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Special Symplectic Spaces

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INTRODUCTION

Symplectic geometry has found numerous applications in dynamics and field theory, see for example Refs. [1], [9], [11], [12], [17], and [23]. The notion, in symplectic geometry, of a Lagrangian subspace (submanifold) of a symplectic space (manifold) was introduced by V. I. Arnold [5] and A. Weinstein [24]. It arises in the analysis of asymptotic behavior of solutions of differential equations, cf. [19], [5], in the study of Fourier integral operators, cf. [13], [14] and in quantization theory, cf. [6], [7]. In most cases the symplectic spaces appearing in applications have the structure of dual pairs and the symplectic manifolds are isomorphic to cotangent bundles. The additional structure present in these geometries allows one to describe Lagrangian subspaces by generating functions. Functions generating canonical transformations have been commonly used in dynamics, see for example Ref. [15]. Generating functions of a wider class of Lagrangian submanifolds have been introduced in [21]. Other examples of generating functions include action functionals in variational formulations and Hamiltonian principal functions, cf. [15]. Both are closely related to differential equations.

A Lagrangian subspace of a symplectic space is a maximal isotropic subspace, however, the converse is not always true [24]. In this paper we study a special class of symplectic spaces, called special symplectic spaces, having the additional structure sufficient for the existence of generating functions of maximal isotropic subspaces. The choice of concepts and the degree of generality are suggested by expected applications to problems in partial differential equations, which are illustrated in a simple case of the Dirichlet and the Neumann problems for a second order linear equation with a formally self-adjoint partial differential operator. The theory developed is, however, directed towards application to linear systems of any order, involving formally self-adjoint matrix partial differential operators, and a much richer class of boundary conditions. The approach to symplectic geometry presented here is algebraic, without any reference to topology in the underlying vector spaces. Taking topology into account one can replace some of the algebraic conditions by topological ones. We are not doing this, however, at the present stage.

In Section 1, we review standard facts about dual pairs of vector spaces and establish the notation. In Section 2, we discuss elementary properties of symplectic spaces and their subspaces. Since we do not consider topology, our definition of a symplectic form generalizes the notion of a weak symplectic form on a Banach space [16].

In Section 3 we discuss the notion of a presymplectic space and its reduction (for a finite dimensional case see [23]). Section 4 contains the definition and properties of linear symplectic relations. Symplectic relations for cotangent bundles of finite dimensional manifolds were first introduced in [13]. These relations between symplectic manifolds, modelled on reflexive Banach spaces, were studied in [21]. Here we study linear symplectic relations, but in arbitrary vector spaces, hence the results of [21] are not directly applicable. In Sections 5 and 6 we introduce special symplectic and special presymplectic spaces and analyze some of their properties. It is shown that a symplectic space structure is isomorphic to that of the product of a dual pair of vector spaces. In Section 7 we discuss the properties of isotropic subspaces of special symplectic spaces, distinguishing a class of isotropic subspaces possessing generating forms. The notion of a generating form is then used in Section 8 to study the composition of symplectic relations. In Sections 9, 10, and 11 the theory developed in the preceding sections is illustrated on problems in partial differential equations. A linear second order formally self-adjoint differential operator A on a bounded domain $\Omega \subset R^n$ is considered. The space of C^∞ solutions of $Af = 0$ defines an isotropic subspace N of the space of Cauchy data on $\partial\Omega$[1] with symplectic form given by Green's second formula. Green's first formula is related to the generating form of N. The subspace N is maximal if A is elliptic; it is also maximal if A is the two-dimensional d'Alembert operator and Ω is a square whose diagonals are characteristics. If the sides of this square are characteristics then the space of solutions of $Af = 0$ is not maximal isotropic. This suggests a connection between a problem for the operator A being well posed, and the maximality of the isotropic subspace defined by

[1] For the sake of simplicity we use here the term "Cauchy data" even in the case when $\partial\Omega$ has characteristic directions.

the space of solutions to $Af = 0$. This connection will also be studied separately. If Ω is a union of two adjacent domains, $\Omega = \operatorname{int}(\bar{\Omega}_1 \cup \bar{\Omega}_2)$, and the solutions of $Af = 0$ give rise to maximal isotropic subspaces N, N_1, and N_2 in the spaces of Cauchy data on $\partial\Omega$, $\partial\Omega_1$, and $\partial\Omega_2$ respectively, then N, as a symplectic relation, is the composition of N_1 and N_2. In Section 12 we study the problem of the composition of symplectic relations corresponding to solutions of $Af = 0$ in Ω, where Ω is the union of an arbitrary number of domains.

2. Dual Pairs of Vector Spaces

Throughout this paper we work in the framework of the category of vector spaces and linear maps. Let E be a vector space, E^* its dual, and e and f any elements of E and E^*, respectively. We denote by $\langle e, f \rangle$ the value of the linear functional f on the vector e. The bilinear form $\langle\ ,\ \rangle$ on E and E^* is called the evaluation form. Given any bilinear form β on a pair of vector spaces E and F we identify it with the induced linear functional on $E \otimes F$, and we denote by $\langle e \otimes f, \beta \rangle$ the value of β on a pair of vectors (e, f), where $e \in E$ and $f \in F$.

A dual pair of vector spaces is a triplet (E, F, β), where E and F are vector spaces and β is a bilinear form on E and F such that

(i) $\langle e \otimes f, \beta \rangle = 0$, for all $f \in F$, implies $e = 0$,

(ii) $\langle e \otimes f, \beta \rangle = 0$, for all $e \in E$, implies $f = 0$.

Let (E, F, β) be a dual pair of vector spaces, and G a subspace of E. The β-orthogonal complement of G, denoted by G', is a subspace of F defined by $f \in G'$ if and only if $\langle e \otimes f, \beta \rangle = 0$ for each $e \in G$. One similarly defines the β-orthogonal complement of a subspace of F. For any subspaces G and H of E (or F) the following statements are true:

(i) $(G')' \supset G$,

(ii) $G \supset H$ implies $H' \supset G'$,

(iii) $(G + H)' = G' \cap H'$, where $G + H$ denotes the subspace spanned by G and H.

3. Symplectic Spaces

Let P be a vector space and ω a bilinear form on P. The transpose ω^T of ω is a bilinear form on P defined for each $p_1, p_2 \in P$ by $\langle p_1 \otimes p_2, \omega^T \rangle = \langle p_2 \otimes p_1, \omega \rangle$. For each $p \in P$, $p \rfloor \omega$ denotes the unique linear functional in P^* such that, for each $p' \in P$, $\langle p', p \rfloor \omega \rangle = \langle p \otimes p', \omega \rangle$. The form ω

is nondegenerate if $p \rfloor \omega = 0$ implies $p = 0$, and it is antisymmetric if $\omega^T = -\omega$. A bilinear form on a vector space is called symplectic if it is antisymmetric and nondegenerate. The definition of a symplectic form on a Banach space, due to A. Weinstein [25], requires that some additional topological conditions should be fulfilled. An antisymmetric nondegenerate bilinear form on a Banach space is called by J. E. Marsden [16] a weak symplectic form. We use here the term symplectic form in the more general sense than in Weinstein's definition for the sake of simplicity.

A symplectic space is a pair (P, ω), where P is a vector space and ω is a symplectic form on P. If (P, ω) is a symplectic space, then (P, P, ω) is a dual pair of vector spaces. An example of a symplectic space is furnished by a dual pair of vector spaces (E, F, β) as follows.

