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Fractal dimension of a random invariant set

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Abstract

In recent years many deterministic parabolic equations have been shown to possess global attractors which, despite being subsets of an infinite-dimensional phase space, are finite-dimensional objects. Debussche showed how to generalize the deterministic theory to show that the random attractors of the corresponding stochastic equations have finite Hausdorff dimension. However, to deduce a parametrization of a 'finite-dimensional' set by a finite number of coordinates a bound on the fractal (upper box-counting) dimension is required. There are non-trivial problems in extending Debussche's techniques to this case, which can be overcome by careful use of the Poincaré recurrence theorem. We prove that under the same conditions as in Debussche's paper and an additional concavity assumption, the fractal dimension enjoys the same bound as the Hausdorff dimension. We apply our theorem to the 2d Navier–Stokes equations with additive noise, and give two results that allow different long-time states to be distinguished by a finite number of observations.

Résumé

Au cours des dernières années, il a été démontré que de nombreuses équations paraboliques déterministes possèdaient des attracteurs globaux qui, tout en étant des sous-ensembles d'un espace de dimension infinie, sont en fait des objets de dimension finie. Debussche a montré comment généraliser la théorie déterministe pour établir que les attracteurs aléatoires des équations stochastiques correspondantes ont une dimension de Hausdorff finie. Cependant, pour déduire une paramétrisation d'un ensemble de dimension finie par un nombre fini de coordonnées, on a besoin d'un majorant de

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la dimension fractale. Des problèmes nontriviaux existent pour généraliser à ce cas les techniques de Debussche ; ils peuvent être surmontés en utilisant le théorème de récurrence de Poincaré. Sous les mêmes conditions que dans l'article de Debussche, nous démontrons que la dimension fractale a une même majorante que la dimension de Hausdorff. Nous appliquons notre théorème aux équations de Navier–Stokes avec bruit additif et nous présentons deux résultats qui, au moyen d'un nombre fini d'observations, permettent de distinguer deux états donnés sur des temps longs.

Keywords: Random invariant set; Fractal dimension; Parabolic equations

1. Introduction

The theory of attractors for deterministic dynamical systems, and in particular for a large class of parabolic partial differential equations, is now well developed (see, for example, the monographs by Babin and Vishik [2], Hale [23], Ladyzhenskaya [27], Robinson [34], Temam [37]). As well as proofs of their existence for an ever-growing number of models, in many cases these objects can be shown to be finite-dimensional.

One would, of course, like to deduce from this that the dynamics 'restricted to the attractor' is, in some sense, also 'finite-dimensional'. Despite the fundamental nature of this question, only partial results in this direction are available in general [18,33,35] and Chapter 16 in [34].

However, it is possible to prove [30,22,24] that a set with finite fractal (more properly 'upper box-counting') dimension d (for a formal definition see Section 2) can be parametrized by $2d + 1$ coordinates:

Theorem 1.1 [24]. *Let H be a Hilbert space, $X \subset H$ a compact set with fractal dimension d, and $N > 2d$ an integer. Then a prevalent*[1] *set of bounded linear functions $L : H \to \mathbb{R}^N$ are one-to-one between X and its image.*

Unfortunately no parametrization is available when it is only known that a set has finite Hausdorff dimension (a counterexample is given by Kan in the appendix of [36]).

Crauel and Flandoli [13] and Crauel, Debussche and Flandoli [15] developed a theory for the existence of random attractors for stochastic systems that closely parallels the deterministic theory. Crauel and Flandoli [14] developed a method for bounding the Hausdorff dimension of attractors for certain systems, but their techniques required the noise to be bounded; Debussche [16] used a 'random squeezing property' (cf. [21]) to bound the Hausdorff dimension without the assumption of bounded noise, a technique generalized to treat the fractal dimension by Langa [28].

However, the best bounds in the deterministic theory come not from a use of the squeezing property, but from the method involving Lyapunov exponents developed by Constantin, Foias and Temam [11]. It is this method that was adapted to the stochastic case by

[1] For a precise definition see Section 6. Here it suffices to say that 'prevalent' is a generalization of 'of full measure' to infinite-dimensional spaces.

Debussche [17] to obtain an upper bound on the Hausdorff dimension. In his paper he remarks that the same arguments could be used to obtain a bound on the fractal dimension of such sets. However, it turns out that there are non-trivial problems in adapting his argument to this case. In this paper these are overcome by a careful use of the Poincaré recurrence theorem.

In Section 2 we address various preliminary problems which are in fact central to the main proof. Indeed, given these tools the main argument follows that in Debussche's paper in [17] a relatively straightforward way.

The main theorem requires a number of (natural) assumptions which, along with the general framework in which we set the problem, are discussed in Section 3. The following section contains the formal statement of the main theorem, along with its proof. We combine the stochastic approach of Debussche with the deterministic argument of Chepyzhov and Vishik [7] which tracks the optimal bound more carefully than the conventional argument. With an additional technical assumption that appears to be satisfied in all interesting applications this leads to a bound on the fractal dimension which agrees with the bound on the Hausdorff dimension (the usual argument produces an additional factor of two).

Section 5.1 illustrates the application of the main theorem, which is proved for the case of a discrete time random dynamical system, to systems evolving in continuous time by treating the 2d stochastic Navier–Stokes equation with an additive white noise.

Section 6 shows how to apply the embedding result of Theorem 1.1, and a related result that allows reconstruction of finite-dimensional sets of analytic functions from measurements of point values, to random systems. As remarked above, such results rely in a fundamental way on the fact that the fractal dimension of the attractor is finite.

2. Preliminaries

In this section we first give a formal definition of the fractal dimension, defined as a certain limit superior as ε tends to zero; the most important result here is that an upper bound on this lim sup can be obtained by considering a sequence of ε_k that tends to zero at some controlled rate. Then in Section 2.2 we prove that the successive excursion times from a set of positive measure in an ergodic system cannot grow faster than linearly.

2.1. Fractal dimension

Let $N(X,\varepsilon)$ denote the minimum number of balls of radius ε required to cover X. Then the fractal dimension is defined as

$$d_{\mathrm{f}}(X) = \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log \varepsilon} \tag{1}$$

(for general results on this dimension see [18,20,32,34]).

The bound on the fractal dimension we prove here would essentially follow from the arguments in Debussche [17] if the limit superior in (1) could be replaced by a straightforward limit. However, "lim sup" is necessary, as there are simple sets for which the limit as $\varepsilon \to 0$ does not exist. For example (cf. Exercise 3.8 in [20]), form a Cantor like set $C = \bigcap_{j=1}^{\infty} C_j$, where C_j is the set at the end of stage j: at stage $2j-1$ remove the middle half 2^{j-1} times, and at stage $2j$ remove the middle third 2^{j-1} times. Considering C_{2j-1}, C itself requires

$$N_{2j-1} := 2^{2^j + 2^{j-1} - 2} \text{ intervals of length } \varepsilon_{2j-1} := 4^{-(2^j-1)} 3^{-(2^{j-1}-1)},$$

to cover it; considering C_{2j}, C requires

$$N_{2j} := 2^{2^{j+1}-2} \text{ intervals of length } \varepsilon_{2j} := 4^{-(2^j-1)} 3^{-(2^j-1)}$$

to cover itself. Therefore

$$\frac{\log N_{2j-1}}{-\log \varepsilon_{2j-1}} = \frac{(2^j + 2^{j-1} - 2)\log 2}{(2^j - 1)\log 4 + (2^{j-1} - 1)\log 3} \to \frac{3\log 2}{2\log 4 + \log 3},$$

while

$$\frac{\log N_{2j}}{-\log \varepsilon_{2j}} = \frac{(2^{j+1} - 2)\log 2}{(2^j - 1)\log 4 + (2^j - 1)\log 3} \to \frac{2\log 2}{\log 4 + \log 3}.$$

