stochastic processes and their

Backward stochastic differential equations with subdifferential operator and related variational inequalities

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Abstract

The existence and uniqueness of the solution of a backward SDE, on a random (possibly infinite) time interval, involving a subdifferential operator is proved. We then obtain a probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities.

Keywords: Backward stochastic equations; Subdifferential operators; Variational inequalities; Viscosity solutions; Probabilistic formulae for PDE

0. Introduction

Backward stochastic differential equations (BSDE) provide probabilistic formulae for the viscosity solution of semilinear partial differential equations (PDE) (see, in particular, Pardoux, 1997; Pardoux and Peng, 1992, and their references). In this paper one gives such formulae for parabolic variational inequalities on the whole space and also for the solution of a Dirichlet problem for an elliptic variational inequality. We restrict ourselves to variational inequalities for PDEs, and not systems of PDEs. The only difficulty in treating general systems concerns the difficulty of giving a definition of viscosity solution for such systems. In the first part of this paper we study BSDEs on a random (possibly infinite) time interval, whose coefficient contains the subdifferential of a convex function.

BSDEs with subdifferential operators include as a special case BSDEs whose solution is reflected at the boundary of a convex subset of $\mathbb{R}^k$. In the one-dimensional case, BDSEs with one-sided reflection have been studied in El Karoui et al. (1997), together with the associated optimal stopping time/optimal control problem, and an obstacle problem for a PDE (also called "variational inequality"). BSDEs with two-sided

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reflection, together with the associated stochastic game of optimal stopping, are studied in Cvitanic and Karatzas (1996). Multi-dimensional BSDEs reflected at the boundary of a convex set is studied in Gegout-Petit and Pardoux (1996). Our BSDEs include this last class as a special case. Also, we prove that the bounded variation process to be added is absolutely continuous, a result which was not formulated for all convex sets in Gegout-Petit and Pardoux (1996). However, our results do not include those in El Karoui et al. (1997) and Cvitanic and Karatzas (1996), since those results allow randomly moving boundaries, while our convex function is fixed. Also, we do not study the stochastic control problem associated with our BSDE.

The paper is organized as follows. The BSDEs and the results concerning them are formulated in Section 1. Section 2 is concerned with a priori estimates for sequences of penalized approximations of our equations. We prove in Section 3 the results stated in Section 1. In Section 4, we prove that the solution of a BSDE provides the unique solution of a certain parabolic variational inequality. Finally, in Section 4 we study the connection between our BSDEs and the Dirichlet problem for an elliptic variational inequality.

1. Backward stochastic variational inequalities: existence and uniqueness results

Let $(\Omega,\mathscr{F},P,\{\mathscr{F}_t:\ t\geqslant 0\})$ be a complete right continuous stochastic basis. We will assume that

$$\mathscr{F}_t=\sigma(\{B_s:\ 0\leqslant s\leqslant t\})\vee\mathscr{N},$$

where $\mathscr{N}$ is the class of P-null sets of $\mathscr{F}$ and B is a d-dimensional standard Brownian motion.

Let $\lambda\in\mathbb{R}$, $k,d\in\mathbb{N}^*$ and τ be a stopping time.

We introduce the notations:

$S_k^{2,\lambda}[0,\tau]$ is the Banach space of continuous $\mathscr{F}_t$-progressively measurable stochastic processes $f:\Omega\times[0,\infty)\to\mathbb{R}^k$ such that

$$\|f\|_S=\left[\mathbb{E}\left(\sup_{0\leqslant t\leqslant\tau}\mathrm{e}^{\lambda t}|f(t)|^2\right)\right]^{1/2}<\infty$$

and $M_k^{2,\lambda}[0,\tau]$ is the Hilbert space of $\mathscr{F}_t$-progressively measurable stochastic processes $f:\Omega\times[0,\infty)\to\mathbb{R}^k$ such that

$$\|f\|_M=\left[\mathbb{E}\left(\int_0^\tau\mathrm{e}^{\lambda s}|f(s)|^2\,\mathrm{d}s\right)\right]^{1/2}<\infty.$$

In the sequel, we shall omit the indices k,λ,τ whenever, respectively, $k=1,\lambda=0$ and $\tau=\infty$. For example, $S^2=S_1^{2,0}[0,\infty)$ and $M_k^2[0,\tau]=M_k^{2,0}[0,\tau]$.

The first goal of this paper is to study the existence and uniqueness of the solution of the backward stochastic differential equation

$$\begin{aligned}&\mathrm{d}Y_t+F(t,Y_t,Z_t)\,\mathrm{d}t\in\partial\varphi(Y_t)\,\mathrm{d}t+Z_t\,\mathrm{d}B_t,\quad 0\leqslant t\leqslant\tau,\\&Y_\tau=\xi,\end{aligned}\tag{1.1}$$

where

(H1) $\tau:\Omega\to[0,\infty)$ is an a.s. finite $\mathscr{F}_t$-stopping time,

the function $F:\Omega\times[0,\infty)\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ satisfies for some $\alpha\in\mathbb{R}$, $\beta,\gamma\geqslant 0$, and η an $\mathscr{F}_t$-progressively measurable process:

$$
(\mathrm{H}_2)\quad
\begin{aligned}
&\text{(i)} && F(\cdot,\cdot,y,z) \text{ is } \mathscr{F}_t\text{-progressively measurable},\\
&\text{(ii)} && y\mapsto F(\omega,t,y,z):\mathbb{R}^k\to\mathbb{R}^k \text{ is continuous},\\
&\text{(iii)} && \langle y-y',F(t,y,z)-F(t,y',z)\rangle\leqslant\alpha|y-y'|^2,\\
& && |F(t,y,z)-F(t,y,z')|\leqslant\beta\|z-z'\|,\\
& && |F(t,y,0)|\leqslant\eta(t)+\gamma|y|
\end{aligned}
$$

for all $t\geqslant 0$, $y,y'\in\mathbb{R}^k$, $z,z'\in\mathbb{R}^{k\times d}$, P-a.s.; $\partial\varphi$ is the subdifferential (see below) of the function $\varphi:\mathbb{R}^k\to]-\infty,+\infty]$ which satisfies

$$
(\mathrm{H}_3)\quad
\begin{aligned}
&\text{(i)} && \varphi \text{ is a proper } (\varphi\not\equiv+\infty)\\
& && \text{convex lower-semicontinuous function},\\
&\text{(ii)} && \varphi(y)\geqslant\varphi(0)=0^2
\end{aligned}
$$

and finally ξ is an $\mathbb{R}^k$-valued $\mathscr{F}_\tau$-measurable random variable, and there exists $\lambda>2\alpha+\beta^2$, such that

$$
(\mathrm{H}_4)\quad
\begin{aligned}
&\text{(i)} && \mathbb{E}[\mathrm{e}^{\lambda\tau}(|\xi|^2+|\varphi(\xi)|)]<\infty,\\
&\text{(ii)} && \mathbb{E}\left(\int_0^\tau \mathrm{e}^{\lambda s}|\eta(s)|^2\right)\mathrm{d}s<\infty.
\end{aligned}
$$

Denote

$$
\begin{aligned}
&\mathrm{Dom}\,\varphi=\{u\in\mathbb{R}^k:\ \varphi(u)<\infty\},\\
&\partial\varphi(u)=\{u^*\in\mathbb{R}^k:\ (u^*,v-u)+\varphi(u)\leqslant\varphi(v),\ \forall v\in\mathbb{R}^k\},\\
&\mathrm{Dom}\,(\partial\varphi)=\{u\in\mathbb{R}^k:\ \partial\varphi(u)\neq\emptyset\},\\
&(u,u^*)\in\partial\varphi\Leftrightarrow u\in\mathrm{Dom}\,(\partial\varphi),\ u^*\in\partial\varphi(u).
\end{aligned}
$$

We remark that the subdifferential operator $\partial\varphi:\mathbb{R}^k\to 2^{\mathbb{R}^k}$ is a maximal monotone operator, i.e. that

$$(u^*-v^*,u-v)\geqslant 0,\quad \forall(u,u^*),(v,v^*)\in\partial\varphi. \tag{1.2}$$

In all what follows, C denotes a constant, which may depend only on λ,α, and β, which may vary from line to line.

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[2] This assumption is not a restriction since we can replace $\varphi(u)$ by $\varphi(u+u_0)-\varphi(u_0)-(u_0^*,u)$ where $(u_0,u_0^*)\in\partial\varphi$.

The main result is given in the following theorem:

Theorem 1.1. *Let the assumptions* (H_1)–(H_4) *be satisfied. Then there exists a unique triple* (Y, Z, U) *such that*

$$Y \in S_k^{2,\lambda}[0,\tau] \cap M_k^{2,\lambda}[0,\tau], \quad Z \in M_{k\times d}^{2,\lambda}[0,\tau], \quad U \in M_k^{2,\lambda}[0,\tau], \tag{1.3a}$$

$$\mathbb{E}\int_0^\tau e^{\lambda s}\varphi(Y_s)\,ds < +\infty, \tag{1.3b}$$

$$(Y_t, U_t) \in \partial\varphi, \ dP \times dt \ \text{a.e. on } [0,\tau], \tag{1.3c}$$

$$Y_t + \int_{t\wedge\tau}^{\tau} U_s\,ds = \xi + \int_{t\wedge\tau}^{\tau} F(s, Y_s, Z_s)\,ds - \int_{t\wedge\tau}^{\tau} Z_s\,dB_s, \quad \forall t \geqslant 0, \ \text{a.s.} \tag{1.3d}$$

Moreover, for any stopping time θ, $0 \leqslant \theta \leqslant \tau$, *this solution satisfies*

$$\mathbb{E}\left[\int_\theta^\tau e^{\lambda s}(|Y_s|^2 + \|Z_s\|^2)\,ds\right] \leqslant C\Gamma_1(\theta,\tau), \tag{1.4a}$$

$$\mathbb{E}\left[\sup_{\theta\leqslant t\leqslant\tau} e^{\lambda t}|Y_t|^2\right] \leqslant C\Gamma_1(\theta,\tau), \tag{1.4b}$$

$$\mathbb{E}[e^{\lambda\theta}\varphi(Y_\theta)] \leqslant C\Gamma_2(\theta,\tau), \tag{1.4c}$$

$$\mathbb{E}\left[\int_\theta^\tau e^{\lambda s}|U_s|^2\,ds\right] \leqslant C\Gamma_2(\theta,\tau), \tag{1.4d}$$

where

$$\Gamma_1(\theta,\tau) = \mathbb{E}\left[e^{\lambda\tau}|\xi|^2 + \int_\theta^\tau e^{\lambda s}|F(s,0,0)|^2\,ds\right], \tag{1.5a}$$

$$\Gamma_2(\theta,\tau) = \mathbb{E}\left[e^{\lambda\tau}(|\xi|^2 + \varphi(\xi)) + \int_\theta^\tau e^{\lambda s}|\eta(s)|^2\,ds\right]. \tag{1.5b}$$

The triple (Y, Z, U) which satisfies Eqs. (1.3a), (1.3b), (1.3c) and (1.3d) will be called a solution of BSDE (1.1) and we shall write $(Y, Z, U) \in \mathrm{BSDE}(\xi, \tau; \varphi, F)$.