Let ω be a bilinear form on $E \oplus F$ defined by

$$\langle (e_1 + f_1) \otimes (e_2 + f_2), \omega \rangle = \langle e_2 \otimes f_1, \beta \rangle - \langle e_1 \otimes f_2, \beta \rangle$$

for all $e_1, e_2 \in E$ and $f_1, f_2 \in F$. Clearly, ω is antisymmetric and nondegenerate, hence $(E \oplus F, \omega)$ is a symplectic space.

Let (P, ω) be a symplectic space, M a subspace of P, and M' the ω-orthogonal complement of M. If $M \subset M'$ then M is called an isotropic subspace of (P, ω). If $M' \subset M$ then M is called a first class subspace of (P, ω), and if $M' \cap M = 0$, M is called a second class subspace of (P, ω); this terminology is an adaptation of that used by P. A. M. Dirac [10]. A maximal isotropic subspace M of (P, ω) is characterized by the condition $M = M'$. For a reflexive Banach space P it has been shown by J. E. Marsden and A. Weinstein that $(M')' = M$ if ω is weak symplectic and M is closed [18], however in general it does not hold. A Lagrangian subspace of (P, ω) is an isotropic subspace possessing an isotropic complement [23]. Every Lagrangian subspace of a symplectic space is maximal isotropic. A necessary and sufficient condition for a maximal isotropic subspace to admit an isotropic complement is given in the following proposition.

PROPOSITION 3.1. *A maximal isotropic subspace N of a symplectic space (P, ω) is Lagrangian if and only if there exists a subspace M of P such that $M + N = P$ and $M' + N = P$.*

Proof. If N is a Lagrangian subspace of (P, ω) then, for some isotropic subspace M, $M + N = P$. But $M \subset M'$, hence $M' + N = P$.

Conversely, let N be a maximal isotropic subspace of (P, ω) and M a subspace of P such that $M + N = M' + N = P$. This implies $N \cap M = N \cap M' = 0$. Hence every vector in P can be uniquely represented as a sum of vectors from M and N, and it can also be uniquely decomposed into a sum of vectors in M' and N. Let L be the subspace of P defined as follows, a vector $p = m + n = m' + n'$, where $m \in M$, $m' \in M'$, $n, n' \in N$, belongs to L if and only if $n + n' = 0$. Clearly, $P = N + L$. Let $p_1 = m_1 + n_1 = m_1' - n_1$ and $p_2 = m_2 + n_2 = m_2' - n_2$, where $m_1, m_2 \in M$, $m_1', m_2' \in M'$, and $n_1, n_2 \in N$, be any two vectors in L. Then,

$$\begin{aligned} \langle p_1 \otimes p_2, \omega \rangle &= \tfrac{1}{4}\langle (m_1 + m_1') \otimes (m_2 + m_2'), \omega \rangle \\ &= \tfrac{1}{4}\langle (m_1 - m_1') \otimes (m_2 - m_2'), \omega \rangle \\ &= \langle n_1 \otimes n_2, \omega \rangle = 0. \end{aligned}$$

Hence L is an isotropic subspace of (P, ω). Therefore N is a Lagrangian subspace of (P, ω). Q.E.D.

4. Presymplectic Spaces

A presymplectic space is a pair (P, ω) where P is a vector space and ω is an antisymmetric bilinear form on P. A presymplectic space (P, ω) is symplectic if ω is nondegenerate.

Let (P, ω) be a presymplectic space and $L = \{p \in P \mid p \rfloor \omega = 0\}$. We denote by P^r the quotient space P/L, the canonical projection by $\rho \colon P \to P^r$, and the bilinear form on P^r defined by

$$\langle \rho(p_1) \otimes \rho(p_2), \omega^r \rangle = \langle p_1 \otimes p_2, \omega \rangle \qquad \text{for all } p_1, p_2 \in P$$

by ω^r. Clearly, ω^r is antisymmetric and nondegenerate. The symplectic space (P^r, ω^r) is called the reduced space of a presymplectic space (P, ω).

Let (P, ω) be a presymplectic space and N a subspace of P. We denote by ω_N the restriction of ω to N. An isotropic subspace of a presymplectic space (P, ω) is a subspace N of P such that $\omega_N = 0$. This definition is an extension of the definition of an isotropic subspace of a symplectic space to the case of a presymplectic space.

5. Symplectic Relations

Let (P_1, ω_1) and (P_2, ω_2) be symplectic spaces. Let $P_1 \times P_2$ denote the product of the vector spaces P_1 and P_2; it is isomorphic to the direct sum $P_1 \oplus P_2$. The bilinear form $\omega_1 \oplus (-\omega_2)$ on $P_1 \times P_2$ is antisymmetric and nondegenerate, hence $(P_1 \times P_2, \omega_1 \oplus (-\omega_2))$ is a symplectic space. A linear symplectic relation from (P_1, ω_1) to (P_2, ω_2) is an isotropic subspace of $(P_1 \times P_2, \omega_1 \oplus (-\omega_2))$. Following the notation of point set theory, we use the symbol "$\circ$" to denote the composition of relations and the symbol

"-1" to denote the inverse. Clearly, a composition of linear symplectic relations is a linear symplectic relation, and the inverse of a linear symplectic relation is a linear symplectic relation. Symplectic relations were first introduced by Hörmander [13].

EXAMPLE 5.1. Let α be a symplectic isomorphism from (P_1, ω_1) to (P_2, ω_2), that is $\alpha: P_1 \to P_2$ is a vector space isomorphism such that, for each $p, p' \in P_1$, $\langle \alpha(p) \otimes \alpha(p'), \omega_2 \rangle = \langle p \otimes p', \omega_1 \rangle$. Then the graph of α is an isotropic subspace of $(P_1 \times P_2, \omega_1 \oplus (-\omega_2))$. Hence the graph of a symplectic isomorphism is a symplectic relation.

EXAMPLE 5.2. Let (P, ω) be a symplectic space, M a subspace of P, ω_M the restriction of ω to M, (P^r, ω^r) the reduced space of a presymplectic space (M, ω_M), and $\rho: M \to P^r$ the canonical projection. We denote by N the graph of ρ imbedded in $P \times P^r$. It is a linear symplectic relation from (P, ω) to (P^r, ω^r) called the reduction of a subspace M in (P, ω). The inverse relation $N^{-1} = \{(p', p) \in P^r \times P \mid (p, p') \in N\}$ is a linear symplectic relation from (P^r, ω^r) to (P, ω) called inverse reduction. The composition $N^{-1} \circ N$ is the graph of the identity map in P^r. The composition $N \circ N^{-1}$ is a linear symplectic relation from (P, ω) to itself such that $(p, p') \in N^{-1} \circ N$ if and only if $p, p' \in M$ and $\rho(p) = \rho(p')$.

LEMMA 5.3. *A linear symplectic relation N from (P_1, ω_1) to (P_2, ω_2) is the graph of a symplectic isomorphism if and only if $pr_1(N) = P_1$ and $pr_2(N) = P_2$, where pr_1 and pr_2 are the first and second projections from $P_1 \times P_2$, respectively.*

Proof. Let N be a linear symplectic relation from (P_1, ω_1) to (P_2, ω_2) such that $pr_1(N) = P_1$ and $pr_2(N) = P_2$. Suppose (p_1', p_2') and (p_1', p_2'') are in N, then $(0, p_2'' - p_2') \in N$. Since $pr_2(N) = P_2$, for each $p_2 \in P_2$, there exists $p_1 \in P_1$ such that $(p_1, p_2) \in N$, and

$$0 = \langle (p_1, p_2) \otimes (0, p_2'' - p_2'), \omega_1 \oplus (-\omega_2) \rangle = -\langle p_2 \otimes (p_2'' - p_2'), \omega_2 \rangle.$$

Hence, $(p_2'' - p_2') \rfloor \omega_2 = 0$ which implies that $p_2'' = p_2'$. Therefore N is the graph of a linear map from P_1 to P_2. Similarly, N^{-1} is the graph of a linear map from P_2 to P_1. Since N is a symplectic relation it follows immediately that it is a graph of a symplectic isomorphism.

