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High-order nonreflecting boundary conditions for the dispersive shallow water equations

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Abstract

Time-dependent dispersive shallow water waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathscr{B}$, and a high-order non-reflecting boundary condition (NRBC) is imposed on $\mathscr{B}$. Then the problem is solved by a finite difference scheme in the finite domain bounded by $\mathscr{B}$. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation of the Higdon conditions, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, a procedure for the automatic choice of the parameters appearing in the NRBC is proposed. The performance of the scheme is demonstrated via a numerical example.

Keywords: Shallow water equations; Non-reflecting boundary conditions; Wave dispersion; High-order method

1. Introduction

In various fields of applications one is often interested in solving a wave problem computationally in a domain which is much smaller than the actual domain where the governing equations hold. Two important examples are regional models in meteorology and problems of scattering from objects in underwater acoustics. See, e.g., the monographs [3,2] for discussion on these problems and on associated solution techniques.

One of the several methods that exist for solving a wave problem in a limited computational domain is that of using nonreflecting boundary conditions (NRBCs). In this method, the original domain is first truncated by introducing an artificial boundary $\mathscr{B}$, which encloses the computational

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domain Ω. Then a special boundary condition is applied on $\mathscr{B}$. This boundary condition must be nonreflecting, namely it should not give rise to reflections when waves that propagate from within Ω impinge on it. Boundary conditions that generate no spurious reflection in all cases are called "perfectly nonreflecting," or "perfectly absorbing," or simply "exact." (See [4] for a review of such NRBCs.) Most NRBCs are approximate, namely they do generate some amount of reflection. However, as long as the reflection error is small (say, of the same order of magnitude as the discretization error) the NRBC is considered adequate. The simplest NRBC is the Sommerfeld-like boundary condition, which has the same form as the Sommerfeld radiation condition that holds at infinity. In the last three decades several improved NRBCs that reduce the spurious reflections have been proposed. See more details in [3].

In some cases, typically in meteorology, the boundary conditions used on $\mathscr{B}$ must also contain information obtained previously from a "global model." The latter is defined over a much larger domain than Ω (e.g., an atmospheric layer above the entire sphere representing the earth) but involves a much coarser discretization. Ways to incorporate such global information in a NRBC are discussed in [2].

A closely related approach to that of using a NRBC is the method of absorbing layers. Here a whole layer adjacent to the truncating boundary $\mathscr{B}$ is artificially introduced. In this layer the governing equations are modified in a special manner to annihilate spurious reflections. Most notably, the perfectly matched layer (PML) is a perfectly absorbing layer for all waves. PML was originally invented in the context of electromagnetic waves [1], but has recently been extended to other fields, including shallow water waves with dispersion and advection [22].

To design a NRBC, one usually assumes that the governing equations in the exterior are linear. This does not prevent the NRBC from being used with nonlinear equations inside Ω. In terms of the complexity of designing accurate NRBCs, one can distinguish between three types of linear wave problems: time-harmonic wave problems, nondispersive time-dependent wave problems, and dispersive wave problems. The prototype governing equations for these problems are, respectively, the Helmholtz equation, the scalar wave equation, and the Klein–Gordon equation. Technically more involved equations, but with similar properties, are of interest in each of the three categories.

The case of time-harmonic waves is, to a large extent, a solved case as far as NRBCs are concerned. Very effective exact and high-order NRBCs are available; see [5,6,30]. The case of time-dependent waves is much more involved. For three-dimensional waves where $\mathscr{B}$ is a sphere, Grote and Keller [9] and Hagstrom and Hariharan [11] constructed exact NRBCs. In two dimensions, Hagstrom and Hariharan [11] proposed a high-order asymptotic NRBC. Dispersive wave problems, in which waves of different frequencies propagate with different speeds, are the most difficult. High-order NRBCs have been constructed in [7,8]. The general practice in meteorology, where waves are dispersive due to the earth rotation (the Coriolis effect), is to use a first-order Sommerfeld-like condition, with a specially chosen phase velocity C. See, for example, the treatment of the lateral boundaries in the regional weather prediction code COAMPS [17]; here C is either specified as a constant or varied adaptively using the Miller–Thorpe method [21].

