denote the category of left H-comodules by ${}^H\mathcal{C}$. Analogously, $\mathcal{C}^H$ denotes the category of right H-comodules.

For two left H-comodules (M, ϱ_M) and (N, ϱ_N), we denote by $\varrho_{M\otimes N}$ the morphism $\varrho_{M\otimes N} : M \otimes N \to H \otimes M \otimes N$ defined by

$$\varrho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

1.5. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. The morphism

$$q_H^B = id_B \wedge (f \circ \lambda_H \circ g) : B \to B$$

is an idempotent in $\mathcal{C}$. As a consequence, we obtain that there exist an epimorphism p_H^B, a monomorphism i_H^B and an object B_H such that $p_H^B \circ i_H^B = id_{B_H}$ and $i_H^B \circ p_H^B = q_H^B$. Note that, if $H = B$ and $f = g = id_H$ we have $q_H^H = \Pi_H^L$ and in this case we denote H_H by H_L, p_H^H by p_L and i_H^H by i_L. Also,

$$B_H \xrightarrow{i_H^B} B \underset{(B\otimes(\Pi_H^L\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B \otimes H, \tag{9}$$

$$B_H \xrightarrow{i_H^B} B \underset{(B\otimes(\overline{\Pi}_H^R\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B \otimes H \tag{10}$$

are equalizer diagrams and

$$B \otimes H \overset{\mu_B \circ (B \otimes f)}{\underset{\mu_B \circ (B \otimes (f \circ \Pi_H^L))}{\rightrightarrows}} B \xrightarrow{p_H^B} B_H, \tag{11}$$

$$B \otimes H \overset{\mu_B \circ (B \otimes f)}{\underset{\mu_B \circ (B \otimes (f \circ \overline{\Pi}_H^L))}{\rightrightarrows}} B \xrightarrow{p_H^B} B_H \tag{12}$$

are coequalizer diagrams (see Propositions 2.1 and 2.2 and Remark 2.3 of [1] for more details). As a consequence, we have:

$$p_H^B \circ \mu_B \circ \left(B \otimes q_H^B\right) = p_H^B \circ \mu_B; \qquad \left(B \otimes q_H^B\right) \circ \delta_B \circ i_H^B = \delta_B \circ i_H^B. \tag{13}$$

It was shown in [1] that $(B_H, \eta_{B_H} = p_H^B \circ \eta_B, \mu_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B))$ is an algebra in $\mathcal{C}$ and $(B_H, \varepsilon_{B_H} = \varepsilon_B \circ i_H^B, \delta_{B_H} = (p_H^B \otimes p_H^B) \circ \delta_B \circ i_H^B)$ is a coalgebra in $\mathcal{C}$. Also, the pair $(B_H, \varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B))$ is a left H-module in $\mathcal{C}$ and $(B_H, \varrho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B)$ is a left H-comodule in $\mathcal{C}$. Moreover, the morphisms φ_{B_H} and ϱ_{B_H} satisfy, respectively, the following equalities (see Proposition 2.4 and Section 3 of [2]):

$$\varphi_{B_H} \circ (H \otimes \eta_{B_H}) = \varphi_{B_H} \circ \left(\Pi_H^L \otimes \eta_{B_H}\right); \tag{14}$$

$$\mu_{B_H} \circ (\varphi_{B_H} \otimes B_H) \circ (H \otimes \eta_{B_H} \otimes B_H) = \varphi_{B_H} \circ \left(\Pi_H^L \otimes B_H\right); \tag{15}$$

$$\varphi_{B_H} \circ (H \otimes \mu_{B_H}) = \mu_{B_H} \circ \varphi_{B_H \otimes B_H}; \tag{16}$$

$$\mu_{B_H} \circ c_{B_H, B_H} \circ \left(\left(\varphi_{B_H} \circ (H \otimes \eta_{B_H})\right) \otimes B_H\right) = \varphi_{B_H} \circ \left(\overline{\Pi}_H^L \otimes B_H\right); \tag{17}$$

$$(H \otimes \varepsilon_{B_H}) \circ \varrho_{B_H} = \left(\Pi_H^L \otimes \varepsilon_{B_H}\right) \circ \varrho_{B_H}; \tag{18}$$

$$(H \otimes \varepsilon_{B_H} \otimes B_H) \circ (\varrho_{B_H} \otimes B_H) \circ \delta_{B_H} = \left(\Pi_H^L \otimes B_H\right) \circ \varrho_{B_H}; \tag{19}$$

$$(H \otimes \delta_{B_H}) \circ \varrho_{B_H} = \varrho_{B_H \otimes B_H} \circ \delta_{B_H}; \tag{20}$$

$$\left(\left((H \otimes \varepsilon_{B_H}) \circ \varrho_{B_H}\right) \otimes B_H\right) \circ c_{B_H, B_H} \circ \delta_{B_H} = \left(\overline{\Pi}_H^L \otimes B_H\right) \circ \varrho_{B_H}. \tag{21}$$

1.6. Let H be a weak Hopf algebra. Let (M, φ_M), (N, φ_N) be left H-modules. Then the morphism

$$\nabla_{M \otimes N} = \varphi_{M \otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$$

is idempotent. In this setting we denote by $M \times N$ the image of $\nabla_{M \otimes N}$ and by $p_{M,N} : M \otimes N \to M \times N$, $i_{M,N} : M \times N \to M \otimes N$ the morphisms such that $i_{M,N} \circ p_{M,N} = \nabla_{M \otimes N}$ and $p_{M,N} \circ i_{M,N} = id_{M \times N}$. Using the definition of $\times$ we obtain that the object $M \times N$ is a left H-module with action $\varphi_{M \times N} = p_{M,N} \circ \varphi_{M \otimes N} \circ (H \otimes i_{M,N}) : H \otimes (M \times N) \to M \times N$ (see [14]).

Lemma 1.7. *Let H be a weak Hopf algebra. Let (M, φ_M), (N, φ_N), (P, φ_P) be left H-modules. Then the following equalities hold*:

$$\varphi_{M \otimes N} \circ (H \otimes \nabla_{M \otimes N}) = \varphi_{M \otimes N}; \tag{22}$$

$$\nabla_{M \otimes N} \circ \varphi_{M \otimes N} = \varphi_{M \otimes N}; \tag{23}$$

$$(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) = (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}); \tag{24}$$

$$\begin{aligned}(M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}) &= (\nabla_{M\otimes N} \otimes P) \circ (M \otimes \nabla_{N\otimes P}) \\ &= (M \otimes \nabla_{N\otimes P}) \circ (\nabla_{M\otimes N} \otimes P).\end{aligned} \tag{25}$$

Proof. The first formula is a consequence of the following computations:

$$\begin{aligned}&\varphi_{M\otimes N} \circ (H \otimes \nabla_{M\otimes N}) \\ &\quad= (\varphi_M \otimes \varphi_N) \circ (\mu_H \otimes M \otimes \mu_H \otimes M) \circ (H \otimes H \otimes c_{H,M} \otimes H \otimes N) \\ &\qquad \circ (H \otimes c_{H,H} \otimes c_{H,M} \otimes N) \circ \big(\delta_H \otimes (\delta_H \circ \eta_H) \otimes M \otimes N\big) \\ &\quad= (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\mu_{H\otimes H} \otimes M \otimes N) \circ \big(\delta_H \otimes (\delta_H \circ \eta_H) \otimes M \otimes N\big) \\ &\quad= \varphi_{M\otimes N} \circ \big(\big(\mu_H \circ (H \otimes \eta_H)\big) \otimes M \otimes N\big) \\ &\quad= \varphi_{M\otimes N}.\end{aligned}$$

Note that the first equality follows from the naturality of the braiding and by the structure of left H-module for M and N. In the second one we use the naturality of the braiding and the third one is a consequence of (a1).