Conversely, if N is the graph of a symplectic isomorphism, then it is a linear symplectic relation, $pr_1(N) = P_1$, and $pr_2(N) = P_2$. Q.E.D.

PROPOSITION 5.4. *Every linear symplectic relation is a composition of reduction, a symplectic isomorphism, and an inverse reduction.*

Proof. Let N be a linear symplectic relation from (P_1, ω_1) to (P_2, ω_2). We denote by K the projection of N to P_1, $K = pr_1(N)$, ω_K the restriction of ω_1 to K, $(P_1{}^r, \omega_1{}^r)$ the reduced space of a presymplectic space (K, ω_K), and by N_1 the symplectic relation from (P_1, ω_1) to $(P_1{}^r, \omega_1{}^r)$ corresponding to the reduction of K in (P_1, ω_1). Similarly, we denote by L the projection of N to P_2, ω_L the restriction of ω_2 to L, $(P_2{}^r, \omega_2{}^r)$ the reduced space of a presymplectic space (L, ω_L), and by N_2 the reduction of L in (P_2, ω_2). Let M be the subspace of $P_1{}^r \times P_2{}^r$ defined by $(p_1', p_2') \in M$ if and only if there exists $(p_1, p_2) \in N$ such that $\rho_1(p_1) = p_1'$ and $\rho_2(p_2) = p_2'$ where $\rho_1: K \to P_1{}^r$ and $\rho_2: L \to P_2{}^r$ are the canonical projections. It is easy to verify using Lemma 5.3 that M is the graph of a symplectic isomorphism from $(P_1{}^r, \omega_1{}^r)$ to $(P_2{}^r, \omega_2{}^r)$. Moreover, $N = N_2^{-1} \circ M \circ N_1$, where N_1 is a reduction, M is a graph of a symplectic isomorphism and N_2^{-1} is an inverse reduction. Q.E.D.

6. Special Symplectic Spaces

Let P be a vector space, and θ a bilinear form on P. We denote by L_θ and R_θ the left and the right characteristic spaces of θ, that is $L_\theta = \{p \in P \mid \langle p \otimes p', \theta\rangle = 0 \text{ for all } p' \in P\}$, and $R_\theta = \{p \in P \mid \langle p' \otimes p, \theta\rangle = 0$ for all $p' \in P\}$. If θ is antisymmetric its left and right characteristic spaces are equal.

A symplectic space (P, ω) is called special if the following additional structure is given: a vector space Q, a vector space epimorphism $\pi: P \to Q$, and a bilinear form θ on P such that (i) $\theta - \theta^T = \omega$, and (ii) $\operatorname{Ker} \pi = R_\theta$. Since the symplectic form ω is determined by θ, we shall denote this special symplectic space by a quadruplet (P, Q, π, θ).

A standard example of a special symplectic space is furnished by a dual pair of vector spaces (E, F, β) as follows. Let $pr_E: E \oplus F \to E$ be the natural projection, and let θ be a bilinear form on $E \oplus F$ defined by $\langle (e + f) \otimes (e' + f'), \theta\rangle = \langle e' \otimes f, \beta\rangle$. Then $\theta - \theta^T$ is a symplectic form on $E \oplus F$, and $(E \oplus F, E, pr_E, \theta)$ is a special symplectic space.

Let (P, Q, π, θ) be a special symplectic space. We denote by η the map from P into Q^* defined as follows, for each $p \in P$, $\eta(p)$ is the unique linear functional on Q such that $\langle \pi(p'), \eta(p)\rangle = \langle p \otimes p', \theta\rangle$ for all $p' \in P$.

Proposition 6.1. *If (P, Q, π, θ) is a special symplectic space then P is a product of Q and $\eta(P)$ with projections $\pi: P \to Q$ and $\eta: P \to \eta(P)$, and $(Q, \eta(P), \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ denotes the evaluation form, is a dual pair of vector spaces.*

Proof. Both maps $\pi\colon P \to Q$ and $\eta\colon P \to \eta(P)$ are epimorphisms. Let $p \in P$ be such that $\pi(p) = 0$ and $\eta(p) = 0$. Then, for each $p' \in P$,

$$\begin{aligned}\langle p' \otimes p, \theta - \theta^T \rangle &= \langle p' \otimes p, \theta \rangle - \langle p \otimes p', \theta \rangle \\ &= \langle \pi(p), \eta(p') \rangle - \langle \pi(p'), \eta(p) \rangle = 0.\end{aligned}$$

Since $\theta - \theta^T$ is nondegenerate, this implies that $p = 0$. Therefore P is a product of Q and $\eta(P)$ with projections π and η, respectively.

Further, if $\langle q, \eta(p) \rangle = 0$, for all $q \in Q$, then $\eta(p) = 0$. Let $q \in Q$ be such that $\langle q, \eta(p) \rangle = 0$ for all $p \in P$. Since P is a product of Q and $\eta(P)$ there exists $p \in P$ such that $\pi(p) = q$ and $\eta(p) = 0$. Then, for each $p' \in P$, $\langle p' \otimes p, \theta - \theta^T \rangle = 0$ which implies that $p = 0$. Hence $q = \pi(p) = 0$. Therefore, $(Q, \eta(P), \langle\ ,\ \rangle)$ is a dual pair of vector spaces. Q.E.D.

7. Special Presymplectic Spaces

A special presymplectic space is a quadruplet (P, Q, π, θ), where P and Q are vector spaces, $\pi\colon P \to Q$ is a vector space epimorphism, and θ is a bilinear form on P such that the characteristic space of $\theta - \theta^T$ is equal to the intersection $L_\theta \cap R_\theta$ of the left and the right characteristic spaces of θ, and $\operatorname{Ker} \pi \subseteq \operatorname{Ker} \pi + (L_\theta \cap R_\theta)$.

Given a special presymplectic space (P, Q, π, θ) let P^r denote the quotient space of P by $L_\theta \cap R_\theta$, $Q^r = Q/\pi(L_\theta \cap R_\theta)$, and $\rho\colon P \to P^r$ and $\sigma\colon Q \to Q^r$ denote the canonical projections. There exists a unique $\pi^r\colon P^r \to Q^r$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & Q \\ \rho\downarrow & & \downarrow\sigma \\ P^r & \xrightarrow{\pi^r} & Q^r \end{array}$$

Let θ^r be a bilinear form on P^r defined by $\langle \rho(p_1) \otimes \rho(p_2), \theta^r \rangle = \langle p_1 \otimes p_2, \theta \rangle$. The antisymmetric part of θ^r is a symplectic form on P^r. Further, $\operatorname{Ker} \pi^r = R_{\theta^r}$. Hence $(P^r, Q^r, \pi^r, \theta^r)$ is a special symplectic space, it is called the reduced space of a special presymplectic space (P, Q, π, θ).

If (P, Q, π, θ) is a special presymplectic space we denote by η a map from P to Q^* defined as follows, for each $p \in P$, $\eta(p)$ is the unique linear functional on Q such that $\langle \pi(p'), \eta(p) \rangle = \langle p \otimes p', \theta \rangle$ for all $p' \in P$. A special presymplectic space (P, Q, π, θ) is special symplectic if P is a product of Q and $\eta(P)$ with the projections π and η, respectively.