We propose a high-order NRBCs scheme, in the context of the two-dimensional nonlinear shallow water equations (SWEs). It is associated with a sequence of NRBCs of increasing order and the Jth-order NRBC is *exact* for any combination of waves that have specified wave number components $(k_x)_j$ and $(k_y)_j$ for $j = 1,\ldots,J$. This methodology originates from the observation that the solution of a dispersive wave problem is an infinite superposition of single waves, each characterized by its wave number components (or, equivalently, by its phase speed components).

We use on the artificial boundary $\mathscr{B}$ one of the *Higdon NRBCs* [16]. For a straight boundary $\mathscr{B}$ normal to the x direction, the Higdon NRBC of order J is

$$H_J: \quad \left[\prod_{j=1}^{J}(\partial_t + C_j\partial_x)\right]\eta(x,y,t) = 0 \quad \text{on } \mathscr{B}. \tag{1}$$

Here, t is time, and the C_j are parameters which have to be chosen and which signify phase speeds in the x direction. The boundary condition (1) is exact for all combinations of waves that propagate with x-direction phase speeds $C_1,\ldots,C_J$.

2. Statement of the problem

Consider the shallow water equations (SWEs) in a semi-infinite channel. For simplicity, we assume that the channel has a flat bottom and that there is no advection, although these assumptions may be removed in future studies. We do take into account rotation (Coriolis) effects. A Cartesian coordinate system (x, y) is introduced such that the channel is parallel to the x direction, as shown in Fig. 1. The width of the channel is denoted b.

The SWEs are (see [26]):

$$\partial_t u + \mu u\partial_x u + \mu v\partial_y u - fv = -g\partial_x\eta, \tag{2}$$

$$\partial_t v + \mu u\partial_x v + \mu v\partial_y v + fu = -g\partial_y\eta, \tag{3}$$

$$\partial_t\eta + \mu u\partial_x\eta + \mu v\partial_y\eta + (h_0 + \mu\eta)\,(\partial_x u + \partial_y v) = 0. \tag{4}$$

Here t is time, $u(x,y,t)$ and $v(x,y,t)$ are the unknown velocities in the x and y directions, h_0 is the given water layer thickness (in the direction normal to the xy plane), $\eta(x,y,t)$ is the unknown water elevation above h_0, f is the Coriolis parameter, and g is the gravity acceleration. We use the following shorthand for partial derivatives:

$$\partial_a^i = \frac{\partial^i}{\partial a^i}.$$

The parameter μ is 1 for the nonlinear SWEs, and is 0 for the linearized SWEs with vanishing mean flow. We shall consider the latter as a special case in the sequel.

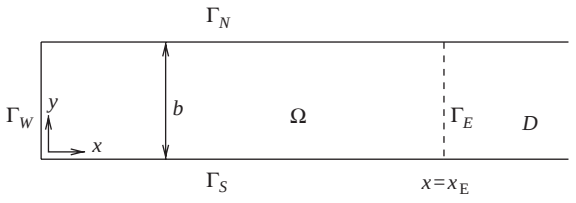

Fig. 1. A semi-infinite channel.

It can be shown (see [10]) that *a single* boundary condition must be imposed along the entire boundary to obtain a well-posed problem. On the south and north channel walls Γ_S and Γ_N we have $v = 0$ (no normal flow). On the west boundary Γ_W we prescribe η using the Dirichlet condition $\eta(0, y, t) = \eta_W(y, t)$, where $\eta_W(y, t)$ is a given function (incoming wave). At $x \to \infty$ the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, initial values for u, v and η are given at time $t = 0$ in the entire domain.

We now truncate the semi-infinite domain by introducing an artificial east boundary Γ_E, located at $x = x_E$ (see Fig. 1). To obtain a well-posed problem in the finite domain Ω we need a single boundary condition on Γ_E. We shall apply a high-order NRBC for the variable η. A discussion on this NRBC follows.