The proof of the second equality is analogous to the first one and we leave the details to the reader. The proof of (24) is the following:

$$\begin{aligned}&(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) \\ &\quad= (\nabla_{M\otimes N} \otimes P) \circ (\varphi_{M\otimes N} \otimes \varphi_P) \circ (H \otimes M \otimes c_{H,N} \otimes P) \circ (H \otimes c_{H,M} \otimes N \otimes P) \\ &\qquad \circ \big((\delta_H \circ \eta_H) \otimes \nabla_{M\otimes N} \otimes P\big) \\ &\quad= (\varphi_{M\otimes N} \otimes \varphi_P) \circ (H \otimes M \otimes c_{H,N} \otimes P) \circ (H \otimes c_{H,M} \otimes N \otimes P) \\ &\qquad \circ \big((\delta_H \circ \eta_H) \otimes M \otimes N \otimes P\big) \\ &\quad= (\varphi_M \otimes \varphi_{N\otimes P}) \circ (H \otimes c_{H,M} \otimes N \otimes P) \circ \big((\delta_H \circ \eta_H) \otimes M \otimes N \otimes P\big) \\ &\quad= (M \otimes \nabla_{N\otimes P}) \circ (\varphi_M \otimes \varphi_{N\otimes P}) \circ (H \otimes c_{H,M} \otimes N \otimes P) \circ \big((\delta_H \circ \eta_H) \otimes M \otimes \nabla_{N\otimes P}\big) \\ &\quad= (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}).\end{aligned}$$

In the last computations, the first equality follows from the definition, the second one by (22) and (23) and the third one by the coalgebra condition for H and the naturality of the braiding. Finally, in the fourth one we use (22) and (23) and the fifth one follows by definition.

We conclude the proof proving (25). In the proof of (24) we obtain the formula:

$$\begin{aligned}&(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) \\ &\quad= (\varphi_M \otimes \varphi_{N\otimes P}) \circ (H \otimes c_{H,M} \otimes N \otimes P) \circ \big((\delta_H \circ \eta_H) \otimes M \otimes N \otimes P\big).\end{aligned}$$

According with this equality and using the naturality of the braiding, (a3) and the condition of left H-module for N, we have:

$$
\begin{aligned}
&(i_{M,N}\otimes P)\circ\nabla_{(M\times N)\otimes P}\circ(p_{M,N}\otimes P)\\
&\quad=(\varphi_M\otimes\varphi_N\otimes\varphi_P)\circ(H\otimes c_{H,M}\otimes c_{H,N}\otimes P)\circ(H\otimes\mu_H\otimes c_{H,M}\otimes N\otimes P)\\
&\qquad\circ\big((\delta_H\circ\eta_H)\otimes(\delta_H\circ\eta_H)\otimes M\otimes N\otimes P\big)\\
&\quad=(\nabla_{M\otimes N}\otimes P)\circ(M\otimes\nabla_{N\otimes P}).
\end{aligned}
$$

Finally, by similar computations we establish the equality

$$(M\otimes i_{N,P})\circ\nabla_{M\otimes(N\times P)}\circ(M\otimes p_{N,P})=(M\otimes\nabla_{N\otimes P})\circ(\nabla_{M\otimes N}\otimes P). \qquad \square$$

Lemma 1.8. *Let $g:B\to H$ and $f:H\to B$ be morphisms of weak Hopf algebras such that $g\circ f=id_H$. Let μ_{B_H} and δ_{B_H} be the product and the coproduct defined in* 1.5. *Then,*

$$\mu_{B_H}\circ\nabla_{B_H\otimes B_H}=\mu_{B_H}; \tag{26}$$

$$\nabla_{B_H\otimes B_H}\circ\delta_{B_H}=\delta_{B_H}. \tag{27}$$

Proof. By (16) we have that $\mu_{B_H}\circ\varphi_{B_H\otimes B_H}=\varphi_{B_H}\circ(H\otimes\mu_{B_H})$. Then,

$$\mu_{B_H}\circ\nabla_{B_H\otimes B_H}=\varphi_{B_H}\circ(\eta_H\otimes\mu_{B_H})=\mu_{B_H}.$$

The equality (27) is obtained in the following way:

$$
\begin{aligned}
&\nabla_{B_H\otimes B_H}\circ\delta_{B_H}\\
&\quad=\big(\big(p_H^B\circ\mu_B\circ\big(f\otimes q_H^B\big)\big)\otimes\big(p_H^B\circ\mu_B\circ\big(f\otimes q_H^B\big)\big)\big)\circ(H\otimes c_{H,B}\otimes B)\\
&\qquad\circ\big((\delta_H\circ\eta_H)\otimes\big(\delta_B\circ i_H^B\big)\big)\\
&\quad=\big(p_H^B\otimes p_H^B\big)\circ(\mu_B\otimes\mu_B)\circ\delta_{B\otimes B}\circ\big(\eta_B\otimes i_H^B\big)\\
&\quad=\big(p_H^B\otimes p_H^B\big)\circ\delta_B\circ\mu_B\circ\big(\eta_B\otimes i_H^B\big)\\
&\quad=\delta_{B_H}.
\end{aligned}
$$

In this calculus the first equality follows by definition. In the second one we use (13) and the properties of f. Finally, the third equality is a consequence of (a1). $\square$

2. Yetter–Drinfeld modules and weak Hopf algebras with projection

Yetter–Drinfeld modules over finite dimensional weak Hopf algebras over fields have been introduced by Böhm in [6]. It is shown in [6] that the category of finite dimensional Yetter–Drinfeld modules is monoidal and in [13] it is proved that this category is isomorphic to the category of finite dimensional modules over the Drinfeld double. In [9], the results of [13] are generalized, using duality results between entwining structures and smash product structures, and more properties are given. Finally in [1] we can find an alternative definition of Yetter–Drinfeld modules (weak Yetter–Drinfeld modules) where the essential difference with the definition introduced by Böhm is to involve the morphism $\varPi_H^R$ in the axioms of Yetter–Drinfeld module.

Definition 2.1. Let H be a weak Hopf algebra. We shall denote by ${}^H_H\mathcal{YD}$ the category of left–left Yetter–Drinfeld modules over H. That is, $M=(M,\varphi_M,\varrho_M)$ is an object in ${}^H_H\mathcal{YD}$ if (M,φ_M) is a left H-module, (M,ϱ_M) is a left H-comodule and

(b1) $(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((\varrho_M\circ\varphi_M)\otimes H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)$
$=(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M)$;

(b2) $(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ\big((\delta_H\circ\eta_H)\otimes\varrho_M\big)=\varrho_M$.

By ${}^H_H\mathcal{WYD}$ we denote the category of left–left weak Yetter–Drinfeld modules over H. That is, $M=(M,\varphi_M,\varrho_M)$ is an object in ${}^H_H\mathcal{WYD}$ if (M,φ_M) is a left H-module, (M,ϱ_M) is a left H-comodule and we have (b2) and

(b3) $(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((\varrho_M\circ\varphi_M)\otimes H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)$
$=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ(\mu_H\otimes\varphi_M\otimes H)\circ(H\otimes c_{H,H}\otimes M\otimes H)$
$\circ\big(\delta_H\otimes\varrho_M\otimes\Pi^R_H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)$.

Let M,N in ${}^H_H\mathcal{YD}$ or in ${}^H_H\mathcal{WYD}$. The morphism $f:M\to N$ is a morphism of left–left-(weak) Yetter–Drinfeld modules if $f\circ\varphi_M=\varphi_N\circ(H\otimes f)$ and $(H\otimes f)\circ\varrho_M=\varrho_N\circ f$.