8. Isotropic Subspaces Generated by Forms

Let (P, Q, π, θ) be a special presymplectic space. A subspace N of P is called an isotropic subspace of (P, Q, π, θ) if N is an isotropic subspace of the presymplectic space $(P, \theta - \theta^T)$.

Let K be a subspace of Q and γ a symmetric bilinear form on K. The subspace N of P defined by $p \in N$ if and only if $\pi(p) \in K$ and, for each $q \in K$, $\langle q, \eta(p)\rangle = \langle q \otimes \pi(p), \gamma\rangle$ is an isotropic subspace of (P, Q, π, θ), since $p, p' \in N$ implies

$$\begin{aligned}\langle p \otimes p', \theta - \theta^T\rangle &= \langle p \otimes p', \theta\rangle - \langle p' \otimes p, \theta\rangle \\ &= \langle \pi(p'), \eta(p)\rangle - \langle \pi(p), \eta(p')\rangle \\ &= \langle \pi(p') \otimes \pi(p), \gamma\rangle - \langle \pi(p) \otimes \pi(p'), \gamma\rangle \\ &= 0.\end{aligned}$$

It is called the isotropic subspace generated by γ, and γ is called the generating form of N. The quadratic function $\frac{1}{2}\langle q \otimes q, \gamma\rangle$, is called the generating function of N.

Conversely, let N be an isotropic subspace of (P, Q, π, θ) and let $K = \pi(N)$. The restriction of θ to N induces a bilinear form γ on K as follows, for each $p, p' \in N$, $\langle \pi(p) \otimes \pi(p'), \gamma\rangle = \langle p \otimes p', \theta\rangle$. Since $\theta - \theta^T$ restricted to N is identically zero, the form γ is symmetric. The form γ generates an isotropic subspace of (P, Q, π, θ) which clearly contains the original isotropic subspace N. Therefore if N is maximal isotropic, it coincides with the isotropic subspace generated by γ. The class of isotropic subspaces generated by forms is in general larger than the class of maximal isotropic subspaces. A sufficient condition for an isotropic subspace of a special symplectic space generated by a form to be maximal or Lagrangian is given in the following propositions.

Proposition 8.1. *Let (P, Q, π, θ) be a special symplectic space, K a subspace of Q such that the orthogonal of the complement of K in the dual pair structure $(Q, \eta(P), \langle\ ,\ \rangle)$ coincides with K, and γ a symmetric bilinear form on K. The isotropic subspace N generated by γ is maximal if $\pi(N) = K$.*

Proof. Let $p \in P$ be such that $\pi(p) \in K$ and $\langle p \otimes p', \theta - \theta^T\rangle = 0$ for all $p' \in N$. This implies that, for each $p' \in N$,

$$\begin{aligned}\langle \pi(p'), \eta(p)\rangle - \langle \pi(p') \otimes \pi(p), \gamma\rangle &= \langle \pi(p'), \eta(p)\rangle - \langle \pi(p), \eta(p')\rangle \\ &= \langle p \otimes p', \theta\rangle - \langle p' \otimes p, \theta\rangle \\ &= 0,\end{aligned}$$

and therefore $p \in N$. If $p \in P$ is such that $\pi(p) \notin K$, then there exists $p' \in P$ such that $\pi(p') = 0$, $\langle q, \eta(p') \rangle = 0$ for all $q \in K$, and $\langle \pi(p), \eta(p') \rangle \neq 0$. The first two of the conditions above imply that $p' \in N$, while the third one gives $\langle p \otimes p', \theta - \theta^T \rangle = -\langle \pi(p), \eta(p') \rangle \neq 0$. Hence there is no isotropic subspace containing N and p. Therefore N is a maximal isotropic subspace of (P, Q, π, θ). Q.E.D.

PROPOSITION 8.2. *Let (P, Q, π, θ) be a special symplectic space, K a subspace of Q and γ a symmetric bilinear form on K. The isotropic subspace N generated by γ is Lagrangian if $\pi(N) = K$ and there exists a subspace L of Q such that $K + L = Q$ and $K' + L' = \eta(P)$, where K' and L' are the orthogonal complements of K and L, respectively, in the dual pair $(Q, \eta(P), \langle\ ,\ \rangle)$.*

Proof. Let M be the isotropic subspace generated by the zero bilinear form on L, that is $p \in M$ if and only if $\pi(p) \in L$ and $\eta(p) \in L'$. Since $\pi(N) = K$, for each $k \in K$, there exists $p \in N$ such that $\pi(p) = k$ and $\eta(p)$ agrees with $k \rfloor \gamma$ on K. The direct sum decomposition $P = K + L$ permits an extension of $k \rfloor \gamma$ to a linear functional $\widetilde{k \rfloor \gamma}$ in $\eta(P)$ in such a way that $\widetilde{k \rfloor \gamma} \in L'$.

Let p be any vector in P. Since P is a product of Q and $\eta(P)$, $Q = K + L$, and $\eta(P) = K' + L'$, there exist $k \in K$, $l \in L$, $r \in K'$ and $s \in L'$ such that $\pi(p) = k + l$ and $\eta(p) = r + s$. Let p_1 be the unique vector in P such that $\pi(p_1) = k$, and $\eta(p_1) = r + \widetilde{k \rfloor \gamma}$. Clearly, $p_1 \in N$. Further, let p_2 be the unique vector in P such that $\pi(p_2) = l$ and $\eta(p_2) = s - \widetilde{k \rfloor \gamma}$. Then $p_2 \in M$, and $p = p_1 + p_2$. Therefore M is an isotropic complement of N in P. Hence N is a Lagrangian subspace of $(P, \theta - \theta^T)$. Q.E.D.

9. SYMPLECTIC RELATIONS GENERATED BY FORMS

Let $(P_1, Q_1, \pi_1, \theta_1)$ and $(P_2, Q_2, \pi_2, \theta_2)$ be special symplectic spaces. Then $(P_1 \times P_2, Q_1 \times Q_2, \pi_1 \times \pi_2, \theta_1 \oplus (-\theta_2))$ is also a special symplectic space. A symplectic relation from $(P_1, Q_1, \pi_1, \theta_1)$ to $(P_2, Q_2, \pi_2, \theta_2)$ is an isotropic subspace N of $(P_1 \times P_2, Q_1 \times Q_2, \pi_1 \times \pi_2, \theta_1 \oplus (-\theta_2))$.

Let for each $i = 1, 2, 3$, $(P_i, Q_i, \pi_i, \theta_i)$ be a special symplectic space, N_{12} a symplectic relation from $(P_1, Q_1, \pi_1, \theta_1)$ to $(P_2, Q_2, \pi_2, \theta_2)$ generated by a symmetric bilinear form γ_{12} on a subspace K_{12} of $Q_1 \times Q_2$, and N_{23} a Lagrangian relation from $(P_2, Q_2, \pi_2, \theta_2)$ to $(P_3, Q_3, \pi_3, \theta_3)$ generated by a symmetric bilinear form γ_{23} on a subspace K_{23} of $Q_2 \times Q_3$. We want to characterize the composite relation $N_{12} \circ N_{23}$ in terms of the generating forms γ_{12} and γ_{23}. The forms γ_{12} and γ_{23} can be extended to bilinear symmetric forms $\tilde{\gamma}_{12}$ and $\tilde{\gamma}_{23}$, on $K = (K_{12} \times Q_3) \cap (Q_1 \times K_{23})$ such that

$$\langle (q_1, q_2, q_3) \otimes (q_1', q_2', q_3'), \tilde{\gamma}_{12} \rangle = \langle (q_1, q_2) \otimes (q_1', q_2'), \gamma_{12} \rangle,$$

and

$$\langle (q_1, q_2, q_3) \otimes (q_1', q_2', q_3'), \tilde{\gamma}_{23} \rangle = \langle (q_2, q_3) \otimes (q_2', q_3'), \gamma_{23} \rangle$$

for all (q_1, q_2, q_3) and (q_1', q_2', q_3') in K. Further, let E_1 be the projection of K_{12} to Q_2, and E_3 be the projection of K_{23} to Q_2. Then $E_1 \cap E_3$ is the projection of K to Q_2. We denote by E_1', E_3', $(E_1 \cap E_3)'$ the orthogonal complements of E_1, E_3, $E_1 \cap E_3$, respectively, in the dual pair $(Q_2, \eta_2(P_2), \langle\ ,\ \rangle)$.

