Wavelet approach to vibratory analysis of surface due to a load moving in the layer

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Abstract

The paper analyses theoretically the surface vibration induced by a point load moving uniformly along a infinitely long beam embedded in a two-dimensional viscoelastic layer. The beam is placed parallel to the traction-free surface and the layer under the beam is assumed to be a half space. The response due to a harmonically varying load is investigated for different load frequencies. The influence of the layer damping and moving load speed on the level of vibrations at the surface is analysed and analytical closed form solutions in the integral form for the displacement amplitude and the amplitude spectra are derived. Approximate displacement values depending on Young's modulus and mass density of layers are obtained. The mathematical model is described by the Euler–Bernoulli beam equation, Navier’s elastodynamic equation of motion for the elastic medium and appropriate boundary and continuity conditions. A special approximation method based on the wavelet theory is used for calculation of the displacements at the surface.

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1. Introduction

The investigation of the railway traffic interaction with the environment has become a very important subject of research in recent years. Due to the existence of fast trains and their influence on the civil constructions, the investigation of the level of vibrations caused by railway traffic plays a significant role in the development of the transport of the future (Metrikine and Vrouwenvelder, 2000; Krylov, 2001; Chen and Huang, 2003).

It is well known that the railway generated ground vibrations can disturb buildings even placed at significant distances (Nelson, 1987). At closer range, the influence of vibrations can be perceptible even when caused by conventional passenger trains (Nelson, 1987; Newland and Hunt, 1991). The recent high-speed trains can reach velocities which increase very dangerously the level of vibrations (Krylov, 1995; Dieterman and Metrikine, 1997) especially when the trains run with a speed close to the critical velocities of the wave propagated in the track-ground structure. This critical velocity, which is normally near the Rayleigh wave velocity...
can be exceeded by the trains operating on a soft soil. The practical measurements show that the Rayleigh velocity in a soft peat ground can be even of the order of 200 km/h (Fryba, 1999). Theoretical investigations (Krylov, 2001) showed that a ground vibration boom appeared when the train moving at the surface exceeds the Rayleigh wave velocity in the supporting soil. This phenomenon can be compared to the sonic boom created by an aircraft crossing the sound barrier and was confirmed experimentally (Krylov, 2001). The technological development foreseen for the train transportation leads to the practical necessity of the investigation of train born surface ground vibrations since the Rayleigh velocity can reach different values specific to the soil properties in the operational zones. One can find in the literature many results concerning railway tracks built on the surface (Dieterman and Metrikine, 1996, 1997; Fryba, 1999; Suiker et al., 1998; Krylov, 2001).

A number of projects dedicated to the development of future high-speed transportation, used existing high-speed and underground trains like the high-speed Swedish X-2000 train at Ledsgard (Takemiya, 2002, 2003), the Bakerloo line of London Underground (Degrande et al., 2006b), the high-speed Shinkansen railway in Japan (Takemiya and Bian, 2006) or the Paris metro network (Clouteau et al., 2004) for the development of new measurement techniques (Hussein and Hunt, 2006) and comprehensive finite element – boundary element models (Degrande et al., 2006a; Forrest and Hunt, 2006a,b; Gupta et al., 2006; Chebli et al., 2007; Hussein and Hunt, 2007) correlated to the actual ground structures in order to bring more insight into the propagation of the traffic and ground induced vibrations, tunnel–soil interactions and the development of efficient measures for vibration reduction.

Many aspects connected with the vibrations caused by trains moving in the tunnels are still open to investigation both theoretically and experimentally. This paper analyses from theoretical point of view the vibrations of a surface generated by a point load moving uniformly along an infinitely long beam inside the layer which is similar to the ground vibrations generated by a train moving in a tunnel. A two-dimensional model together with an efficient wavelet approximation were developed for the investigation of the displacements when the parameters describing the viscoelastic properties of the half space under the tunnel and the layer above it vary within realistic intervals. It is shown that, in certain cases, the displacement at the surface caused by a train moving with the velocity near the critical value can reach even 12 mm when the tunnel is placed at the depth of 12 m (Metrikine and Vrouwenvelder, 2000). Therefore, the analysis of the dynamic behaviour of the surface under the load moving inside the layer is of paramount importance for the train tunnels in construction engineering.

The model described in this paper assumes that the layer under the beam is a half space. This assumption is more realistic than that of a finite layer presented in previous studies (Metrikine and Vrouwenvelder, 2000). The depth of the lower layer has an important effect on the system characteristic as well as on the level of accuracy of numerical approximations. Practical calculations show that classical numerical methods of integration do not give results which are accurate enough. For some systems of parameters the complexity of calculations prevents the effective analysis of the system due to increased computational time.