In what follows we will make use of an equivalent definition of the fractal dimension:

Lemma 2.1. *Let $M(X, \varepsilon)$ denote the minimum number of balls of radius ε with centres in X that are required to cover X. Then*

$$d_{\mathrm{f}}(X) = \limsup_{\varepsilon \to 0} \frac{\log M(X, \varepsilon)}{-\log \varepsilon}. \tag{2}$$

Proof. Denote by $\delta_{\mathrm{f}}(X)$ the right-hand side of (2). Then it is clear that $N(X, \varepsilon) \leqslant M(X, \varepsilon)$, and so that $d_{\mathrm{f}}(X) \leqslant \delta_{\mathrm{f}}(X)$. In order to prove the reverse inequality consider a cover of X by $N(X, \varepsilon)$ balls of radius ε, $B(x_i, \varepsilon)$. Discarding any unnecessary balls from this cover, each ball $B(x_i, \varepsilon)$ must contain a point $y_i \in X$. Since

$$B(y_i, 2\varepsilon) \supset B(x_i, \varepsilon)$$

it follows that $M(X, 2\varepsilon) \leqslant N(X, \varepsilon)$, and so

$$\frac{\log M(X, 2\varepsilon)}{-\log(2\varepsilon)} \leqslant \frac{\log N(X, \varepsilon)}{-\log 2 - \log \varepsilon}$$

which yields $\delta_{\mathrm{f}}(X) \leqslant d_{\mathrm{f}}(X)$ and hence (2). $\square$

In fact we will want to take the limit (superior) through a sequence of ε_k that tend to zero in a potentially non-uniform way, allowing in addition for some irregularities. The following lemma will be sufficient for our purposes.

Lemma 2.2. *Fix a strictly increasing sequence* $\{\tau_k\}_{k=1}^{\infty}$ *of positive integers satisfying,*

$$\limsup_{k\to\infty}\frac{\tau_{k+1}}{\tau_k}=1, \tag{3}$$

and let ε_k *be a sequence such that for some* $\beta>0$*, given any* $\delta>0$ *there exist* c_δ*,* C_δ*, and* k_δ *for which*

$$c_\delta \mathrm{e}^{-(\beta+\delta)\tau_k}\leqslant \varepsilon_k\leqslant C_\delta \mathrm{e}^{-(\beta-\delta)\tau_k},$$

for all $k\geqslant k_\delta$*. Then*

$$d_{\mathrm{f}}(X)=\limsup_{k\to\infty}\frac{\log N(X,\varepsilon_k)}{-\log\varepsilon_k}. \tag{4}$$

Proof. As a first step we prove (4) with ε_k replaced by $\varepsilon_k=c\mathrm{e}^{-\alpha\tau_k}$. To this end, given $\varepsilon>0$ let k be such that $\varepsilon_{k+1}\leqslant\varepsilon<\varepsilon_k$; then we have:

$$\begin{aligned}\frac{\log N(X,\varepsilon)}{-\log\varepsilon}\leqslant\frac{\log N(X,\varepsilon_{k+1})}{-\log\varepsilon_k}&=\frac{\log N(X,\varepsilon_{k+1})}{-\log\varepsilon_{k+1}}\frac{-\log\varepsilon_{k+1}}{-\log\varepsilon_k}\\&=\frac{\log N(X,\varepsilon_{k+1})}{-\log\varepsilon_{k+1}}\frac{\alpha\tau_{k+1}-\log c}{\alpha\tau_k-\log c},\end{aligned}$$

and so, using (3),

$$d_{\mathrm{f}}(X)\leqslant\limsup_{k\to\infty}\frac{\log N(X,\varepsilon_k)}{-\log\varepsilon_k}. \tag{5}$$

Now to prove (4), fix $\varepsilon>0$, choose $\delta>0$ such that

$$\frac{\beta+\delta}{\beta-\delta}<1+\varepsilon,$$

and then find k'_δ such that for $k\geqslant k'_\delta$,

$$\frac{-\log c_\delta+(\beta+\delta)\tau_k}{-\log C_\delta+(\beta-\delta)\tau_k}<1+\varepsilon. \tag{6}$$

Then for all $k\geqslant\max(k_\delta,k'_\delta)$,

$$\begin{aligned}\frac{\log N(X,\varepsilon_k)}{-\log\varepsilon_k}&\leqslant\frac{\log N(X,c_\delta\mathrm{e}^{-(\beta+\delta)\tau_k})}{-\log C_\delta+(\beta-\delta)\tau_k}\\&=\frac{\log N(X,c_\delta\mathrm{e}^{-(\beta+\delta)\tau_k})}{-\log c_\delta+(\beta+\delta)\tau_k}\left(\frac{-\log c_\delta+(\beta+\delta)\tau_k}{-\log C_\delta+(\beta-\delta)\tau_k}\right)\\&<(1+\varepsilon)\frac{\log N(X,c_\delta\mathrm{e}^{-(\beta+\delta)\tau_k})}{-\log(c_\delta\mathrm{e}^{-(\beta+\delta)\tau_k})},\end{aligned}$$

using (6) in the final step. It follows using (5) with $c = c_\delta$ and $\alpha = \beta + \delta$ that

$$\limsup_{k\to\infty} \frac{\log N(X, \varepsilon_k)}{-\log \varepsilon_k} \leqslant (1+\varepsilon) d_{\mathrm{f}}(X). \tag{7}$$

Since this holds for any $\varepsilon > 0$ we obtain (4). □

In the proof of the main theorem we keep track of the 'ε-approximate γ volume',

$$V_\gamma(X, \varepsilon) := \varepsilon^\gamma N(X, \varepsilon). \tag{8}$$

We note the following simple corollary:

Corollary 2.3. *Let ε_k be a sequence as in the statement of Lemma* 2.2*, and suppose that*

$$\lim_{k\to 0} V_\gamma(X, \varepsilon_k) = 0.$$

Then $d_{\mathrm{f}}(X) \leqslant \gamma$.

2.2. Return times

In order to complete the proof of the main theorem we will need to make use of the Poincaré recurrence theorem to guarantee that the attractor can frequently be covered by some fixed number of ε balls. In itself this is not enough, and we will require some additional information on the growth of successive return times. The following lemma shows that, asymptotically, successive excursion times grow slowly ($\leqslant \varepsilon n$ for any $\varepsilon > 0$), and hence that successive return times also grow slowly.

Lemma 2.4. *Let $T : \Omega \to \Omega$ be an ergodic transformation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose that $\mathbb{P}(A) > 0$. Denote by $\tau_n(\cdot) : A \to \mathbb{Z}_{>0}$ the nth return time to A, i.e.,*

$$\tau_n(\omega) = \min\bigl\{ j > \tau_{n-1} \colon T^j(\omega) \in A \bigr\},$$

where $\tau_0 := 0$. Denote by δ_n the length of the nth excursion, i.e.,

$$\delta_n = \tau_n - \tau_{n-1};$$

then for each $\varepsilon > 0$, for $\mathbb{P}$-almost every element $\omega \in A$ there exists an $N_{\omega,\varepsilon}$ such that

$$\delta_n \leqslant \varepsilon n \quad \text{for all } n \geqslant N_{\omega,\varepsilon}. \tag{9}$$

Note that it follows that for any $\varepsilon > 0$ the sequence of return times eventually satisfies $\tau_k + 1 \leqslant \tau_{k+1} \leqslant (1+\varepsilon)\tau_k$, and thus in particular that

$$\limsup_{k\to\infty} \frac{\tau_{k+1}}{\tau_k} = 1,$$

as required by (3) in Lemma 2.2.