3. Higdon's NRBCs

On the artificial boundary Γ_E we use one of the Higdon NRBCs [16]. These NRBCs were presented and analyzed in a sequence of papers [12–15] for nondispersive acoustic and elastic waves, and were extended in [16] for dispersive waves. Their main advantages are as follows:

1. The Higdon NRBCs are very *general*, namely they apply to a variety of wave problems, in one, two and three dimensions and in various configurations.
2. They form a *sequence* of NRBCs of increasing order. This enables one, in principle (leaving implementational issues aside for the moment), to obtain solutions with unlimited accuracy.
3. The Higdon NRBCs can be used, without any difficulty, for *dispersive* wave problems and for problems with layers. Most other available NRBCs are either designed for nondispersive media (as in acoustics and electromagnetics) or are of low order (as in meteorology and oceanography).
4. For certain choices of the parameters, the Higdon NRBCs are equivalent to NRBCs that are derived from rational approximation of the dispersion relation (the Engquist–Majda conditions being the most well-known example). This has been proved by Higdon in [16] and in earlier papers. Thus, the Higdon NRBCs can be viewed as generalization of rational-approximation NRBCs.

The scheme used here was developed in [7] and is different than the original Higdon scheme [16] in the following ways:

1. The discrete Higdon conditions were developed in the literature up to third order only, because of their algebraic complexity which increases rapidly with the order. Here we use the implementation to an *arbitrarily high order* given in [7]. The order of the scheme is simply an input parameter.
2. The original Higdon conditions were applied to the Klein–Gordon linear wave equation and to the elastic equations. Here we show how to apply them to the SWEs (2)–(4).
3. The Higdon NRBCs involve some parameters which must be chosen. Higdon [16] discusses some general guidelines for their manual a priori choice by the user. We shall choose these parameters *automatically* as detailed in [7].

The Higdon NRBC of order J is given by (1). To see its basic property, consider a wave which satisfies the linearized SWEs (Eqs. (2)–(4) with $\mu = 0$; see, e.g., [26,

p. 77]. It has the form

$$\eta = \eta_0 Y(y)\cos(kx - \omega t + \psi), \tag{5}$$

where

$$\omega^2 = \begin{cases} C_0^2\left(k^2 + \dfrac{n^2\pi^2}{b^2}\right) + f^2, & n = 1,2,\ldots, \\ C_0^2 k^2, & n = 0, \end{cases} \tag{6}$$

$$Y(y) = \begin{cases} \cos\dfrac{n\pi y}{b} - \dfrac{bf}{n\pi C_x}\sin\dfrac{n\pi y}{b}, & n = 1,2,\ldots, \\ \exp(-f y/C_0), & n = 0, \end{cases} \tag{7}$$

$$C_0 = \sqrt{h_0 g}, \quad C_x = \frac{\omega}{k}. \tag{8}$$

In (5)–(8), η_0 is the wave amplitude, ψ is its phase, k is the x-component wave number, ω is the wave frequency, C_0 is the reference wave speed (which is both the phase speed and the group speed for the nondispersive case $f = 0$), and C_x is the x-direction phase velocity. Eq. (6) is the dispersion relation. The solutions corresponding to the modes $n = 1,2,\ldots$ are Poincaré waves, whereas the solution corresponding to $n = 0$ is the Kelvin wave. These complete the set of all wave solutions for wave number k and mode n. There are also solutions that decay exponentially in the x direction. However, Higdon's NRBCs ignore them. They are usually not of great concern, since the decaying modes are expected to be insignificant at Γ_E, provided that Γ_E is sufficiently far away from where the waves are generated.

Now, it is easy to verify that if one of the C_j's in (1) is equal to C_x, then the wave (5) satisfies the boundary condition (1) exactly.

We make a few observations:

- From (6) and (8) we have always $C_x \geqslant C_0$; hence one should take $C_j \geqslant C_0$. In general, the solution consists of an infinite number of waves of the form (5) with different phase speeds.
- The first-order condition H_1 is a Sommerfeld-like boundary condition. If we set $C_1 = C_0$ we get the classical Sommerfeld-like NRBC. A lot of work in the meteorological literature is based on using H_1 with a specially chosen C_1. Pearson [25] used a special but constant value of C_1, while in the scheme devised by Orlanski [24] and in later improved schemes [19,21,27,32] the C_1 changes dynamically and locally in each time step based on the solution from the previous time step. Some of the limited-area weather prediction codes used today are based on such schemes, e.g., COAMPS [17]. See also the recent papers [18,23,28] where several such schemes are compared.
- The condition H_J involves up to Jth-order normal and temporal derivatives.
- It is easy to show (see [16] for a similar setting) that when a wave of the form (5) impinges on the boundary Γ_E where the NRBC H_J is imposed, the resulting *reflection coefficient* is

