Controllability for parabolic equations with nonlinear memory

Qiang Tao, Hang Gao

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

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1. Introduction and main results

Let \( \Omega \in \mathbb{R}^N \), \( N \in \mathbb{N} \), be a bounded domain with smooth boundary \( \partial \Omega \). We consider the following nonlinear parabolic system

\[
\begin{align*}
\frac{dy}{dt} - \Delta y &= \int_0^t a(x, s)f(y(x, s))ds + \chi_{\omega} u, & \text{in } Q_T = \Omega \times (0, T), \\
y(x, t) &= 0, & \text{on } \Sigma_T = \partial \Omega \times (0, T), \\
y(x, 0) &= y_0(x), & \text{in } \Omega,
\end{align*}
\]

(1.1)

where \( a \in L^\infty(Q_T) \) is a given function, \( f \in C(\mathbb{R}) \) is a locally Lipschitz continuous function and there exists a function \( g \in C^1(\mathbb{R}) \) such that \( |f(s)| \leq g(s) \), for any \( s \in \mathbb{R} \), \( u \in L^\infty(Q_T) \) is a control that acts on the non-empty set \( \omega \subset \Omega \), \( \chi_{\omega} \) is the characteristic function of \( \omega \), and \( y_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \).

Such equations describe diffusion phenomena with nonlocal reaction terms in time and they arise in many fields such as heat conduction in materials with memory, population dynamics, nuclear reactors, etc. (See for instance [1–3].) In this paper, we are concerned with the controllability and the lack of controllability of (1.1).

We say that system (1.1) is approximately controllable at time \( T > 0 \), if for any \( y_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \), target \( y_d \in L^2(\Omega) \) and \( \varepsilon > 0 \), there exists a corresponding control function \( u \in L^\infty(Q_T) \) such that \( \| y(T) - y_d \|_{L^2(\Omega)} < \varepsilon \). System (1.1) is null controllable at time \( T > 0 \), if for any \( y_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \), there exists a corresponding control function \( u \in L^\infty(Q_T) \) such that \( y(x, T) = 0 \) a.e. in \( \Omega \).

Controllability and noncontrollability of a control system \( y_t - \Delta y + f(y) = \chi_{\omega} u \) is of great interest to many people (see [4–10] and the references therein). It is well known that if \( f \in C(\mathbb{R}) \) is a globally Lipschitz continuous function, the system is null controllable and approximately controllable. Most of these results are established by applying the fixed-point argument and the fact that such semilinear equations can be viewed as “linear equations” with the coefficients uniformly..
bounded in some sense. If \( f \in C(\mathbb{R}) \) is a locally Lipschitz continuous function, Fernández-Cara and Zuazua [9] showed the controllability of the system with a superlinear term. They proved that the system is null controllable and approximately controllable at any time provided the nonlinear term \( f(y) \) is such that \( |f(s)| \) grows slower than \( |s| \log^{3/2}(1 + |s|) \) as \( |s| \to \infty \), and for some functions \( f \) that behave at infinity like \( |s| \log^p(1 + |s|) \) with \( p > 2 \), controllability does not hold.

Controlability of systems with nonlinear memory has been studied by some authors. Sakthivel et al. [11] obtained the null controllability of the system \( y_t - \Delta y = \int_0^1 k(t, s)f(y(x, s))ds + \chi_\omega u \) with sublinear memory, i.e. \( f(y) \) is a globally Lipschitz continuous function. The proof relies on a Carleman inequality which requires that the memory kernel \( k(t, s) \) is sufficiently smooth and has support about \( t \) in \( (t_0, t_1) \) where \( 0 < t_0 < t_1 < T \). In [12], the authors showed the similar results for systems with mixed and Neumann boundaries. However, as far as we know, few works are concerned with the controllability and noncontrollability of system (1.1) with nonlinear memory having a superlinear growth. This is the precise problem which we consider in this paper. We shall prove that systems fail to be controllable with power-like nonlinear memory, i.e. in the more restrictive class of nonlinear terms growing at infinity like \( |s|^p \) with \( p > 1 \). In other words, no matter what control function is chosen, making use of a localized estimate in \( \Omega \setminus \tilde{\omega} \), we can see that the blow-up phenomena will still happen. On the other hand, we shall show that for initial data and target, both of which vanish identically in exterior domain of \( \omega \), system (1.1) is approximately controllable at any time \( T \) and the system is also null controllable with a class of smooth initial data. These results can be extended to other general equations.