Proposition 1.1. *Under the conditions of Theorem* 1.1, *if* $(Y, Z, U) \in \mathrm{BSDE}(\xi, \tau; \varphi, F)$ *and* $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathrm{BSDE}(\tilde{\xi}, \tau; \varphi, \tilde{F})$, *we have*

$$\mathbb{E}\left[\int_0^\tau e^{\lambda s}(|Y_s - \tilde{Y}_s|^2 + \|Z_s - \tilde{Z}_s\|^2)\,ds\right] \leqslant C\Delta(\tau), \tag{1.6a}$$

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant\tau} e^{\lambda t}|Y_t - \tilde{Y}_t|^2\right] \leqslant C\Delta(\tau), \tag{1.6b}$$

where

$$\Delta(\tau) = \mathbb{E}\left[e^{\lambda\tau}|\xi - \tilde{\xi}|^2 + \int_0^\tau e^{\lambda s}|F(s, Y_s, Z_s) - \tilde{F}(s, Y_s, Z_s)|^2\,ds\right]. \tag{1.7}$$

Remark 1.1. In the case where $\tau = T$ is a finite fixed number, the same results hold, with the same assumptions except that we need not assume that $\lambda > 2\alpha + \beta^2$, and we can choose $\lambda = 0$.

Corollary 1.1. *Let assumptions* (H_2), (H_3) *and*

$$\lambda > 2\alpha + \beta^2, \tag{1.8a}$$

$$\mathbb{E}\int_0^\infty e^{\lambda s}|\eta(s)|^2\,ds < \infty \tag{1.8b}$$

be satisfied. Then there exists a unique triple $(Y,Z,U) \in (S_k^{2,\lambda} \cap M_k^{2,\lambda}) \times M_{k\times d}^{2,\lambda} \times M_k^{2,\lambda}$ *such that*: $\forall T > 0$

$$Y_t + \int_t^T U_s\,ds = Y_T + \int_t^T F(s,Y_s,Z_s)\,ds - \int_t^T Z_s\,dB_s, \quad \forall 0 \leqslant t \leqslant T,\ P - a.s. \tag{1.9a}$$

$$\lim_{t\to\infty} \mathbb{E}(e^{\lambda t}|Y_t|^2) = 0, \tag{1.9b}$$

$$(Y_t, U_t) \in \partial\varphi,\ dP \times dt\ a.e. \tag{1.9c}$$

Moreover,

$$\mathbb{E}\left[\left(\sup_{t\geqslant 0} e^{\lambda t}|Y_t|^2\right) + \int_0^\infty e^{\lambda s}(|Y_s|^2 + \|Z_s\|^2)\,ds\right] \leqslant C\mathbb{E}\int_0^\infty e^{\lambda s}|F(s,0,0)|^2\,ds, \tag{1.10a}$$

$$\sup_{t\geqslant 0} \mathbb{E}e^{\lambda t}\varphi(Y_t) + \mathbb{E}\int_0^\infty e^{\lambda s}|U_s|^2 \leqslant C\mathbb{E}\int_0^\infty e^{\lambda s}|\eta(s)|^2\,ds. \tag{1.10b}$$

2. A priori estimates on a penalized equation

The existence result for Theorem 1.1 will be obtained via an approximation of the function φ by a convex C^1-function φ_ε, $\varepsilon > 0$, defined by

$$\begin{aligned}\varphi_\varepsilon(u) &= \inf\{\tfrac{1}{2}|u-v|^2 + \varepsilon\varphi(v):\ v \in \mathbb{R}^k\}\\ &= \tfrac{1}{2}|u - J_\varepsilon u|^2 + \varepsilon\varphi(J_\varepsilon u),\end{aligned} \tag{2.1}$$

where $J_\varepsilon u = (I + \varepsilon\partial\varphi)^{-1}(u)$. For the reader's convenience we mention some properties of this approximation (see Barbu, 1976 or Brezis, 1973 for more details):

$$\frac{1}{\varepsilon}D\varphi_\varepsilon(u) = \frac{1}{\varepsilon}\partial\varphi_\varepsilon(u) = \frac{1}{\varepsilon}(u - J_\varepsilon u) \in \partial\varphi(J_\varepsilon u), \tag{2.2a}$$

$$|J_\varepsilon u - J_\varepsilon v| \leqslant |u - v| \quad \text{and} \quad \lim_{\delta\searrow 0} J_\delta u = \mathrm{Pr}_{\overline{\mathrm{Dom}\,\varphi}}(u) \tag{2.2b}$$

for all $u, v \in \mathbb{R}^k$, $\varepsilon > 0$.

We first note that the convexity of φ_ε implies that for all $u \in \mathbb{R}^k$,

$$\varphi_\varepsilon(0) \geqslant \varphi_\varepsilon(u) + (D\varphi_\varepsilon(u), -u).$$

But from (H_3 – ii) and the definition of φ_ε it follows easily that $\varphi_\varepsilon(u)\geqslant 0=\varphi_\varepsilon(0)$. Hence, for all $u\in\mathbb{R}^k$,

$$0\leqslant\varphi_\varepsilon(u)\leqslant(D\varphi_\varepsilon(u),u). \tag{2.2c}$$

By Eq. (2.2a) and the monotonicity of the operator $\partial\varphi$ we have

$$\begin{aligned}0 &\leqslant \left(\frac{1}{\varepsilon}D\varphi_\varepsilon(u)-\frac{1}{\delta}D\varphi_\delta(v),J_\varepsilon u-J_\delta v\right)\\ &=\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(u)-\frac{1}{\delta}D\varphi_\delta(v),u-v\right)-\frac{1}{\varepsilon}|D\varphi_\varepsilon(u)|^2\\ &\quad-\frac{1}{\delta}|D\varphi_\delta(v)|^2+\left(\frac{1}{\varepsilon}+\frac{1}{\delta}\right)(D\varphi_\varepsilon(u),D\varphi_\delta(v))\end{aligned}$$

and then

$$\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(u)-\frac{1}{\delta}D\varphi_\delta(v),u-v\right)\geqslant-\left(\frac{1}{\varepsilon}+\frac{1}{\delta}\right)|D\varphi_\varepsilon(u)|\times|D\varphi_\delta(v)| \tag{2.3}$$

for all $u,v\in\mathbb{R}^k$, $\varepsilon,\delta>0$. Consider the approximating equation

$$Y_t^\varepsilon+\frac{1}{\varepsilon}\int_{t\wedge\tau}^{\tau}D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s=\xi+\int_{t\wedge\tau}^{\tau}F(s,Y_z^\varepsilon,Z_s^\varepsilon)\,\mathrm{d}s-\int_{t\wedge\tau}^{\tau}Z_s^\varepsilon\,\mathrm{d}B_s,\quad\forall t\geqslant 0,\ P-a.s. \tag{2.4}$$

It follows from the results in Darling and Pardoux (1997) that Eq. (2.4) has a unique solution $(Y^\varepsilon,Z^\varepsilon)\in(S_k^{2,\lambda}[0,\tau]\cap M_k^{2,\lambda}[0,\tau])\times M_{k\times d}^{2,\lambda}[0,\tau]$.

Proposition 2.1. *Let assumptions* (H_1)–(H_4) *be satisfied and let* θ *be a stopping time such that* $0\leqslant\theta\leqslant\tau$. *Then*

$$\mathbb{E}\left[\sup_{\theta\leqslant t\leqslant\tau}\mathrm{e}^{\lambda t}|Y_t^\varepsilon|^2+\int_\theta^\tau\mathrm{e}^{\lambda s}(|Y_s^\varepsilon|^2+\|Z_s^\varepsilon\|^2)\,\mathrm{d}s\right]\leqslant C\Gamma_1(\theta,\tau) \tag{2.5}$$

with Γ_1 *defined by Eq.* (1.5a).

Proof. Itô's formula for $\mathrm{e}^{\lambda t}|Y_t^\varepsilon|^2$ yields

$$\begin{aligned}&\mathrm{e}^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau}^\varepsilon|^2+\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(\lambda|Y_s^\varepsilon|^2+\|Z_s^\varepsilon\|^2)\,\mathrm{d}s+\frac{2}{\varepsilon}\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(D\varphi_\varepsilon(Y_s^\varepsilon),Y_s^\varepsilon)\,\mathrm{d}s\\ &\quad=\mathrm{e}^{\lambda\tau}|\xi|^2+2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(F(s,Y_s^\varepsilon,Z_s^\varepsilon),Y_s^\varepsilon)\,\mathrm{d}s-2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(Y_s^\varepsilon,Z_s^\varepsilon\,\mathrm{d}B_s).\end{aligned}$$

But from Eq. (2.2c)

$$\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(y),y\right)\geqslant 0$$

and from Schwarz's inequality

$$2(F(s,y,z),y)\leqslant(2\alpha+(1+r)\beta^2+r)|y|^2+\frac{1}{1+r}\|z\|^2+\frac{1}{r}|F(s,0,0)|^2.$$

Hence,

$$e^{\lambda(t\wedge\tau)}|Y^{\varepsilon}_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{\tau}e^{\lambda s}\left[(\lambda-2\alpha-\beta^2-r(1+\beta^2))|Y^{\varepsilon}_s|^2+\frac{r}{1+r}\|Z^{\varepsilon}_s\|^2\right]\mathrm{d}s$$

$$\leqslant e^{\lambda\tau}|\xi|^2+\frac{1}{r}\int_{t\wedge\tau}^{\tau}e^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s-2\int_{t\wedge\tau}^{\tau}e^{\lambda s}(Y^{\varepsilon}_s,Z^{\varepsilon}_s\,\mathrm{d}B_s),\quad\forall t\geqslant 0,\ \text{a.s.}$$

We choose

$$\lambda>2\alpha+\beta^2,\qquad 0<r<\frac{\lambda-(2\alpha+\beta^2)}{1+\beta^2}\wedge 1. \tag{2.6}$$

The result without the sup in the expectation follows by taking the expectation in the above inequality. Finally, the result follows by a combination with Burkholder–Davis–Gundy's inequality.