Proof. Define the first return map $R : A \to A$ by:

$$R(\omega) = T^{\tau_1(\omega)}(\omega).$$

It is a standard result (see [31], for example) that this induced transformation R is once again measure-preserving. Observe that

$$\begin{aligned}\frac{\delta_n}{n} = \frac{\tau_1(R^{n-1}\omega)}{n} &= \frac{1}{n}\sum_{j=0}^{n-1}\tau_1(R^j\omega) - \frac{n-1}{n}\frac{1}{n-1}\sum_{j=0}^{n-2}\tau_1(R^j\omega) \\ &\to \mathbb{E}(\tau_1) - \mathbb{E}(\tau_1) = 0,\end{aligned}$$

using the ergodic theorem. The bound in (9) follows immediately. □

3. Assumptions

This section introduces our main assumptions and in particular defines the expansion factors that play a central role.

3.1. The underlying random dynamical system

We consider a random dynamical system on a Hilbert space H with norm $|\cdot|$, driven by a noise that lies in an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose time evolution is governed by a measure-preserving ergodic transformation θ: the time evolution of an element $u \in H$ driven by noise ω is given by the sequence:

$$u,\ S(\omega)u,\ S(\theta\omega)S(\omega)u,\ S(\theta^2\omega)S(\theta\omega)S(\omega)u,\ \ldots.$$

For simplicity of notation we denote:

$$S^n(\omega) = S(\theta^{n-1}\omega)S(\theta^{n-2}\omega)\cdots S(\theta\omega)S(\omega). \tag{10}$$

We are interested in the fractal dimension of a compact set that is invariant under the stochastic flow, i.e., a random set $A(\omega)$ that is compact for each ω, satisfies

$$S(\omega)A(\omega) = A(\theta\omega) \quad \mathbb{P}\text{-a.s.},$$

and for which the mapping $\omega \mapsto \operatorname{dist}(x, A(\omega))$ is measurable for any $x \in H$.

3.2. Fractal dimension is a.s. constant for Lipschitz S

We note here that under the above conditions the fractal dimension of the random attractor is almost surely constant, provided we assume in addition that S is Lipschitz from H into H. Indeed, we can follow the argument of Crauel and Flandoli [14], noting that the fractal dimension can be defined by analogy with the Hausdorff dimension as

$$d_{\mathrm{f}}(Y)=\inf\bigl\{s\geqslant 0\colon\ \mu_{\mathrm{f}}(Y,s)=0\bigr\},$$

where

$$\mu_{\mathrm{f}}(Y,s)=\limsup_{\varepsilon\to 0}V_{\varepsilon}(Y,s)$$

(recall that $V_\varepsilon(Y,s)=\varepsilon^s N(Y,\varepsilon)$, see (8)). Since the map $Y\mapsto V_\varepsilon(Y,s)$ is Borel measurable (cf. Lemma 3.6 in [14]) for any $\varepsilon>0$ and $s>0$, it follows that $Y\mapsto\mu_{\mathrm{f}}(Y,s)$ is also Borel measurable. It then follows (Lemma 4.2 in [14]) that the map $\omega\mapsto d_{\mathrm{f}}(A(\omega))$ is measurable. The non-increasing nature of the fractal dimension under Lipschitz maps (see [20], for example) implies that

$$d_{\mathrm{f}}\bigl(A(\theta\omega)\bigr)\leqslant d_{\mathrm{f}}\bigl(A(\omega)\bigr),$$

and since θ is ergodic, this implies (see Remark 2 after Theorem 16 in [38]) that $d_{\mathrm{f}}(A(\omega))$ is constant $\mathbb{P}$-a.s.

3.3. The linearization and its expansion factors

Our main assumptions reproduce those of Debussche: First we assume that the cocycle is almost surely uniformly differentiable on $A(\omega)$, i.e., for all $u\in A(\omega)$ there exists a linear map $DS(\omega,u)$ from H to H satisfying:

$$\bigl|S(\omega)(u+h)-S(\omega)u-DS(\omega,u)h\bigr|\leqslant K(\omega)|h|^{1+\alpha},\tag{11}$$

where $\alpha>0$ is fixed and $K(\omega)$ is a random variable such that $K(\omega)\geqslant 1$ for all $\omega\in\Omega$ and $\mathbb{E}(\ln K)<\infty$.

Given a bounded linear operator L, we define:

$$\alpha_n(L)=\sup_{G\subset H\colon\ \dim G=n}\ \inf_{\phi\in G\colon\ |\phi|=1}|L\phi|,$$

and

$$\omega_n(L)=\alpha_1(L)\cdots\alpha_n(L).$$

The numbers $\alpha_n(L)$, the linear expansion factors, are the eigenvalues of $(L^*L)^{1/2}$ arranged in decreasing order: they are the semiaxes of the ellipse obtained by applying L to the unit ball in H; $\omega_n(L)$ are the expansion factors for n-dimensional volumes.

Since we are concerned with coverings by a collection of balls of fixed radius, the following lemma from [7, Chapter III, Lemma 2.2] will be needed. In essence it says that to cover an ellipse with semiaxes $\{\alpha_i\}$ with balls of radius r, it suffices to cover the j-dimensional ellipse whose semiaxes are $\alpha_1, \dots, \alpha_j$, with $\alpha_j \geqslant r > \alpha_{j+1}$. The number of balls required is then proportional to the volume of this j-dimensional ellipse, essentially ω_j, divided by the j-volume of the ball, r^j.

Proposition 3.1. *Let E be an ellipsoid whose semiaxes have lengths $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_j \geqslant \cdots$. Then for any $r < \alpha_1$ the number of balls of radius $\sqrt{2}r$ needed to cover E is less than*

$$7^j \frac{\omega_j}{r^j},$$

where j is the largest integer such that $r \leqslant \alpha_j$.

3.4. Assumptions on the expansion factors

We assume that for each $j = 1, 2, \dots$ there exists an integrable random variable $\bar{\omega}_j$ such that $\mathbb{P}$-a.s.

$$\omega_j\big(DS(u, \omega)\big) \leqslant \bar{\omega}_j(\omega) \quad \text{for all } u \in A(\omega),$$

with $\mathbb{E} \ln \bar{\omega}_j < \infty$. We also assume the existence of integrable random variables $\bar{\alpha}_1$ and $\bar{\alpha}_d$ such that $\mathbb{P}$-a.s.

$$\alpha_j\big(DS(\omega, u)\big) \leqslant \bar{\alpha}_j(\omega) \quad \text{for all } u \in A(\omega),\ j = 1, \dots, d,$$

$\bar{\alpha}_1 \geqslant 1$ and $\mathbb{E} \ln \bar{\alpha}_j < \infty$, $j = 1, \dots, d$.

4. The main theorem

We now state and prove the main theorem. Until Eq. (25) the argument is a combination of that in [17] and that of [7]. The proof is concluded using the results of Section 2.

Theorem 4.1. *Let the assumptions of Section* 3 *hold. Suppose that*

$$\mathbb{E} \ln \bar{\omega}_d < 0.$$

Then $\mathbb{P}$-a.s.

$$d_{\mathrm{f}}\big(A(\omega)\big) \leqslant \gamma$$

for any γ such that

$$\gamma > \frac{\mathbb{E}[\max_{1 \leqslant j \leqslant d}(dq_j - jq_d)]}{-\mathbb{E}q_d}, \tag{12}$$

where $q_j = \log\bar{\omega}_j$.

Proof. First we note that it follows from (12) that

$$\mathbb{E}\left(\log\left[\bar{\omega}_d^{\gamma/d}\max_{1\leqslant j\leqslant d}\frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}\right]\right) < 0. \tag{13}$$

(A direct generalization of the ‘standard method’ used to bound the fractal dimension (e.g., [37]) would require (13) to hold for $\gamma = d$. We obtain the extra freedom in γ by following Chepyzhov and Vishik [7].)