Note that if H is a Hopf algebra ${}^H_H\mathcal{WYD}={}^H_H\mathcal{YD}$. In the weak Hopf algebra case we have the following:

Proposition 2.2. *Let H be a weak Hopf algebra.*

(i) ${}^H_H\mathcal{YD}\subset{}^H_H\mathcal{WYD}$.
(ii) *Suppose that the antipode of H is an isomorphism and let $M=(M,\varphi_M,\varrho_M)\in|{}^H_H\mathcal{WYD}|$. Then, $M=(M,\varphi_M,\varrho_M)\in|{}^H_H\mathcal{YD}|$ if and only if*

$$\varphi_M\circ\big((\overline{\Pi}^L_H\circ\overline{\Pi}^R_H)\otimes M\big)\circ\varrho_M=id_M. \tag{28}$$

Proof. We first prove (i). Let $M=(M,\varphi_M,\varrho_M)\in|{}^H_H\mathcal{YD}|$. Then,

$$\begin{aligned}
&(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ(\mu_H\otimes\varphi_M\otimes H)\circ(H\otimes c_{H,H}\otimes M\otimes H)\\
&\quad\circ\big(\delta_H\otimes\varrho_M\otimes\Pi^R_H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)\\
&=(\mu_H\otimes M)\circ(\mu_H\otimes c_{M,H})\circ(H\otimes c_{M,H}\otimes H)\circ\big((\varrho_M\circ\varphi_M)\otimes H\otimes H\big)\\
&\quad\circ\big(H\otimes c_{H,M}\otimes\Pi^R_H\big)\circ(\delta_H\otimes c_{H,M})\circ(\delta_H\otimes M)\\
&=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((\varrho_M\circ\varphi_M)\otimes\big(id_H\wedge\Pi^R_H\big)\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)\\
&=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((\varrho_M\circ\varphi_M)\otimes H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)
\end{aligned}$$

and, as a consequence, $M\in|{}^H_H\mathcal{WYD}|$.

In the last computations, the first one follows by (b1), the second one by the naturality of the braiding and in the third one we use the equality

$$id_H \wedge \Pi_H^R = id_H. \tag{29}$$

Now we prove (ii). Note that if M is a left H-module and a left H-comodule, the following identity is always true:

$$\begin{aligned}
&(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,H} \otimes M \otimes H) \circ \big(\delta_H \otimes \varrho_M \otimes \Pi_H^R\big) \\
&\quad \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M) \\
&\quad = (\mu_H \otimes \varphi_M) \circ \big(H \otimes c_{H,H} \otimes \big(\varphi_M \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes M\big) \circ \varrho_M\big)\big) \circ (\delta_H \otimes \varrho_M).
\end{aligned} \tag{30}$$

Thus, by (30) and (b3), if M satisfies (28), we obtain that M is a left–left Yetter–Drinfeld module.

Conversely, if M is a left–left weak Yetter–Drinfeld module such that M is a left–left Yetter–Drinfeld module, composing with $\varepsilon_H \otimes M$ on (b2) we find that:

$$\begin{aligned}
id_M = {} & \big((\varepsilon_H \circ \mu_H) \otimes M\big) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,H} \otimes M \otimes H) \\
& \circ \big(\delta_H \otimes \varrho_M \otimes \Pi_H^R\big) \circ (H \otimes c_{H,M}) \circ \big((\delta_H \circ \eta_H) \otimes M\big),
\end{aligned} \tag{31}$$

and then, by (30), we have the following identity

$$\begin{aligned}
id_M = {} & \big((\varepsilon_H \circ \mu_H) \otimes \varphi_M\big) \circ \big(H \otimes c_{H,H} \otimes \big(\varphi_M \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes M\big) \circ \varrho_M\big)\big) \\
& \circ \big((\delta_H \circ \eta_H) \otimes \varrho_M\big).
\end{aligned} \tag{32}$$

Therefore, this establishes the formula (28) because

$$\begin{aligned}
&\big((\varepsilon_H \circ \mu_H) \otimes \varphi_M\big) \circ \big(H \otimes c_{H,H} \otimes \big(\varphi_M \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes M\big) \circ \varrho_M\big)\big) \circ \big((\delta_H \circ \eta_H) \otimes \varrho_M\big) \\
&\quad = \varphi_M \circ \big(\big(\big(\Pi_H^L \wedge \big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big)\big)\big) \otimes M\big) \circ \varrho_M \\
&\quad = \varphi_M \circ \big(\big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \wedge \Pi_H^L\big) \otimes M\big) \circ \varrho_M \\
&\quad = \varphi_M \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes \big(\varphi_M \circ \big(\Pi_H^L \otimes H\big) \circ \varrho_M\big)\big) \circ \varrho_M \\
&\quad = \varphi_M \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes M\big) \circ \varrho_M.
\end{aligned}$$

In these calculus, the first equality follows by definition of Π_H^L and in the second one we use that, if the antipode is an isomorphism,

$$\Pi_H^L \wedge \big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) = \big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \wedge \Pi_H^L. \tag{33}$$

The third equality follows from the structure of left H-module of M and, finally, in the fourth one we apply

$$\Pi_H^L \wedge id_H = id_H. \qquad \square \tag{34}$$

Proposition 2.3. *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, $(B_H, \varphi_{B_H}, \varrho_{B_H})$ belongs to ${}^H_H\mathcal{YD}$.*

Proof. In Proposition 2.8 of [1] we prove that $(B_H, \varphi_{B_H}, \varrho_{B_H})$ is an object in the category ${}^H_H\mathcal{WYD}$. Then, by Proposition 2.2, we only need to show that B_H satisfies (28) or equivalently

$$i_H^B \circ \varphi_{B_H} \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes B_H\big) \circ \varrho_{B_H} = i_H^B.$$

Indeed:

$$\begin{aligned}
& i_H^B \circ \varphi_{B_H} \circ \big(\big(\overline{\Pi}_H^L \circ \overline{\Pi}_H^R\big) \otimes B_H\big) \circ \varrho_{B_H} \\
&\quad = q_H^B \circ \mu_B \circ \big(\big(f \circ \overline{\Pi}_H^L \circ \overline{\Pi}_H^R \circ g\big) \otimes q_H^B\big) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\lambda_B \otimes B)\big)\big) \circ \big(\big(\delta_B \circ \overline{\Pi}_B^L \circ \overline{\Pi}_B^R \circ f \circ g\big) \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\lambda_B \otimes B)\big)\big) \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ \big(B \otimes \overline{\Pi}_B^L\big)\big) \otimes B\big) \\
&\qquad \circ \big((\delta_B \circ \eta_B) \otimes \big(\overline{\Pi}_B^R \circ f \circ g\big) \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\mu_B \otimes B)\big)\big) \circ \big(B \otimes \lambda_B \otimes \big(\lambda_B \circ \overline{\Pi}_B^L \circ \overline{\Pi}_B^R \circ f \circ g\big) \otimes B\big) \\
&\qquad \circ \big((\delta_B \circ \eta_B) \otimes \big(\delta_B \circ i_H^B\big)\big) \\
&\quad = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\mu_B \otimes B)\big)\big) \circ \big(B \otimes \lambda_B \otimes \big(\overline{\Pi}_B^R \circ f \circ g\big) \otimes B\big) \\
&\qquad \circ \big((\delta_B \circ \eta_B) \otimes \big(\delta_B \circ i_H^B\big)\big) \\
&\quad = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\mu_B \otimes B)\big)\big) \circ \big(B \otimes \lambda_B \otimes \overline{\Pi}_B^R \otimes B\big) \circ \big((\delta_B \circ \eta_B) \otimes \big(\delta_B \circ i_H^B\big)\big) \\
&\quad = \mu_B \circ \big(\mu_B \otimes \big(\mu_B \circ (\lambda_B \otimes B)\big)\big) \circ (B \otimes c_{B,B} \otimes B) \\
&\qquad \circ \big((\delta_B \circ \eta_B) \otimes \big(c_{B,B} \circ \big(\overline{\Pi}_B^R \otimes B\big) \circ \delta_B \circ i_H^B\big)\big) \\
&\quad = \mu_B \circ (B \otimes \mu_B) \circ \big(B \otimes \big(\lambda_B \circ \Pi_B^L\big) \otimes B\big) \circ (\delta_B \otimes B) \circ c_{B,B} \circ \big(\overline{\Pi}_B^R \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ c_{B,B} \circ (B \otimes \mu_B) \circ \big(\overline{\Pi}_B^R \otimes B \otimes \big(\lambda_B \circ \Pi_B^L\big)\big) \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ c_{B,B} \circ (B \otimes \mu_B) \circ \big(\overline{\Pi}_B^R \otimes B \otimes \big(\lambda_B \circ f \circ \Pi_H^L \circ g\big)\big) \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ c_{B,B} \circ \big(\overline{\Pi}_B^R \otimes q_H^B\big) \circ \delta_B \circ i_H^B \\
&\quad = \mu_B \circ c_{B,B} \circ \big(\overline{\Pi}_B^R \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = i_H^B.
\end{aligned}$$