In this paper, a special analytical approximation method based on the wavelet theory (Wang et al., 2003) has been developed for the calculation of the displacements. This method is using the coiflet filter coefficients and allows to obtain accurate results with less computational effort than in the classical numerical integration. The mathematical model is composed of the Euler–Bernoulli equation of motion for the beam, Navier’s elastodynamic equation of motion for the medium and appropriate boundary and interface conditions. The main result of this work is the analysis of the influence of layer damping and moving load speed on the level of vibrations at the surface. The harmonically varying load, which can be related, for example, to the wagon or wheels vibrations, is discussed for different values of the load frequency. This type of load leads to stationary vibrations and therefore one could apply the Fourier transform for the analysis of the problem.

The second section of the paper describes the theoretical two-dimensional model for the load moving in the layer along with the boundary and continuity conditions. This system is solved in a general manner in the third section and the closed form solutions for the displacement amplitude and the amplitude spectra are determined. The fourth section provides fundamental information about wavelets and describes the wavelet approximation method of the Fourier and the inverse Fourier transform by using the coiflets which is one of family of the orthogonal wavelets. In Sections 5 and 6, the solutions for the harmonic load are given. The application of the wavelet approximation is presented in detail along with the analysis of these solutions,
and the results for the case of relatively low, sub-critical velocity. The efficiency of this type of approximation is analysed for both the horizontal and vertical displacements. A parametric analysis of the system behaviour is carried out by varying the mass density, Young’s modulus, load velocity and load frequency in Section 7. Some aspects regarding the correlation of the moving load critical velocity, in relation to the ground vibration boom phenomenon for different load frequencies, are presented in Section 8. The paper ends with a discussion of the modelling aspects and the application of the wavelet approximation.

2. Model description

The model consists of an infinitely long beam located in a two-dimensional layer with infinite lower half space and a uniformly moving load (Fig. 1). The upper layer has a thickness $h$ in the $z$ direction.

The equation of the beam vertical motions can be written by using the Euler–Bernoulli theory as

$$EI \frac{\partial^4 W}{\partial x^4} + \rho_B \frac{\partial^2 W}{\partial t^2} = P(t) \delta(x - Vt) + a[\sigma_{zz}(x, h^-, t) - \sigma_{zz}(x, h^+, t)],$$

where $P(t)$ is the vertical point load, $W(x, t)$ is the vertical displacement of the beam, $\sigma_{zz}(x, z, t)$ is the vertical stress, $EI$ and $\rho_B$ are the bending stiffness and the mass per unit length of the beam, $\delta(\cdot)$ is the Dirac delta function (Miklowitz, 1978; Achenbach, 1984; Fryba, 1999), $a$ is a characteristic length associated with the length of the structure in $y$ direction (the thickness of the beam in $y$ direction).

By the assumption of a small viscosity in the layers the equation of motion can be written as

$$(\lambda + \mu) \nabla_{x,z}(\nabla_{x,z} u) + \mu \nabla_{x,z}^2 u = \frac{\rho^2 u}{\partial t^2},$$

where $u(x, z, t) = [u(x, z, t), 0, w(x, z, t)]$ is the displacement vector for the layer, $\lambda + \lambda^* \partial / \partial t$ and $\mu + \mu^* \partial / \partial t$ are operators used to describe the viscoelastic behaviour of the layer, $\lambda$, $\mu$ are Lame’ constants and $\rho$ is the mass density of the layer.

The boundary and continuity (interface) conditions are given as:

$$u(x, h^-, t) = 0, \quad u(x, h^+, t) = 0, \quad (3a)$$

$$w(x, h^-) = W(x, t), \quad w(x, h^+, t) = W(x, t), \quad (3b)$$

$$\sigma_{zz}(x, 0, t) = 0, \quad \sigma_{zz}(x, 0, t) = 0, \quad (3c)$$

$$\lim_{z \to \infty} u(x, z, t) = 0, \quad \lim_{z \to \infty} w(x, z, t) = 0. \quad (3d)$$

The physical interpretation of these conditions are: the beam does not move horizontally (3a), the beam and the layer displacement are the same at the interfaces $z = h^+ = \lim_{z \to h^+} z = h^- = \lim_{z \to h^-} z$ (3b), the surface of the soil is traction-free (3c), and the radiated waves vanish at the infinity (3d).

The system of Eqs. (1)–(3) is sufficient for obtaining the steady state response of the surface (Metrikine and Vrouwenvelder, 2000) and the initial conditions are not necessary.