For fixed values of d and γ, by considering multiple iterates of $S(\omega)$ if necessary (for the nth iterate of $S(\omega)$, the expansion factor $\bar{\omega}_j$ can be replaced by $n\bar{\omega}_j$, see [17]) we can and will assume the stronger condition:

$$\mathbb{E}\left(\log\left[\bar{\omega}_d^{\gamma/d}\max_{1\leqslant j\leqslant d}\frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}\right]\right) < -\gamma\ln 2 - d\ln 7, \tag{14}$$

or more compactly,

$$\mathbb{E}\log\Omega_\gamma < 0 \quad \text{where } \Omega_\gamma(\omega) := 2^\gamma 7^d \bar{\omega}_d^{\gamma/d}\max_{1\leqslant j\leqslant d}\frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}. \tag{15}$$

Preliminary considerations

In the spirit of Debussche [17], we define the measurable set:

$$J(\eta) = \left\{\omega\in\Omega:\ K(\omega)\eta^\alpha \leqslant \left(2-\sqrt{2}\right)\bar{\alpha}_d(\omega)\right\},$$

and note that $\mathbb{P}(J(\eta)) \to 1$ as $\eta \to 0$. We also introduce the random variable:

$$\tau_\eta(\omega) = \begin{cases} \Omega_\gamma^{1/\gamma}, & \omega\in J(\eta),\\ \bar{\alpha}_1(\omega) + K(\omega), & \omega\notin J(\eta). \end{cases} \tag{16}$$

By the dominated convergence theorem, as $\eta \to 0$ we have:

$$\mathbb{E}(\ln\tau_\eta) \to \mathbb{E}\left(\frac{1}{\gamma}\ln\Omega_\gamma\right) = -\theta, \tag{17}$$

for some $\theta > 0$. Now fix some $\eta_0 > 0$ such that

$$\mathbb{E}(\ln\tau_\eta) < -\theta/2 \tag{18}$$

for all $\eta \leqslant \eta_0$.

The image of one covering under $S(\omega)$

First consider a covering of $A(\omega)$ by balls of radius $\varepsilon < \eta_0$,

$$A(\omega) \subset \bigcup_{j=1}^{N} B(u_i, \varepsilon),$$

where $u_i \in A(\omega)$ (see Lemma 2.1). In the argument we keep track of the "total approximate γ-volume",

$$V_\gamma(X, \varepsilon) := \varepsilon^\gamma N(X, \varepsilon),$$

of a covering of $A(\omega)$: if $\lim_{k\to\infty} V_\gamma(X, \varepsilon_k) = 0$ for a sequence ε_k as in Lemma 2.2 then $d_{\mathrm{f}}(X) \leqslant \gamma$ (this was Corollary 2.3).

Under application of $S(\omega)$ we have:

$$A(\theta\omega) \subset \bigcup_{j=1}^{N} S(\omega)B(u_i, \varepsilon).$$

Clearly

$$S(\omega)B(u_i, \varepsilon) \subset S(\omega)u_i + DS(\omega, u_i)B(0, \varepsilon) + B\big(0, K(\omega)\varepsilon^{1+\alpha}\big). \tag{19}$$

We now consider 3 cases in turn: in each case we show that

$$V_\gamma\big(A(\theta\omega), \tau_\eta(\omega)\varepsilon\big) \leqslant \tau_\eta(\omega)^\gamma V_\gamma\big(A(\omega), \varepsilon\big). \tag{20}$$

Case 1: $\omega \notin J(\eta)$

In this case,

$$\begin{aligned} S(\omega)B(u_i, \varepsilon) &\subset B\big(S(\omega)u_i, \big[\alpha_1\big(DS(\omega, u_i)\big) + K(\omega)\varepsilon^\alpha\big]\varepsilon\big) \\ &\subset B\big(S(\omega)u_i, \big[\bar{\alpha}_1(\omega) + K(\omega)\varepsilon^\alpha\big]\varepsilon\big) \\ &\subset B\big(S(\omega)u_i, \big[\bar{\alpha}_1(\omega) + K(\omega)\big]\varepsilon\big). \end{aligned}$$

And we have

$$N\big(A(\theta\omega), \big[\bar{\alpha}_1(\omega) + K(\omega)\big]\varepsilon\big) \leqslant N\big(A(\omega), \varepsilon\big),$$

i.e.,

$$N\big(A(\theta\omega), \tau_\eta(\omega)\varepsilon\big) \leqslant N\big(A(\omega), \varepsilon\big),$$

and so (20) holds.

Case 2: $\omega \in J(\eta)$: $K(\omega)\varepsilon^\alpha < (2-\sqrt{2})\bar{\alpha}_d(\omega)$

Case 2(a): $\alpha_1(DS(\omega, u_i)) < \bar{\alpha}_d(\omega)$

In the unlikely event that $\alpha_1(DS(\omega, u_i)) < \bar{\alpha}_d(\omega)$ (which is possible since α_1 is the *local* rate of expansion, while $\bar{\alpha}_d$ is a *global* upper bound on the contraction rate) then we can return to (19) and write:

$$DS(\omega, u_i)B(0, \varepsilon) \subset B\big(0, \alpha_1\big(DS(\omega, u_i)\big)\varepsilon\big) \subset B\big(0, \bar{\alpha}_d(\omega)\varepsilon\big),$$

so that

$$S(\omega)B(u_i, \varepsilon) \subset B\big(S(\omega)u_i, 2\bar{\alpha}_d(\omega)\varepsilon\big),$$

and thus

$$V_\gamma\big(A(\theta\omega), 2\bar{\alpha}_d(\omega)\varepsilon\big) \leqslant \big(2\bar{\alpha}_d(\omega)\big)^\gamma V_\gamma\big(A(\omega), \varepsilon\big).$$

Now, note that $(2\bar{\alpha}_d)^\gamma \leqslant \Omega_\gamma$; indeed,

$$(2\bar{\alpha}_d)^\gamma \leqslant 2^\gamma \bar{\omega}_d^{\gamma/d} \leqslant 7^{-d}\left(\max_{1\leqslant j\leqslant d} \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}\right)^{-1} \Omega_\gamma \leqslant 7^{-d}\Omega_\gamma. \tag{21}$$

It follows that

$$V_\gamma\big(A(\theta\omega), \Omega_\gamma^{1/\gamma}\varepsilon\big) \leqslant \Omega_\gamma V_\gamma\big(A(\omega), \varepsilon\big),$$

which gives (20) once more.

Case 2(b): $\alpha_1(DS(\omega, u_i)) \geqslant \bar{\alpha}_d(\omega)$

In this more likely case, when $\alpha_1(DS(\omega, u_i)) \geqslant \bar{\alpha}_d(\omega)$, we will use Lemma 3.1: the number of balls of radius $\sqrt{2}\,\bar{\alpha}_d(\omega)\varepsilon$ required to cover $DS(\omega, u_i)B(0, \varepsilon)$ is bounded by

$$7^j \frac{\omega_j(DS(\omega, u))}{\bar{\alpha}_d(\omega)^j},$$

where j is the largest integer such that $\bar{\alpha}_d(\omega) \leqslant \alpha_j(\omega)$. Since $\alpha_d(DS(\omega, u)) \leqslant \bar{\alpha}_d(\omega)$ for every $u \in A(\omega)$, $j \leqslant d-1$. Thus no more than

$$N := 7^j \max_{1\leqslant j\leqslant d} \frac{\bar{\omega}_j(\omega)}{\bar{\alpha}_d(\omega)^j}$$

balls are needed to cover $DS(\omega, u_i)B(0, \varepsilon)$. It follows that $S(\omega)B(u, \varepsilon)$ can be covered by N balls of radius:

$$\big[\sqrt{2}\bar{\alpha}_d(\omega) + K(\omega)\varepsilon^\alpha\big]\varepsilon \leqslant 2\bar{\alpha}_d\varepsilon \leqslant \Omega_\gamma^{1/\gamma}\varepsilon.$$