For any \( \theta \in (0, 1) \) and any \( k, l \in \mathbb{N} \), we denote Hölder spaces

\[
C^{k+\theta, l+\theta} (\Omega_T) = \left\{ f \in C^k(\Omega) ; \sup_{|\beta|=k, (x,t) \neq (y,s)} \frac{|\partial^\beta_x f(x, t) - \partial^\beta_x f(y, s)|}{(|x-y| + |t-s|^{1/2})^\theta} < +\infty \right\}
\]

and

\[
C^{2, \theta} (\Omega) = \left\{ f \in C^2(\Omega) ; \sup_{|\beta|=2} \frac{|\partial^\beta_x f(x) - \partial^\beta_x f(y)|}{|x-y|^{\theta}} < +\infty \right\},
\]

both of which are Banach spaces with canonical norms. Throughout this paper, we study the weak solution of system (1.1). Let \( U = C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^\infty(\Omega_T) \) for some \( T > 0 \). As in [13], we give the definition of the weak solution to (1.1) as follows:

**Definition 1.1.** A function \( y \) is called a weak solution of (1.1) on \([0, T]\), if \( y \in U \) and satisfies

\[
\begin{align*}
\iint_{\Omega_T} y \psi_t \, dx \, dt - \iint_{\Omega_T} \nabla y \nabla \psi \, dx \, dt + \int_{\Omega_T} y_0 \psi(x, 0) \, dx &+ \iiint_{\Omega_T} \left( \int_0^t a(x, s)f(y(x, s)) \, ds \right) \psi \, dx \, dt \\
+ \iint_{\Omega_T} \chi_\omega u \psi \, dx \, dt &= 0
\end{align*}
\]

for any \( \psi \in U \cap W^{1, 1}(\Omega_T) \); \( \psi(T) = 0 \).

The main results of this paper are the following

**Theorem 1.1.** Assume that there exists a constant \( \mu_0 > 0 \), such that \( a \geq \mu_0 \) and \( y_0(x) \neq 0 \) in \( \Omega \setminus \tilde{\omega} \). If the function \( f \) satisfies \( f(s) \geq C_0 |s|^p \) with \( p > 1 \) and \( C_0 \) is a positive constant, then system (1.1) fails to be controllable.

**Theorem 1.2.** Let \( f(0) = 0 \), then for any \( y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega) \) with \( \{ x \in \Omega ; y_0(x) \neq 0 \} \subset \omega \) and \( y_d \in L^2(\Omega) \) with \( \{ x \in \Omega ; y_d(x) \neq 0 \} \subset \omega \), system (1.1) is approximately controllable at time \( T \).

**Theorem 1.3.** Let \( f(0) = 0 \), then for any \( y_0 \in C^{2, \frac{1}{2}}(\Omega) \) satisfying the first order compatibility condition with \( \text{supp} y_0 \subset \omega \), system (1.1) is null controllable at time \( T \).

**Remark 1.1.** With the method we prove Theorems 1.2 and 1.3, for a class of initial data and targets, one can easily get the controllability for more general equation \( y_t - \Delta y = f(x, t, y, \nabla y) + \chi_\omega u \) under appropriate assumptions on \( f \) to guarantee the uniqueness of a solution.

The rest of the article is organized as follows. In Section 2, we investigate the local existence and uniqueness of the solution of (1.1). Sections 3 and 4 are devoted to the study of noncontrollability and controllability results respectively.

**2. Local existence and uniqueness**

Since system (1.1) is nonlinear, we need to give the existence and uniqueness of the local weak solution to (1.1). In order to prove the existence, we consider the following system.
\[ \begin{aligned}
\begin{cases}
  y_t - \Delta y = \int_0^t a(x, s)f(y(x, s))ds + h(x, t), & \text{in } Q_T, \\
  y(x, t) = 0, & \text{on } \Sigma_T, \\
  y(x, 0) = y_0(x), & \text{in } \Omega,
\end{cases}
\end{aligned} \tag{2.1} \]

where \( h \) is a \( L^\infty \) function, \( a \) and \( f \) are defined as in (1.1).

The following two comparison principles are the main tools of our work.