$$R = \prod_{j=1}^{J}\left|\frac{C_j - C_x}{C_j + C_x}\right|. \tag{9}$$

Again we see that if $C_j = C_x$ for one of the j's then $R = 0$, namely there is no reflection and the NRBC is exact. Moreover, we see that the reflection coefficient is a product of J factors, *each of which is smaller than 1*. This implies that the reflection coefficient becomes smaller as the order J increases regardless of the choice made for the parameters C_j. Of course, a good choice for the C_j would lead to better accuracy with a lower order J, but even if we miss the correct C_j's considerably (say, if we make the simplest choice $C_j = C_0$ for $j = 1, \ldots, J$), we are still guaranteed to reduce the spurious reflection as we increase the order J. This is an important property of the Higdon's NRBCs and is the reason for their robustness.

3.1. Discretization of Higdon's NRBCs

The Higdon condition H_J is a product of J operators of the form $\partial_t + C_j \partial_x$. Consider the following finite difference (FD) approximations:

$$\partial_t \simeq \frac{I - S_t^-}{\Delta t}, \quad \partial_x \simeq \frac{I - S_x^-}{\Delta x}. \tag{10}$$

In (10), Δt and Δx are, respectively, the time-step size and grid spacing in the x direction, I is the identity operator, and S_t^- and S_x^- are shift operators defined by

$$S_t^- \eta_{pq}^n = \eta_{pq}^{n-1}, \quad S_x^- \eta_{pq}^n = \eta_{p-1,q}^n. \tag{11}$$

Here and elsewhere, η_{pq}^n is the FD approximation of $\eta(x, y, t)$ at grid point (x_p, y_q) and at time t_n. We use (10) in (1) to obtain

$$\left[\prod_{j=1}^{J} \left(\frac{I - S_t^-}{\Delta t} + C_j \frac{I - S_x^-}{\Delta x} \right) \right] \eta_{\mathrm{E}q}^n = 0. \tag{12}$$

Here, the index E correspond to a grid point on the boundary Γ_{E}. Higdon has solved this difference equation (and also a slightly more involved equation that is based on time- and space-averaging approximations for ∂_x and ∂_t) for $J \leqslant 3$ to obtain an explicit formula for $\eta_{\mathrm{E}q}^n$. This formula is used to find the current values on the boundary Γ_{E} *after* the solution in the interior points and on the other boundaries has been updated. The formula for $J = 2$ is found in [15], and the one for $J = 3$ appears in the appendix of [14]. The algebraic complexity of these formulas increases rapidly with the order J. We have implemented the Higdon NRBCs *to any order* using a simple algorithm, see [7].

Our procedure for implementing the Higdon NRBCs can easily be modified to admit improved approximations. The main feature that has to be changed in the algorithm is the *base* to which the counting decimal integer m is transformed.

4. The interior scheme

We consider explicit FD interior discretization schemes for the SWEs (2)–(4) to be used in conjunction with the H_J condition. The interaction between the H_J condition and the interior scheme

is of concern, since simple choices for an explicit interior scheme turn out to give rise to long-time instabilities. We have tried the Miller–Pearce time integration [20], Leap-Frog [29], a version of semi-implicit time integration [31] and the MacCormack scheme [29,2] (which is equivalent for linear problems to the Lax–Wendroff scheme). They are all stable for a sufficiently small time step when used with the boundary condition H_1, but they all become unstable for $J \geqslant 2$. The instability appears earlier in time when J becomes larger.