In the last computations, the first equality follows by definition and in the second one we use (13) and

$$f \circ \overline{\Pi}_H^L \circ \overline{\Pi}_H^R \circ g = \overline{\Pi}_B^L \circ \overline{\Pi}_B^R \circ f \circ g; \tag{35}$$

$$q_H^B \circ \mu_B \circ \big(f \otimes i_H^B\big) = \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\lambda_B \otimes B)\big)\big) \circ (\delta_B \otimes B) \circ \big(f \otimes i_H^B\big). \tag{36}$$

The third equality is a consequence of

$$\delta_B \circ \overline{\Pi}_B^L = \big(B \otimes (\mu_B \circ c_{B,B})\big) \circ \big((\delta_B \circ \eta_B) \otimes \overline{\Pi}_B^L\big) \tag{37}$$

and in the fourth one we apply (1). Using (6) and (4) we can obtain the fifth equality and the sixth one is by

$$\overline{\Pi}_B^R \circ f \circ g = \overline{\Pi}_B^R. \tag{38}$$

The seventh equality is due to the naturality of the braiding and the eight one follows from

$$\big(B \otimes \Pi_B^L\big) \circ \delta_B = (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ \big((\delta_B \circ \eta_B) \otimes B\big). \tag{39}$$

In the ninth equality we use the naturality of the braiding and the tenth follows from

$$f \circ \Pi_H^L \circ g = \Pi_B^L. \tag{40}$$

Finally, the eleventh one follows by the coalgebra condition for B and by (9), the twelfth one by (13) and in the last one we use

$$\mu_B \circ c_{B,B} \circ \big(\overline{\Pi}_B^R \otimes B\big) \circ \delta_B = id_B. \qquad \square \tag{41}$$

2.4. It is a well-know fact that, if the antipode of a weak Hopf algebra H is an isomorphism, ${}_H^H\mathcal{YD}$ is a braided monoidal category (see Proposition 2.7 of [13] for modules over a field K or Theorem 2.6 of [9] for modules over a commutative ring). In the following lines we give a brief resume of the braided monoidal structure that we can construct in the category ${}_H^H\mathcal{YD}$.

For two left–left Yetter–Drinfeld modules $M = (M, \varphi_M, \varrho_M)$, $M = (N, \varphi_N, \varrho_N)$ the tensor product is defined as object as in 1.6. As a consequence $M \times N$ is a left–left Yetter–Drinfeld module with the following action and coaction:

$$\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}); \tag{42}$$

$$\varrho_{M\times N} = (H \otimes p_{M,N}) \circ \varrho_{M\otimes N} \circ i_{M,N}. \tag{43}$$

The base object is $H_L = \mathrm{Im}(\Pi_H^L)$ or, equivalently, the equalizer of δ_H and $\zeta_H^1 = (H \otimes \Pi_H^L) \circ \delta_H$ (see (9)) or the equalizer of δ_H and $\zeta_H^2 = (H \otimes \overline{\Pi}_H^R) \circ \delta_H$ (see (10)). The structure of left–left Yetter–Drinfeld module for H_L is the one derived of the following morphisms:

$$\varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L), \qquad \varrho_{H_L} = (H \otimes p_L) \circ \delta_H \circ i_L. \tag{44}$$

The unit constrains are:

$$l_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L,M} : H_L \times M \to M; \tag{45}$$

$$r_M = \varphi_M \circ c_{M,H} \circ \big(M \otimes \big(\overline{\Pi}_H^L \circ i_L\big)\big) \circ i_{M,H_L} : M \times H_L \to M. \tag{46}$$

These morphisms are isomorphisms with inverses:

$$l_M^{-1} = p_{H_L,M} \circ (p_L \otimes \varphi_M) \circ \big((\delta_H \circ \eta_H) \otimes M\big) : M \to H_L \times M; \tag{47}$$

$$r_M^{-1} = p_{M,H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ \big((\delta_H \circ \eta_H) \otimes M\big) : M \to M \times H_L. \tag{48}$$

If M, N, P are objects in the category ${}_H^H\mathcal{YD}$, the associativity constrains are defined by

$$a_{M,N,P} = p_{M\times N,P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,N\times P} : M \times (N \times P) \to (M \times N) \times P; \tag{49}$$

where the inverse is the morphism:

$$a_{M,N,P}^{-1} = p_{M,N\times P} \circ (M \otimes p_{N,P}) \circ (i_{M,N} \otimes P) \circ i_{M\times N,P} : (M \times N) \times P \to M \times (N \times P). \tag{50}$$

If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M',N'} \circ (\gamma \otimes \phi) \circ i_{M,N} : M \times N \to M' \times N' \tag{51}$$

is a morphism in ${}_H^H\mathcal{YD}$ and

$$(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi), \tag{52}$$

where $\gamma' : M' \to M''$ and $\phi' : N' \to N''$ are morphisms in ${}_H^H\mathcal{YD}$.

Finally, the braiding is

$$\tau_{M,N} = p_{N,M} \circ t_{M,N} \circ i_{M,N} : M \times N \to N \times M \tag{53}$$

where $t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \to N \otimes M$. The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau_{M,N}^{-1} = p_{M,N} \circ t'_{M,N} \circ i_{N,M} : N \times M \to M \times N \tag{54}$$

where $t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M)$.

2.5. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Using the properties of f and (9) we have

$$\begin{aligned}
\big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ f \circ i_L &= \big(f \otimes \Pi_H^L\big) \circ \delta_H \circ i_L \\
&= (f \otimes H) \circ \delta_H \circ i_L \\
&= (B \otimes g) \circ \delta_B \circ f \circ i_L.
\end{aligned}$$

Then, there exists an unique morphism $u_{B_H}: H_L \to B_H$ commuting the following diagram

$$\begin{tikzcd} H_L \arrow[r, "i_L"] \arrow[d, "u_{B_H}"'] & H \arrow[r, shift left, "\delta_H"] \arrow[r, shift right, "(H\otimes \Pi_H^L)\circ\delta_H"'] \arrow[d, "f"] & H\otimes H \arrow[d, "f\otimes H"] \\ B_H \arrow[r, "i_H^B"'] & B \arrow[r, shift left, "(B\otimes g)\circ\delta_B"] \arrow[r, shift right, "(B\otimes(\Pi_H^L\circ g))\circ\delta_B"'] & B\otimes H \end{tikzcd}$$

and therefore

$$u_{B_H} = p_H^B \circ f \circ i_L. \tag{55}$$

Analogously, we have that $p_L \circ g \circ \mu_B \circ (B \otimes f) = p_L \circ g \circ \mu_B \circ (B \otimes (f \circ \Pi_H^L))$. Thus, by (11), there exists an unique morphism $e_{B_H}: B_H \to H_L$ such that $p_L \circ g = e_{B_H} \circ p_H^B$ and as a consequence:

$$e_{B_H} = p_L \circ g \circ i_H^B. \tag{56}$$

The morphisms u_{B_H} and e_{B_H} are morphisms in ${}_H^H\mathcal{YD}$ because by the usual arguments we have:

$$\begin{aligned} &u_{B_H} \circ \varphi_{H_L} \\ &\quad = p_H^B \circ f \circ \Pi_H^L \circ \mu_H \circ (H \otimes i_L) \\ &\quad = p_H^B \circ q_H^B \circ f \circ \mu_H \circ (H \otimes i_L) \\ &\quad = p_H^B \circ f \circ \mu_H \circ (H \otimes i_L) \\ &\quad = p_H^B \circ \mu_B \circ \big(f \otimes \big(q_H^B \circ f \circ i_L\big)\big) \\ &\quad = \varphi_{B_H} \circ (H \otimes u_{B_H}), \end{aligned}$$

$$\begin{aligned} &\varrho_{B_H} \circ u_{B_H} \\ &\quad = \big(g \otimes p_H^B\big) \circ \delta_B \circ q_H^B \circ f \circ i_L \\ &\quad = \big(g \otimes p_H^B\big) \circ \delta_B \circ f \circ i_L \\ &\quad = \big(H \otimes \big(p_H^B \circ f\big)\big) \circ \delta_H \circ i_L \\ &\quad = \big(H \otimes \big(p_H^B \circ f \circ \Pi_H^L\big)\big) \circ \delta_H \circ i_L \\ &\quad = (H \otimes u_{B_H}) \circ \varrho_{H_L}, \end{aligned}$$

$$\begin{aligned} &e_{B_H} \circ \varphi_{B_H} \\ &\quad = p_L \circ g \circ q_H^B \circ \mu_B \circ \big(f \otimes i_H^B\big) \\ &\quad = p_L \circ \Pi_H^L \circ g \circ \mu_B \circ \big(f \otimes i_H^B\big) \\ &\quad = p_L \circ g \circ \mu_B \circ \big(f \otimes i_H^B\big) \\ &\quad = p_L \circ \mu_H \circ \big(H \otimes \big(\Pi_H^L \circ g \circ i_H^B\big)\big) \\ &\quad = \varphi_{H_L} \circ (H \otimes e_{B_H}), \end{aligned}$$

$$\begin{aligned} &\varrho_{H_L} \circ e_{B_H} \\ &\quad = (H \otimes p_L) \circ \delta_H \circ \Pi_H^L \circ g \circ i_H^B \\ &\quad = (H \otimes p_L) \circ \delta_H \circ g \circ q_H^B \circ i_H^B \\ &\quad = (H \otimes p_L) \circ \delta_H \circ g \circ i_H^B \\ &\quad = \big(g \otimes \big(p_L \circ g \circ \Pi_B^L\big)\big) \circ \delta_B \circ i_H^B \\ &\quad = (H \otimes e_{B_H}) \circ \varrho_{B_H}. \end{aligned}$$

Now take the morphism $m_{B_H \times B_H}: B_H \times B_H \to B_H$ defined by

$$m_{B_H \times B_H} = \mu_{B_H} \circ i_{B_H, B_H} \tag{57}$$

where μ_{B_H} is the product defined in 1.5. Then, this morphism belongs to the category of left–left Yetter–Drinfeld modules. Indeed, by (16) and (26) we have the following:

$$\begin{aligned} m_{B_H} \circ \varphi_{B_H \times B_H} &= \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ \varphi_{B_H \otimes B_H} \circ (H \otimes i_{B_H,B_H}) \\ &= \mu_{B_H} \circ \varphi_{B_H \otimes B_H} \circ (H \otimes i_{B_H,B_H}) \\ &= \varphi_{B_H} \circ (H \otimes m_{B_H}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\varrho_{B_H} \circ m_{B_H} \\ &\quad = \left(g \otimes p_H^B\right) \circ \delta_B \circ \mu_B \circ \left(i_H^B \otimes i_H^B\right) \circ i_{B_H,B_H} \\ &\quad = \left(g \otimes p_H^B\right) \circ \left(\mu_B \otimes \left(\mu_B \circ \left(q_H^B \otimes q_H^B\right)\right)\right) \circ \delta_{B \otimes B} \circ \left(i_H^B \otimes i_H^B\right) \circ i_{B_H,B_H} \\ &\quad = \mu_{H \otimes B_H} \circ (\varrho_{B_H} \otimes \varrho_{B_H}) \circ i_{B_H,B_H} \\ &\quad = \left(\mu_H \otimes \left(\mu_{B_H} \circ \nabla_{B_H \otimes B_H}\right)\right) \circ (H \otimes c_{B_H,H} \otimes B_H) \circ (\varrho_{B_H} \otimes \varrho_{B_H}) \circ i_{B_H,B_H} \\ &\quad = (H \otimes m_{B_H}) \circ \varrho_{B_H \times B_H}. \end{aligned}$$

In the last computations, the second equality follows from (13) and in the fourth one we use (26).

Similarly to m_{B_H}, define the morphism $\Delta_{B_H} : B_H \to B_H \times B_H$ by

$$\Delta_{B_H} = p_{B_H,B_H} \circ \delta_{B_H}, \tag{58}$$

where δ_{B_H} is the morphism defined in 1.5. We claim that Δ_{B_H} is in the category ${}^H_H\mathcal{YD}$. The proof of this assertion is similar with the one developed for m_{B_H} and we leave the details to the reader.

For the morphisms u_{B_H}, e_{B_H}, m_{B_H} and Δ_{B_H} we have the following result:

Proposition 2.6. *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, we have the following*:

(i) (B_H, u_{B_H}, m_{B_H}) *is an algebra in* ${}^H_H\mathcal{YD}$.
(ii) $(B_H, e_{B_H}, \Delta_{B_H})$ *is a coalgebra in* ${}^H_H\mathcal{YD}$.

Proof. First, note that by 2.5, u_{B_H}, m_{B_H}, e_{B_H} and Δ_{B_H} are morphisms in the category ${}^H_H\mathcal{YD}$. We prove (i). The proof for (ii) is similar and we leave it to the reader.

First we check the unit properties.

$$\begin{aligned} &m_{B_H} \circ (u_{B_H} \times B_H) \circ l_{B_H}^{-1} \\ &\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (u_{B_H} \otimes B_H) \circ \nabla_{H_L \otimes B_H} \circ (p_L \otimes \varphi_{B_H}) \circ \left((\delta_H \circ \eta_H) \otimes B_H\right) \\ &\quad = p_H^B \circ \mu_B \circ \left(\left(q_H^B \circ \Pi_B^L\right) \otimes B\right) \circ \mu_{B \otimes B} \circ \left((\delta_B \circ \eta_B) \otimes B \otimes \mu_B\right) \circ \left((\delta_B \circ \eta_B) \otimes i_H^B\right) \\ &\quad = p_H^B \circ \mu_B \circ \left(\left(\mu_B \circ \left(\left(q_H^B \circ \Pi_B^L \circ \mu_B \circ \left(B \otimes \Pi_B^L\right)\right) \otimes \mu_B\right) \circ \delta_{B \otimes B} \circ (\eta_B \otimes \eta_B)\right) \otimes i_H^B\right) \\ &\quad = p_H^B \circ \mu_B \circ \left(\left(\mu_B \circ \left(\left(q_H^B \circ \Pi_B^L\right) \otimes B\right) \circ \delta_B \circ \eta_B\right) \otimes i_H^B\right) \\ &\quad = p_H^B \circ \mu_B \circ \left(\left(\left(\Pi_B^L \wedge id_B\right) \circ \eta_B\right) \otimes i_H^B\right) \\ &\quad = p_H^B \circ \mu_B \circ \left(\eta_B \otimes i_H^B\right) = id_{B_H}. \end{aligned}$$

In these computations, the first equality follows by definition, the second one by (26), (13) and the properties of f and the third one by the naturality of the braiding and the associativity of μ_B. In the fourth equality we use (13), for $H = B$ and $f = g = id_H$, (a1) and the unit condition for μ_B. Finally, in the fifth one we apply

$$q_H^B \circ \Pi_B^L = \Pi_B^L \tag{59}$$

and the sixth one is a consequence of (34).