Indeed, the first step yields, in particular, that

$$\mathbb{E}\int_0^{\tau}e^{\lambda s}\|Z^{\varepsilon}_s\|^2\,\mathrm{d}s\leqslant C,$$

and one then obtains

$$\sup_{\theta\leqslant t\leqslant\tau}e^{\lambda t}|Y^{\varepsilon}_t|^2\leqslant e^{\lambda\tau}|\xi|^2+\frac{1}{r}\int_{\theta}^{\tau}e^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s+2\sup_{\theta\leqslant t\leqslant\tau}\left|\int_t^{\tau}e^{\lambda s}(Y^{\varepsilon}_s,Z^{\varepsilon}_s\,\mathrm{d}B_s)\right|.$$

Then, from Burkholder–Davis–Gundy's inequality,

$$\mathbb{E}\left(\sup_{\theta\leqslant t\leqslant\tau}e^{\lambda t}|Y^{\varepsilon}_t|^2\right)\leqslant C_1+2\mathbb{E}\left(\sup_{\theta\leqslant t\leqslant\tau}\left|\int_t^{\tau}e^{\lambda s}(Y^{\varepsilon}_s,Z^{\varepsilon}_s\,\mathrm{d}B_s)\right|\right)$$

$$\leqslant C_1+\frac{1}{2}\mathbb{E}\left(\sup_{\theta\leqslant t\leqslant\tau}e^{\lambda t}|Y^{\varepsilon}_t|^2\right)+C_2\mathbb{E}\int_{\theta}^{\tau}e^{\lambda s}\|Z^{\varepsilon}_s\|^2\,\mathrm{d}s$$

and the result follows. □

Proposition 2.2. *Under the conditions of Proposition* 2.1, *there exists a positive constant C such that for any stopping time* θ,

$$\mathbb{E}\int_{\theta}^{\tau}e^{\lambda s}\left(\frac{1}{\varepsilon}|D\varphi_{\varepsilon}(Y^{\varepsilon}_s)|\right)^2\mathrm{d}s\leqslant C\Gamma_2(\theta,\tau), \tag{2.7a}$$

$$\mathbb{E}e^{\lambda\theta}\varphi(J_{\varepsilon}Y^{\varepsilon}_{\theta})+\mathbb{E}\int_{\theta}^{\tau}e^{\lambda s}\varphi(J_{\varepsilon}Y^{\varepsilon}_{\theta})\,\mathrm{d}s\leqslant C\Gamma_2(\theta,\tau), \tag{2.7b}$$

$$\mathbb{E}(e^{\lambda\theta}|Y^{\varepsilon}_{\theta}-J_{\varepsilon}(Y^{\varepsilon}_{\theta})|^2)\leqslant\varepsilon^2C\Gamma_2(\theta,\tau), \tag{2.7c}$$

where $\Gamma_2(\theta,\tau)$ *is given by* Eq. (1.5b).

Proof. Writing the subdifferential inequality

$$e^{\lambda s}\varphi_{\varepsilon}(Y^{\varepsilon}_s)\geqslant(e^{\lambda s}-e^{\lambda r})\varphi_{\varepsilon}(Y^{\varepsilon}_s)+e^{\lambda r}\varphi_{\varepsilon}(Y^{\varepsilon}_r)+e^{\lambda r}(D\varphi_{\varepsilon}(Y^{\varepsilon}_r),Y^{\varepsilon}_s-Y^{\varepsilon}_r)$$

for $s=t_{i+1}\wedge\tau$, $r=t_i\wedge\tau$, where $t=t_0<t_1<t_2<\cdots$ and $t_{i+1}-t_i=1/n$, summing up over i, and passing to the limit as $n\to\infty$, we deduce:

$$\begin{aligned}
&\mathrm{e}^{\lambda t\wedge\tau}\varphi_\varepsilon(Y^\varepsilon_{t\wedge\tau})+\int_{t\wedge\tau}^{\tau}\lambda\mathrm{e}^{\lambda s}\varphi_\varepsilon(Y^\varepsilon_s)\,\mathrm{d}s+\frac{1}{\varepsilon}\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}|D\varphi_\varepsilon(Y^\varepsilon_s)|^2\,\mathrm{d}s\\
&\quad\leqslant\mathrm{e}^{\lambda\tau}\varphi_\varepsilon(\xi)+\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(D\varphi_\varepsilon(Y^\varepsilon_s),F(s,Y^\varepsilon_s,Z^\varepsilon_s))\,\mathrm{d}s\\
&\qquad-\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(D\varphi_\varepsilon(Y^\varepsilon_s),Z^\varepsilon_s\,\mathrm{d}B_s),\quad\forall t\geqslant 0,\ \text{a.s.}
\end{aligned}\tag{2.8}$$

The result follows by combining this with the following inequalities and Eq. (2.5) (the right side of the second inequality follows from Eq. (2.2c))

$$\tfrac{1}{2}|D\varphi_\varepsilon(y)|^2+\varepsilon\varphi(J_\varepsilon y)=\varphi_\varepsilon(y),\ \ \varepsilon\varphi(J_\varepsilon y)\leqslant\varphi_\varepsilon(y),$$

$$-\lambda\varphi_\varepsilon(y)\leqslant|\lambda|\varphi_\varepsilon(y)\leqslant|\lambda|(D\varphi_\varepsilon(y),y),$$

$$\varphi_\varepsilon(\xi)\leqslant\varepsilon\varphi(\xi),$$

$$\begin{aligned}
(D\varphi_\varepsilon(y),|\lambda|y+F(s,y,z))&\leqslant\frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2+\frac{\varepsilon}{2}(|\lambda||y|+|F(s,y,z)|)^2\\
&\leqslant\frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2+\varepsilon(|\lambda|^2|y|^2+|F(s,y,z)|^2)\\
&\leqslant\frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2+\varepsilon[(|\lambda|^2|y|^2+4(\beta^2\|z\|^2\\
&\quad+\gamma^2|y|^2+\eta^2(s))].
\end{aligned}$$

Proposition 2.3. *Let assumptions* (H$_1$)–(H$_4$) *be satisfied and* $\varepsilon,\delta>0$. *Then*

$$\mathbb{E}\left[\int_0^\tau\mathrm{e}^{\lambda s}(|Y^\varepsilon_s-Y^\delta_s|^2+\|Z^\varepsilon_s-Z^\delta_s\|^2)\,\mathrm{d}s\right]\leqslant(\varepsilon+\delta)C\Gamma(\tau),\tag{2.9a}$$

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant\tau}\mathrm{e}^{\lambda t}|Y^\varepsilon_t-Y^\delta_t|^2\right)\leqslant(\varepsilon+\delta)C\Gamma(\tau),\tag{2.9b}$$

where

$$\Gamma(\tau)=\mathbb{E}\left[\mathrm{e}^{\lambda\tau}(|\xi|^2+\varphi(\xi))+\int_0^\tau\mathrm{e}^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s\right].\tag{2.10}$$

Proof. By Itô's formula

$$\begin{aligned}
&\mathrm{e}^{\lambda(t\wedge\tau)}|Y^\varepsilon_{t\wedge\tau}-Y^\delta_{t\wedge\delta}|^2+\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}[\lambda|Y^\varepsilon_s-Y^\delta_s|^2+\|Z^\varepsilon_s-Z^\delta_s\|^2]\,\mathrm{d}s\\
&\quad+2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}\left(Y^\varepsilon_s-Y^\delta_s,\frac{1}{\varepsilon}D\varphi_\varepsilon(Y^\varepsilon_s)-\frac{1}{\delta}D\varphi_\delta(Y^\delta_s)\right)\,\mathrm{d}s
\end{aligned}$$

$$= 2\int_{t\wedge\tau}^{\tau}(Y_s^\varepsilon - Y_s^\delta, F(s,Y_s^\varepsilon,Z_s^\varepsilon) - F(s,Y_s^\delta,Z_s^\delta))\,\mathrm{d}s$$

$$-2\int_{t\wedge\tau}^{\tau}(Y_s^\varepsilon - Y_s^\delta,(Z_s^\varepsilon - Z_s^\delta)\,\mathrm{d}B_s).$$

We have, moreover,

$$2(Y^\varepsilon - Y^\delta, F(s,Y^\varepsilon,Z^\varepsilon) - F(s,Y^\delta,Z^\delta))$$

$$\leqslant (2\alpha + (1+r)\beta^2)|Y^\varepsilon - Y^\delta|^2 + \frac{1}{1+r}\|Z^\varepsilon - Z^\delta\|^2$$

and by Eq. (2.3b) it follows that

$$\mathrm{e}^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau}^\varepsilon - Y_{t\wedge\delta}^\delta|^2 + \int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}[(\lambda - 2\alpha - \beta^2 - r\beta^2)|Y_s^\varepsilon - Y_s^\delta|^2$$

$$+\frac{r}{1+r}\|Z_s^\varepsilon - Z_s^\delta\|^2]\,\mathrm{d}s \leqslant 2\left(\frac{1}{\varepsilon}+\frac{1}{\delta}\right)\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}|D\varphi_\varepsilon Y_s^\varepsilon||D\varphi_\delta(Y_s^\delta)|\,\mathrm{d}s$$

$$-2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(Y_s^\varepsilon - Y_s^\delta,(Z_s^\varepsilon - Z_s^\delta)\,\mathrm{d}B_s). \quad (2.11)$$

Now, from Eq. (2.7a)

$$2\left(\frac{1}{\varepsilon}+\frac{1}{\delta}\right)\mathbb{E}\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}|D\varphi_\varepsilon Y_s^\varepsilon||D\varphi_\delta(Y_s^\delta)|\,\mathrm{d}s \leqslant C(\varepsilon+\delta)\Gamma(\tau).$$

Eq. (2.9a) then follows by taking the expectation in Eq. (2.11), and Eq. (2.9b) follows from Eqs. (2.11), (2.9a) and Burkholder–Davis–Gundy's inequality.

3. Proofs of the existence and uniqueness of the solution

We begin with the

Proof of Proposition 1.1. From Itô's formula we have

$$\mathrm{e}^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau} - \tilde{Y}_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(\lambda|Y_s - \tilde{Y}_s|^2 + \|Z_s - \tilde{Z}_s\|^2)\,\mathrm{d}s$$

$$+2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(U_s - \tilde{U}_s, Y_s - \tilde{Y}_s)\,\mathrm{d}s$$

$$= \mathrm{e}^{\lambda\tau}|\xi - \tilde{\xi}|^2 + 2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(Y_s - \tilde{Y}_s, F(s,Y_s,Z_s) - \tilde{F}(s,\tilde{Y}_s,\tilde{Z}_s))\,\mathrm{d}s$$

$$-2\int_{t\wedge\tau}^{\tau}\mathrm{e}^{\lambda s}(Y_s - \tilde{Y}_s,(Z_s - \tilde{Z}_s)\,\mathrm{d}B_s). \quad (3.1)$$

But

$$2(U_s - \tilde{U}_s, Y_s - \tilde{Y}_s) \geqslant 0, \quad dP \times ds \quad \text{a.e.},$$

$$2(Y - \tilde{Y}, F(s,Y,Z) - \tilde{F}(s,\tilde{Y},\tilde{Z})) \leqslant (2\tilde{\alpha} + (1+r)\tilde{\beta}^2 + r)|Y - \tilde{Y}|^2 + \frac{1}{1+r}\|Z - \tilde{Z}\|^2 + \frac{1}{r}|F(s,Y,Z) - \tilde{F}(s,Y,Z)|^2$$

with r given by Eq. (2.6), where α, β are replaced by $\tilde{\alpha}$, $\tilde{\beta}$. With these inequalities and Eq. (3.1), taking the expectation, we clearly have Eq. (1.6a). Then in a standard manner from Eqs. (3.1) and (1.6a), via Burkholder–Davis–Gundy's inequality, we obtain easily Eq. (1.6b).