Higdon [16] has proved, in the context of the scalar Klein–Gordon equation,

$$\partial_t^2 \eta - C_0^2 \nabla^2 \eta + f^2 \eta = 0, \tag{13}$$

that the discrete NRBCs (12) are stable if the interior scheme is the standard *second-order centered* difference scheme

$$\eta_{pq}^{n+1} = 2\eta_{pq}^{n} - \eta_{pq}^{n-1} + \left(\frac{C_0 \Delta t}{\Delta x}\right)^2 (\eta_{p+1,q}^{n} - 2\eta_{p,q}^{n} + \eta_{p-1,q}^{n}) + \left(\frac{C_0 \Delta t}{\Delta y}\right)^2 (\eta_{p,q+1}^{n} - 2\eta_{p,q}^{n} + \eta_{p,q-1}^{n}) - (f \Delta t)^2 \eta_{p,q}^{n}. \tag{14}$$

Now we shall show how the SWEs (2)–(4) can be discretized in such a way as to mimic (14) and to lead to a stable scheme.

First, we define the new variables

$$V^+ = h_0(\partial_x u + \partial_y v), \quad V^- = h_0(\partial_x v - \partial_y u). \tag{15}$$

From the SWEs (2)–(4) we obtain equations which involve these two variables. By differentiating (2) and (3) with respect to x and to y, respectively, and then summing the results, we get the equation

$$\partial_t V^+ - f V^- + g h_0 \nabla^2 \eta = N_1, \tag{16}$$

where

$$N_1 = \mu h_0 [\partial_x (u \partial_x u) + \partial_x (v \partial_y u) + \partial_y (u \partial_x v) + \partial_y (v \partial_y v)]. \tag{17}$$

Note that N_1 is the nonlinear part of Eq. (16). Similarly, we differentiate (3) and (2) with respect to x and to y, respectively, and then subtract the second from the first to obtain

$$\partial_t V^- + f V^+ = N_2, \tag{18}$$

where

$$N_2 = \mu h_0 [\partial_x (u \partial_x v) + \partial_x (v \partial_y v) - \partial_y (u \partial_x u) - \partial_y (v \partial_y u)]. \tag{19}$$

We write (4) as

$$\partial_t \eta + V^+ = N_3, \tag{20}$$

where

$$N_3 = \mu [\partial_x (u \eta) + \partial_y (v \eta)]. \tag{21}$$

Finally, we also consider the time derivative of Eq. (20), namely

$$\partial_{tt}\eta + \partial_t V^+ = \partial_t N_3. \tag{22}$$

Now we base the interior scheme on Eqs. (16), (18), (20) and (22). First, we discretize (20) to obtain an updating formula for V^+:

$$(V^+)_{pq}^{n+1} = N_3^n - \frac{\eta_{pq}^n - \eta_{pq}^{n-1}}{\Delta t}. \tag{23}$$

The notation N_3^n means that we calculate all the variables appearing in the expression for N_3 at time step n. We shall discuss the discretization of the spatial derivatives in N_3 later. Then we use (18) to update $V_t^- \equiv \partial_t V^-$:

$$(V_t^-)_{pq}^{n+1} = N_2^n - f(V^+)_{pq}^{n+1}. \tag{24}$$

Next we integrate (24) to update V^-:

$$(V^-)_{pq}^{n+1} = (V^-)_{pq}^n + \Delta t(V_t^-)_{pq}^{n+1}. \tag{25}$$

Now we use (16) to update $V_t^+ \equiv \partial_t V^+$. We use second-order central differences in space to approximate $\nabla^2\eta$:

$$(V_t^+)_{pq}^{n+1} = N_1^n + f(V^-)_{pq}^{n+1} - gh_0\left(\frac{\eta_{p+1,q}^n - 2\eta_{p,q}^n + \eta_{p-1,q}^n}{\Delta x^2} + \frac{\eta_{p,q+1}^n - 2\eta_{p,q}^n + \eta_{p,q-1}^n}{\Delta y^2}\right). \tag{26}$$

Finally we use Eq. (22) to update η. We use second-order central differences in time to approximate $\partial_{tt}\eta$:

$$\eta_{pq}^{n+1} = 2\eta_{pq}^n - \eta_{pq}^{n-1} - \Delta t^2(V_t^+)_{pq}^{n+1} + \Delta t(N_3^n - N_3^{n-1}). \tag{27}$$

After η_{pq}^{n+1} is known, the updated values for u and v, i.e., u_{pq}^{n+1} and v_{pq}^{n+1} may be found in a number of ways. We have chosen to integrate the original SWEs (2) and (3) using a forward FD approximation in time to obtain these values.