PROPOSITION 9.1. *Let* $(E_1 \cap E_3)' = E_1' + E_3'$, *then* $(p_1, p_3) \in P_1 \times P_3$ *belongs to* $N_{12} \circ N_{23}$ *if and only if there exists* $q_2 \in Q_2$ *such that* $(\pi_1(p_1), q_2, \pi_3(p_3)) \in K$ *and, for all* $(k_1, k_2, k_3) \in K$,

$$\langle (k_1, k_2, k_3) \otimes (\pi_1(p_1), q_2, \pi_3(p_3)), \tilde{\gamma}_{12} + \tilde{\gamma}_{23} \rangle = \langle k_1, \eta_1(p_1) \rangle - \langle k_3, \eta_3(p_3) \rangle.$$

Proof. Let η_{ij} denote the map from $P_i \times P_j$ to $(Q_i \times Q_j)^*$ induced by the bilinear form $\theta_i \oplus (-\theta_j)$, that is, for each $(p_i, p_j) \in P_i \times P_j$ and each $(q_i, q_j) \in Q_i \times Q_j$,

$$\begin{aligned} \langle (q_i, q_j), \eta_{ij}(p_i, p_j) \rangle &= \langle q_i, \eta_i(p_i) \rangle - \langle q_j, \eta_j(p_j) \rangle \\ &= \langle p_i' \otimes p_i, \theta_i \rangle - \langle p_j' \otimes p_j, \theta_j \rangle, \end{aligned}$$

where p_i and p_j are any vectors in P_i and P_j, respectively, such that $\pi_i(p_i') = q_i$ and $\pi_j(p_j') = q_j$. If $(p_1, p_3) \in N_{12} \circ N_{23}$, then there exists $p_2 \in P_2$ such that $(p_1, p_2) \in N_{12}$ and $(p_2, p_3) \in N_{23}$. Clearly, $(\pi_1(p_1), \pi_2(p_2), \pi_3(p_3)) \in K$ and, for each $(k_1, k_2, k_3) \in K$,

$$\begin{aligned} &\langle (k_1, k_2, k_3) \otimes (\pi_1(p_1), \pi_2(p_2), \pi_3(p_3)), \tilde{\gamma}_{12} + \tilde{\gamma}_{23} \rangle \\ &\quad = \langle (k_1, k_2) \otimes (\pi_1(p_1), \pi_2(p_2)), \gamma_{12} \rangle + \langle (k_2, k_3) \otimes (\pi_2(p_2), \pi_3(p_3)), \gamma_{23} \rangle \\ &\quad = \langle (k_1, k_2), \eta_{12}(p_1, p_2) \rangle - \langle (k_2, k_3), \eta_{23}(p_2, p_3) \rangle \\ &\quad = \langle k_1, \eta_1(p_1) \rangle - \langle k_3, \eta_3(p_3) \rangle. \end{aligned}$$

Conversely, let for some $(p_1, p_3) \in P_1 \times P_3$ there exists $q_2 \in Q_2$ such that $(\pi_1(p_1), q_2, \pi_3(p_3)) \in K$ and, for all $(k_1, k_2, k_3) \in K$,

$$\langle (k_1, k_2, k_3) \otimes (\pi_1(p_1), q_2, \pi_3(p_3)), \tilde{\gamma}_{12} + \tilde{\gamma}_{23} \rangle = \langle k_1, \eta_1(p_1) \rangle - \langle k_3, \eta_3(p_3) \rangle.$$

Then there exists p_2' and p_2'' in P_2 such that $\pi_2(p_2') = \pi_2(p_2'') = q_2$, $(p_1, p_2') \in N_{12}$ and $(p_2'', p_3) \in N_{23}$. Hence, for each $(k_1, k_2, k_3) \in K$,

$$\langle (k_1, k_2, k_3) \otimes (\pi_1(p_1), q_2, \pi_3(p_3)), \tilde{\gamma}_{12} \rangle = \langle k_1, \eta_1(p_1) \rangle - \langle k_2, \eta_2(p_2') \rangle$$

and

$$\langle (k_1, k_2, k_3) \otimes (\pi_1(p_1), q_2, \pi_3(p_3)), \tilde{\gamma}_{23} \rangle = \langle k_2, \eta_2(p_2'') \rangle - \langle k_3, \eta_3(p_3) \rangle.$$

Therefore, for all $(k_1, k_2, k_3) \in K$, $\langle k_2, \eta_2(p_2' - p_2'') \rangle = 0$, and so $\eta_2(p_2' - p_2'') \in (E_1 \cap E_3)'$. Since $(E_1 \cap E_3)' = E_1' + E_3'$ there exist r' and r''

in P_2 such that $\pi_2(r') = \pi_2(r'') = 0$, $\eta(r') \in E_1'$, $\eta(r'') \in E_3'$, and $p_2' - p_2'' = r' - r''$. Let $p_2 = p_2' - r' = p_2'' - r''$. Then for each $(k_1, k_2) \in K_{12}$,

$$\begin{aligned}\langle (k_1, k_2) \otimes \eta_{12}(p_1, p_2)\rangle &= \langle k_1, \eta_1(p_1)\rangle - \langle k_2, \eta_2(p_2)\rangle \\ &= \langle k_1, \eta_1(p_1)\rangle - \langle k_2, \eta_2(p_2')\rangle + \langle k_2, \eta_2(r')\rangle \\ &= \langle k_1, \eta_1(p_1)\rangle - \langle k_2, \eta_2(p_2')\rangle \\ &= \langle (k_1, k_2) \otimes (\pi_1(p_1), \pi_2(p_2)), \gamma_{12}\rangle.\end{aligned}$$

Hence $(p_1, p_2) \in N_{12}$. Similarly, $(p_2, p_3) \in N_{23}$. Therefore $(p_1, p_3) \in N_{12} \circ N_{23}$.
Q.E.D.

Let $N_{12} \circ N_{23}$ be generated by a symmetric bilinear form γ_{13} on a subspace K_{13} of $Q_1 \times Q_3$. In order to determine K_{13} and γ_{13} in terms of γ_{12} and γ_{23} let us consider the following dual pair of vector spaces

$$\langle Q_1 \times Q_2 \times Q_3, \eta_1(P_1) \times \eta_2(P_2) \times \eta_3(P_3), \beta\rangle,$$

where

$$\begin{aligned}&\langle (q_1, q_2, q_3) \otimes (\eta_1(p_1), \eta_2(p_2), \eta_3(p_3)), \beta\rangle \\ &\quad = \langle q_1, \eta_1(p_1)\rangle - \langle q_2, \eta_2(p_2)\rangle + \langle q_3, \eta_3(p_3)\rangle.\end{aligned}$$

We denote by $(K_{12} \times 0)'$ and $(0 \times K_{23})'$ the β-orthogonal complements of $K_{12} \times 0$ and $0 \times K_{23}$, respectively.