3. Analytical solution

The problem can be solved by introducing Lame’ potentials (Eringen and Suhubi, 1975; Achenbach, 1984) and by using the Fourier transforms. One can describe the layer motion, according to the theory of Lame’ potentials, in terms of the following scalar and vector functions:

![Fig. 1. Geometry of the problem – half space under the beam.](image-url)
\[ \varphi = \varphi(x, z, t), \quad \Psi = [0, -\varphi(x, z, t), 0]. \] (4)

The equation of motion (2) is often difficult to integrate and usually a transformation is applied to the dependent variable \( u \) leading to a system of differential equations which is relatively easy to solve. The Lame' potentials, giving the general representation of the displacement vector, assume a relatively simple form in the case of two-dimensional problems of elastodynamics. The displacement and the stress components can be written as

\[ u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \varphi}{\partial z} - \frac{\partial \psi}{\partial x}, \] (5)

\[ \sigma_{zz} = \hat{\lambda} \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + 2\mu \left( \frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right), \quad \sigma_{xz} = \mu \left( 2 \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right). \] (6)

Applying this method to the governing Eq. (2) leads to a system of two scalar equations for the layer's motion:

\[ \frac{\partial^2 \varphi}{\partial t^2} - \left( \frac{c_1^2}{c_t^2} + \frac{\lambda^* + 2\mu^*}{\rho} \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = 0 \] (7)

and

\[ \frac{\partial^2 \psi}{\partial t^2} - \left( \frac{c_1^2}{c_t^2} + \frac{\mu^*}{\rho} \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) = 0, \] (8)

where \( c_1 = \sqrt{(\lambda + 2\mu)/\rho} \) and \( c_T = \sqrt{\mu/\rho} \) are velocities of the longitudinal and the shear waves in the layer.

For the processes analysed, it is suitable to apply the integral Fourier transform, with respect to the time \( t \) and the space variable \( \tau \), defined as follows

\[ \tilde{f}(k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, t) e^{i(\omega t - kx)} \, dx \, dt, \] (9)

\[ f(x, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k, \omega) e^{-i(\omega t - kx)} \, d\omega \, dk. \] (10)

By applying these transforms one can formulate the equations, the boundary and continuity conditions in the transform domain. The Eqs. (7) and (8) for the layer motion become:

\[ \frac{d^2 \tilde{\varphi}}{dz^2} - R_1^2 \tilde{\varphi} = 0, \quad \frac{d^2 \tilde{\psi}}{dz^2} - R_1^2 \tilde{\psi} = 0, \] (11)

and from Eq. (1), the dynamics of the beam is described by:

\[ \left( EI k^4 - \rho_0 \omega^2 \right) \tilde{W}(k, \omega) = \tilde{P}(\omega - i\kappa) + a(\tilde{\sigma}_w(k, h^-, \omega) - \tilde{\sigma}_w(k, h^+, \omega)). \] (12)

The new coefficients which appear in Eq. (11) are connected with the velocities \( c_1 \) and \( c_T \) of the longitudinal and the shear waves in the layer, respectively:

\[ R_1^2 = k^2 - \omega^2 / \left( c_1^2 - i\omega(\lambda^* + 2\mu^*)/\rho \right), \quad R_T^2 = k^2 - \omega^2 / \left( c_T^2 - i\omega\mu^*/\rho \right), \] (13)

and \( \tilde{P} \) denotes the following integral

\[ \tilde{P}(\omega) = \int_{-\infty}^{\infty} P(t) e^{i\omega t} \, dt. \] (14)

The interface and the boundary conditions related to Eqs. (3a)–(3c) and (3d) take the following form:

\[ \tilde{u}(k, h^-, \omega) = 0, \quad \tilde{u}(k, h^+, \omega) = 0, \] (15a)

\[ \tilde{w}(k, h^-, \omega) = \tilde{W}(k, \omega), \quad \tilde{w}(k, h^+, \omega) = \tilde{W}(k, \omega), \] (15b)

\[ \tilde{\sigma}_{xz}(k, 0, \omega) = 0, \quad \tilde{\sigma}_{xz}(k, 0, \omega) = 0, \] (15c)

\[ \lim_{z \to -\infty} \tilde{u}(k, z, \omega) = 0, \quad \lim_{z \to -\infty} \tilde{w}(k, z, \omega) = 0. \] (15d)
The equations for the displacements and the stress components in terms of the potentials are rearranged in the form:

\[
\ddot{u}(k, z, \omega) = \frac{d}{d z} \ddot{\varphi}(k, z) - k^2 \ddot{\psi}(k, z), \quad \ddot{w}(k, z, \omega) = \frac{d}{d z} \ddot{\psi}(k, z) + \ddot{\varphi}(k, z),
\]