**Lemma 2.1.** If \( y \in C(Q_T) \cap C^{2,1}(Q_T) \) satisfies
\[ \begin{aligned}
\begin{cases}
  y_t - \Delta y \leq \int_0^t a(x, s)y(x, s)ds, & \text{in } Q_T, \\
  y(x, t) \leq 0, & \text{on } \Sigma_T, \\
  y(x, 0) \leq 0, & \text{in } \Omega,
\end{cases}
\end{aligned} \tag{2.2} \]

where \( a \in C(Q_T) \), then \( y \leq 0 \) for \( (x, t) \in Q_T \).

The proof is standard. For completeness, we show the details.

**Proof.** Set \( v(x, t) = e^{c_0 t}y \), where \( c_0 < 0 \) is to be determined later. Then
\[ v_t - \Delta v - c_0v - e^{c_0 t} \int_0^t a(x, s)e^{-c_0 s}v(x, s)ds \leq 0. \tag{2.3} \]

Suppose that \( v \) achieves its positive maximum in \( Q_T \) at \( (x_0, t_0) \in \Omega \times (0, T] \). Then we have
\[ v(x_0, t_0) > 0, \quad \Delta v(x_0, t_0) \leq 0, \quad \text{and } \quad v_t(x_0, t_0) \geq 0. \]

Thus,
\[ \left[ v_t - \Delta v - c_0v - e^{c_0 t} \int_0^t a(x, s)e^{-c_0 s}v(x, s)ds \right] (x_0, t_0) \geq -c_0v(x_0, t_0) - e^{c_0 t_0} \int_0^{t_0} a(x_0, s)e^{-c_0 s}v(x_0, s)ds \]
\[ \geq -c_0v(x_0, t_0) - e^{c_0 t_0} \int_0^{t_0} a(x_0, s)e^{-c_0 s}v(x_0, s)ds \]
\[ \geq -(c_0 + e^{c_0 t_0} \| a \|_{L^\infty(Q_T)})v(x_0, t_0). \]

Choosing \( c_0 < 0 \) appropriate small such that \( c_0 + e^{c_0 t_0} \| a \|_{L^\infty(Q_T)} < 0 \), we arrive at a contradiction to (2.3). \( \square \)

The following version of the comparison theorem is used in showing the existence of a local solution.

**Theorem 2.1.** Let \( f \in C^1(\mathbb{R}) \). Suppose \( u, v \in C(Q_T) \cap C^{2,1}(Q_T) \) satisfy
\[ \begin{aligned}
\begin{cases}
  u_t - \Delta u \geq \int_0^t a(x, s)f(u)ds + h(x, t), & \text{in } Q_T, \\
  v_t - \Delta v \leq \int_0^t a(x, s)f(v)ds + h(x, t), & \text{in } Q_T, \\
  u(x, t) \geq v(x, t), & \text{on } \Sigma_T, \\
  u(x, 0) \geq v(x, 0), & \text{in } \Omega,
\end{cases}
\end{aligned} \tag{2.4} \]

where \( a \in C(Q_T) \), then \( u \geq v \) for \( (x, t) \in Q_T \).

**Proof.** By (2.4), we can obtain
\[ \begin{aligned}
\begin{cases}
  (v - u)_t - \Delta (v - u) \leq \int_0^t a(x, s)(f(v) - f(u))ds, & \text{in } Q_T, \\
  v - u \leq 0, & \text{on } \Sigma_T, \\
  v(x, 0) - u(x, 0) \leq 0, & \text{in } \Omega.
\end{cases}
\end{aligned} \]

Let \( \varphi = v - u \) and \( \tilde{a} \) be the continuous function defined by
\[ \tilde{a}(x, s) = a(x, s) \times \begin{cases}
  \frac{f(v) - f(u)}{v - u}, & v \neq u, \\
  f'(u), & v = u.
\end{cases} \]

Then \( \varphi \) satisfies
\[
\begin{aligned}
\left\{ \begin{array}{l}
\varphi_t - \Delta \varphi \leq \int_0^t a(x, s) \varphi \, ds, & \text{in } Q_T, \\
\varphi \leq 0, & \text{on } \Sigma_T, \\
\varphi(x, 0) \leq 0, & \text{in } \Omega.
\end{array} \right.
\end{aligned}
\]

By Lemma 2.1, we obtain that \( \varphi \leq 0 \) in \( Q_T \). The conclusion of the theorem follows immediately. \( \square \)

Based on the comparison theorem, it follows the existence and uniqueness theorem.