Thus the contribution of $S(\omega)B(u,\varepsilon)$ to $V_\gamma(A(\theta\omega),\Omega_\gamma^{1/\gamma}\varepsilon)$ is bounded by:

$$\begin{aligned}\big(2\bar{\alpha}_d(\omega)\varepsilon\big)^\gamma 7^j \max_{1\leqslant j\leqslant d-1}\frac{\bar{\omega}_j(\omega)}{\bar{\alpha}_d(\omega)^j} &= 2^\gamma 7^j\varepsilon^\gamma\left[\bar{\omega}_d^{\gamma/d}\max_{1\leqslant j\leqslant d-1}\frac{\bar{\omega}_j(\omega)}{\bar{\omega}_d(\omega)^{j/d}}\right]\\ &= 2^\gamma 7^j\Omega_\gamma\varepsilon^\gamma = \Omega_\gamma\,\varepsilon^\gamma,\end{aligned}$$

where Ω_γ was defined in (14). Thus we have, as above,

$$V_\gamma\big(A(\theta\omega),\Omega_\gamma^{1/\gamma}\varepsilon\big)\leqslant\Omega_\gamma V_\gamma\big(A(\omega),\varepsilon\big).$$

Iterated coverings

Whatever the status of ω (w.r.t. $J(\eta_0)$) we have obtained (20):

$$V_\gamma\big(A(\theta\omega),\tau_\eta(\omega)\varepsilon\big)\leqslant\tau_\eta(\omega)^\gamma V_\gamma\big(A(\omega),\varepsilon\big),$$

where τ_η is defined in (16). Replacing ω by $\theta^{-1}\omega$ gives,

$$V_\gamma\big(A(\omega),\tau_\eta(\theta^{-1}\omega)\varepsilon\big)\leqslant\tau_\eta(\theta^{-1}\omega)^\gamma V_\gamma\big(A(\theta^{-1}\omega),\varepsilon\big). \tag{22}$$

We would like to iterate (22) to obtain:

$$V_\gamma\left(A(\omega),\left[\prod_{j=1}^k\tau_\eta(\theta^{-j}\omega)\right]\varepsilon\right)\leqslant\left(\prod_{j=1}^k\tau_\eta(\theta^{-j}\omega)\right)^\gamma V_\gamma\big(A(\theta^{-k}\omega),\varepsilon\big), \tag{23}$$

but we need to ensure that we can keep $\tau_\eta(\theta^{-1}\omega)\varepsilon$ (and successive iterates) below η_0.

To see that this is possible, given some $\varepsilon_0(\omega)>0$ consider the sequence

$$\varepsilon_k(\omega)=\left(\prod_{j=1}^k\tau_\eta(\theta^{-j}\omega)\right)\varepsilon_0(\omega).$$

Using ergodicity, we have:

$$\frac{1}{k}\sum_{j=1}^k\ln\big(\tau_\eta(\theta^{-j}\omega)\big)\to\mathbb{E}(\ln\tau_\eta)=-\beta<-\theta/2\quad\text{as } k\to\infty,$$

and so

(i) there exists a $k(\omega)$ such that

$$\sum_{j=1}^k\ln\big(\tau_\eta(\theta^{-j}\omega)\big)<0$$

for all $k \geqslant k(\omega)$,

(ii) for any choice of $\delta > 0$, there exists a k_δ such that for $k \geqslant k_\delta$ we have,

$$-\beta - \delta < \frac{1}{k}\sum_{j=1}^{k} \ln\bigl(\tau_\eta(\theta^l\omega)\bigr) < -\beta + \delta.$$

It follows that we can choose $\varepsilon_0(\omega)$ such that

(i) $\varepsilon_k(\omega) < \eta_0$ for all $k = 0, 1, 2, \ldots,$ and

(ii) given any $\delta > 0$ we have,

$$\varepsilon_0(\omega)\mathrm{e}^{-(\beta+\delta)k} \leqslant \varepsilon_k(\omega) \leqslant \varepsilon_0(\omega)\mathrm{e}^{-(\beta-\delta)k} \tag{24}$$

for all $k \geqslant k_\delta$.

We can therefore iterate (22) starting with $\varepsilon = \varepsilon_0(\omega)$ to obtain:

$$V_\gamma\bigl(A(\omega), \varepsilon_k(\omega)\bigr) \leqslant \left(\prod_{j=1}^{k} \tau_\eta(\theta^{-j}\omega)\right)^{\gamma} V_\gamma\bigl(A(\theta^{-k}\omega), \varepsilon_0(\omega)\bigr).$$

Since for k sufficiently large, we have:

$$\frac{1}{k}\sum_{j=0}^{k} \ln\bigl(\tau_\eta(\theta^{-j}\omega)\bigr) < -\beta/2,$$

we set $\zeta = \mathrm{e}^{-\beta\gamma/2} < 1$ and obtain, for all k sufficiently large,

$$V_\gamma\bigl(A(\omega), \varepsilon_k(\omega)\bigr) \leqslant \zeta^k V_\gamma\bigl(A(\theta^{-k}\omega), \varepsilon_0(\omega)\bigr). \tag{25}$$

Taking the limit using the Poincaré recurrence theorem

We would like to take the limit as $k \to \infty$ in (25), but we do not know that $V_\gamma(A(\theta^{-k}\omega), \varepsilon_0(\omega))$ is bounded. Indeed, in general one would only expect a sub-exponential bound on the radius of $A(\theta^{-k}\omega)$ (see [9], for example), and this does not translate readily[2] into a bound on $N(A(\theta^{-k}\omega), \varepsilon)$.

Instead we use the Poincaré recurrence theorem (see [38], for example) to find a sequence of times for which $V_\gamma(A(\theta^{-k}\omega), \varepsilon_0(\omega))$ is bounded, and control the length of the excursions using Lemma 2.4.

[2] One could assume a sub-exponential bound on the radius of $A(\theta^{-k}\omega)$ in H^1 and translate this into a bound on the number of ε-balls in L^2 required to cover A (see [19], for example). However, the resulting estimate grows much too rapidly to be of any use here.

For each ω, $A(\omega)$ is compact, so for each fixed η we know that $V_\gamma(A(\omega),\varepsilon)$ is finite. Choose $M>0$ and consider the set:

$$\Omega_M=\{\omega\colon\ V_\gamma\bigl(A(\omega),\eta_0\bigr)\leqslant M\}.$$

For any M sufficiently large this set has positive measure: we choose and fix one such M. It follows from the Poincaré recurrence theorem that for $\mathbb{P}$ almost every $\omega\in\Omega$ there is a sequence $k_j\to\infty$ (which can depend on ω) such that $\theta^{-k_j}\omega\in\Omega_M$.

For this sequence k_j it follows from (25) that

$$V_\gamma\bigl(A(\omega),\varepsilon_{k_j}(\omega)\bigr)\to 0\quad\text{as } j\to\infty$$

and hence that

$$\limsup_{j\to\infty}\frac{\log N(A(\omega),\varepsilon_{k_j}(\omega))}{-\log\varepsilon_{k_j}(\omega)}\leqslant\gamma.$$

Lemma 2.4 shows that k_j satisfies

$$\limsup_{j\to\infty}\frac{k_{j+1}}{k_j}=1;$$

thus $\varepsilon_{k_j}(\omega)$ satisfies the asymptotic condition (3) of Lemma 2.2, and so we finally obtain:

$$d_{\mathrm{f}}\bigl(A(\omega)\bigr)\leqslant\gamma.\qquad\square$$

We now give a corollary of Theorem 4.2 that allows for a simple bound on the fractal dimension. Since the hypothesis of the theorem is satisfied in most applications, we obtain the same bound on the fractal dimension as on the Hausdorff dimension. The argument is adapted from [7], see also [6].