(16a)

\[
\ddot{\sigma}_{zz}(k, z, \omega) = \ddot{\lambda}(k) \left( \frac{d^2 \varphi}{d z^2} - k^2 \varphi(k, z) \right) + 2 \ddot{\mu}(k) \left( \frac{d^2 \psi}{d z^2} - ik \frac{d \psi}{d z} \right),
\]

(17a)

\[
\ddot{\sigma}_{zz}(k, z, \omega) = \ddot{\mu}(k) \left( 2ik \frac{d \varphi}{d z} + \frac{d^3 \psi}{d z^3} + k^2 \ddot{\psi} \right).
\]

In the following, the two layers of the soil surrounding the beam are assumed to have different material properties. Parameters \( \rho_1, E_1 \) and \( \rho_2, E_2 \) denote the mass density and the Young modulus of the layer above the beam and under the beam, respectively. Then the solutions for potentials can be written as:

\[
\ddot{\varphi}_j = A_{1+n}(k, \omega) e^{R_{Lj} z} + A_{2+n}(k, \omega) e^{-R_{Lj} z},
\]

(18a)

\[
\ddot{\psi}_j = A_{3+n}(k, \omega) e^{R_{Tj} z} + A_{4+n}(k, \omega) e^{-R_{Tj} z},
\]

(18b)

where \( j = 1, n = 0 \) are for the upper layer (\( z \in [0, h] \)) and \( j = 2, n = 4 \) are for the lower layer (\( z > h \)). \( R_{Lj} \) and \( R_{Tj} \) denote the roots of the characteristic equations of Eqs. (11) for the upper layer (\( j = 1 \)) and the lower one (\( j = 2 \)).

Using (18a) and (18b) one can write the solutions for displacements and stresses in both layers:

\[
\ddot{u}_j = ik \left( A_{1+n}(k, \omega) e^{R_{Lj} z} + A_{2+n}(k, \omega) e^{-R_{Lj} z} \right) + R_{Tj} \left( A_{3+n}(k, \omega) e^{R_{Tj} z} - A_{4+n}(k, \omega) e^{-R_{Tj} z} \right),
\]

\[
\ddot{w}_j = R_{Lj} \left( A_{1+n}(k, \omega) e^{R_{Lj} z} - A_{2+n}(k, \omega) e^{-R_{Lj} z} \right) - ik \left( A_{3+n}(k, \omega) e^{R_{Tj} z} + A_{4+n}(k, \omega) e^{-R_{Tj} z} \right),
\]

\[
\ddot{\sigma}_{zzj} = \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) (R_{Lj})^2 \ddot{\varphi}_j - \ddot{\lambda}_j k^2 \left( A_{1+n}(k, \omega) e^{R_{Lj} z} + A_{2+n}(k, \omega) e^{-R_{Lj} z} \right)
\]

\[
- 2ik \ddot{\mu}_j R_{Tj} \left[ A_{3+n}(k, \omega) e^{R_{Tj} z} - A_{4+n}(k, \omega) e^{-R_{Tj} z} \right] + k^2 \left( R_{Tj} \right)^2
\]

\[
\times \left( A_{3+n}(k, \omega) e^{R_{Tj} z} + A_{4+n}(k, \omega) e^{-R_{Tj} z} \right).
\]

(19)

Substituting formulas (19) into the transformed equation for the beam motion (12) and applying boundary and interface conditions (15a)–(15c) and (15d) one obtains a linear system of six algebraic equations with respect to \( A_j(k, \omega) \), which can be written as:

\[
B \cdot X = \ddot{P}(\omega - i k) \cdot F,
\]

(20)

where

\[
B = [b_{ij}], \quad F = [1, 0, 0, 0, 0, 0]^T, \quad X = [A_1, \ldots, A_6, A_8]^T
\]

(21)

and

\[
b_{1j} = \left[ a \left( \ddot{\lambda}_j k^2 - \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) R_{Lj}^2 \right) g_{Lj, 1}, a \left( \ddot{\lambda}_j k^2 - \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) R_{Lj}^2 \right) g_{Lj, 1}^{-1}, 2a \ddot{\mu}_j ik R_{Tj} g_{Tj, 1},
\]

\[
-2a \ddot{\mu}_j ik R_{Tj} g_{Tj, 1}^{-1}, (a \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) R_{Lj}^2 - \ddot{\lambda}_j k^2 - \gamma R_{Lj}^2) g_{Lj, 2}^{-1}, (2a \ddot{\mu}_j ik R_{Tj} - \gamma ik) g_{Tj, 2}^{-1} \right],
\]