**Theorem 2.2.** For any \( y_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \), there exists \( T_1 \in (0, +\infty) \) such that (2.1) has a unique solution \( y \in C([0, T_1]; L^2(\Omega)) \cap L^2(0, T_1; H^1_0(\Omega)) \cap L^\infty(\Omega_T) \).

**Proof.** Consider the sequence of problems

\[
\begin{aligned}
&y_t - \Delta y = \int_0^t a_n(x, s) f_n(y(x, s)) \, ds + h_n(x, t), & \text{in } Q_T, \\
y(x, t) = 0, & \text{on } \Sigma_T, \\
y(x, 0) = y_{0n}(x), & \text{in } \Omega,
\end{aligned}
\]

where \( f_n \in C^1(\mathbb{R}), f_n \to f \) uniformly on bounded subsets of \( \mathbb{R} \) and \( |f_n(y)| \leq g(y); y_{0n} \in C^\infty(\Omega) \), such that \( \|y_{0n}\|_{L^\infty(\Omega)} \leq \|y_0\|_{L^\infty(\Omega)}, \|y_{0n}\|_{H^1(\Omega)} \leq \|y_0\|_{H^1(\Omega)} \) and \( y_{0n} \to y_0 \) strongly in \( H^1(\Omega) \) (see Chapter 2 of [14] and Chapter 4 of [15]).

It is well known that (2.5) has a locally classical solution \( y_n \) with \( T_n^* \in (0, T) \) as the maximum existence time in \( L^\infty \) norm sense (see [16, 17]). Now we claim that there exist \( T_0 \in (0, T), T_1 \in (0, T_0) \) and a constant \( M_0 \) such that

\[
T_n^* \geq T_0 \quad \text{and} \quad \|y_n\|_{L^\infty(\Omega_{T_1})} \leq M_0 \quad \text{for all } n.
\]

In fact, let \( v^\pm \) be the solutions of the ordinary differential equations

\[
\begin{aligned}
&\frac{dv^\pm}{dt} = \pm M_1 \int_0^t g(v^\pm) \, ds \pm M_2, \\
v^\pm(0) = \pm \|y_0\|_{L^\infty(\Omega)}.
\end{aligned}
\]

By standard theory, there exists \( T^* \in (0, T) \) such that \( v^\pm \) exist on \([0, T^*] \) and \( T^* \) depends on \( g \) and \( \|y_0\|_{L^\infty(\Omega)} \). We will show that \( T_n^* \geq T^* \). If \( T_n^* < T^* \), by Theorem 2.1, we have

\[
|y_n(x, t)| \leq \max\{v^+(t), -v^-(t)\}, \quad 0 < t < T_n^*, x \in \Omega.
\]

Then, let \( t \to T_n^{*-} \). We arrive at a contradiction. Thus, we obtain that

\[
|y_n(x, t)| \leq \max\{v^+(t), -v^-(t)\}, \quad 0 < t < T^*, x \in \Omega,
\]

and we can choose \( T_0 = T^* \).

Setting \( T_1 = \min\{\frac{T^*}{2}, T_0\} \). After taking \( M_0 = \max\{v^+(T_1), -v^-(T_1)\} \), we obtain (2.6).

Multiplying (2.5) by \( y_n \) and integrating over \( \Omega \), we get

\[
\frac{1}{2} \int_{Q_{T_1}} y_n^2 \, dx + \int_{Q_{T_1}} |\nabla y_n|^2 \, dx = \int_{Q_{T_1}} y_n \left( \int_0^t a_{nf_n}(y_n) \, ds \right) \, dx + \int_{Q_{T_1}} h_n y_n \, dx.
\]

Integrating over \((0, t)\) for \( 0 < t < T_1 \) and using (2.6), we may derive

\[
\|\nabla y_n\|_{L^2(Q_{T_1})} \leq C, \tag{2.7}
\]

where here and below \( C \) denotes the constants independent of \( n \).