Proof of Theorem 1.1. Uniqueness is a consequence of Proposition 1.1. The existence of the solution (Y,Z,U) is obtained as limit of the triple $(Y_s^\varepsilon, Z_s^\varepsilon, \frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon))$.

From Proposition 2.3 we have

$$\begin{aligned} &\exists Y \in S_k^{2,\lambda}[0,\tau] \cap M_k^{2,\lambda}[0,\tau], \quad Z \in M_{k\times d}^{2,\lambda} \text{ s.t.} \\ &\lim_{\varepsilon \searrow 0} Y^\varepsilon = Y \text{ in } S_k^{2,\lambda}[0,\tau] \cap M_k^{2,\lambda}[0,\tau], \\ &\lim_{\varepsilon \searrow 0} Z^\varepsilon = Z \text{ in } M_{k\times d}^{2,\lambda}, \end{aligned} \tag{3.2}$$

and Eqs. (1.4a) and (1.4b) follows by passing to the limit in Eq. (2.5). Also, from Eqs. (2.7a) and (2.7c) we have

$$\lim_{\varepsilon \searrow 0} J_\varepsilon(Y^\varepsilon) = Y \text{ in } M_k^{2,\lambda}[0,\tau],$$
$$\lim_{\varepsilon \searrow 0} \mathbb{E}(e^{\lambda\theta}|J_\varepsilon(Y_\theta^\varepsilon) - Y_\theta|^2) = 0$$

for any stopping time θ, $0 \leqslant \theta \leqslant \tau$.

Eqs. (1.3b) and (1.4c) follow from Eqs. (2.7b), (2.9b) and the fact that φ is l.s.c. Hence, the limit pair (Y,Z) satisfies Eqs. (1.3a), (1.3b) and (1.4a)–(1.4c).

For each $\varepsilon > 0$, define $U_t^\varepsilon = (1/\varepsilon)D\varphi_\varepsilon(Y_t^\varepsilon)$ and $\bar{U}_t^\varepsilon = \int_0^t U_s^\varepsilon\, ds$. It follows from our convergence results and Eq. (2.4) that there exists a progressively measurable $\mathbb{R}^k$-valued process $\{\bar{U}_t,\ 0 \leqslant t \leqslant \tau\}$ such that for all $T > 0$,

$$E\left(\sup_{0\leqslant t\leqslant T\wedge\tau} |\bar{U}_t^\varepsilon - \bar{U}_t|^2\right) \to 0, \quad \varepsilon \to 0.$$

Moreover, from Eq. (2.7a),

$$\sup_{\varepsilon>0} E\int_0^\tau e^{\lambda t}|U_t^\varepsilon|^2\, dt < \infty.$$

From this, it follows that for each $T > 0$, $\bar{U}^\varepsilon$ is bounded in the space $L^2(\Omega; H^1(0, T\wedge\tau))$, and – at least along a susequence – it converges weakly to a limit in that space. The limit is necessarily $\bar{U}$, hence the whole sequence converges weakly, and $\bar{U} \in L^2(\Omega; H^1(0, T\wedge\tau))$, in particular, it is a.s. absolutely continuous, $\bar{U}_t$ takes the form $\bar{U}_t = \int_0^t U_s\, ds$, where $\{U_t,\ 0 \leqslant t \leqslant \tau\}$ is progressively measurable. Now, Eq. (1.4d) follows from the above inequality and Fatou's lemma. Moreover, it follows e.g. from Lemma 5.8 in Gegout-Petit and Pardoux (1996) that for all $0\leqslant a<b\leqslant T$, $V\in M_k^2(a,b)$,

$$\int_{a\wedge\tau}^{b\wedge\tau}(U_t^\varepsilon, V_t - Y_t^\varepsilon)\,\mathrm{d}t \to \int_{a\wedge\tau}^{a\wedge\tau}(U_t, V_t - Y_t)\,\mathrm{d}t$$

in probability, and from Eq. (2.7a) $\int_{a\wedge\tau}^{b\wedge\tau}(U_t^\varepsilon, J_\varepsilon(Y_t^\varepsilon) - Y_t^\varepsilon)\,\mathrm{d}t \to 0$. Now, since $U_t^\varepsilon \in \partial\varphi(J_\varepsilon(Y_t^\varepsilon))$,

$$\int_{a\wedge\tau}^{b\wedge\tau}(U_t^\varepsilon, V_t - J_\varepsilon(Y_t^\varepsilon))\,\mathrm{d}t + \int_{a\wedge\tau}^{b\wedge\tau}\varphi(J_\varepsilon(Y_t^\varepsilon))\,\mathrm{d}t \leqslant \int_{a\wedge\tau}^{b\wedge\tau}\varphi(V_t)\,\mathrm{d}t,$$

and taking the lim inf in probability in the above, we obtain that

$$\int_{a\wedge\tau}^{b\wedge\tau}(U_t, V_t - Y_t)\,\mathrm{d}t + \int_{a\wedge\tau}^{b\wedge\tau}\varphi(Y_t)\,\mathrm{d}t \leqslant \int_{a\wedge\tau}^{b\wedge\tau}\varphi(V_t)\,\mathrm{d}t.$$

Since a, b and the process V are arbitrary, this establishes Eq. (1.3c). Eqs. (1.3d) has also been proved.

Proof of Corollary 1.1. For each $n\geqslant 1$, let $(Y^n, Z^n, U^n)\in \mathrm{BSDE}(0,n;\varphi,F)$. From the estimate (1.4) in Theorem 1.1 we have

$$\mathbb{E}\int_0^n e^{\lambda s}(|Y_s^n|^2 + \|Z_s^n\|^2)\,\mathrm{d}s \leqslant C_1\mathbb{E}\int_0^\infty e^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s,$$

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant n}(e^{\lambda s}|Y_s^n|^2)\right] \leqslant C_1\mathbb{E}\int_0^\infty e^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s,$$

$$\mathbb{E}[e^{\lambda t}\varphi(Y_t^n)] \leqslant C_2\mathbb{E}\int_0^\infty e^{\lambda s}|\eta(s)|^2\,\mathrm{d}s,$$

$$\mathbb{E}\left[\int_0^n e^{\lambda s}|U_s^n|^2\,\mathrm{d}s\right] \leqslant C_2\mathbb{E}\int_0^\infty e^{\lambda s}|\eta(s)|^2\,\mathrm{d}s,$$

and $Y_s^n = Y_n^n = 0, Z_s^n = 0, U_s^n = 0$, for $s>n$.

Let $m>n$. We have

$$Y_t^m + \int_t^n U_s^m\,\mathrm{d}s = Y_n^m + \int_t^n F(s, Y_s^m, Z_s^m)\,\mathrm{d}s - \int_t^n Z_s^m\,\mathrm{d}B_s$$

for all $t\in[0,n]$, ω-a.s., and from Proposition 1.1

$$\mathbb{E}\left[\int_0^n e^{\lambda s}(|Y_s^n - Y_s^m|^2 + \|Z_s^n - Z_s^m\|^2)\,\mathrm{d}s\right] \leqslant Ce^{\lambda n}\mathbb{E}|Y_n^m|^2,$$

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant n} e^{\lambda s}|Y_s^n - Y_s^m|^2\right) \leqslant Ce^{\lambda n}\mathbb{E}|Y_n^m|^2.$$

From Eq. (1.4b),

$$e^{\lambda T}\mathbb{E}(|Y_T^m|^2) \leqslant \mathbb{E}\left(\sup_{T\leqslant t\leqslant m} e^{\lambda s}|Y_s^m|^2\right) \leqslant C_1\mathbb{E}\left(\int_T^\infty e^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s\right) \to 0$$

as $T=n\to\infty$. Hence, $\exists Y\in S_k^{2,\lambda}\cap M_k^{2,\lambda}$, $Z\in M_{k\times d}^{2,\lambda}$ and $U\in M_k^{2,\lambda}$ such that as $n\to\infty$ for all $T>0$,

$$\begin{aligned}
&Y^n\to Y \quad \text{in } S_k^{2,\lambda}(0,T)\cap M_k^{2,\lambda}(0,T),\\
&e^{\lambda}T\mathbb{E}|Y_T|^2\leqslant C\mathbb{E}\int_T^\infty e^{\lambda t}|F(t,0,0)|^2\,dt,\\
&Z^n\to Z \quad \text{in } M_{k\times d}^{2,\lambda}(0,T),\\
&\bar{U}^n\to \bar{U} \quad \text{in } S_k^{2,\lambda}(0,T),
\end{aligned}$$

where $\bar{U}_t^n=\int_0^t U_s^n\,ds$, $\bar{U}$ is absolutely continuous and (Y,Z,U) satisfies the assertions of Corollary 1.1, where $U=d\bar{U}/dt$.

The solution is unique since if (Y,Z,U) and $(\tilde{Y},\tilde{Z},\tilde{U})$ are two solutions of Eqs. (1.9a), (1.9b) and (1.9c) then from Proposition 1.1

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant s\leqslant n} e^{\lambda s}|Y_s-\tilde{Y}_s|^2\right)+\mathbb{E}\int_0^n e^{\lambda s}|Y_s-\tilde{Y}_s|^2\,ds\\
&\quad+\mathbb{E}\int_0^n e^{\lambda s}\|Z_s-\tilde{Z}_s\|^2\,ds\leqslant C_1\mathbb{E}(e^{\lambda n}|Y_n-\tilde{Y}_n|^2)
\end{aligned}$$

and for $n\to\infty$ we get $Y=\tilde{Y}$, $Z=\tilde{Z}$; U is uniquely defined by Eq. (1.9a).

4. Connection with parabolic variational inequalities

In this section we will show that the BSDE studied in the previous sections allows us to give a probabilistic representation of solutions of a parabolic variational inequality.

Let $(\Omega,\mathscr{F},P,\mathscr{F}_t,B_t)_{t\geqslant 0}$ be a $\mathbb{R}^d$-valued Wiener process, $\mathscr{F}_t=\sigma(\{B_s\colon 0\leqslant s\leqslant t\})\vee\mathscr{N}$, and

$$\begin{aligned}
&b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d,\quad \sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d\\
&\text{be continuous mappings such that}\\
&|b(t,x)-b(t,\tilde{x})|+\|\sigma(t,x)-\sigma(t,\tilde{x})\|\leqslant L_1|x-\tilde{x}|,\\
&\forall t\in[0,T],\ \forall x,\tilde{x}\in\mathbb{R}^d
\end{aligned}\tag{4.1}$$

(for some constant $L_1>0$).