Corollary 4.2. *Let the assumptions of Section* 3 *hold. Suppose that* $\ln\bar{\omega}_j\leqslant\phi_j(\omega)$, *where* ϕ_j *is a concave function of* j *for each* ω, *and*

$$\mathbb{E}\ln\bar{\omega}_n<0\tag{26}$$

for some $n\in\mathbb{Z}$. *Then for* $\mathbb{P}$*-almost every* ω:

$$d_{\mathrm{f}}\bigl(A(\omega)\bigr)\leqslant n.$$

Proof. The key observation is that there exist positive random variables α and β with

$$0<\mathbb{E}\alpha,\mathbb{E}\beta<+\infty,$$

such that

$$q_j\leqslant-\alpha j+\beta,$$

for all $j = 1, 2, \ldots$. Indeed, for each fixed ω choose α, β such that $q = -\alpha n + \beta$ is the straight line through (n, ϕ_n) and the point $(n-1, \phi_{n-1})$. Since $j \mapsto \phi_j$ is concave, all points (j, ϕ_j) lie below the line $q = -\alpha n + \beta$; thus we can replace the bound

$$\ln \bar{\omega}_j \leqslant \phi_j(\omega)$$

by

$$\ln \bar{\omega}_j \leqslant -\alpha j + \beta.$$

Since the argument leading to (12) takes into account only the upper bounds $\bar{\omega}_j$, we can simply replace q_j by the upper bound $-\alpha j + \beta$. It follows that we must take:

$$\begin{aligned}\gamma &> \frac{\mathbb{E}[\max_{1 \leqslant j \leqslant d}(d(-\alpha j + \beta) - j(-\alpha d + \beta))]}{\mathbb{E}(\alpha d - \beta)} \\ &= \frac{\mathbb{E}[\max_{1 \leqslant j \leqslant d}(d - j)\beta]}{d\,\mathbb{E}\alpha - \mathbb{E}\beta} \leqslant \frac{d\mathbb{E}\beta}{d\mathbb{E}\alpha - \mathbb{E}\beta}.\end{aligned}$$

However, $d > n$ is arbitrary, so we can let $d \to \infty$ and show that $d_{\mathrm{f}}(A(\omega))$ is bounded above by γ for any $\gamma > \mathbb{E}\beta/\mathbb{E}\alpha$. Since $\mathbb{E}q_n = -n\mathbb{E}\alpha + \mathbb{E}\beta$, we have $\mathbb{E}\beta/\mathbb{E}\alpha < n$ and so $d_{\mathrm{f}}(A(\omega)) \leqslant n$. $\square$

5. Application of the theorem to stochastic PDEs

In this section we discuss the application of our theorem to stochastic PDEs: we treat the 2d Navier–Stokes equations with an additive noise in some detail, and then recall previous results for stochastic reaction–diffusion equations.

Consider a stochastic PDE (or ODE) evolving in continuous time that generates a cocycle $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$, such that at time t the solution starting at u_0 with noise ω is given by:

$$\varphi(t, \omega)u_0,$$

and the cocycle rule of composition,

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega)\varphi(s, \omega),$$

holds for all $t, s \geqslant 0$, where ϑ is a two-sided shift on Ω. For more details see [1], for example.

We apply our theorem by taking $S(\omega) := \varphi(T, \omega)$ and $\theta = \vartheta_T$ for some suitable choice of T. Note that the cocycle rule of composition reproduces the composition rule (10) for $S(\omega)$.

5.1. The 2d Navier–Stokes equation with additive noise

Crauel et al. [15] proved the existence of a global attractor for the model:

$$\mathrm{d}u + \big(-\nu\Delta u + (u\cdot\nabla)u + \nabla p\big)\,\mathrm{d}t = f\,\mathrm{d}t + \sum_{j=1}^{m}\phi_j\,\mathrm{d}W_j(t), \tag{27}$$

with $\nabla\cdot u = 0$, where $f, \phi_j \in L^2$ and $W_j(t)$ are independent one-dimensional Brownian motions, and boundary conditions are periodic on $Q = [0, L]^2$.

In order to cast this equation in its standard functional form, let $\mathcal{P}$ be the space of trigonometric polynomials $u(x)$ in $\mathbb{R}^2$ of period L in both directions and values in $\mathbb{R}^2$ such that $\nabla\cdot u = 0$ and $\int_Q u\,\mathrm{d}x = 0$. Define:

$$H = \text{closure of } \mathcal{P} \text{ in } \big(L^2(Q)\big)^2$$

and

$$V = \text{closure of } \mathcal{P} \text{ in } \big(H^1(Q)\big)^2.$$

Equipped with the $(L^2(Q))^2$ norm $|\cdot|$, H is a Hilbert space.

In the standard way (see [37] or [12], for example) we rewrite Eq. (27) as a stochastic evolution equation on H: Letting Π denote the orthogonal projection from in $(L^2(Q))^2$ onto H, we define the Stokes operator $A = -\Pi\Delta$ and the bilinear form $B(u,u) = \Pi[(u\cdot\nabla)u]$. This bilinear form satisfies the orthogonality property $(B(u,v),v) = 0$ for all $u, v \in V$. Eq. (27) then becomes:

$$\mathrm{d}u + \big[\nu Au + B(u,u) - f\big]\,\mathrm{d}t = \varepsilon w_j\,\mathrm{d}W(t),$$

where for simplicity, following Flandoli and Langa [21], we have taken $m = 1$ and $\phi = w_j$, one of the eigenfunctions of the Stokes operator.

We show that the random attractor for this equation has finite upper fractal dimension for every choice of f and ε, and that the dimension estimate reduces to the deterministic estimate:

$$d_{\mathrm{f}}(A) \leqslant c\frac{|f|}{\nu^2\lambda_1},$$

as $\varepsilon \to 0$ (λ_1 is the first eigenvalue of the Stokes operator A).

To find bounds on $\omega_d(DS(T;u,\omega))$, we use the trace formula due to Constantin, Foias and Temam [11] (see also Chapter V of [37]). It is relatively simple to show that $S(\omega) := \varphi(T,\omega)$ is almost surely uniformly differentiable on $A(\omega)$ in the appropriate sense, and that $D\varphi(T;\omega,u_0)h$ is the solution of the linearized equation:

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(u,U) + B(U,u) = 0, \quad U(0) = \mathrm{Id}_H, \tag{28}$$

where $u(t)$ is the solution of (27) with $u(0)=u_0$ (cf. Theorem 13.20 in [34], for example). Writing (28) as

$$\mathrm{d}U/\mathrm{d}t = L\big(t, u(t)\big)U,$$

where

$$L(u)\phi = \nu\Delta\phi - B(u,\phi) - B(\phi,u),$$

the trace formula asserts that the volume expansion factors at time T are given by:

$$q_d(T;\omega,x) = \ln\omega_d\big(D\varphi(T;\omega,x)\big) = \sup_{P(0)}\left(\operatorname{Tr}\int_0^T L\big(s; x(s)\big)P(s)\,\mathrm{d}s\right),$$

where $P(0)$ is an orthogonal projector of rank d onto the space spanned by n orthonormal elements $\{\phi_j\}_{j=1}^d$ of H, and $P(t)$ the projector onto the space spanned by the images of the vectors ϕ_j under the linearized flow $D\varphi(t,\omega;x)$.

We can therefore bound $\omega_d(D\varphi(T;\omega,x))$ by bounding,

$$\operatorname{Tr} L\big(s; x(s)\big)P,$$

uniformly over all rank d projectors P and all $0 \leqslant s \leqslant T$.