Multiplying (2.5) by \( y_{nt} \) and integrating over \( Q_{T_1} \), we have

\[
\begin{aligned}
&\int_{Q_{T_1}} y_{nt}^2 \, dx + \frac{1}{2} \int_{Q_{T_1}} \frac{d}{dt} |\nabla y_n|^2 \, dx dt \\
&= \int_{Q_{T_1}} y_n \left( \int_0^t a_{nf_n}(y_n) \, ds \right) \, dx dt + \int_{Q_{T_1}} h_n y_n \, dx dt \\
&\leq \frac{1}{2} \int_{Q_{T_1}} y_{nt}^2 \, dx dt + C \left( \int_{Q_{T_1}} \left( \int_0^t a_{nf_n}(y_n) \, ds \right)^2 \, dx dt + \int_{Q_{T_1}} h_n^2 \, dx dt \right).
\end{aligned}
\]
It follows
\[
\int_0^\infty \int_{Q_{\tau_1}} y_n^2 \, dx \, dt \leq C \left( \int_\Omega |\nabla y_0|^2 \, dx + \int_0^\infty \int_{Q_{\tau_1}} \left( \int_0^t a_{\phi_n} (y_n) \, ds \right)^2 \, dx \, dr + \int_0^\infty 1_{Q_{\tau_1}} \, h_n^2 \, dx \, dr \right),
\]
then we obtain
\[
\int_0^\infty \int_{Q_{\tau_1}} y_n^2 \, dx \, dt \leq C.
\]

Inequalities (2.7) and (2.8) imply that there are a subsequence \( \{ n_k \}_{k=1}^\infty \) of \( \{ n \}_{n=1}^\infty \) and a function \( y \in C([0, T_1]; L^2(\Omega)) \cap L^2(0, T_1; H^1_0(\Omega)) \) such that
\[
y_{n_k} \to y, \quad a_n f_{n_k} (y_n(x, t)) \to af(y) \quad \text{a.e. on } Q_{\tau_1},
\]
\[
\nabla y_{n_k} \to \nabla y \quad \text{weakly in } (L^2(Q_{\tau_1})), \quad y_{n_k t} \to y_t \quad \text{weakly in } (L^2(Q_{\tau_1})),
\]
as \( n_k \to \infty \). Thus, the existence follows by a standard limiting process.

Now, we prove the uniqueness of the solution. Let \( y_1, y_2 \) be two solutions of (2.1). From the definition of a weak solution, we have
\[
\frac{1}{2} \int_\Omega (y_1 - y_2)^2 (t) \, dx \leq \|a\|_{L^\infty} \int_\Omega \int_0^t \left( \int_0^\tau |f(y_1) - f(y_2)| \, d\tau \right) |y_1 - y_2| \, dx \, dt
\]
\[
\leq \|a\|_{L^\infty} T_1 \int_\Omega \int_0^t |y_1 - y_2|^2 \, dx \, dt.
\]
Recalling Gronwall’s inequality, we arrive at
\[
\int_\Omega (y_1 - y_2)^2 (t) \, dx = 0,
\]
that is \( y_1 = y_2 \), which completes the proof of Theorem 2.2.

3. Proof of Theorem 1.1

In the section, we give the proof of the lack of controllability for the system (1.1) with superlinear growth memory. We prove a localized estimate in \( \Omega \backslash \bar{\omega} \) which shows that the control cannot compensate the blow-up phenomena occurring in \( \Omega \backslash \bar{\omega} \) by multiplying a smooth potential. To this end, we show a necessary lemma.

Lemma 3.1 (Jensen’s Inequality). Assume \( \Phi \) is convex on \([0, +\infty)\), \( g(x) \geq 0 \) on \( \mathbb{R}^n \) and \( \int_{\mathbb{R}^n} g(x) \, dx > 0 \), \( f(x) \geq 0 \), then
\[
\Phi \left( \int_{\mathbb{R}^n} g(x) f(x) \, dx \right) \int_{\mathbb{R}^n} g(x) \, dx \leq \int_{\mathbb{R}^n} g(x) \Phi(f(x)) \, dx \int_{\mathbb{R}^n} g(x) \, dx.
\]

Proof of Theorem 1.1. Let \( \omega' \subset \Omega \) be a subdomain of \( \Omega \) such that \( \omega \subset \omega' \), \( \partial \omega' \in C^\infty \). Denote by \( \rho \in C^\infty(\overline{\Omega}) \) a function such that
\[
\rho|_{\omega} = 0, \quad \rho|_{\omega'} = 0, \quad 0 < \rho(x) < 1 \quad \forall x \in \Omega \backslash \omega'.
\]
Here and below, we denote different positive constants by \( C, C_i, C'_i, i = 1, 2, \ldots, 6 \), which depend only on the known quantities.