For each $(t,x)\in[0,T]\times\mathbb{R}^d$, let $\{X_s^{tx},\ 0\leqslant s\leqslant T\}$ be the unique solution of the SDE

$$X_s^{tx}=x+\int_t^{t\vee s} b(r,X_r^{tx})\,dr+\int_t^{t\vee s}\sigma(r,X_r^{tx})\,dB_r. \tag{4.2}$$

We have (see Friedman, 1976) for $t\in[0,T]$; $x,x'\in\mathbb{R}^d$:

$$X_s^{tx}=x,\quad \forall s\in[0,t], \tag{4.3a}$$

$$X_s^{tx}\in S_d^p[0,T],\quad \forall p\geqslant 2, \tag{4.3b}$$

$$\mathbb{E}\left(\sup_{s\in[0,T]}|X_s^{tx}|^p\right)\leqslant C(1+|x|^p), \tag{4.3c}$$

$$\mathbb{E}\left(\sup_{s\in[0,T]}|X_s^{tx}-X_s^{t'x'}|^p\right)\leqslant C(1+|x|^p+|x'|^p)(|t-t'|^{p/2}+|x-x'|^p), \tag{4.3d}$$

where $C=C(p,T,L_1,K_1)$, $K_1=\sup_{t\in[0,T]}\{|b(t,0)|+|\sigma(t,0)|\}$.

We now consider the BSDE Eq. (1.1) in the case $k=1$, with the data $(\xi,\tau;\varphi,F)$ of the form

$$\begin{aligned}&\tau=T,\\ &\xi(\omega)=g(X_T^{tx}(\omega)),\\ &F(\omega,s,y,z)=f(s,X_s^{tx}(\omega),y,z),\end{aligned} \tag{4.4}$$

where g,f satisfies

$$\begin{aligned}&g\in C(\mathbb{R}^d;\mathbb{R})\text{ and }\exists M>0,\ \exists q\in\mathbb{N}\text{ such that}\\ &|g(x)|\leqslant M(1+|x|^q),\ \text{for all } x\in\mathbb{R}^d,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&f\in C([0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)\text{ and}\\ &\exists\alpha,\beta,\gamma>0,\ p\in\mathbb{N}\text{ such that}\\ &|f(t,x,y,0)|\leqslant\gamma(1+|x|^p+|y|)\\ &(y-\tilde y)(f(t,x,y,z)-f(t,x,\tilde y,z))\leqslant\alpha|y-\tilde y|^2,\\ &|f(t,x,t,z)-f(t,x,y,\tilde z)|\leqslant\beta|z-\tilde z|\end{aligned} \tag{4.6}$$

for all $t\in[0,T]$; $x\in\mathbb{R}^d$; $y,\tilde y\in\mathbb{R}$; $z,\tilde z\in\mathbb{R}^d$, and

$$\varphi:\mathbb{R}\to[0,+\infty]\text{ is a proper, convex l.s.c. function, s.t.}$$

$$\varphi(y)\geqslant\varphi(0)=0 \tag{4.7}$$

and

$$\begin{aligned}&\exists M>0,\exists m\in\mathbb{N}^*\text{ such that}\\ &|\varphi(g(x))|\leqslant M(1+|x|^m),\quad\forall x\in\mathbb{R}^d.\end{aligned} \tag{4.8}$$

For each $t\geqslant0$ we denote by $\{\mathscr{F}_s^t,\ s\in[t,T]\}$ the natural filtration of the Brownian motion $\{B_s-B_t,\ s\in[t,T]\}$ argumented with the P-null sets of $\mathscr{F}$.

Under the assumptions (4.4)–(4.8) it follows from Theorem 1.1 (see Remark 1.1) that for each $(t,x)\in[0,T]\times\mathbb{R}^d$ there exists a unique $\mathscr{F}_s^t$-progressively measurable triple $(Y^{tx},Z^{tx},U^{tx})\in S^2[t,T]\times M_d^2[t,T]\times M^2[t,T]$ such that

$$\begin{aligned}&Y_s^{tx}+\int_s^T U_r^{tx}\,\mathrm{d}r=g(X_T^{tx})+\int_s^T f(r,X_r^{tx},Y_r^{tx},Z_r^{tx})\,\mathrm{d}s-\int_s^T Z_r^{tx}\,\mathrm{d}B_r,\\ &\quad\forall s\in[t,T],\ P-\text{a.s.}\end{aligned} \tag{4.9}$$

and

$$(Y_s^{tx}, U_s^{tx}) \in \partial\varphi, \quad dP \times ds \quad \text{a.e. on } \Omega \times [t,T]. \tag{4.10}$$

We shall extend $Y_s^{tx}, Z_s^{tx}, U_s^{tx}$, for $s \in [0,T]$ by choosing $Y_s^{tx} = Y_t^{tx}$, $Z_s^{tx} = 0$, $U_s^{tx} = 0$, $\forall s \in [0,t]$. Hence

$$Y_s^{tx} + \int_s^T U_r^{tx}\, dr = g(X_T^{tx}) + \int_s^T 1_{[t,T]}(r) f(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx})\, dr$$
$$- \int_s^T Z_r^{tx}\, dB_r, \ \forall s \in [0,T], \quad \text{a.s.}$$

and Eq. (4.10) is satisfied a.e. on $\Omega \times [t,T]$.

Proposition 4.1. *Under assumptions* (4.1),(4.4)–(4.8) *we have*

$$\mathbb{E}\left(\sup_{s\in[0,T]} |Y_s^{tx}|^2\right) \leqslant C(1+|x|^p) \tag{4.11}$$

and

$$\mathbb{E}\left(\sup_{s\in[0,T]} |Y_s^{tx} - Y_s^{t'x'}|^2\right) \leqslant C[\mathbb{E}|g(X_T^{tx}) - g(X_T^{t'x'})|^2$$
$$+ \mathbb{E}\int_0^T |1_{[t,T]}(r) f(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx})$$
$$- 1_{[t',T]}(r) f(r, X_r^{t'x'}, Y_r^{tx}, Z_r^{tx})|^2\, dr] \tag{4.12}$$

for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$ ($C>0$ *and* $p \in \mathbb{N}$ *are constants independent of* $t,t' \in [0,T]$ *and* $x,x' \in \mathbb{R}^d$).

Proof. From inequality (1.4b), with $(\theta = t, \tau = T)$ in Theorem 1.1,

$$\mathbb{E}\sup_{s\in[t,T]} |Y_s^{tx}|^2 \leqslant C\left(\mathbb{E}|g(X_T^{tx})|^2 + \mathbb{E}\int_t^T |f(r, X_r^{tx}, 0, 0)|^2\, dr\right),$$

where $C>0$ is independent of $t \in [0,T]$ and $x \in \mathbb{R}^d$, which yields Eq. (4.11) using the assumptions on f and g and Eq. (4.3c).

Eq. (4.12) follows from Eq. (1.6b) in Proposition 1.1. □

We define

$$u(t,x) = Y_t^{tx}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.14}$$

which is a deterministic quantity since Y_t^{tx} is $\mathscr{F}_t^t$-measurable, and $\mathscr{F}_t^t$ is a trivial σ-algebra.

Corollary 4.1. *Under assumptions* (4.1) *and* (4.4)–(4.8) *the function u satisfies*:

$$u(t,x) \in \operatorname{Dom} \varphi, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.15a}$$

$$|u(t,x)| \leqslant C(T)(1+|x|^{p/2}), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.15b}$$

$$u \in C([0,T] \times \mathbb{R}^d), \tag{4.15c}$$

where $C(T) > 0$, $p \in \mathbb{N}$ *are constants independent of t and x.*

Proof. We have $\varphi(u(t,x)) = \mathbb{E}\varphi(Y_t^{tx}) < +\infty$; Eq. (4.15a) follows, Eq. (4.15b) follows from Eq. (4.11).

Let $(t_n, x_n) \to (t,x)$. Then

$$\begin{aligned} |u(t_n,x_n) - u(t,x)|^2 &= \mathbb{E}|Y_{t_n}^{t_n x_n} - Y_t^{tx}|^2 \\ &\leqslant 2\mathbb{E} \sup_{s \in [0,T]} |Y_s^{t_n x_n} - Y_s^{tx}|^2 + 2\mathbb{E}|Y_{t_n}^{tx} - Y_t^{tx}|^2. \end{aligned}$$

Using Eqs. (4.12), (4.3c) and (4.3d), we obtain that $u(t_n x_n) \to u(t,x)$ as $(t_n, x_n) \to (t,x)$. □

In the sequel, we shall prove that the function u defined by Eq. (4.14) is a viscosity solution of the parabolic variational inequality (PVI):

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} + \mathscr{L}_t u(t,x) + f(t,x,u(t,x),(\nabla u \sigma)(t,x)) \in \partial\varphi(u(t,x)), \\ &t \in [0,T], \quad x \in \mathbb{R}^d, \\ &u(T,x) = g(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{4.16}$$

where

$$\mathscr{L}_t = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x) \frac{\partial}{\partial x_i}.$$

Remark that at every point $y \in \operatorname{Dom} \varphi$

$$\partial\varphi(y) = [\varphi'_-(y), \varphi'_+(y)],$$

where $\varphi'_-(y)$ and $\varphi'_+(y)$ are the left derivative and the right derivative, respectively, at the point y.

We shall define the notion of viscosity solution in the language of sub- and super-jets, following Crandall–Ishii–Lions (1992). $S(d)$ will denote below the set of $d \times d$ symmetric non-negative matrices.

Definition 4.1. Let $u \in C([0,T] \times \mathbb{R}^d)$ and $(t,x) \in [0,T] \times \mathbb{R}^d$. We denote by $\mathscr{P}^{2+}u(t,x)$ (the parabolic superjet of u at (t,x)) the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times S(d)$ which are such that

$$\begin{aligned} u(s,y) \leqslant{} & u(t,x) + p(s-t) + (q, y-x) \\ & + \tfrac{1}{2}(X(y-x), y-x) + o(|s-t| + |y-x|^2). \end{aligned}$$

$\mathscr{P}^{2-}u(t,x)$ (the parabolic subjet of u at (t,x)) is defined similarly as the set of triples $(p,q,X)\in\mathbb{R}\times\mathbb{R}^d\times S(d)$ which are such that

$$\begin{aligned} u(s,y) \geqslant\ & u(t,x)+p(s-t)+(q,y-x) \\ & +\tfrac{1}{2}(X(y-x),y-x)+\mathrm{o}(|s-t|+|y-x|^2). \end{aligned}$$

We can give now the definition of a viscosity solution of the parabolic variational inequality (4.16):

Definition 4.2. Let $u\in C([0,T]\times\mathbb{R}^d)$ which satisfies $u(T,x)=g(x)$.