We therefore need to estimate:

$$\begin{aligned}\sum_{j=1}^n \big\langle L(u)\phi_j, \phi_j\big\rangle &= \sum_{j=1}^n \big\langle \nu\Delta\phi_j - B(u,\phi_j) - B(\phi_j,u), \phi_j\big\rangle \\ &= \sum_{j=1}^n \nu\langle\Delta\phi_j,\phi_j\rangle + \sum_{j=1}^n b(\phi_j,u,\phi_j).\end{aligned}$$

Following the standard argument, using the Lieb–Thirring inequality (see [37], for example), we obtain:

$$\sum_{j=1}^n \big\langle L(u)\phi_j,\phi_j\big\rangle \leqslant -c\nu\lambda_1 n^2 + \frac{c}{\nu}|Du|^2.$$

It follows that

$$\begin{aligned} q_n(T;\omega,x) &\leqslant \int_0^T \left(-c\nu\lambda_1 n^2 + \frac{c}{\nu}\big|Du(t)\big|^2\,\mathrm{d}r\right) \\ &= T\left(-c\nu\lambda_1 n^2 + \frac{c}{\nu}\frac{1}{T}\int_0^T \big|Du(t)\big|^2\,\mathrm{d}t\right). \end{aligned} \tag{29}$$

Thus, given a fixed choice of T, we have an estimate for q_n of the form,

$$q_n \leqslant T(-\alpha n^2 + \beta),$$

where $\alpha = c\nu\lambda_1$ is deterministic, and

$$\beta = \frac{c}{\nu}\frac{1}{T}\int_0^T \left|Du(t)\right|^2 \mathrm{d}t.$$

The problem therefore reduces to bounding the expectation of the time integral of $|Du|^2$.

We note here that in the deterministic case it is easy to show that

$$\lim_{T\to\infty}\frac{\nu}{T}\int_0^T \left|Du(s)\right|^2 \mathrm{d}s \leqslant \frac{|f|^2}{\nu\lambda_1}. \tag{30}$$

Obtaining a bound on this quantity in the stochastic case can be done by carefully following the analysis in [21], the end result being that, uniformly for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$,

$$\frac{\nu}{T}\int_0^T \left|Du(t)\right|^2 \mathrm{d}t \leqslant T^{-1}R(\omega) + \varepsilon\, M(T,\omega) + (1+\varepsilon)\frac{|f|^2}{\nu\lambda_1},$$

where $\mathbb{E}\, R = \varrho < +\infty$, and $\mathbb{E}M(T,\cdot) = \mu(T) < +\infty$. (It is possible to obtain explicit, but unpleasant, expressions for ϱ and $\mu(T)$.)

Therefore,

$$q_n(T;\omega,x) \leqslant T(-\alpha n^2 + \beta),$$

where $\alpha = c\nu\lambda_1$ and

$$\mathbb{E}\beta \leqslant \frac{c}{\nu^2}\left(T^{-1}\tilde{\varrho} + \varepsilon\mu(T) + (1+\varepsilon)\frac{|f|^2}{\nu\lambda_1}\right).$$

It follows that for any choice of the parameters, n can be chosen sufficiently large,

$$n^2 \geqslant \frac{c}{\nu^3\lambda_1}\left(T^{-1}\varrho + \varepsilon\mu(T) + (1+\varepsilon)\frac{|f|^2}{\nu\lambda_1}\right),$$

to ensure that $\mathbb{E}q_n < 0$. (Given the explicit forms for ϱ and μ one would naturally optimize over T to minimize n.) Since the estimate for q_n is concave for every ω, Corollary 4.2 now guarantees that $d_{\mathrm{f}}(A(\omega)) \leqslant n$ for $\mathbb{P}$-a.e. ω.

We note in particular, given any $\delta > 0$, we can choose first T and then ε_0 to ensure that for all $\varepsilon \leqslant \varepsilon_0$,

$$\mathbb{E}q_n \leqslant T\left(-c\nu\lambda_1 n^2 + \frac{c}{\nu^2}\frac{|f|^2}{\nu\lambda_1} + c\nu\lambda_1\delta^2\right),$$

i.e.,

$$d_{\mathrm{f}}\big(A(\omega)\big) \leqslant c\frac{|f|}{\nu^2\lambda_1} + \delta,$$

showing that for small ε the estimate is close to that in the deterministic case.

5.2. Reaction–diffusion equations

We also mention here the bounds on the Hausdorff dimension of the random attractors of certain reaction–diffusion equations obtained by Debussche [17] and Caraballo, Langa and Robinson [3,4]. Since the estimates required to obtain the Hausdorff and fractal dimensions are identical, previous calculations now yield the same bounds on the fractal dimension of these attractors.

For the equation:

$$\mathrm{d}u = \big(\Delta u + \beta u - u^3\big)\,\mathrm{d}t + \varepsilon\phi\,\mathrm{d}W_t \quad \text{with } u|_{\partial U} = 0,$$

on a bounded domain $U \subset \mathbb{R}^d$ with $\phi \in D(A)$ (for $d \leqslant 4$), the analysis in Debussche's paper show that $d_{\mathrm{f}}(A) \leqslant c\beta^{d/2}$, an estimate of the same order as in the deterministic case. However, we note here that it has recently been shown that in fact the random attractor for this equation consists of a single random point, $A(\omega) = \{a(\omega)\}$, and hence has dimension zero ([5]; see [10] for a related result for Neumann boundary conditions).

The same equation with a multiplicative noise,

$$\mathrm{d}u = \big(\Delta u + \beta u - u^3\big)\,\mathrm{d}t + \sigma u \circ \mathrm{d}W_t \quad \text{with } u|_{\partial U} = 0,$$

also has $d_{\mathrm{f}}(A) \leqslant c\beta^{d/2}$ [5]. In this case the attractor does not collapse to a point: its dimension is bounded below by $c'\beta^{d/2}$, showing that as in the deterministic case [4],

$$d_{\mathrm{f}}(A) \sim \beta^{d/2}.$$

In particular it is interesting to note that this dimension estimate does not depend on σ, the level of the noise.

6. Distinguishing experimental observations

Suppose that a particular experiment is governed by a random dynamical system. Then comparing the observations in two different experiments involves two different realizations of the noise. The results here guarantee a rich choice of measurements that will distinguish between distinct states of the system, even allowing for the different realizations of the noise.

More mathematically, we suppose that the evolution of the physical system is governed by a random dynamical system that has a finite-dimensional random attractor. Then 'most' choices of observation function (in some precise sense) will distinguish between all points in

$$\left\{A(\omega)\colon\ \omega = \theta_t\omega_1 \text{ or } \theta_t\omega_2,\ t \in \mathbb{R}\right\},$$

with $\mathbb{P} \times \mathbb{P}$ probability one. We follow and expand on the approach in [29], which treated similar problems for non-autonomous dynamical systems.

6.1. Abstract linear embeddings

The first result is based on the embedding theorem due to Hunt and Kaloshin [24] discussed in the introduction (Theorem 1.1). Their result uses the concept of 'prevalence', which generalizes the notion of 'almost every' from finite to infinite-dimensional spaces and was introduced by Hunt, Sauer and Yorke [25].

Definition 6.1. A Borel subset S of a normed linear space V is *prevalent* if there exists a compactly supported probability measure μ such that $\mu(S + v) = 1$ for all $v \in V$.

For a more intuitive version of the definition, if we set $E = \mathrm{supp}(\mu)$ then E can be thought of as a 'probe set', which consists of 'allowable perturbations' with which, given a $v \in V$, we 'probe' and test whether $v + e \in E$ for μ-almost every $e \in E$.

Note that

(i) If V is finite-dimensional then this corresponds (via the Fubini theorem) to S being a set whose complement has zero measure;
(ii) If S is prevalent then S is dense in V;
(iii) The countable intersection of prevalent sets is itself prevalent.