It is easy to show that in the linear case, and with zero initial conditions, the updating formula for η, Eq. (27), coincides with formula (14) for the Klein–Gordon equation. Indeed, in this case $(V^-)^{n+1} = f\eta^n$, and using (23)–(27) without the nonlinear terms leads exactly to formula (14). Thus stability is guaranteed in this case.

In the nonlinear case, we have to calculate the quantities N_1^n, N_2^n and N_3^n. These involve first- and second-order spatial derivatives. All these derivatives may be calculated using second-order centered differences.

5. A numerical example

We apply the new scheme to a simple test problem whose exact solution is synthesized a priori. We consider the linearized SWEs in a two-dimensional uniform semi-infinite channel or wave guide.

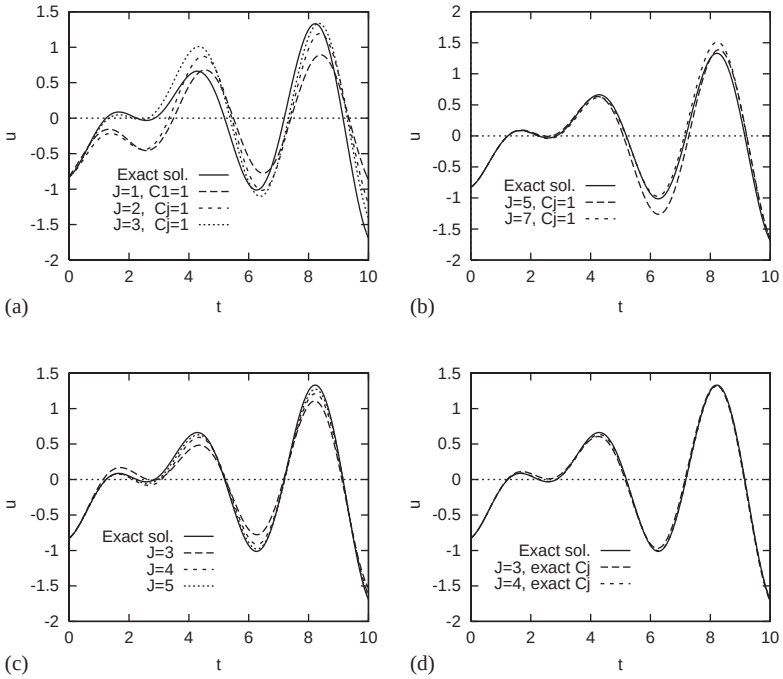

Fig. 2. Solution of the three-wave test problem: u at the point $x = 5,\ y = 2.75$ (on Γ_E) as a function of time. (a) Exact solution and the H_1, H_2 and H_3 solutions with $C_j = 1$. (b) Exact solution and the H_5 and H_7 solutions with $C_j = 1$. (c) Exact solution and the H_3, H_4 and H_5 solutions with automatically chosen C_j. (d) Exact solution and the H_3 and H_4 solutions with the exact C_j.

A Cartesian coordinate system (x, y) is introduced such that the wave guide is parallel to the x direction. The width of the wave guide is denoted b. We set $b = 5,\ C_0 = 1$ and $f = 0.5$. The boundary function $\eta_W(y,t)$ on Γ_W and the initial conditions are those that correspond to a solution $\eta(x,y,t)$ which is a linear combination of three waves of the form (5), i.e.,

$$\eta = \sum_{m=1}^{3} A_m \cos\frac{n_m \pi y}{b} \cos(k_m x - \omega_m t). \tag{28}$$

The parameters chosen in (28) are: $A_m = 1,1,1;\ n_m = 1,2,2;\ \omega_m = 0.81, 1.37, 1.68$. This corresponds to the three phase velocities: $C_x/C_0 = 7.61, 6.27, 1.69$. The k_m in (28) are obtained from the ω_m and the n_m via the dispersion relation (6).