(a) u is a viscosity subsolution of (4.16) if:

$$u(t,x)\in \mathrm{Dom}\,\varphi, \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d$$

and at any point $(t,x)\in(0,T)\times\mathbb{R}^d$, for any $(p,q,X)\in\mathscr{P}^{2+}u(t,x)$

$$\begin{aligned} & -p-\tfrac{1}{2}\mathrm{Tr}((\sigma\sigma^*)(t,x)X)-(b(t,x),q)-f(t,x,u(t,x),q\sigma(t,x)) \\ & \quad \leqslant -\varphi'_-(u(t,x)). \end{aligned} \tag{4.17}$$

(b) u is a viscosity supersolution of Eq. (4.16) if:

$$u(t,x)\in \mathrm{Dom}\,\varphi, \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d,$$

and at any point $(t,x)\in(0,T)\times\mathbb{R}^d$, for any $(p,q,X)\in\mathscr{P}^{2-}u(t,x)$

$$\begin{aligned} & -p-\tfrac{1}{2}\mathrm{Tr}\,(\sigma\sigma^*(t,x)X)-(b(t,x),q)-f(t,x,u(t,x),q\sigma(t,x)) \\ & \quad \geqslant -\varphi'_+(u(t,x)). \end{aligned} \tag{4.18}$$

(c) u is a viscosity solution of Eq. (4.16) if it is both a viscosity sub- and super-solution.

Theorem 4.1. *Let assumptions* (4.1) *and* (4.4)–(4.8) *be satisfied. Then the function* $u(t,x)$ *defined by Eq.* (4.14) *is a viscosity solution of Eq.* (4.16).

Proof. For each $(t,x)\in[0,T]\times\mathbb{R}^d$, $\varepsilon\in]0,1]$, let $(Y^{tx}_{\varepsilon;s},Z^{tx}_{\varepsilon;s})$, $s\in[t,T]$, the solution of BSDE

$$Y^{tx}_{\varepsilon;s}+\int_s^T \frac{1}{\varepsilon}D\varphi_\varepsilon(Y^{tx}_{\varepsilon;r})\,\mathrm{d}r=g(X^{tx}_T)+\int_s^T f(r,X^{tx}_r,Y^{tx}_{\varepsilon;r},Z^{tx}_{\varepsilon;r})\,\mathrm{d}r-\int_s^T Z^{tx}_{\varepsilon;r}\,\mathrm{d}B_r.$$

It is known (see Pardoux, 1997) that

$$u_\varepsilon(t,x)=Y^{tx}_{\varepsilon;t}, \quad t\in[0,T],\ x\in\mathbb{R}^d$$

is the viscosity solution of the parabolic differential equation:

$$\begin{aligned} & \frac{\partial u_\varepsilon(t,x)}{\partial t}+L_t u_\varepsilon(t,x)+f(t,x,u_\varepsilon(t,x),(\nabla u_\varepsilon\sigma)(t,x))=\frac{1}{\varepsilon}D\varphi_\varepsilon(u_\varepsilon(t,x)), \\ & u_\varepsilon(T,x)=g(x), \quad t\in[0,T],\ x\in\mathbb{R}^d. \end{aligned} \tag{4.19}$$

From Proposition 2.3 we have

$$|u_\varepsilon(t,x)-u(t,x)|^2 \leqslant \mathbb{E}\sup_{s\in[t,T]}|Y^{tx}_{\varepsilon;s}-Y^{tx}_s|^2 \leqslant C(1+|x|^p)\varepsilon$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ ($C>0$ and $p\in\mathbb{N}$ are constants independent of ε and $(t,x)\in[0,T]\times\mathbb{R}^d$).

First, we shall show that u is a subsolution. From Lemma 6.1 in Crandall–Ishii–Lions (1992), if $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(p,q,X)\in\mathscr{P}^{2+}u(t,x)$, then there exist sequences

$$\varepsilon_n\searrow 0$$
$$(t_n,x_n)\in[0,T]\times\mathbb{R}^d,$$
$$(p_n,q_n,X_n)\in\mathscr{P}^{2+}u_{\varepsilon_n}(t_n,x_n),$$

such that

$$(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),p_n,q_n,X_n)\to(t,x,u(t,x),p,q,X)\quad\text{as } n\to\infty.$$

But for any n:

$$\begin{aligned}&-p_n-\tfrac{1}{2}\mathrm{Tr}\,((\sigma\sigma^*)(t_n,x_n)X_n)-(b(t_n,x_n),q_n)\\&\quad-f(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),q_n\sigma(t_n,x_n))\leqslant\frac{1}{\varepsilon_n}D\varphi_{\varepsilon_n}(u_{\varepsilon_n}(t_n,x_n)).\end{aligned}\tag{4.20}$$

We can assume that $u(t,x)>\inf(\mathrm{Dom}\,\varphi)$ since for $u(t,x)=\inf(\mathrm{Dom}\,\varphi)$ we have $\varphi'_-(u(t,x))=-\infty$ and inequality (4.17) in Definition 4.2 is clearly satisfied.

Let $y\in\mathrm{Dom}\,\varphi$, $y<u(t,x)$. The uniformly convergence $u_\varepsilon\to u$ on compacts implies that $\exists n_0=n_0(t,x,y)>0$ such that $y<u_{\varepsilon_n}(t_n,x_n)$, $\forall n\geqslant n_0$.

We multiply Eq. (4.20) by $u_{\varepsilon_n}(t_n,x_n)-y$, one follows:

$$\begin{aligned}&[-p_n-\tfrac{1}{2}\mathrm{Tr}\,((\sigma\sigma^*)(t_n,x_n)X_n)-(b(t_n,x_n),q_n)\\&\quad-f(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),q_n\sigma(t_n,x_n)](u_{\varepsilon_n}(t_n,x_n))-y)\\&\quad+\varphi(J_{\varepsilon_n}(u_{\varepsilon_n}(t_n,x_n)))\leqslant\varphi(y).\end{aligned}\tag{4.21}$$

Passing to $\liminf_{n\to\infty}$ in Eq. (4.21) we obtain

$$\begin{aligned}&[-p-\tfrac{1}{2}\mathrm{Tr}((\sigma\sigma^*)(t,x)X)-(b(t,x),q)f(t,x,u(t,x),q\sigma(t,x))](u(t,x)-y)\\&\quad+\varphi(u(t,x))\leqslant\varphi(y),\end{aligned}$$

hence,

$$\begin{aligned}&-p-\tfrac{1}{2}\mathrm{Tr}\,((\sigma\sigma^*)(t,x)X)-(b(t,x),q)-f(t,x,u(t,x),q\sigma(t,x))\\&\quad\leqslant-\frac{\varphi(u(t,x))-\varphi(y)}{u(t,x)-y},\end{aligned}$$

for all $y<u(t,x)$, which implies Eq. (4.17).

Let us show that u is a supersolution. Similarly, given $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(p,q,X)\in\mathscr{P}^{2-}u(t,x)$ there exist the sequences

$$\varepsilon_n\searrow 0$$
$$(t_n,x_n)\in[0,T]\times\mathbb{R}^d,$$
$$(p_n,q_n,X_n)\in\mathscr{P}^{2-}u_{\varepsilon_n}(t_n,x_n),$$

such that

$$(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),\,p_n,q_n,X_n)\to(t,x,u(t,x),\,p,q,X)\quad\text{as } n\to\infty.$$

For any n:

$$\begin{aligned}&-p-\tfrac{1}{2}\mathrm{Tr}\,((\sigma\sigma^*)(t_n,x_n)X_n)-(b(t_n,x_n),q_n)\\&\quad-f(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),q_n\sigma(t_n,x_n))\geqslant-\frac{1}{\varepsilon_n}D\varphi_{\varepsilon_n}(u_{\varepsilon_n}(t_n,x_n)).\end{aligned}\tag{4.22}$$

We can assume that $u(t,x)<\sup(\mathrm{Dom}\,\varphi)$ since for $u(t,x)=\sup(\mathrm{Dom}\,\varphi)$ we have $\varphi'_+(u(t,x))=+\infty$ and Eq. (4.18) is satisfied. Let $y\in\mathrm{Dom}\,\varphi$, $u(t,x)<y$. Then there exists $n_0=n_0(t,x,y)>0$ such that $u_{\varepsilon_n}(t_n,x_n)<y$, $\forall n\geqslant n_0$.

We multiply Eq. (4.22) by $y-u_{\varepsilon_n}(t_n,x_n)$, and we have

$$\begin{aligned}&[-p_n-\tfrac{1}{2}\mathrm{Tr}((\sigma\sigma^*)(t_n,x_n)X_n)-(b(t_n,x_n),q_n)\\&\quad-f(t_n,x_n,u_{\varepsilon_n}(t_n,x_n),q_n\sigma(t_n,x_n))](y-u_{\varepsilon_n}(t_n,x_n)))\\&\quad\geqslant\varphi(J_{\varepsilon_n}(u_{\varepsilon_n}(t_n,x_n)))-\varphi(y),\ \ \forall y>u(t,x),\end{aligned}$$

from where passing to $\liminf_{n\to\infty}$ inequality (4.18) follows. □

We can now improve Eq. (4.15a).

Corollary 4.2. (a) $u(t,x)\in\mathrm{Dom}\,(\partial\varphi)$, $\forall(t,x)\in[0,T]\times\mathbb{R}^d$.

(b) $Y_s^{tx}\in\mathrm{Dom}\,(\partial\varphi)$, $\forall s\in[0,T]$, *P-a.s.* $\omega\in\Omega$.

Proof. (b) follows from (a) since $Y_s^{t,x}=Y_s^{s,X_s^{tx}}=u(s,X_s^{tx})$.

To prove (a) we have two cases.

(c_1) $\mathrm{Dom}\,(\partial\varphi)=\mathrm{Dom}\,\varphi$ and in this case, by Eq. (4.15a), $u(t,x)\in\mathrm{Dom}\,(\partial\varphi)$; $\forall(t,x)\in[0,T\times\mathbb{R}^d$.