Multiplying (1.1) by \( \rho^k \) with \( k > \frac{2p}{p-1} \) and integrating by parts with respect to variable \( x \), we can get
\[
\frac{d}{dt} \int_\Omega \rho^k y \, dx = \int_\Omega (\Delta \rho^k) y \, dx + \int_0^t \int_\Omega \rho^k a(x, s) f(y) \, dx \, ds.
\]
As for \( |\Delta \rho^k| \leq C \rho^{k-2}, \quad a \geq \mu_0 \) and \( f(s) \geq C_0 |s|^p \), we have
\[
\frac{d}{dt} \int_\Omega \rho^k y \, dx \geq -C \int_\Omega \rho^{k-2} |y| \, dx + C_0 \mu_0 \int_0^t \int_\Omega \rho^k |y|^p \, dx \, ds.
\]
According to Hölder’s inequality, \( k > \frac{2p}{p-1} \) and \( 0 < \rho < 1 \), we obtain
\[
\int_\Omega \rho^{k-2} |y| \, dx \leq C \left( \int_\Omega \rho^{(k-2)p} |y|^p \, dx \right)^{1/p} \leq C \left( \int_\Omega \rho^k |y|^p \, dx \right)^{1/p}.
\]
Theorem 2.2

Choosing 1 < r < p and from (3.1), we get
\[
\frac{d}{dt} \int_\Omega \rho^k y dx \geq -C \left( \int_\Omega \rho^k |y|^p dx \right)^{1/p} + C_0 \mu_0 \int_0^t \int_\Omega \rho^k |y|^p dx ds.
\]  
(3.2)

Define the functions
\[
\eta(t) = \int_\Omega \rho^k(x) y(x, t) dx \quad \text{and} \quad z(t) = \int_\Omega \int_\Omega \rho^k(x) |y(x, s)|^p dx ds, \quad 0 \leq t < T.
\]

It is easy to see that
\[
z'(t) = \int_\Omega \rho^k(x) |y(x, t)|^p dx, \quad \eta(0) = \int_\Omega \rho^k(x) y_0(x) dx \quad \text{and} \quad z(0) = 0.
\]

There exists \( y_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \), such that \( \eta(0) > 0 \) (we can take \( y_0 > 0 \) and \( y_0 |_{\partial \Omega} = 0 \) for example). Then Theorem 2.2 implies that for any \( u \in L^\infty(\Omega_T) \), there exists a \( t_0 > 0 \) small enough, such that \( \eta(t) > 0 \), \( t \in (0, t_0] \). In view of (3.2), we have
\[
\eta' \geq -C(z')^{1/p} + C_0 \mu_0 z, \quad 0 \leq t < T.
\]  
(3.3)

It follows from Jensen's inequality that
\[
z'(t) = \int_\Omega \rho^k(x) |y(x, t)|^p dx \geq \left( \int_\Omega \rho^k(x) |y(x, t)| dx \right)^p \geq \left( \int_\Omega \rho^k(x) y(x, t) dx \right)^p = |\eta|^p, \quad 0 \leq t < T.
\]  
(3.4)

Thus, we have \( z(t) > 0 \) for \( 0 < t < T \), which implies that \( z(t_0) > 0 \).

Then we consider the system of differential inequalities and take \( t_0 \) as the origin of time.

\[
\begin{cases}
C(z')^{1/p} + \eta' \geq C_0 \mu_0 z, & t \geq t_0.
\end{cases}
\]  
(3.5)

Choosing \( 1 < r < p \) and from (3.5), we can get \( (z')^{1/p} \geq |\eta| \) and \( (z')^{r/p} \geq |\eta|^r \). Adding \( (z')^{1/p} + (z')^{r/p} \) on both sides for the first inequalities of (3.5), we obtain
\[
(1 + C)(z')^{1/p} + (z')^{r/p} + \eta' \geq C_0 \mu_0 z + |\eta|^r + |\eta|.
\]  
(3.6)

Making use of Young's inequality, we have
\[
(1 + C)(z')^{1/p} \leq (z')^{r/p} + C_1.
\]

Substituting it into (3.6), we get
\[
2(z')^{r/p} + \eta' \geq C_0 \mu_0 z + |\eta|^r + |\eta| - C_1.
\]  
(3.7)