On the other hand,

$$\begin{aligned}
&m_{B_H} \circ (B_H \times u_{B_H}) \circ r_{B_H}^{-1} \\
&\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ \big(B_H \otimes \big(p_H^B \circ f \circ i_L\big)\big) \circ \nabla_{B_H \otimes H_L} \circ \big(\big(p_H^B \circ \mu_B \circ \big(f \otimes i_H^B\big)\big) \otimes p_L\big) \\
&\qquad \circ (H \otimes c_{H,B_H}) \circ \big((\delta_H \circ \eta_H) \otimes B_H\big) \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(q_H^B \circ \mu_B \circ (\mu_B \otimes B)\big) \otimes \mu_B\big) \circ (B \otimes B \otimes c_{B,B} \otimes B) \circ (B \otimes c_{B,B} \otimes c_{B,B}) \\
&\qquad \circ \big((\delta_B \circ \eta_B) \otimes (\delta_B \circ \eta_B) \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big(q_H^B \otimes \Pi_B^L\big) \circ (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ \big((\delta_B \circ \eta_B) \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big(q_H^B \otimes \Pi_B^L\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big(B \otimes \big(f \circ (\lambda_H \wedge id_H \wedge \lambda_H) \circ g\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ q_H^B \circ i_H^B \\
&\quad = id_{B_H}.
\end{aligned}$$

The second equality follows by (13), (26), the properties of f and

$$\big(H \otimes \Pi_H^L\big) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H. \tag{60}$$

In the third equality we use the naturality of the braiding, (a1) and the unity condition for μ_B. The fourth one is a consequence of the following identity:

$$(\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ \big((\delta_B \circ \eta_B) \otimes B\big) = \big(B \otimes \Pi_B^L\big) \circ \delta_B. \tag{61}$$

Finally, the sixth equality follows from (a4-3) and (40).

Let us show that the product m_{B_H} is associative:

$$\begin{aligned}
&m_{B_H} \circ (m_{B_H} \times B_H) \circ a_{B_H,B_H,B_H} \\
&\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (\mu_{B_H} \otimes B_H) \circ (i_{B_H,B_H} \otimes B_H) \circ \nabla_{(B_H \times B_H) \otimes B_H} \circ (p_{B_H,B_H} \otimes B_H) \\
&\qquad \circ (B_H \otimes i_{B_H,B_H}) \circ i_{B_H,B_H \times B_H} \\
&\quad = \mu_{B_H} \circ (\mu_{B_H} \otimes B_H) \circ (B_H \otimes i_{B_H,B_H}) \circ \nabla_{B_H \otimes (B_H \times B_H)} \circ i_{B_H,B_H \times B_H} \\
&\quad = \mu_{B_H} \circ (\mu_{B_H} \otimes B_H) \circ (B_H \otimes i_{B_H,B_H}) \circ i_{B_H,B_H \times B_H} \\
&\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (B_H \otimes m_{B_H}) \circ i_{B_H,B_H \times B_H} \\
&\quad = m_{B_H} \circ (B_H \times m_{B_H}).
\end{aligned}$$

In the last equalities, the second one follows by (26) and (24). In the fourth one we use (26). □

2.7. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Let Θ_H^B be the morphism $\Theta_H^B = ((f \circ g) \wedge \lambda_B) \circ i_H^B : B_H \to B$. Following Proposition 2.9 of [1] we have that

$$(B \otimes g) \circ \delta_B \circ \Theta_H^B = \big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ \Theta_H^B \tag{62}$$

and, as a consequence, there exists an unique morphism $\lambda_{B_H} : B_H \to B_H$ such that $i_H^B \circ \lambda_{B_H} = \Theta_H^B$. Therefore,

$$\lambda_{B_H} = p_H^B \circ \Theta_H^B. \tag{63}$$

The morphism λ_{B_H} belongs to the category of left–left Yetter–Drinfeld modules. Indeed, λ_{B_H} is a morphism of left H-modules:

$$\begin{aligned}
&\lambda_{B_H} \circ \varphi_{B_H} \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \lambda_B\big) \circ (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ \big(B \otimes q_H^B \otimes (f \circ \lambda_H \circ g)\big) \\
&\qquad \circ (\delta_B \otimes B) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ \big((f \circ g) \otimes (f \circ \lambda_B \circ g) \otimes \big(\lambda_B \circ q_H^B\big)\big) \circ (B \otimes c_{B,B}) \\
&\qquad \circ (\delta_B \otimes B) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \big(\lambda_B \circ \mu_B \circ \big(q_H^B \otimes (f \circ g)\big) \circ \delta_B\big)\big) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \big(\lambda_B \circ \mu_B \circ \big(B \otimes \big(f \circ \Pi_H^R \circ g\big)\big) \circ \delta_B\big)\big) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big) \\
&\quad = p_H^B \circ \big((f \circ g) \wedge \lambda_B\big) \circ \mu_B \circ \big(f \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(\mu_B \circ \big(f \otimes (f \circ g)\big)\big) \otimes \big(\mu_B \circ c_{B,B} \circ \big((\lambda_B \circ f) \otimes \lambda_B\big)\big)\big) \circ \delta_{H \otimes B} \circ \big(H \otimes i_H^B\big) \\
&\quad = p_H^B \circ \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ (\lambda_B \otimes B)\big)\big) \circ \big((\delta_B \circ f) \otimes \big(i_H^B \circ \lambda_{B_H}\big)\big) \\
&\quad = p_H^B \circ q_H^B \circ \mu_B \circ \big(f \otimes \big(i_H^B \circ \lambda_{B_H}\big)\big) \\
&\quad = \varphi_{B_H} \circ (H \otimes \lambda_{B_H}).
\end{aligned}$$

In the last computations, the first equality follows from

$$\delta_B \circ q_H^B = (\mu_B \otimes B) \circ \big(B \otimes \big(c_{B,B} \circ \big(q_H^B \otimes (f \circ \lambda_H \circ g)\big)\big)\big) \circ (\delta_B \otimes B) \circ \delta_B \tag{64}$$

and the second one by the properties of f and g. In the third and the fourth ones we use (1), the associativity of the coproduct δ_B and the properties of f and g. The fifth one follows from

$$id_B \wedge \big(f \circ \Pi_H^R \circ g\big) = id_B, \tag{65}$$

the sixth one is a consequence of (a1) and (1) and the seventh one is by definition of λ_{B_H}. Finally, in the eighth one we use

$$q_H^B \circ \mu_B = \mu_B \circ (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ (B \otimes \lambda_B \otimes B) \circ (\delta_B \otimes B) \tag{66}$$

and the ninth one follows from the idempotent character of q_H^B.