(c_2) $\mathrm{Dom}\,(\partial\varphi)\neq\mathrm{Dom}\,\varphi$. Let $b\in\mathrm{Dom}\,\varphi\backslash\mathrm{Dom}\,(\partial\varphi)$. Then $b=\sup(\mathrm{Dom}\,\varphi)$ or $b=\inf\mathrm{Dom}\,\varphi$. If $b=\sup(\mathrm{Dom}\,\varphi)$ and $u(t,x)=b$, then $(0,0,0)\in\mathscr{P}^{2+}u(t,x)$ since

$$u(s,y)\leqslant u(t,x)+\mathrm{o}(|s-t|+|y-x|^2)$$

and from Eq. (4.17) it follows $\varphi'_-(b)=\varphi'_-(u(t,x))<\infty$ and consequently $b\in\mathrm{Dom}\,(\partial\varphi)$; a contradiction which shows that $u(t,x)<b$. We argue similarly in the case $b=\inf(\mathrm{Dom}\,\varphi)$. □

In order to establish a uniqueness result, we need to impose the following additional assumption. For each $R>0$ there exists a continuous function $m_R:\mathbb{R}_+\to\mathbb{R}_+$, $m_R(0)=0$ such that

$$\begin{aligned}&|f(t,x,r,p)-f(t,y,r,p)|\leqslant m_R(|x-y|(1+|p|)),\\&\quad\forall t\in[0,T],\quad|x|,|y|\leqslant R,\quad p\in\mathbb{R}^d.\end{aligned}\tag{4.23}$$

Theorem 4.2. *Under assumptions* (4.1), (4.4)–(4.8) *and* (4.23) *the PVI* (4.16) *has a unique viscosity solution in the class of continuous functions which grow at most polynomially at infinity.*

Proof. The existence is proved by Theorem 4.1. The proof of uniqueness is based on the ideas in El Karoui et al. (1997). It suffices to show that if u is a subsolution and v a supersolution such that $u(T,x)=v(T,x)=g(x),\ x\in\mathbb{R}^d$, then $u\leqslant v$.

We perform the transformation

$$\bar{u}(t,x):=u(t,x)\mathrm{e}^{\lambda t}(1+|x|^2)^{-k/2},$$
$$\bar{v}(t,x):=\left(v(t,x)+\frac{\varepsilon}{t}\right)\mathrm{e}^{\lambda t}(1+|x|^2)^{-k/2}$$

as in the proof of Theorem 8.6 in El Karoui et al. (1997). For the simplicity of notations, we will write below u,v instead of $\bar{u},\bar{v}$. Hence, the (transformed) u and v satisfy (in the viscosity sense)

$$-\frac{\partial u}{\partial t}+F(t,x,u(t,x),Du(t,x),D^2u(t,x))\leqslant -\varphi'_-(e^{-\lambda t}\zeta(x)u(t,x)),$$
$$-\frac{\partial v}{\partial t}+F(t,x,v(t,x),Dv(t,x),D^2v(t,x))\geqslant\frac{\varepsilon}{t^2}-\varphi'_+\left(e^{-\lambda t}\zeta(x)\left(v(t,x)-\frac{\varepsilon}{t}\right)\right)$$

with F defined as in El Karoui et al. (1997) and $\zeta(x)=(1+|x|^2)^{k/2}$. Exactly as in El Karoui et al. (1997), we need only to show that for any $R>0$, if $B_R:=\{|x|<R\}$,

$$\sup_{(0,T)\times B_R}(u-v)^+\leqslant\sup_{(0,T]\times\partial B_R}(u-v)^+,$$

since the right-hand side tends to zero as $R\to\infty$. To prove this fact we assume there exists $R,\delta>0$ such that for some $(t_0,x_0)\in(0,T)\times B_R$

$$\delta=u(t_0,x_0)-v(t_0,x_0)=\sup_{(0,T)\times B_R}(u-v)^+>\sup_{(0,T]\times\partial B_R}(u-v)^+\geqslant 0,$$

and we shall arrive at a contradiction.

We define $(\hat{t},\hat{x},\hat{y})$ as being a point in $[0,T]\times\bar{B}_R\times\bar{B}_R$ where the function

$$\Phi_\alpha(t,x,y)=u(t,x)-v(t,x)-\frac{\alpha}{2}|x-y|^2$$

achieves its maximum. Then by Lemma 8.7 from El Karoui et al. (1997):

$$\text{for } \alpha \text{ large enough}, (\hat{t},\hat{x},\hat{y})\in(0,T)\times B_R\times B_R, \tag{4.24a}$$

$$\alpha|\hat{x}-\hat{y}|^2\to 0\quad\text{and}\quad|\hat{x}-\hat{y}|^2\to 0\quad\text{as }\alpha\to\infty, \tag{4.24b}$$

$$u(\hat{t},\hat{x})\geqslant v(\hat{t},\hat{y})+\delta. \tag{4.24c}$$

Then for α large enough

$$\mathrm{e}^{-\lambda\hat{t}}\zeta(\hat{x})u(\hat{t},\hat{x})>\mathrm{e}^{-\lambda\hat{t}}\zeta(\hat{y})\left(v(\hat{t},\hat{y})-\frac{\varepsilon}{\hat{t}}\right)$$

and, consequently,

$$-\varphi'_-(\mathrm{e}^{-\lambda\hat{t}}\zeta(\hat{x})u(\hat{t},\hat{x}))\leqslant-\varphi'_+\left(\mathrm{e}^{-\lambda\hat{t}}\zeta(\hat{y})\left(v(\hat{t},\hat{y})-\frac{\varepsilon}{\hat{t}}\right)\right)$$

and the proof continues exactly as in El Karoui et al. (1997). □

5. Connection with elliptic variational inequalities

We consider the following elliptic variational inequality (EVI):

$$\begin{aligned} &-\mathscr{L}u(x)+\partial\varphi(u(x))\ni f(x,u(x),(\nabla u\sigma)(x)), \quad x\in D,\\ &u|_{\partial D}=g, \end{aligned} \tag{5.1}$$

or equivalently:

$$\begin{aligned} &\mathscr{L}u(x)+f(x,u(x))(\nabla u\sigma)(x)\in[\varphi'_-(u(x)),\varphi'_+(u(x))], \quad x\in D,\\ &u|_{\partial D}(x)=g(x),\ x\in\partial D. \end{aligned} \tag{5.1'}$$

Here D is a bounded domain of $\mathbb{R}^d$ of the form

$$D=\{x\in\mathbb{R}^d:\ \phi(x)>0\}, \tag{5.2}$$

where $\phi\in C^2(\mathbb{R}^d)$, $|\nabla\phi(x)|\neq 0$, $\forall x\in\partial D\subset\{x\in\mathbb{R}^d:\ \phi(x)=0\}$.

We assume that

$$g\in C(\mathbb{R}^d), \tag{5.3}$$

$f\in C(\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)$ and $\exists\alpha\in\mathbb{R}$, $\exists\gamma,\beta\geqslant 0$ such that

$$|f(x,y,0)|\leqslant\gamma(1+|y|), \tag{5.4a}$$

$$(y-\tilde{y})(f(x,y,z)-f(x,\tilde{y},z))\leqslant\alpha|y-\tilde{y}|^2, \tag{5.4b}$$

$$|f(x,y,z)-f(x,y,\tilde{z})|\leqslant\beta|z-\tilde{z}|; \tag{5.4c}$$

for all $x\in\mathbb{R}^d$, $y,\tilde{y}\in\mathbb{R}$, $z,\tilde{z}\in\mathbb{R}^d$, and

$\varphi:\mathbb{R}\to]-\infty,+\infty]$ is a proper convex l.s.c. function s.t. :

$$\varphi(y)\geqslant\varphi(0)=0, \tag{5.5a}$$

$$\exists M>0:\ |\varphi(g(x))|\leqslant M, \quad \forall x\in\bar{D}, \tag{5.5b}$$

and $\mathscr{L}$ is the infinitesimal generator of the Markov diffusion process X_t:

$$X_t^x=x+\int_0^t b(X_s^x)\,\mathrm{d}s+\int_0^t\sigma(X_s^x)\,\mathrm{d}B_s, \quad \forall t\geqslant 0,$$

i.e.

$$\mathscr{L}=\frac{1}{2}\sum_{i,j=1}^d(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}.$$

Here $(\Omega,\mathscr{F},P,(\mathscr{F}_t)_{t\geqslant 0},B_t)$ is a d-dimensional Brownian motion as in Section 4 and

$$b:\mathbb{R}^d\to\mathbb{R}^d,\ \sigma:\mathbb{R}^d\to\mathbb{R}^{d\times d} \text{ are Lipschitz continuous on } \bar{D}. \tag{5.6}$$

Define the stopping-time: $\tau_x = \inf\{t \geqslant 0: X_t(t) \notin \bar{D}\}$. We assume that

$$P(\tau_x < \infty) = 1, \quad \forall x \in \bar{D}, \tag{5.7a}$$

$$\Gamma = \{x \in \partial D: P(\tau_x > 0) = 0\} \text{ is a closed subset of } \partial D, \tag{5.7b}$$

$$\sup_{x \in \bar{D}} \mathbb{E}(e^{\lambda \tau_x}) < \infty \quad \text{for some } \lambda > 2\alpha + \beta^2.^3 \tag{5.7c}$$

Consider now for each $x \in \bar{D}$ the one-dimensional BSDE:

$$Y_t^x + \int_{t\wedge\tau_x}^{\tau_x} U_s^x \, ds = g(X_{\tau_x}^x) + \int_{t\wedge\tau_x}^{\tau_x} f(X_s^x, Y_s^x, Z_s^x) \, ds - \int_{t\wedge\tau_x}^{\tau_x} Z_s^x \, dB_s, \quad \forall t \geqslant 0, \ \omega - \text{a.s.}, \tag{5.8}$$

$$(Y_{t\wedge\tau_x}^x, U_{t\wedge\tau_x}^x) \in \partial\varphi, \quad \text{a.e. on } \Omega \times [0, \tau].$$

It follows from Theorem 1.1 that the BSDE (5.8) has a unique solution

$$(Y^x, Z^x, U^x) \in (S^{2,\lambda}[0, \tau_x] \cap M^{2,\lambda}[0, \tau_x]) \times M_d^{2,\lambda}[0, \tau_x] \times M^{2,\lambda}[0, \tau_x].$$

As in Darling and Pardoux (1997) we can show that

$$x \mapsto \tau_x \quad \text{is a.s. continuous}, \tag{5.9a}$$

$$u(x) = Y_0^x, \quad x \in \bar{D}, \text{ is a determinist continuous function}, \tag{5.9b}$$

$$Y_t^x = u(X_t^x), \quad 0 \leqslant t \leqslant \tau_x, \text{ a.s.} \tag{5.9c}$$

Proposition 5.1. *If the Dirichlet problem* (5.1) *has a classical solution* $u \in C^2(D) \cap C(\bar{D})$; *then* $u(x) = Y_0^x$, $x \in D$, *where* (Y^x, Z^x, U^x) *is the solution of* BSDE (5.8).