Choosing \( m \) small enough such that \( 0 < m < \min \left\{ \frac{r}{p}, \frac{p - r}{p} \right\} \), by Young's inequality again, we have
\[
2(z')^{r/p} = 2 \frac{(z')^{r/p}}{z^m} z^m \leq C_2 \frac{z'}{z^m} \leq C_2 \frac{z'}{z^m} + \epsilon z \frac{m}{p - r} \leq C_2 (z^{1-m/p})' + \epsilon z + C_3,
\]
where \( \epsilon > 0 \) is appropriately small. Let \( \beta = 1 - \frac{mp}{r} \). By substituting the above inequality into (3.7), we get, for some small \( \epsilon \leq \frac{C_0 \mu_0}{2} \),
\[
C_2 (z^\beta)' + \eta' \geq C_0 \mu_0 z - \epsilon z + |\eta|^r + |\eta| - C_1 - C_3
\]
\[
\geq \frac{C_0 \mu_0}{2} z + |\eta|^r + |\eta| - C_4.
\]  
(3.8)

Now, we set \( \delta = \frac{p_0}{q_0} \), where \( p_0 \) is an integer and \( q_0 \) is an odd integer, such that \( 1 < \delta \leq \min \left\{ r, \frac{1}{p} \right\} \). Then, it suffices to verify \( |\eta'| + |\eta| \geq |\eta|^\delta \). From (3.8), it follows that
\[
(C_2 z^\beta + \eta') \geq |\eta|^\delta + \frac{C_0 \mu_0}{2} (z^{1-x^\beta}) z^{x^\beta} - C_4
\]
\[
\geq |\eta|^\delta + \left[ \frac{C_0 \mu_0}{2(C_2 z_{(t_0)}^{1-x^\beta})} \right] (\kappa) z^{x^\beta} - C_4.
\]
Let $C_5 = \min \left\{ 1, \frac{\sigma \rho_0}{\mu C_2^2} \xi(t_0)^{1-\beta \delta} \right\}$. Thus
\[
(C_2'z^\beta + \eta(t))' \geq C_5((C_2'z^\beta)^\delta + |\eta|^\delta) - C_4 \geq C_0((C_2'z^\beta) + |\eta|^\delta) - C_4 \\
\geq C_0((C_2'z^\beta) + \eta(t)) - C_4.
\]

Define $\xi(t) = C_2'z(t)^\beta + \eta(t)$ with the constant $C_2'$ large enough such that $C_0 \xi(t_0)^\delta > C_4$ and $\xi_0 = C_2'z(t_0)^\beta + \eta(t_0)$. We find that
\[
\begin{cases}
\xi' \geq C_0 \xi^\delta - C_4, \\
\xi(t_0) = \xi_0.
\end{cases}
\]
(3.9)

We may assume that $\xi \in C^1([t_0, T^*))$ and we will show that $T^* < +\infty$.

From (3.9) we can see the function $\xi$ is nondecreasing. Set $G(\xi_0; s) = \int_{t_0}^s \frac{1}{C_0 \xi^\delta - C_4} \, d\sigma$ for any $s \geq \xi_0$. Then we have
\[
\frac{dG(\xi_0; \xi(t))}{ds} = \frac{\xi'(t)}{C_0 \xi^\delta - C_4} \geq 1,
\]
(3.10)

and
\[
G(\xi_0; +\infty) = \int_{\xi_0}^{+\infty} \frac{1}{C_0 \xi^\delta - C_4} \, d\sigma < +\infty.
\]
(3.11)

It follows from (3.10) that
\[
G(\xi_0; \xi(t)) - G(\xi_0; \xi_0) = G(\xi_0; \xi(t)) \geq t - t_0, \quad \forall t \in [t_0, T^*).
\]
(3.12)

Combining (3.11) with (3.12), we deduce that $\xi$ blows up in finite time, and therefore, $y$ blows up in finite time. In fact, we also find the upper bound for the maximal time of existence
\[
T^* \leq \int_{\xi_0}^{+\infty} \frac{1}{C_0 \xi^\delta - C_4} \, d\sigma.
\]

Obviously, as $\xi_0 \to +\infty$, $(C_2' \to +\infty)$, the blow-up time of $\xi$ tends to zero and so is $y$. This completes the proof of Theorem 1.1. □

4. Proof of Theorems 1.2 and 1.3

In this section, we present the proof of the controllability for the system (1.1). First, we show the approximate controllability result.