(22a)

\[
b_{2j} = [0, 0, 0, 0, ik g_{Lj, 2}^{-1}, -R_{Tj} g_{Tj, 2}^{-1}],
\]

(22b)

\[
b_{3j} = [ik g_{Lj, 1}, ik g_{Lj, 1}^{-1}, R_{Tj} g_{Tj, 1}, -R_{Tj} g_{Tj, 1}^{-1}, 0, 0],
\]

(22c)

\[
b_{4j} = [-R_{Lj} g_{Lj, 1}, R_{Lj} g_{Lj, 1}^{-1}, ik g_{Tj}, ik g_{Tj}^{-1}, -R_{Lj} g_{Lj, 1}^{-1}, -ik g_{Tj}^{-1}],
\]

(22d)

\[
b_{5j} = \left[ \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) R_{Lj}^2 - \ddot{\lambda}_j k^2, \left( \ddot{\lambda}_j + 2 \ddot{\mu}_j \right) R_{Lj}^2 - \ddot{\lambda}_j k^2, -2 \ddot{\mu}_j ik R_{Tj}, 2 \ddot{\mu}_j ik R_{Tj}, 0, 0 \right],
\]

(22e)

\[
b_{6j} = \left[ 2 \ddot{\mu}_j ik R_{Lj}, -2 \ddot{\mu}_j ik R_{Lj}, \ddot{\mu}_j (R_{Tj}^2 + k^2), \ddot{\mu}_j (R_{Tj}^2 + k^2), 0, 0 \right],
\]

(22f)
\[
\begin{align*}
g_{L1} &= e^{\beta k_{L1}}, \quad g_{T1} = e^{\beta k_{T1}}, \quad g_{L2} = e^{\beta k_{L2}}, \quad g_{T2} = e^{\beta k_{T2}}, \quad \gamma = EI k^4 - \rho_B \omega^2. \quad (23)
\end{align*}
\]

According to the Cramer’s rule, the solution of the system of algebraic equations (20) can be expressed as

\[
A_j(k, \omega) = \frac{\tilde{P}(\omega - \nu k)}{D(k, \omega)}, \quad (24)
\]

where \(D\) is a determinant of the matrix \(B\) and \(D_j\) is a determinant of modified matrix \(B\) with \(j\)th column replaced by the vector \(F\) (Eq. (21)).

By substituting the formulas (24) into the Eq. (19) one obtains the solution for the displacements and stresses in the transform domain. The displacements on the surface (\(z = 0\)) can be written as:

\[
\begin{align*}
\tilde{u}_1(k, 0, \omega) &= \tilde{P}(\omega - \nu k)\tilde{u}_0^1(k, \omega), \quad (25a) \\
\tilde{w}_1(k, 0, \omega) &= \tilde{P}(\omega - \nu k)\tilde{w}_0^1(k, \omega), \quad (25b)
\end{align*}
\]

where

\[
\begin{align*}
\tilde{u}_0^1(k, \omega) &= \left( ik(D_1 + D_2) + R_{T1}(D_3 - D_4) \right) / D, \\
\tilde{w}_0^1(k, \omega) &= \left( R_{L1}(D_1 - D_2) - ik(D_3 + D_4) \right) / D. \quad (26a, 26b)
\end{align*}
\]

By applying the inverse transform (10) to the Eqs. (25a) and (25b) one obtains the solution for the displacement at the surface in the physical domain:

\[
\mathbf{u}(x, 0, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \tilde{u}_1(k, 0, \omega), \tilde{w}_1(k, 0, \omega) \right] e^{-i(\omega t - kx)} \, dk \, d\omega. \quad (27)
\]

In order to analyze the steady state response at the surface for the considered type of the moving load, it is sufficient to study the vibrations of any point on the surface, and without loss of generality, the point \(x = 0\) will be used for the analysis. The amplitude spectrum of vibrations is determined as:

\[
\mathbf{u}_f(f) = \int_{-\infty}^{\infty} \mathbf{u}(0, 0, t) e^{-2\pi i ft} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{P}(-2\pi f - \nu k) \left[ \tilde{u}_0^1(k, -2\pi f), \tilde{w}_0^1(k, -2\pi f) \right] \, dk \quad (28)
\]

and consequently, one can write the formula for the displacement at the point \(x = 0\) as

\[
\mathbf{u}(0, 0, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{P}(\omega - \nu k) \left[ \tilde{u}_0^1(k, \omega), \tilde{w}_0^1(k, \omega) \right] e^{-i\omega t} \, dk \, d\omega, \quad (29)
\]

where \(\tilde{P}(\omega - \nu k)\) is defined by Eq. (14). In the analytical calculations, the following interpretation (Korn and Korn, 1961) of the Dirac delta function was used:

\[
\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, dt. \quad (30)
\]