PROPOSITION 9.2. *Let $N_{12} \circ N_{23}$ be generated by a symmetric bilinear form γ_{13} on a subspace K_{13} of $Q_1 \times Q_3$, and let $[(K_{12} \times 0) \cap (0 \times K_{23})]' = (K_{12} \times 0)' + (K_{23} \times 0)'$, then*

(i) *(q_1, q_3) belongs to K_{13} if and only if there exists $q_2 \in Q_2$ such that $(q_1, q_2, q_3) \in K$ and, for all $k_2 \in Q_2$ such that $(0, k_2, 0) \in K$,*

$$\langle (0, k_2, 0) \otimes (q_1, q_2, q_3), \tilde{\gamma}_{12} + \tilde{\gamma}_{23}\rangle = 0;$$

(ii) *for each (k_1, k_3) and (q_1, q_3) in K_{13},*

$$\langle (k_1, k_3) \otimes (q_1, q_3), \gamma_{13}\rangle = \langle (k_1, k_2, k_3) \otimes (q_1, q_2, q_3), \tilde{\gamma}_{12} + \tilde{\gamma}_{23}\rangle,$$

where q_2 is any vector in Q_2 such that (q_1, q_2, q_3) satisfies the condition (i), *and k_2 is any vector in Q_2 such that $(k_1, k_2, k_3) \in K$.*

This proposition is essentially a linearized version of Proposition 4.4 of Ref. [21], with the difference that more general spaces than reflexive Banach spaces are allowed here, and that is why the condition

$$[(K_{12} \times 0) \cap (0 \times K_{23})]' = (K_{12} \times 0)' + (0 \times K_{13})'$$

is needed. Its proof is analogous to the proof of Proposition 4.4 in Ref. [21].

10. Second Order Self-Adjoint Differential Equations

Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise C^∞ boundary $\partial\Omega$. That is

$$\partial\Omega = \bigcup_{l=1}^{k} \bar{S}_l ,$$

where the S_l's are pairwise disjoint $(n-1)$-dimensional C^∞ submanifolds of $\mathbb{R}^n$, and $\bar{S}_l$ denotes the closure of S_l in $\mathbb{R}^n$. For each $l = 1, 2,..., k$, we denote by n_l the unit normal to S_l pointing towards the exterior of Ω, and, by dS_l the induced surface element in S_l. Let us introduce the notation, $E = C^\infty(\partial\Omega)$, $F = C^\infty(\bar{\Omega})$, and

$$H = \bigtimes_{l=1}^{k} C^\infty(\bar{S}_l).$$

The spaces F and $F \times H$ are dual with respect to a bilinear form β defined as follows, for each $f \in F$ and $(g, (h_l)) \in F \times H$,

$$\langle f \otimes (g, (h_l)), \beta\rangle = \sum_{l=1}^{k} \int_{S_l} fh_l \, dS_l - \int_{\Omega} fg \, dx.$$

Similarly, the spaces E and H are dual with respect to a bilinear form β^r, defined by

$$\langle e \otimes (h_l), \beta^r\rangle = \sum_{l=1}^{k} \int_{S_l} eh_l \, dS_l ,$$

for each $e \in E$ and each $(h_l) \in H$. The dual pair structure $(F, F \times H, \beta)$ gives rise to a special symplectic space $(F \times F \times H, F, \pi, \theta)$, where $\pi: F \times F \times H \to F$ is the projection onto the first factor and

$$\langle (f, g, (h_l)) \otimes (f', g', (h_l')), \theta\rangle = \langle f' \otimes (g, (h_l)), \beta\rangle.$$

Similarly, the dual pair structure (E, H, β^r) gives rise to a special symplectic space $(E \times H, E, \pi^r, \theta^r)$ where $\pi^r: E \times H \to E$ is the projection to the first factor and

$$\langle (e, (h_l)) \otimes (e', (h_l')), \theta^r\rangle = \langle e' \otimes (h_l), \beta^r\rangle.$$

We denote by ω the antisymmetric part of θ and by ω^r the antisymmetric part of θ^r, i.e., $\omega = \theta - \theta^T$, $\omega^r = \theta^r - \theta^{rT}$.

The subspace $M = F \times 0 \times H$ of $F \times F \times H$ is a first class subspace of the symplectic space $(F \times F \times H, \omega)$ and (M, F, π_M, θ_M), where π_M is the restriction of π to M and θ_M is the restriction of θ to M, is a special presymplectic space. The reduced space of (M, F, π_M, θ_M) is isomorphic to $(E \times H, E, \pi^r, \theta^r)$. We denote by $\rho: M \to E \times H$ the canonical projection associating to each $(f, 0, (h_l)) \in M$, $\rho(f, 0, (h_l)) = (f \mid \partial\Omega, (h_l)) \in E \times H$.

Let A be a linear self-adjoint differential operator on $\bar{\Omega}$. For each $f \in F$,

$$Af = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) + af,$$

where $a, a_{ij} \in F$ and $a_{ij} = a_{ji}$. For any $f, f' \in F$ we have the following Green's formulae

$$\sum_{l=1}^{k} \int_{S_l} \left(\sum_{i,j=1}^{n} f a_{ij} \frac{\partial f'}{\partial x_j} n_l^i\right) dS_l - \int_\Omega f A f' \, dx = \int_\Omega \left(\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} a_{ij} \frac{\partial f'}{\partial x_j} - aff'\right) dx \qquad \text{(I)}$$

and

$$\sum_{l=1}^{k} \int_{S_l} \sum_{i,j=1}^{n} \left(f a_{ij} \frac{\partial f'}{\partial x_j} - f' a_{ij} \frac{\partial f}{\partial x_j}\right) n_l^i \, dS_l - \int_\Omega (fAf' - f'Af)\, dx = 0 \quad \text{(II)}$$

where n_l^i denotes the ith component of a unit normal n_l. Let N be the subspace of $F \times F \times H$ defined by $(f, g, (h_l)) \in N$ if and only if $Af = g$ and

$$\sum_{i,j=1}^{k} a_{ij} \frac{\partial f}{\partial x_j} n_l^i = h_l,$$

for each $l = 1, 2, \ldots, k$. From the Green's formula (II) it follows that N is an isotropic subspace of $(F \times F \times H, F, \pi, \theta)$. Further, $0 \times F \times H$ is also an isotropic subspace of $(F \times F \times H, F, \pi, \theta)$ and $N + 0 \times F \times H = F \times F \times H$. Hence N is Lagrangian. Clearly, $\pi(N) = F$, and the Green's formula (I) shows that N is generated by a symmetric bilinear form γ on F defined by the right hand side of (I), i.e.,

$$\langle f \otimes f', \gamma \rangle = +\int_\Omega \left(\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} a_{ij} \frac{\partial f}{\partial x_j} - aff'\right) dx.$$

Let us consider now a differential equation $Af = 0$. This amounts to the study of the intersection of N with $M = F \times 0 \times H$. Clearly, $N \cap M$ is an isotropic subspace of the special presymplectic space (M, F, π_M, θ_M) and, $\pi(M) = F = \pi(N)$, the form γ also generates $N \cap M$ in the special presymplectic structure (M, F, π_M, θ_M). The study of Dirichlet problems for the differential equation $Af = 0$ corresponds to the transition to the reduced space $(E \times H, E, \pi^r, \theta^r)$. The projection $\rho(N)$ of N to $E \times H$ is an isotropic subspace of $(E \times H, E, \pi^r, \theta^r)$, and it consists of these $(e, (h_l)) \in E \times H$ for which there exists $f \in F$ such that $Af = 0, f \mid \partial\Omega = e$, and

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial f}{\partial x_j} n_l^i = h_l .$$

The question whether $\rho(N)$ is generated by some bilinear form can be answered in special cases after a more thorough examination of the operator A.