We introduce the artificial boundary Γ_E (see Fig. 1) at $x_E = 5$. Thus, the computational domain Ω is a 5×5 square. In Ω we use a uniform grid with 21×21 points. We discretize the linearized SWEs in Ω using the explicit central-difference FD interior scheme. On Γ_E we impose the Higdon NRBC implemented in its high-order form. The time-step size is $\Delta t = 0.025$, which is smaller than the CFL limit and thus guarantees stability.

In Figs. 2(a)–(d), we plot the solution η at the point $x = 5,\ y = 2.75$ (located on Γ_E) as a function of time. In each of the four figures the exact solution is compared to a number of numerical solutions obtained with different NRBC schemes, namely with different choices of the order J and

the parameters C_j. First we choose $C_j = 1$ for all j. Fig. 2(a) shows the H_1, H_2 and H_3 solutions. Their accuracy is poor, although the H_3 solution is significantly better than the other two. Fig. 2(b) shows the H_5 and H_7 solutions. The H_7 solution is quite accurate in the entire time interval shown. Thus, if the C_j's are not specially chosen, we need the order of the Higdon NRBC to be as high as 7 for high accuracy.

Now we employ the procedure to automatically choose the C_j's. Fig. 2(c) shows the resulting H_3, H_4 and H_5 solutions. We see that in this case the approach of the numerical solutions to the exact solution is monotone. Moreover, for $J = 5$ we get about the same level of accuracy as we did with $J = 7$ when all the C_j had the value one (Fig. 2(b)). For additional reference, we show in Fig. 2(d) the H_3 solution obtained with the C_j corresponding to the three phase velocities C_x of the exact solution. It is about as accurate as the H_5 solution in Fig. 2(c). We also show the H_4 solution obtained with the exact C_1, C_2, C_3 and with $C_4 = 1$. The numerical solution is indistinguishable from the exact solution. In this case not only the NRBC is exact, but we gain additional accuracy on the boundary due to the increased order of the FD scheme.

This example demonstrates, albeit in a simplified setting, that the same level of accuracy obtained with parameter values C_j that are well estimated can be achieved with ill-chosen parameter values but with an increased order J. Of course, increasing the order to ensure high accuracy is computationally expensive, and therefore it is usually beneficial to use our algorithm to choose the parameters C_j.

6. Nonzero advection

When using the Higdon NRBC (1) with the SWEs, it has been assumed that the SWEs are linearized (at least in the exterior domain D) about the state of zero mean flow (no advection). Now, suppose that the SWEs are linearized about a state corresponding to a *nonzero* mean flow. For simplicity, let us assume that this mean flow is constant in space and time. If the component of the advection velocity in the x direction (the direction normal to Γ_E) is U_0, then in the nondispersive case, the Sommerfeld-like condition $(\partial_t + C_0\partial_x)\eta = 0$ simply becomes $(\partial_t + (U_0 + C_0)\partial_x)\eta = 0$ (see, e.g., [2]). We can infer from this, that the Higdon NRBCs (1) should be replaced by

$$H_J: \quad \left[\prod_{j=1}^{J}(\partial_t + \hat{C}_j\partial_x)\right]\eta = 0 \quad \text{on } \Gamma_E, \quad \hat{C}_j = U_0 + C_j. \tag{29}$$

Thus, the Higdon conditions remain unchanged in the presence of advection, except that the parameter multiplying the x derivative in each operator factor stands for the *total* phase velocity in the x direction (and not the perturbed velocity).

The interior scheme discussed in Section 4 and the scheme for selecting the parameters can be extended to the advective case without difficulty.