On the other hand, λ_{B_H} is a morphism of left H-comodules:

$$\begin{aligned}
&\varrho_{B_H} \circ \lambda_{B_H} \\
&\quad = \big(g \otimes p_H^B\big) \circ \delta_B \circ \mu_B \circ \big((f \circ g) \otimes \lambda_B\big) \circ \delta_B \circ i_H^B \\
&\quad = \big(\big(\mu_H \circ (H \otimes g)\big) \otimes \big(p_H^B \circ \mu_B\big)\big) \circ (H \otimes c_{B,B} \otimes B) \\
&\qquad \circ \big(\big(\big(g \otimes (f \circ g)\big) \circ \delta_B\big) \otimes \big((\lambda_B \otimes \lambda_B) \circ c_{B,B} \circ \delta_B\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = (\mu_H \circ B_H) \circ (H \otimes c_{B_H,H}) \circ \big(g \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big) \otimes (g \circ \lambda_B)\big)\big) \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = \big((g \circ \mu_B) \otimes B_H\big) \circ (B \otimes c_{B_H,H}) \circ \big(B \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big) \circ q_H^B\big) \otimes \lambda_B\big) \\
&\qquad \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = \big((g \circ \mu_B) \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big)\big)\big) \circ (B \otimes c_{B,B}) \circ \big(B \otimes q_H^B \otimes \lambda_B\big) \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = \big(g \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big)\big)\big) \circ \delta_B \circ q_H^B \circ i_H^B \\
&\quad = \big(g \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big)\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = \big(g \otimes \big(p_H^B \circ \big((f \circ g) \wedge \lambda_B\big) \circ q_H^B\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = (H \otimes \lambda_{B_H}) \circ \varrho_{B_H}.
\end{aligned}$$

The first equality follows from definition of λ_{B_H} and the second one by (1), (a1) and the properties of f and g. In the third one we use the coassociativity of δ_B and the fourth one is a consequence of the definition of λ_{B_H}. The fifth one follows by the naturality of the braiding and in the sixth one we use the following equality

$$(g \otimes B) \circ \delta_B \circ q_H^B = \big((g \circ \mu_B) \otimes B\big) \circ \big(B \otimes \big(c_{B,B} \circ \big(q_H^B \otimes \lambda_B\big)\big)\big) \circ (\delta_B \otimes B) \circ \delta_B \tag{67}$$

derived directly from (64). In the seventh one we apply the idempotent character of q_H^B and the eighth one is a consequence of (13). Finally, the ninth one follows from definition.

The remainder of this section will be devoted to the proof of the main theorem of this paper.

Theorem 2.8. *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras satisfying the equality $g \circ f = id_H$ and suppose that the antipode of H is an isomorphism. Let u_{B_H}, m_{B_H}, e_{B_H}, Δ_{B_H}, λ_{B_H} be the morphisms defined in* 2.5 *and* 2.7 *respectively. Then $(B_H, u_{B_H}, m_{B_H}, e_{B_H}, \Delta_{B_H}, \lambda_{B_H})$ is a Hopf algebra in the category of left–left Yetter–Drinfeld modules.*

Proof. By Proposition 2.6 we know that (B_H, u_{B_H}, m_{B_H}) is an algebra and $(B_H, e_{B_H}, \Delta_{B_H})$ is a coalgebra in ${}^H_H\mathcal{YD}$.

First we prove that m_{B_H} is a coalgebra morphism. That is:

(c1) $\Delta_{B_H} \circ m_{B_H} = (m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H\times B_H} \circ \big(B_H \times a^{-1}_{B_H,B_H,B_H}\big)$
$\circ \big(B_H \times (\tau_{B_H,B_H} \times B_H)\big) \circ (B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H\times B_H}$
$\circ (\Delta_{B_H} \times \Delta_{B_H})$,

(c2) $e_{B_H} \circ m_{B_H} = l_{H_L} \circ (e_{B_H} \times e_{B_H})$.

Indeed:

$$\begin{aligned}
&(m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H\times B_H} \circ \big(B_H \times a^{-1}_{B_H,B_H,B_H}\big) \circ \big(B_H \times (\tau_{B_H,B_H} \times B_H)\big) \\
&\quad \circ (B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H\times B_H} \circ (\Delta_{B_H} \times \Delta_{B_H}) \\
&= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes i_{B_H,B_H} \otimes B_H) \circ (\nabla_{B_H\otimes(B_H\times B_H)} \otimes B_H) \\
&\quad \circ (B_H \otimes \nabla_{(B_H\times B_H)\otimes B_H}) \circ \big(B_H \otimes (p_{B_H,B_H} \circ t_{B_H,B_H} \circ i_{B_H,B_H}) \otimes B_H\big) \\
&\quad \circ (B_H \otimes \nabla_{(B_H\times B_H)\otimes B_H}) \\
&\quad (\nabla_{B_H\otimes(B_H\times B_H)} \otimes B_H) \circ (B_H \otimes p_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H} \\
&= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ \big(B_H \otimes (\nabla_{B_H\otimes B_H} \circ t_{B_H,B_H} \circ \nabla_{B_H\otimes B_H}) \otimes B_H\big) \circ (\delta_{B_H} \otimes \delta_{B_H}) \\
&\quad \circ i_{B_H,B_H} \\
&= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes t_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H} \\
&= p_{B_H,B_H} \circ \delta_{B_H} \circ \mu_{B_H} \circ i_{B_H,B_H} \\
&= \Delta_{B_H} \circ m_{B_H}.
\end{aligned}$$

In the last computations, the first and the second equalities follow from (24)–(26), and (27). In the third one we use the following result: if M is a left–left Yetter–Drinfeld module then

$$t_{M,M} \circ \nabla_{M\otimes M} = t_{M,M}, \qquad \nabla_{M\otimes M} \circ t_{M,M} = t_{M,M}. \tag{68}$$

The fourth equality follows from Proposition 2.9 of [1] and, finally, the fifth one follows by definition.

On the other hand,

$$\begin{aligned}
&l_{H_L} \circ (e_{B_H} \times e_{B_H}) \\
&\quad = p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L\otimes H_L} \circ (p_L \otimes p_L) \circ \big(\big(g \circ i_H^B\big) \otimes \big(g \circ i_H^B\big)\big) \circ i_{B_H,B_H} \\
&\quad = p_L \circ \mu_H \circ \big(\big(\Pi_H^L \circ g \circ i_H^B\big) \otimes \big(\Pi_H^L \circ g \circ i_H^B\big)\big) \circ i_{B_H,B_H} \\
&\quad = p_L \circ \mu_H \circ \big(\big(g \circ q_H^B \circ i_H^B\big) \otimes \big(g \circ q_H^B \circ i_H^B\big)\big) \circ i_{B_H,B_H} \\
&\quad = p_L \circ \mu_H \circ \big(\big(g \circ i_H^B\big) \otimes \big(g \circ i_H^B\big)\big) \circ i_{B_H,B_H} \\
&\quad = p_L \circ g \circ i_H^B \circ \mu_{B_H} \circ i_{B_H,B_H} \\
&\quad = e_{B_H} \circ m_{B_H}.
\end{aligned}$$

The first equality follows from definition, the second one from

$$p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L \otimes H_L} \circ (p_L \otimes p_L) = p_L \circ \mu_H \circ \left(\Pi_H^L \otimes \Pi_H^L\right) \tag{69}$$

and the third one from

$$\Pi_H^L \circ g = g \circ q_H^B. \tag{70}$$

Finally, the fourth one follows from the idempotent character of q_H^B, the fifth one from the properties of g and the definition of μ_{B_H} and the sixth one from definition.