We discuss two special cases in the subsequent sections.

11. Elliptic Boundary Value Problems

In addition to the hypotheses of the preceding section let us assume that A is an elliptic operator. We denote by K the projection of $\rho(N)$ to E, $K = \pi^r(\rho(N))$, that means K is the subspace of all admissible C^∞ Dirichlet data on $\partial\Omega$ for the equation $Af = 0$. Let L be the subspace of H defined as follows, $(h_l) \in L$ if and only if there exists $f \in F$ such that $Af = 0$, $f \mid \partial\Omega = 0$ and

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial f}{\partial x_j} n_l^i = h_l , \qquad l = 1, 2, \ldots, k.$$

Since A is elliptic L is a finite dimensional subspace of H. From Fredholm's theorems it follows also that $e \in K$ if and only if

$$\sum_{l=1}^{k} \int_{S_l} eh_l \, dS_l = 0$$

for all $(h_l) \in L$, cf. Ref. [20]. Therefore K is the β^r-orthogonal complement of L. Since L is finite dimensional, $K' = (L')' = L$ [8]. To show that $\rho(N)$ is maximal isotropic let us consider $(e, (h_l)) \in E \times H$ such that

$$\langle (e, (h_l)) \otimes (e', (h_l')), \omega^r \rangle = 0 \qquad \text{for all } (e', (h_l')) \in \rho(N).$$

Then, restricting this condition to $0 \times L \subset \rho(N)$, we get

$$\begin{aligned} 0 = \langle (e, (h_l)) \otimes (0, (h_l')), \theta^r - \theta^{rT} \rangle &= \langle (0, (h_l')) \otimes (e, (h_l)), \theta^r \rangle \\ &= \langle e \otimes (h_l'), \beta^r \rangle \end{aligned}$$

for all $(h_l') \in L$. Hence $e \in L' = K$, and there exists $f \in F$ such that $Af = 0$ and $f \mid \partial\Omega = e$. Let $(\tilde{h}_l) \in H$ be defined as follows, for each $l = 1, 2, \ldots, k$,

$$\tilde{h}^l = h_l - \sum_{i,j=1}^{n} a_{ij} \frac{\partial f}{\partial x_j} n_l^i, \qquad l = 1, 2, \ldots, k.$$

Then, for each $(e', (h_l')) \in N$, $\langle (0, (\tilde{h}_l)) \otimes (e', (h_l')), \omega^r \rangle = 0$ and therefore $\langle e' \otimes (\tilde{h}_l), \beta^r \rangle = 0$ for all $e' \in K$. Hence $(\tilde{h}_l) \in K' = L$, and there exists $\tilde{f} \in F$ such that $A\tilde{f} = 0$, $\tilde{f} \mid \partial\Omega = 0$, and

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial \tilde{f}}{\partial x_j} n_l^i = \tilde{h}_l , \qquad l = 1, 2, \ldots, k.$$

Further, $f + \tilde{f} \in F$ and it satisfies the following conditions, $A(f + \tilde{f}) = 0$, $(f + \tilde{f}) \mid \partial\Omega = e$, and

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial (f + \tilde{f})}{\partial x_j} n_l^i = h_l , \qquad l = 1, 2, \ldots, k,$$

which implies that $(e, (h_l)) \in \rho(N)$. Therefore $\rho(N)$ is a maximal isotropic subspace of $(E \times H, E, \pi^r, \theta^r)$. Every maximal isotropic subspace of a special symplectic space is generated by a bilinear symmetric form.

The generating form for $\rho(N)$ is obtained from that for $N \cap M$ if one expresses the integral

$$\int_\Omega \left(\sum_{i,j} \frac{\partial f}{\partial x_i} a_{ij} \frac{\partial f'}{\partial x_j} - aff' \right) dx$$

in terms of the boundary values of the functions f and f'. It is a generalization of the Hamilton principal function [15] to the case of a partial differential equation. The suitable spaces for elliptic differential operators are Sobolev spaces. The analysis given above extends easily to this case, yielding an additional result that $\rho(N)$ is Lagrangian since every maximal isotropic subspace of a Hilbert space is Lagrangian (A. Weinstein [24]).

12. The Wave Equation in $\mathbb{R}^2$

Suppose now that Ω is the interior of the square in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, and let $Af = \partial^2 f/\partial x^2 - \partial^2 f/\partial y^2$. The boundary of Ω consists of four segments of straight lines. Using the d'Alembert formula for the solution of the wave equation one can verify that $\rho(N)$ is a maximal isotropic subspace of $(E \times H, E, \pi^r, \theta^r)$ with the projection $K = \pi^r(\rho(N))$ uniquely characterized by the condition: $e \in K$ if and only if, for each $z \in [0, 1]$, $e(z, 0) + e(1 - z, 1) - e(0, z) - e(1, 1 - z) = 0$.

The generating function $\frac{1}{2}\langle e \otimes e, \gamma^r \rangle$ of $\rho(N)$ can be computed by expressing the generating function of $N \cap M$, given in this case by

$$\tfrac{1}{2}\langle f \otimes f, \gamma \rangle = \tfrac{1}{2} \int_\Omega \left| (\partial f/\partial x)^2 - (\partial f/\partial y)^2 \right| dx\, dy,$$

in terms of the restriction of f to Ω. This gives, for each $e \in K$,

$$\tfrac{1}{2}\langle e \otimes e, \gamma^r\rangle$$
$$= \tfrac{1}{2}\int_0^1 \{e(0, z)\, e_x(z, 0) - e(z, 0)\, e_y(0, z) + e(z, 1)\, e_y(1, z) - e(1, z)\, e_x(z, 1)\}\, dz$$

where e_x and e_y denote the derivatives of e with respect to the coordinates x and y, respectively.

It should be noted that if we took Ω to be the interior of a characteristic square, then $\rho(N)$ would not be a maximal isotropic subspace of $(E \times H, E, \pi^r, \theta^r)$.

Similar boundary value problems for the wave equation have been studied by R. A. Aleksandrian [2], [3], [4] and S. L. Sobolev [22].

13. Lagrangian Structure Over a Graph

In previous sections we have associated to any region in $\mathbb{R}^n$ a special symplectic space and to a differential equation in this region an isotropic subspace. If we have two adjacent regions then the isotropic subspace corresponding to the differential equation in the union of these regions can be obtained by composition from the isotropic subspaces corresponding to each region separately. In many cases, however, one encounters more general types of relations than binary symplectic relations. We give here a general framework for this type of relations and illustrate it by a discussion of a differential equation on a simplicial complex in $\mathbb{R}^n$.

A graph is a pair (V, S) where V is a set and S is a subset of $V \times V$ such that $(u, v) \in S$ implies $u \neq v$ and $(v, u) \in S$. If (V, S) is a graph and U is a subset of V we denote by BU the subset of S defined by $BU = \{(u, v) \in S \mid u \in U, v \notin U\}$.

A Lagrangian structure over a graph (V, S) consists of the following:

(i) A family $\mathscr{U}$ of subsets of V such that, for each $U \in \mathscr{U}$, BU is a finite nonempty set.

(ii) A family of special symplectic spaces $(P_U, Q_U, \pi_U, \theta_U)$ and their isotropic subspaces N_U generated by symmetric forms γ_U on $K_U \subset Q_U$, indexed over $\mathscr{U}$.

(iii) A family of special symplectic spaces $(P_{uv}, Q_{uv}, \pi_{uv}, \theta_{uv})$, indexed over S.