4. Wavelet approach

The classical methods of numerical integration are sufficient for solving the model with the bounded layer supporting the beam (Metrikine and Vrouwenvelder, 2000). However, there is a strong dependence of the complexity of the model dynamics on this layer’s thickness which leads to difficulties in the numerical simulations. The problem of the appropriate choice of the calculation method for displacements becomes very important when the layer supporting the beam is a half space. The values of the vector displacement can be calculated relatively simply by direct numerical integration only in the case of the horizontal component. In order to alleviate the numerical difficulties in calculations of the vertical displacement, one can use an analytical approximation based on a wavelet expansion of functions in the transform domain. This method allows to observe, due to the properties of wavelet bases, more detailed features of solution which can be lost during the process of numerical integration. The multiresolution character of wavelets allows to reconstruct the solution from its transform keeping all the most important features of the original function (Mallat, 1998). In addition, the wavelet approximation works in the function space and therefore gives a continuous solution
which is proper for each point in some interval of the independent variable whereas the numerical calculation estimates the function based on results calculated for a discrete set of points. When the integrand displays complex variations some important information regarding the characteristics of the system might be filtered out in the process of sampling the analysed function domain.

One can write, in terms of the wavelet theory, the two-scale relations of the scaling function and the wavelet function, respectively:

\[
\Phi(x) = \sum_{k=0}^{M} p_k \Phi(2x - k), \quad (31)
\]

\[
\Psi(x) = \sum_{k=0}^{M} q_k \Phi(2x - k), \quad (32)
\]

where the sequences \( p_k \), \( q_k \) are filter coefficients (Daubechies, 1993), and \( M \) is an integer which is characteristic for their accuracy. The functions \( \Phi \) and \( \Psi \) are called the scaling function and the wavelet, respectively.

One can obtain the refinement equations of \( \Phi \) and \( \Psi \) in the transform domain (Hong et al., 2005) by taking the Fourier transform

\[
\tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x} \, dx \quad (33)
\]

of Eqs. (31) and (32):

\[
\tilde{\Phi}(\omega) = P \left( e^{-\frac{\omega}{2}} \right) \tilde{\Phi}(\frac{\omega}{2}), \quad (34)
\]

\[
\tilde{\Psi}(\omega) = Q \left( e^{-\frac{\omega}{2}} \right) \tilde{\Phi}(\frac{\omega}{2}), \quad (35)
\]

where \( P \) and \( Q \) are polynomials defined as:

\[
P(z) = \frac{1}{2} \sum_{k=0}^{M} p_k z^k, \quad Q(z) = \frac{1}{2} \sum_{k=0}^{M} q_k z^k. \quad (36)
\]

Applying Eq. (34) recursively:

\[
\tilde{\Phi}(\omega) = P \left( e^{-\frac{\omega}{2}} \right) \tilde{\Phi}(\frac{\omega}{2}) = P \left( e^{-\frac{\omega}{4}} \right) \tilde{\Phi}(\frac{\omega}{4}) = P \left( e^{-\frac{\omega}{8}} \right) \tilde{\Phi}(\frac{\omega}{8}) = \cdots \quad (37)
\]

and with the assumption \( \tilde{\Phi}(0) = 1 \) (Wang et al., 2003) one obtains

\[
\tilde{\Phi}(\omega) = \prod_{k=1}^{\infty} P \left( e^{-\omega/2^k} \right). \quad (38)
\]

Combining Eqs. (35) and (38) leads to:

\[
\tilde{\Psi}(\omega) = Q \left( e^{-\frac{\omega}{2}} \right) \prod_{k=1}^{\infty} P \left( e^{-\omega/2^{k+1}} \right). \quad (39)
\]

The relations (38) and (39) describe the low pass filter and the high pass filter, respectively.

The coefficients \( p_k \) of the filter which was used in numerical calculations are listed in Table 1 (Wang et al., 2003). This filter is one of the low pass filters from a class of non-symmetric orthogonal wavelet functions called coiflets (Beylkin et al., 1991). This family of wavelets was specially constructed for applications in numerical analysis due to its property of vanishing moments condition (Daubechies, 1988; Beylkin et al., 1991; Daubechies, 1993), both for the scaling function and the wavelet function:

\[
\int_{-\infty}^{+\infty} t^k \Psi(t) \, dt = 0 \quad \text{for } k = 0, 1, \ldots, N - 1, \quad (40)
\]

\[
\int_{-\infty}^{+\infty} \Phi(t) \, dt = 1, \quad \int_{-\infty}^{+\infty} t^k \Phi(t) \, dt = 0 \quad \text{for } k = 1, \ldots, N - 1, \quad (41)
\]
where \( N \) is a positive integer.