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References

[1] J.P. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, J. Comput. Phys. 127 (1994) 185–200.
[2] D.R. Durran, Numerical Methods for Wave Equations in Geophysical Fluid Dynamics, Springer, New York, 1999.
[3] D. Givoli, Numerical Methods for Problems in Infinite Domains, Elsevier, Amsterdam, 1992.
[4] D. Givoli, Exact representations on artificial interfaces and applications in mechanics, Appl. Mech. Rev. 52 (1999) 333–349.
[5] D. Givoli, Recent advances in the DtN FE method, Arch. Comput. Methods Eng. 6 (1999) 71–116.
[6] D. Givoli, High-order non-reflecting boundary conditions without high-order derivatives, J. Comput. Phys. 170 (2001) 849–870.
[7] D. Givoli, B. Neta, High-order non-reflecting boundary conditions for dispersive waves, Wave Motion 37 (2003) 257–271.
[8] D. Givoli, B. Neta, High-order non-reflecting boundary scheme for time-dependent Waves, J. Comput. Phys. 186 (2003) 24–46.
[9] M.J. Grote, J.B. Keller, Exact nonreflecting boundary conditions for the time dependent wave equation, SIAM J. Appl. Math. 55 (1995) 280–297.
[10] B. Gustafsson, H.-O. Kreiss, J. Oliger, Time Dependent Problems and Difference Methods, Wiley, New York, 1995.
[11] T. Hagstrom, S.I. Hariharan, A formulation of asymptotic and exact boundary conditions using local operators, Appl. Numer. Math. 27 (1998) 403–416.
[12] R.L. Higdon, Absorbing boundary conditions for difference approximations to the multi-dimensional wave equation, Math. Comp. 47 (1986) 437–459.
[13] R.L. Higdon, Numerical absorbing boundary conditions for the wave equation, Math. Comp. 49 (1987) 65–90.
[14] R.L. Higdon, Absorbing boundary conditions for elastic waves, Geophysics 56 (1991) 231–241.
[15] R.L. Higdon, Absorbing boundary conditions for acoustic and elastic waves in stratified media, J. Comput. Phys. 101 (1992) 386–418.
[16] R.L. Higdon, Radiation boundary conditions for dispersive waves, SIAM J. Numer. Anal. 31 (1994) 64–100.
[17] R.M. Hodur, The naval research laboratory's coupled ocean/atmosphere mesoscale prediction system (COAMPS), Monthly Weather Rev. 125 (1997) 1414–1430.
[18] T.G. Jensen, Open boundary conditions in stratified ocean models, J. Marine Systems 16 (1998) 297–322.
[19] J.B. Klemp, D.K. Lilly, Numerical simulation of hydrostatic mountain waves, J. Atmos. Sci. 35 (1978) 78–107.
[20] M.J. Miller, R.P. Pearce, A three dimensional primitive equation model of cumulonimbus convection, Quart. J. Roy. Meteorol. Soc. 100 (1974) 133–154.
[21] M.J. Miller, A.J. Thorpe, Radiation conditions for the lateral boundaries of limited-area numerical models, Quart. J. Roy. Meteoroi. Soc. 107 (1981) 615–628.
[22] I.M. Navon, B. Neta, M.Y. Hussaini, A perfectly matched layer formulation for the nonlinear shallow water equations models: the split equation approach, Monthly Weather Rev., to appear.
[23] F.S.B.F. Oliveira, Improvement on open boundaries on a time dependent numerical model of wave propagation in the nearshore region, Ocean Eng. 28 (2000) 95–115.
[24] I. Orlanski, A simple boundary condition for unbounded hyperbolic flows, J. Comput. Phys. 21 (1976) 251–269.
[25] R.A. Pearson, Consistent boundary conditions for the numerical models of systems that admit dispersive waves, J. Atmos. Sci. 31 (1974) 1418–1489.
[26] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1987.
[27] W.H. Raymond, H.L. Kuo, A radiation boundary condition for multi-dimensional flows, Quart. J. Roy. Meteorol. Soc. 110 (1984) 535–551.
[28] X. Ren, K.H. Wang, K.R. Jin, Open boundary conditions for obliquely propagating nonlinear shallow-water waves in a wave channel, Comput. Fluids 26 (1997) 269–278.

[29] J.C. Strikwerda, Finite difference schemes and partial differential equations, Wadsworth & Brooks, Pacific Grove, CA, 1989.
[30] S.V. Tsynkov, Numerical solution of problems on unbounded domains, a review, Appl. Numer. Math. 27 (1998) 465–532.
[31] C.B. Vreugdenhil, Numerical methods for shallow water flow, Kluwer, Dordrecht, 1994.
[32] M. Wurtele, J. Paegle, A. Sielecki, The use of open boundary conditions with the Storm-Surge equations, Monthly Weather Rev. 99 (1971) 537–544.