Proof. Let $u^*(x) = \mathscr{L}u(x) + f(x, u(x), (\nabla u\sigma)(x)) \in \partial\varphi(u(x)), x \in D$. Applying Itô's formula to $e^{-\lambda t}u(X_t^x)$ we have

$$e^{-\lambda(t\wedge\tau_x)}u(X_{t\wedge\tau_x}) + \int_{t\wedge\tau_x}^{\tau_x} e^{-\lambda s}[-\lambda u(X_s) + \mathscr{L}u(X_s)] \, ds = e^{-\lambda\tau_x}u(X_{\tau_x}) + \int_{t\wedge\tau_x}^{\tau_x} e^{-\lambda s}(\nabla u(X_s), \sigma(X_s) \, dB_s),$$

and, consequently,

$$e^{-\lambda(t\wedge\tau_x)}u(X_{t\wedge\tau_x}) + \int_{t\wedge\tau_x}^{\tau_x} e^{-\lambda s}u^*(X_s) \, ds = e^{-\lambda\tau_x}g(X_{\tau_x}) + \int_{t\wedge\tau_x}^{\tau_x} e^{-\lambda s}[\lambda u(X_s) + f(X_s, u(X_s), (\nabla u\sigma)(X_s))] \, ds + \int_{t\wedge\tau_x}^{\tau_x} e^{-\lambda s}(\nabla u(X_s), \sigma(X_s) \, dB_s).$$

[3] If for some $1 \leqslant i \leqslant d, \inf_{x \in D}(\sigma\sigma^*)_{ii}(x) > 0$, then $\exists \lambda > 0$ such that Eq. (5.7c) holds (see Stroock and Varadhan, 1972).

Hence, by uniqueness:

$$Y_t^x = u(X_{t\wedge\tau_x}^x),\quad Z_t^x = (\nabla u\sigma)(X_{t\wedge\tau_x}^x),\quad U_t^x = u^*(X_{t\wedge\tau_x}^x). \qquad \square$$

Under the assumptions given above, we cannot hope for a classical solution to exist in general. That is why we define the notion of viscosity solution. $\mathscr{P}^{2+}u(x)$ (the elliptic superjet) and $\mathscr{P}^{2-}u(x)$ (the elliptic subjet) are defined similarly as in Definition 4.1. Let $u\in C(\bar{D})$ and $x\in\bar{D}$; then $(q,X)\in\mathscr{P}_{\bar{D}}^{2,+}u(x)$ if

$$u(y)\leqslant u(x)+(q,y-x)+\tfrac{1}{2}(X(y-x),(y-x))+o(|y-x|^2),\quad \forall y\in\bar{D}$$

and $(q,X)\in\mathscr{P}_{\bar{D}}^{2-}u(x)$ if

$$u(y)\geqslant u(x)+(q,y-x)+\tfrac{1}{2}(X(y-x),(y-x))+o(|y-x|^2),\quad \forall y\in\bar{D}.$$

Definition 5.1. (a) A function $u\in C(\bar{D})$ is a viscosity subsolution of Eq. (5.1) if $\forall x\in\bar{D}$, $\forall (q,X)\in\mathscr{P}_{\bar{D}}^{2+}u(x)$,

$$u(x)\in \operatorname{Dom}\varphi, \tag{5.10a}$$

$$\begin{aligned} V_-(x;q,X) &\stackrel{\text{def}}{=} -\tfrac{1}{2}\operatorname{Tr}((\sigma\sigma^*)(x)X)-(b(x),q)\\ &\quad -f(x,u(x),q\sigma(x)+\varphi'_-(u(x))\leqslant 0 \text{ if } x\in D, \end{aligned} \tag{5.10b}$$

$$\min\{V_-(x;q,X),u(x)-g(x)\}\leqslant 0 \text{ if } x\in\partial D. \tag{5.10c}$$

(b) $u\in C(\bar{D})$ is a viscosity supersolution of Eq. (5.1), if $\forall x\in\bar{D}$, $\forall (q,X)\in\mathscr{P}_{\bar{D}}^{2-}u(x)$,

$$u(x)\in \operatorname{Dom}\varphi, \tag{5.11a}$$

$$\begin{aligned} V_+(x;q,X) &\stackrel{\text{def}}{=} -\tfrac{1}{2}\operatorname{Tr}((\sigma\sigma^*)(x)X)-(b(x),q)\\ &\quad -f(x,u(x),q\sigma(x))+\varphi'_+(u(x))\geqslant 0 \quad \text{if } x\in D, \end{aligned} \tag{5.11b}$$

$$\max\{V_+(x;q,X),u(x)-g(x)\}\geqslant 0 \quad \text{if } x\in\partial D. \tag{5.11c}$$

(c) $u\in C(\bar{D})$ is a viscosity solution of Eq. (5.1) if it is both a viscosity subsolution and a viscosity supersolution.

Theorem 5.1. *Under assumptions* (5.2)–(5.7) *the function* $u\in C(\bar{D})$ *given by* $u(x)=Y_0^x$ *is a viscosity solution of* Eq. (5.1). *Moreover*, $u(x)\in\operatorname{Dom}(\partial\varphi)$, $\forall x\in\bar{D}$,.

Proof. Assuming that u is a viscosity solution of Eq. (5.1), we deduce as in Corollary 4.2 that $u(x)\in\operatorname{Dom}(\partial\varphi)$. In order to prove that $u(x)=Y_0^x$ is a viscosity solution we could use as in the previous section an argument based on penalization.

Let us, however, give a direct proof of the fact that u is a viscosity subsolution.

Let $x\in\bar D$ and $(q,X)\in\mathscr{P}_{\bar D}^{2+}u(x)$. From the 0–1 law, there are two possible cases:
(a) $\tau_x(\omega)=0$ a.s. Then $x\in\partial D$, $u(x)=Y_0^x=g(x)$ and consequently (5.10a) is satisfied.
(b) $\tau_x>0$ a.s. We want to show that in this case $V_-(x;q,X)\leqslant 0$, which will conclude the proof.

Suppose this is not the case. Then $V_-(x;q,X)>0$. It follows by continuity of f,u,b and σ, left continuity and monotonicity of φ'_- that there exists $\varepsilon>0$, $\delta>0$ such that for all $|y-x|\leqslant\delta$,

$$\begin{aligned}&-\tfrac{1}{2}\mathrm{Tr}[\sigma\sigma^*(y)(X+\varepsilon I)]-(b(y),q+(X+\varepsilon I)(y-x))\\&\quad -f(y,u(y),[q+(X+\varepsilon I)(y-x)]\sigma(y))+\varphi'_-(u(y))>0.\end{aligned}\tag{5.13}$$

Now, since $(q,X)\in\mathscr{P}_{\bar D}^{2+}u(x)$ there exists $0<\delta'\leqslant\delta$ such that $u(y)<\psi(y)$, for all $y\in\bar D$ such that $|y-x|\leqslant\delta'$, where

$$\psi(y):=u(x)+(q,y-x)+\tfrac{1}{2}((X+\varepsilon I)(y-x),y-x)).$$

Let

$$v:=\inf\{t>0;|X_t^x-x|\geqslant\delta'\}\wedge\tau_x\wedge 1.$$

We note that

$$(\bar Y_t,\bar Z_t):=(Y_{t\wedge v}^x,1_{[0,v]}(t)Z_t^x),\quad 0\leqslant t\leqslant 1,$$

solves the BSDE

$$\begin{aligned}&\bar Y_t=u(X_v^x)+\int_t^1 1_{[0,v]}(s)[f(X_s^x,u(X_s^x),\bar Z_s)-U_s^x]\,\mathrm{d}s-\int_t^1\bar Z_s\,\mathrm{d}B_s,\\&(\bar Y_t,U_t^x)\in\partial\varphi,\mathrm{d}P\times\mathrm{d}t\ \text{a.e. on}\ [0,v].\end{aligned}$$

Moreover, it follows from Itô's formula that

$$(\hat Y_t,\hat Z_t):=(\psi(X_{t\wedge v}^x),1_{[0,v]}(t)(\nabla\psi\,\sigma)(X_t^x)),\quad 0\leqslant t\leqslant 1,$$

satisfies

$$\hat Y_t=\psi(X_v^x)-\int_t^1 1_{[0,v]}(s)\mathscr{L}\psi(X_s^x)\,\mathrm{d}s-\int_t^1\hat Z_s\,\mathrm{d}B_s,\quad 0\leqslant t\leqslant 1.$$

Let $(\tilde Y_t,\tilde Z_t):=(\hat Y_t-\bar Y_t,\hat Z_t-\bar Z_t)$. We have

$$\begin{aligned}\tilde Y_t=\psi(X_v^x)-u(X_v^x)&+\int_t^1 1_{[0,v]}(s)[-\mathscr{L}\psi(X_s^x)-f(X_s^x,u(X_s^x),\bar Z_s)+U_s^x]\,\mathrm{d}s\\&-\int_t^1\tilde Z_s\,\mathrm{d}B_s,0\leqslant t\leqslant 1.\end{aligned}$$

Let

$$\bar\beta_s:=[\mathscr{L}\psi(X_s^x)+f(X_s^x,u(X_s^x),\bar Z_s)]1_{[0,v]}(s),$$

$$\hat\beta_s:=[\mathscr{L}\psi(X_s^x)+f(X_s^x,u(X_s^x),\hat Z_s)]1_{[0,v]}(s).$$

Since $|\hat{\beta}_s - \bar{\beta}_s| \leqslant C|\hat{Z}_s - \bar{Z}_s|$, there exists a bounded d-dimensional progressively measurable process $\{\gamma_s; 0\leqslant s\leqslant 1\}$ such that

$$\hat{\beta}_s - \bar{\beta}_s = (\gamma_s, \tilde{Z}_s).$$

Now,

$$\tilde{Y}_t = \psi(X_v^x) - u(X_v^x) + \int_t^1 [-\hat{\beta}_s + U_s^x + (\gamma_s, \tilde{Z}_s)]\,\mathrm{d}s - \int_t^1 \tilde{Z}_s\,\mathrm{d}B_s.$$

It is easily seen (see e.g. the proof of Theorem 1.6 in Pardoux, 1997) that $\tilde{Y}_0$ takes the form

$$\tilde{Y}_0 = E\left[\Gamma_v(\psi(X_v^x) - u(X_v^x)) + \int_0^v \Gamma_s(U_s^x - \hat{\beta}_s)\,\mathrm{d}s\right],$$

where $\Gamma_t = \exp(\int_0^t \langle\gamma_s, \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t |\gamma_s|^2\,\mathrm{d}s)$.

We first note that $(Y_t^x, U_t^x)\in\partial\varphi$ implies that

$$\varphi'_-(u(X_t^x)) \leqslant U_t^x$$

and this holds $\mathrm{d}P\times\mathrm{d}t$ a.e. Moreover, the choice of δ' and v implies that

$$u(X_v^x) < \psi(X_v^x),$$

$v>0$ a.e. and for $0\leqslant t\leqslant v$, it follows from Eq. (5.13) that

$$\hat{\beta}_t < \varphi'_-(u(X_t^x)).$$

All these inequalities and the above formula for $\tilde{Y}_0$ imply that $\tilde{Y}_0 > 0$, i.e. $\psi(x) > u(x)$, which contradicts the definition of ψ. Hence, $V_-(x;q,X)\leqslant 0$. □

Remark 5.1. Under appropriate additional assumptions, namely Eq. (4.23) and the fact that α is large enough, one can show that the above elliptic variational inequality has a unique viscosity solution, adapting the proof in Crandall–Ishii–Lions (1992).

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