The difference between the energy spectrum \(|\tilde{\Phi}(\omega)|^2\) of that filter and the energy spectrum of the perfect low pass filter can be reduced to less than 0.3% in the frequency domain \( [0, \frac{\pi}{2}] \) as seen in Fig. 2(a) and (b) meaning that the scaling function based on these coefficients has good low pass characteristic near zero. Using the filter with this level of accuracy, allows obtaining accurate enough approximate solutions (Wang et al., 2003; Koziol et al., 2006).

One can write the expansion of a function \( f(x) \in L^2(\mathbb{R}) \) into the following series:

\[
f(x) = P_n f(x) + \sum_{j=0}^{\infty} Q_j f(x) = \sum_{k=-\infty}^{+\infty} c_{n,k} \Phi_{n,k}(x) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{+\infty} d_{j,k} \Psi_{j,k}(x), \tag{42}
\]

where \( P_n, Q_j \) are the project operators (Meyer, 1992) which are connected, due to the multiresolution analysis (Daubechies, 1988; Daubechies, 1993), by the following equations:

\[
P_{j+1} = P_j + Q_j, \tag{43}
\]

\[
f(x) = \lim_{n \to \infty} P_n f(x) = \sum_{j=-\infty}^{+\infty} Q_j f(x). \tag{44}
\]

The parameters \( c_{n,k} \) and \( d_{j,k} \) in Eq. (42) (called multiresolution coefficients) are defined as:

\[
c_{n,k} = 2^{n/2} \int_{-\infty}^{+\infty} f(x) \Phi(2^n x - k) \, dx = \int_{-\infty}^{+\infty} f(x) \Phi_{n,k}(x) \, dx, \tag{45}
\]

\[
d_{j,k} = 2^{j/2} \int_{-\infty}^{+\infty} f(x) \Psi(2^j x - k) \, dx = \int_{-\infty}^{+\infty} f(x) \Psi_{j,k}(x) \, dx. \tag{46}
\]

When applying the inverse Fourier transform

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{i\omega x} \, d\omega \tag{47}
\]

to Eq. (42) with \( \tilde{f}(\omega) \) changed for \( f(x) \) one can write

\[
f(x) = \frac{2^{-(n+2)/2}}{\pi} \tilde{\Phi}(-x2^{-n}) \sum_{k=-\infty}^{+\infty} c_{n,k} e^{ik2^{-n}} \frac{1}{2\pi} \sum_{j=0}^{+\infty} 2^{-j} \tilde{\Psi}(-x2^{-j}) \sum_{k=-\infty}^{+\infty} d_{j,k} e^{ik2^{-j}}. \tag{48}
\]
The original function \( f(x) \) can be calculated from \( \tilde{f}(\omega) \) by using Eq. (48) if the coefficients \( c_{n,k} \) and \( d_{j,k} \) are known. The calculation of the multiresolution coefficients is simplified when using the coiflets (Beylkin et al., 1991). One can write approximating equations determined by using the refinement equations (Eqs. (34) and (35)) and the characteristic of the coiflets (Wang et al., 2003)

\[
c_{n,k} \approx 2^{-n/2} \tilde{f}((k + \tilde{M})2^{-n})
\]

and

\[
d_{j,k} \approx 2^{-j/2-1} \sum_{m=0}^{M} (-1)^m \tilde{p}_{M-m} f((\tilde{M}+m+2k)2^{-j-1}),
\]

where

\[
\tilde{M} = \frac{1}{2} \sum_{k=0}^{M} k p_k.
\]

Using Eqs. (48)–(50) leads to the approximating formula for the inverse Fourier transform:

\[
f(x) = \lim_{n \to \infty} f_n(x),
\]

where

\[
f_n(x) = \frac{2^{-n-1}}{\pi} \tilde{\Phi}(-x2^{-n}) \sum_{k=-\infty}^{\infty} \tilde{f}((k + \tilde{M})2^{-n})e^{i\kappa2^{-n}}.
\]

The approximation \( f_n(x) \approx f(x) \) can be adapted for the calculation of the vertical and the horizontal components of the displacement vector for the system considered in this paper. In order to determine the inversion \( f_n(x) \), a proper range of the variable \( \omega \) must be used. This range \([\omega_{\text{min}}, \omega_{\text{max}}]\) should cover the set of \( \omega \) for which \( \tilde{f}(\omega) \) has a significant influence on the characteristic of the original function. The equations describing the range of summation in Eq. (53) have the following form (Wang et al., 2003):

\[
k_{\text{min}} = \omega_{\text{min}} 2^n - N_c + 2, \quad k_{\text{max}} = \omega_{\text{max}} 2^n - 1,
\]

where \( N_c \) is the number of filter coefficients (see Table 1).