(iv) A family of vector space epimorphisms $\sigma_{Uuv}\colon P_U \to P_{uv}$ and $\tau_{Uuv}\colon Q_U \to Q_{uv}$ defined for each $U \in \mathscr{U}$ and each $(u, v) \in BU$.

The following conditions are satisfied:

(a) $P_{uv} = P_{vu}$, $Q_{uv} = Q_{vu}$, $\pi_{uv} = \pi_{vu}$, $\theta_{uv} + \theta_{vu} = 0$.

(b) The following diagram commutes

$$\begin{array}{ccc} P_U & \xrightarrow{\sigma_{Uuv}} & P_{uv} \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_{uv}} \\ Q_U & \xrightarrow{\tau_{Uuv}} & Q_{uv} \end{array}$$

(c) $P_U \ni p = 0$ if and only if $\sigma_{Uuv}(p) = 0$ for all $(u, v) \in BU$, $Q_U \ni q = 0$ if and only if $\tau_{Uuv}(q) = 0$ for all $(u, v) \in BU$.

(d) For each p, p' in P_U,

$$\langle p \otimes p', \mathscr{V}_U \rangle = \sum_{(u,v)\in BU} \langle \sigma_{Uuv}(p) \otimes \sigma_{Uuv}(p'), \theta_{uv} \rangle.$$

(e) For each $U \in \mathscr{U}$ and each finite family $\mathscr{X} \subset \mathscr{U}$ consisting of pairwise disjoint subsets of U covering U, the isotropic subspace N_U is related to the isotropic subspaces N_X, $X \in \mathscr{X}$, as follows: $p_U \in N_U$ if and only if, for each $X \in \mathscr{X}$, there exists a $p_X \in N_X$ such that

(α) $\sigma_{Xuv}(p_X) = \sigma_{Yvu}(p_Y)$, whenever $(u, v) \in BX$, $(v, u) \in BY$ and $X, Y \in \mathscr{X}$;

(β) $\sigma_{Xuv}(p_X) = \sigma_{Uuv}(p_U)$, whenever $(u, v) \in BX \cap BU$ and $X \in \mathscr{X}$.

Given $U \in \mathscr{U}$ and a finite family $\mathscr{X} \subset \mathscr{U}$ consisting of pairwise disjoint subsets of U covering U, we denote by K the subspace of $\times_{X\in\mathscr{X}} K_X$ such that $(q_X) \in K$ if and only if $\tau_{Xuv}(q_X) = \tau_{Yvu}(q_Y)$ whenever $(u, v) \in BX$ and $(v, u) \in BY$. Under the assumptions analogous to those in Proposition 9.2 the subspace K_U of Q_U is uniquely characterized by the following condition, $q_U \in K_U$ if and only if there exists $(q_X) \in K$ such that $\tau_{Xuv}(q_X) = \tau_{Uuv}(q_U)$ whenever $(u, v) \in BX \cap BU$ and, for all $(k_X) \in K$ such that $\tau_{Xuv}(k_X) = 0$ whenever $(u, v) \in BX \cap BU$,

$$\sum_{X\in\mathscr{X}} \langle k_X \otimes q_X, \gamma_X \rangle = 0.$$

Further, the generating form γ_U of N_U is given as follows. For each k_U and q_U in K_U,

$$\langle k_U \otimes q_U, \gamma_U \rangle = \sum_{X\in\mathscr{X}} \langle k_X \otimes q_X, \gamma_X \rangle,$$

where (k_X) and (q_X) are any elements of K satisfying together with (k_U) and (q_U), respectively, the conditions given above.

Consider a simplicial complex in $\mathbb{R}^n$ such that $\mathbb{R}^n$ is the space of the complex. Let V be the set of all n-simplices in the complex and let S be the subset of $V \times V$ consisting of all pairs of different n-simplices in the complex having an $(n - 1)$-simplex as a common face. Then (V, S) is a graph. Let $\mathscr{U}$ be the family of all finite, nonempty subsets of V. For each $U \in \mathscr{U}$, BU is finite and nonempty. Let, for each $U \in \mathscr{U}$, Ω_U be the interior of the union of the spaces of all n-simplices in U. Clearly, Ω_U is open and relatively compact. Let for each $(u, v) \in BU$, S_{uv} be the interior of the $(n - 1)$-simplex which is the common face of $u \in U$ and $v \notin U$. Then

$$\partial\Omega_U = \bigcup_{(u,v)\in BU} \bar{S}_{uv} .$$

Let, for each $(u, v) \in S$, $Q_{uv} = C^\infty(\bar{S}_{uv})$, $P_{uv} = C^\infty(\bar{S}_{uv}) \times C^\infty(\bar{S}_{uv})$, $\pi_{uv}: P_{uv} \to Q_{uv}$ be the projection onto the first factor. Further, let n_{uv} be the unit normal to S_{uv} directed towards the interior of the n-simplex v, and let dS_{uv} be the surface element in S_{uv} induced by the volume in $\mathbb{R}^n$ and the orientation of the normal n_{uv}. Then $n_{uv} = -n_{vu}$ and $dS_{uv} = -dS_{vu}$. Let θ_{uv} be the bilinear form on P_{uv} defined for each (e, h), $(e', h') \in P_{uv}$ by

$$\langle (e, h) \otimes (e', h'), \theta_{uv}\rangle = \int_{S_{uv}} e'h \, dS_{uv} .$$

Clearly $\theta_{uv} = -\theta_{vu}$ and $(P_{uv}, Q_{uv}, \pi_{uv}, \theta_{uv})$ is a special symplectic space.

For each $U \in \mathscr{U}$, let $Q_U = C^\infty(\partial\Omega_U)$, $H_U = \times_{BU} C^\infty(\bar{S}_{uv})$, $P_U = Q_U \times H_U$, $\pi_U: P_U \to Q_U$ be the projection onto the first factor, and let $\mathscr{V}_U$ be the bilinear form on P_U defined, for each (e, h_{uv}) and (e', h'_{uv}) in P_U, by

$$\langle (e, h_{uv}) \otimes (e', h'_{uv}), \mathscr{V}_U\rangle = \sum_{BU} \int_{S_{uv}} e'h_{uv} \, dS_{uv} .$$

Then $(P_U, Q_U, \pi_U, \mathscr{V}_U)$ is a special symplectic space.

For each $(u, v) \in BU$ we denote by $\tau_{Uuv}: Q_U \to Q_{uv}$ the restriction map, i.e. for each $e \in Q_U = C^\infty(\partial\Omega_U)$, $\tau_{Uuv}(e)$ is the restriction of e to $S_{uv} \subset \partial\Omega_U$, and by $\sigma_{Uuv}: P_U \to P_{uv}$ the map defined by $\sigma_{Uuv}(e, h_{uv}) = (\tau_{Uuv}(e), h_{uv})$. Both maps are vector space epimorphisms and the conditions (a), (b), (c), and (d) are satisfied.

Let $A: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be a linear self-adjoint elliptic differential operator of second order, given by

$$Af = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) + af.$$

For each $U \in \mathscr{U}$ we denote by N_U the isotropic subspace of $(P_U, Q_U, \pi_U, \mathscr{V}_U)$ defined as follows: $(e, h_{uv}) \in N_U$ if and only if there exists an $f \in C^\infty(\bar{\Omega}_U)$ such that $Af = 0$, $f \mid \partial\Omega_U = C$ and, for each $(u, v) \in BU$,

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial f}{\partial x_j} n_{uv}^i = h_{uv} .$$

Then the family $\{N_U\}_{U \in \mathscr{U}}$ satisfies the condition (e). Hence this construction gives an example of a Lagrangian structure over a graph as defined above.

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