Proof of Theorem 1.2. Let $T > 0$, $y_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ with $x \in \Omega; y_0(x) \neq 0 \subset \omega$ and $y_\delta \in L^2(\Omega)$ with $x \in \Omega; y_\delta(x) \neq 0 \subset \omega$ be given. Since $C_0^2(\Omega)$ is dense in $L^\infty(\Omega) \cap H^1(\Omega)$ and $L^2(\Omega)$ and by the continuous dependence of the solution with respect to the initial data, we only need to consider the case $y_0 \in C_0^2(\Omega)$ with supp $y_0 \subset \omega$, and $y_\delta \in C_0^2(\Omega)$ with supp $y_\delta \subset \omega$. We set $\omega_2 = \text{supp} y_0 \cup \text{supp} y_\delta$ and choose $\omega_1$ satisfies $\omega_2 \subset \omega_1 \subset \omega$ and mes $(\omega_1 \setminus \omega_2) < \epsilon_0$ where $\epsilon_0$ is a constant small enough.

Now, let us deal with a traditional linear control system
\[
\begin{aligned}
\xi_1 - \Delta \xi &= \chi_0 g(x, t), & \text{in } Q_T, \\
\xi(x, t) &= 0, & \text{on } \Sigma_T, \\
\xi(x, 0) &= y_0(x), & \text{in } \Omega,
\end{aligned}
\]
(4.1)

where $g(x, t)$ is a control function. It is well known that (see [7]) system (4.1) is approximately controllable, that is for any initial data $y_0 \in L^2(\Omega)$, target $y_\delta \in L^2(\Omega)$ and $\epsilon' > 0$, there exists a control $g \in L^\infty(Q_T)$ such that the solution of (4.1) satisfies $\|\xi(T) - y_\delta\|L^2(\Omega) < \epsilon'$. By the standard mollifier technique and the regularity of solution, for any $y_0 \in C_0^2(\Omega)$ with supp $y_0 \subset \omega$, $y_\delta \in C_0^2(\Omega)$ with supp $y_\delta \subset \omega$ and any $\epsilon > 0$, we can get a function $\tilde{g} \in \mathcal{C}^\frac{1}{2, 1}(\overline{Q_T})$ such that there exists $\tilde{\xi} \in C(\overline{Q_T}) \cap \mathcal{C}^\frac{1}{2, 1}(\overline{Q_T})$ solution of (4.1) and satisfying $\|\tilde{\xi}(T) - y_\delta\|L^2(\Omega) < \frac{\epsilon}{2}$.

Then, we choose $\phi \in \mathcal{C}_0^\infty(\Omega)$ with supp $\phi \subset \omega_1$ and $\phi \equiv 1$ in $\omega_2$. Denote $\tilde{y} = \phi \tilde{\xi}$. It is easy to see that
\[
\|\tilde{y}(T) - y_\delta\|L^2(\Omega) = \|\phi \tilde{\xi}(T) - y_\delta\|L^2(\Omega)
\leq \|\phi \tilde{\xi}(T) - \tilde{\xi}(T)\|L^2(\omega_1) + \|\phi \tilde{\xi}(T) - y_\delta\|L^2(\omega_2)
\leq \frac{\epsilon}{2} + C\epsilon_0
\leq \epsilon,
\]
provided $\epsilon_0$ is sufficiently small. We also have $\bar{y}(x, 0) = y_0$ in $\Omega$ and $\bar{y}(x, t) = 0$ on $\Sigma_T$. By using a simple calculation, we can see that if we take $u = \phi \tilde{g} - \Delta \phi \tilde{\zeta} - 2(\nabla \phi \nabla \tilde{\zeta}) - \int_0^t a(x, s)f(\phi \tilde{\zeta}) ds$ as the control of (1.1), then $u \in L^{\infty}(Q_T)$ and $\bar{y}$ is exactly the corresponding solution of system (1.1). Indeed, by the uniqueness of solution, we have shown that system (1.1) is approximately controllable for all $T > 0$. □

We are now in a position to present a null controllability result.

**Proof of Theorem 1.3.** As the argument in Theorem 1.2, and by the null controllability for (4.1) with smooth control function (see [18]), we can obtain the null controllability result for (1.1) with smooth initial data. The control function is $u = \phi g - \Delta \phi \zeta - 2(\nabla \phi \nabla \zeta) - \int_0^t a(x, s)f(\phi \zeta) ds$, where $\zeta \in C(Q_T) \cap C^2_{1,1}(Q_T)$ is the solution of (4.1) with control $g \in C^{\frac{1}{2},\frac{1}{2}}(Q_T)$. □

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**References**