To finish the proof we only need to show

$$m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H} = l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H}.$$

We begin by proving

$$l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = u_{B_H} \circ e_{B_H}, \tag{71}$$

$$\begin{aligned}
&l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} \\
&\quad = p_H^B \circ \mu_B \circ (f \otimes B) \circ \left(i_L \otimes i_H^B\right) \circ \nabla_{H_L \otimes B_H} \circ \left(p_L \otimes p_H^B\right) \circ (g \otimes f) \circ \left(i_H^B \otimes i_L\right) \circ \nabla_{B_H \otimes H_L} \\
&\qquad \circ \left(p_H^B \otimes p_L\right) \circ \left(\left(\mu_B \circ \left(f \otimes i_H^B\right)\right) \otimes H\right) \circ (H \otimes c_{H,B_H}) \circ \left((\delta_H \circ \eta_H) \otimes B_H\right) \\
&\quad = p_H^B \circ \mu_B \circ \left(\left(\Pi_B^L \wedge \Pi_B^L\right) \otimes \Pi_B^L\right) \circ \left(\left(f \circ g \circ q_H^B\right) \otimes \left(\mu_B \circ \left(\Pi_B^L \otimes \left(f \circ g \circ \Pi_B^L\right)\right)\right)\right) \\
&\qquad \circ (\delta_B \otimes B) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \left(\left(\Pi_B^L \circ f \circ g \circ q_H^B\right) \otimes \left(f \circ \Pi_H^L \circ g \circ \Pi_B^L\right)\right) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ f \circ \mu_H \circ \left(\Pi_H^L \otimes \Pi_H^L\right) \circ \delta_H \circ g \circ i_H^B \\
&\quad = p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&\quad = u_{B_H} \circ e_{B_H}.
\end{aligned}$$

The first equality follows from definition, the second one from

$$\left(\left(\mu_B \circ \left(f \otimes i_H^B\right)\right) \otimes H\right) \circ (H \otimes c_{H,B_H}) \circ \left((\delta_H \circ \eta_H) \otimes B_H\right) = \left(B \otimes \left(g \circ \Pi_B^L\right)\right) \circ \delta_B \circ i_H^B, \tag{72}$$

$$\left(i_H^B \otimes i_L\right) \circ \nabla_{B_H \otimes H_L} \circ \left(p_H^B \otimes p_L\right) = \left(q_H^B \otimes \left(\Pi_H^L \circ g \circ \mu_B\right)\right) \circ \left(B \otimes \Pi_B^L \otimes f\right) \circ (\delta_B \otimes H), \tag{73}$$

$$\begin{aligned}
&\left(i_L \otimes i_H^B\right) \circ \nabla_{H_L \otimes B_H} \circ \left(p_L \otimes p_H^B\right) \\
&\quad = \left(\left(\Pi_H^L \circ g\right) \otimes \left(q_H^B \circ \mu_B\right)\right) \circ \left(B \otimes \Pi_B^L \otimes B\right) \circ \left((\delta_B \circ f) \otimes B\right),
\end{aligned} \tag{74}$$

(13) and (40). In the third one we use (40) and

$$\Pi_B^L \wedge \Pi_B^L = \Pi_B^L. \tag{75}$$

The fourth one follows from (70) and from the idempotent character of Π_H^L. Finally, in the fifth one we apply (75) for $B = H$.

On the other hand,

$$\begin{aligned}
&m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H} \\
&\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H} \\
&\quad = \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&\quad = \big((\varepsilon_{B_H} \circ \mu_{B_H}) \otimes B_H\big) \circ (B_H \otimes t_{B_H,B_H}) \circ \big((\delta_{B_H} \circ \eta_{B_H}) \otimes B_H\big) \\
&\quad = \big(\big(\varepsilon_B \circ q_H^B \circ \mu_B\big) \otimes p_H^B\big) \circ \big(\big(\mu_B \circ \big(q_H^B \otimes (f \circ g)\big) \circ \delta_B\big) \otimes c_{B,B}\big) \circ \big(\big(\delta_B \circ q_H^B \circ \eta_B\big) \otimes i_H^B\big) \\
&\quad = p_H^B \circ \Pi_B^L \circ i_H^B \\
&\quad = p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&\quad = u_{B_H} \circ e_{B_H}.
\end{aligned}$$

In these computations, the first equality follows from definition, the second one from (26) and (27), the third one from (4-1) of Proposition 2.9 of [1] and the fourth one is a consequence of (13) and the coassociativity of δ_B. The fifth equality follows from

$$\mu_B \circ \big(q_H^B \otimes (f \circ g)\big) \circ \delta_B = id_B, \tag{76}$$

and

$$q_H^B \circ \eta_B = \eta_B, \qquad \varepsilon_B \circ q_H^B = \varepsilon_B. \tag{77}$$

In the sixth one we use (40) and the last one follows from definition.

Finally, using similar arguments and (4-2) of Proposition 2.9 of [1] we obtain:

$$\begin{aligned}
&m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H} \\
&\quad = \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (B_H \otimes \lambda_{B_H}) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H} \\
&\quad = \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&\quad = \big(B_H \otimes (\varepsilon_{B_H} \circ \mu_{B_H})\big) \circ (t_{B_H,B_H} \otimes B_H) \circ \big(B_H \otimes (\delta_{B_H} \circ \eta_{B_H})\big) \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \Pi_B^R\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \big(f \circ \Pi_H^R \circ g\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ f \circ \big(id_H \wedge \Pi_H^R\big) \circ g \circ i_H^B \\
&\quad = p_H^B \circ f \circ g \circ i_H^B \\
&\quad = p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&\quad = u_{B_H} \circ e_{B_H}. \qquad \square
\end{aligned}$$

2.9. Let $g:B\to H$ and $f:H\to B$ be morphisms of weak Hopf algebras satisfying the equality $g\circ f=id_H$. Let $B_H\times H$ be the image of the idempotent morphism $\nabla_{B_H\otimes H}$. Then, we can define the following morphisms:

$$\eta_{B_H\times H}=p_{B_H\otimes H}\circ(\eta_{B_H}\otimes\eta_H):K\to B_H\times H,$$

$$\mu_{B_H\times H}:(B_H\times H)\otimes(B_H\times H)\to B_H\times H,$$

$$\begin{aligned}\mu_{B_H\times H}:={}&p_{B_H,H}\circ(\mu_{B_H}\otimes\mu_H)\\&\circ\big(B_H\otimes\big((\varphi_{B_H}\otimes H)\circ(H\otimes c_{H,B_H})\circ(\delta_H\otimes B_H)\big)\otimes H\big)\circ(i_{B_H,H}\otimes i_{B_H,H}),\end{aligned}$$

$$\varepsilon_{B_H\times H}=(\varepsilon_{B_H}\otimes\varepsilon_H)\circ i_{B_H,H}:B_H\times H\to K,$$

$$\delta_{B_H\times H}:B_H\times H\to(B_H\times H)\otimes(B_H\times H),$$

$$\begin{aligned}\delta_{B_H\times H}:={}&(p_{B_H,H}\otimes p_{B_H,H})\circ\big(B_H\otimes\big((\mu_H\otimes B_H)\circ(H\otimes c_{H,B_H})\circ(\varrho_{B_H}\otimes H)\big)\otimes H\big)\\&\circ(\delta_{B_H}\otimes\delta_H)\circ i_{B_H,H},\end{aligned}$$

where $\mu_{B_H\times H}$ is the weak version of the smash product and $\delta_{B_H\times H}$ the weak version of smash coproduct.

Finally, using the last theorem and Theorem 4.1 of [2] we obtain the complete version of Radford's Theorem linking weak Hopf algebras with projection and Hopf algebras in the category of Yetter–Drinfeld modules over H.

Theorem 2.10. *Let H, B be weak Hopf algebras in $\mathcal{C}$. Let $g:B\to H$ and $f:H\to B$ be morphisms of weak Hopf algebras such that $g\circ f=id_H$ and suppose that the antipode of H is an isomorphism. Then there exists a Hopf algebra B_H living in the braided monoidal category ${}^H_H\mathcal{YD}$ such that B is isomorphic to $B_H\times H$ as weak Hopf algebras, being the* (*co*)*algebra structure in $B_H\times H$ the smash* (*co*)*product, that is the* (*co*)*product defined in* 2.9. *The expression for the antipode of $B_H\times H$ is*

$$\begin{aligned}\lambda_{B_H\times H}:={}&p_{B_H,H}\circ(\varphi_{B_H}\otimes H)\\&\circ(H\otimes c_{H,B_H})\circ\big((\delta_H\circ\lambda_H\circ\mu_H)\otimes\lambda_{B_H}\big)\circ(H\otimes c_{B_H,H})\\&\circ(\varrho_{B_H}\otimes H)\circ i_{B_H,H}.\end{aligned}$$

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