For convenience we restate Hunt and Kaloshin's theorem here:

Theorem 6.2 [24]. *Let H be a Hilbert space, $X \subset H$ a compact set with fractal dimension d, and $N > 2d$ an integer. Then a prevalent set of bounded linear functions $L : H \to \mathbb{R}^N$ are one-to-one between X and its image.*

With the danger of labouring the point, the theorem says that there is a subset $E \subset \mathcal{L}(H, \mathbb{R}^N)$, 'the probe set', such that for every $L \in \mathcal{L}(H, \mathbb{R}^N)$, $L + e$ is one-to-one between X and its image for μ-almost every $e \in E$. It is important to remark here that the probe space E can be chosen to be independent of X (if not of $d_{\mathrm{f}}(X)$).

We now show that a prevalent set of bounded linear functions will distinguish between elements of random attractors. We denote by $\mathbb{A}(\omega)$ the entire history of the random attractor over a particular realization,

$$\mathbb{A}(\omega) := \bigcup_{t\in\mathbb{R}} A(\theta_t\omega).$$

In what follows we will use the shorthand 'L is one-to-one on X' to mean that L is one-to-one between X and its image.

Theorem 6.3. *Suppose that $\{A(\omega)\}$ is a compact random set for a random dynamical system for which the map,*

$$(t, \omega, u) \mapsto \varphi(t, \omega)u,$$

is Lipschitz continuous in u and α-Hölder continuous in t. Suppose that

$$d_{\mathrm{f}}\big(A(\omega)\big) = d < \infty \quad \mathbb{P}\text{-a.s.}$$

Let $N > 2(d+1)/\alpha$ be an integer. Then there is a prevalent set $\mathcal{G}$ of bounded linear functions $L : H \to \mathbb{R}^N$ such that if $L \in \mathcal{G}$, L is one-to-one on

$$\mathbb{A}(\omega_1) \cup \mathbb{A}(\omega_2)$$

with $\mathbb{P} \times \mathbb{P}$ probability one.

Proof. Denote the full $\mathbb{P}$-measure set of ω for which $d_{\mathrm{f}}(A(\omega)) = d$ by $\widetilde{\Omega}$, and fix $\omega \in \widetilde{\Omega}$. The set

$$\mathbb{A}_n(\omega) = \bigcup_{-n\leqslant t\leqslant n} A(\theta_t\omega)$$

is the image of the $(d+1)$-dimensional set $[-n, n] \times A(\omega)$ under the map $\varphi(t, \omega)$. Since the fractal dimension of $f(X)$ is bounded above by $d_{\mathrm{f}}(X)/\alpha$ when f is α-Hölder, see [20] or [18], for example, it follows that $d_{\mathrm{f}}(\mathbb{A}_n) \leqslant (d+1)/\alpha$. It follows from Theorem 6.2 that if N is an integer with $N > 2(d+1)/\alpha$ then for each n a prevalent set $\mathcal{G}_n$ of bounded linear maps $L : H \to \mathbb{R}^N$ are one-to-one on $\mathbb{A}_n(\omega)$.

The countable intersection $\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n$ is still prevalent (by (iii) above), and consists of bounded linear maps that are one-to-one on $\mathbb{A}(\omega)$. Indeed, if not there must be two elements $u, v \in \mathbb{A}(\omega)$ and an $L \in \mathcal{G}_\infty$ such that $Lu = Lv$. But since we must have $u, v \in \mathbb{A}_n(\omega)$ for some n, and $L \in \mathcal{G}_n$, this cannot be.

Now, it is clear that given a choice of two realizations $\omega_1, \omega_2 \in \widetilde{\Omega}$, for every $L \in \mathcal{L}(H, \mathbb{R}^N)$, μ-almost every choice of $e \in E$ makes $L + e$ one-to-one on $\mathbb{A}(\omega_1) \cup \mathbb{A}(\omega_2)$. For each $L \in \mathcal{L}(H, \mathbb{R}^N)$, denote by $\mathbb{G}_L$ the set of all $(\omega_1, \omega_2, e) \in \Omega \times \Omega \times E$ for which $L + e$ is one-to-one on $\mathbb{A}(\omega_1) \cup \mathbb{A}(\omega_2)$.

We have

$$\int_\Omega \int_\Omega \int_E \chi(\mathbb{G}_L)\,\mathrm{d}\mu\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mathbb{P} = 1,$$

where $\chi(\mathbb{G}_L)$ is the characteristic function of $\mathbb{G}_L$. Fubini's theorem allows us to change the order of integration,

$$\int_E \int_\Omega \int_\Omega \chi(\mathbb{G}_L)\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mu = 1.$$

It follows that μ-almost every choice of $e \in E$ makes $L + e$ one-to-one on $\mathbb{A}(\omega_1) \cup \mathbb{A}(\omega_2)$ with $\mathbb{P} \times \mathbb{P}$ probability one. Since this is true for every $L \in \mathcal{L}(H, \mathbb{R}^N)$, the theorem follows. □

6.2. Point measurements

We now give a result allowing for more physical observations, provided that the attractor consists of analytic functions. The deterministic version of the result is as follows:

Theorem 6.4 [26]. *Let U be a bounded open subset of $\mathbb{R}^n$, and let X be a compact subset of $L^2(U, \mathbb{R}^d)$ with finite fractal dimension d that consists of real analytic functions*[3] *so that, in particular, for each $r \in \mathbb{N}$ and for every compact subset K of U, X is a bounded subset of $C^r(K, \mathbb{R}^d)$. Then for $k \geqslant 16d + 1$ Lebesgue-almost every set $\mathbf{x} = (x_1, \ldots, x_k)$ of k points in U makes the map $E_\mathbf{x}$, defined by,*

$$E_\mathbf{x}[u] = \big(u(x_1), \ldots, u(x_k)\big),$$

one-to-one between X and its image.

In the light of Theorem 6.3, the stochastic version of this result is unsurprising:

Theorem 6.5. *Let $\{A(\omega)\}$ be the a compact random set such that for $\mathbb{P}$-a.e. ω, $\mathbb{A}_n(\omega)$ satisfies the conditions of Theorem* 6.4 *for every $n \in \mathbb{Z}_+$. Then for $k \geqslant 16d + 1$, almost every choice of* $\mathbf{x}$ *is one-to-one between*

$$\bigcup_{t\in\mathbb{R}} A(\theta_t\omega_1) \;\cup\; \bigcup_{t\in\mathbb{R}} A(\theta_t\omega_2) \tag{31}$$

and its image with $\mathbb{P} \times \mathbb{P}$ probability one.

[3] This can in fact be weakened: the requirement is that the attractor consists of C^∞ functions with derivatives bounded uniformly in every compact $K \subset U$ as in the statement of Hu theorem, and that $u - v$ has finite order of vanishing for every pair of distinct elements $u, v \in X$.

We will omit the proof, which is a simplified version of that of Theorem 6.3. As in Theorem 6.3, the condition required on the dimension $\mathbb{A}_n$ could be satisfied if $d_{\mathrm{f}}(A(\omega)) \leqslant \tilde{d}$ and $A(\omega)$ is invariant for a random dynamical system for which φ is Lipschitz on H and α-Hölder in time. Then we could take $d = (\tilde{d}+1)/\alpha$. The analyticity properties need to hold in a uniform way over each $\mathbb{A}_n$; this is usually the case in applications (see, for example, [8]).

7. Conclusion

We have shown that the random attractors that arise in the random dynamical systems generated by certain stochastic PDEs enjoy the same estimates on their fractal dimension as those on their Hausdorff dimension.

As a particular example we have obtained a bound on the dimension of the 2d Navier–Stokes equations with a particular form of additive noise. It is an interesting open problem to obtain similar bounds for more general additive noise, and for multiplicative noise.

One consequence of our results is that a single finite-dimensional linear map can be used to embed most realizations of the random attractor into a finite-dimensional space (Theorem 6.3). It is therefore natural to ask whether the dynamics restricted to the random attractor can be captured by a finite-dimensional random dynamical system. However, even in the deterministic case this question has not been satisfactorily settled (see, e.g., Chapter 16 in [34] and [33,35]).

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