This approximation method (Wang et al., 2003) was used previously for the derivation of the inverse Laplace transform and the investigation of the dynamic behaviour of a beam resting on random foundation (Koziol and Hryniewicz, 2006).

5. Harmonic load

The analysis of the harmonic load is very important for the study of the effect of the train vibrations on the response of the structure. The load with the magnitude varying harmonically can be written as
\[ P(t) = P_0 \cos(\Omega t) \]. Using the Eqs. (28) and (29), the representation (30) and analytical properties of the Dirac delta function (Korn and Korn, 1961; Miklowitz, 1978; Achenbach, 1984) along with the features of the Fourier integral, one can write after some calculations

\[ \tilde{P}(\omega - kV) = \int_{-\infty}^{+\infty} P_0 \cos(\Omega t)e^{i(\omega - kV)t} \, dt = 2\pi P_0 (\delta(\omega - kV + \Omega) + \delta(\omega - kV - \Omega)) \]

and hence the Eqs. (28) and (29) can be rewritten as:

\[ u_j(f) = -\frac{P_0}{2V} \left\{ \tilde{u}_0^i \left( \frac{\Omega - 2\pi f}{V}, -2\pi f \right), \tilde{w}_0^i \left( \frac{\Omega - 2\pi f}{V}, -2\pi f \right) \right\} + \left\{ \tilde{u}_0 \left( -\Omega - 2\pi f \right), -2\pi f \right\}, \tilde{w}_0 \left( -\Omega - 2\pi f \right), -2\pi f \right\} \]

(56)

for the amplitude spectra and

\[ u(0,0,t) = \frac{P_0}{4\pi} \int_{-\infty}^{+\infty} \left\{ \tilde{u}_0^i(k,kV - \Omega), \tilde{w}_0^i(k,kV - \Omega) \right\} e^{-i(kV - \Omega)t} + \left\{ \tilde{u}_0^i(k,kV + \Omega), \tilde{w}_0^i(k,kV + \Omega) \right\} e^{-i(kV + \Omega)t} \right\} \, dk \]

(57)

for the displacement vector.

The case of harmonically varying load shows a strong response of the structure for even relatively small velocities of the moving load. This property is due to the following equation describing the relationship between the angular frequency \( \omega \) of radiated wave and the load frequency \( \Omega \):

\[ \omega = kV \pm \Omega. \]

The complexity of the response for the harmonic load causes several difficulties in the numerical analysis. The train moving along the rail generates a vertical force which has a complex dependence on time and frequency, but the most important part of the energy is placed near the zero frequency, when the transform domain is taken into account. This fact can be treated as the consequence of the gravity force causing the constant pressure on the rail track and be idealised by the case of a constant load \( P(t) = P_0 \) uniformly moving along the beam. One should note that the constant load \( P(t) = P_0 \) can be treated as a special case of harmonically varying load \( P(t) = P_0 \cos(\Omega t) \) with the load frequency \( \Omega = 0 \). For this limit case, the integrand in Eq. (57) has strong singularity, preventing the numerical solution for the displacements.

The Eq. (58) describes the kinematic invariant which is very important in the analysis of the dispersion curves and the spectral analysis of the load moving problems (Fryba, 1999). In the present paper the discussion of the dispersion curves as well as an analysis in the frequency domain were omitted, and the main attention was directed to the study of the response amplitude.

### 6. Wavelet approximation of displacement

The wavelet approximation described in Section 4 represents an efficient method for calculations of the displacement at the surface for this model. The comparison with the numerical results for the case with relatively low sub-critical velocity 30 m/s is carried out for its validation and demonstration of its features.

One can notice that the Eq. (57) can be transformed into

\[ u(0,0,t) = -\frac{P_0}{4\pi V} \int_{-\infty}^{+\infty} \left\{ \tilde{u}_0^i \left( \frac{\Omega - \omega_1}{V}, -\omega_1 \right), \tilde{w}_0^i \left( \frac{\Omega - \omega_1}{V}, -\omega_1 \right) \right\} e^{i\omega_1 t} \, d\omega_1 \]

\[-\frac{P_0}{4\pi V} \int_{-\infty}^{+\infty} \left\{ \tilde{u}_0 \left( -\Omega - \omega_2 \right), -\omega_2 \right\}, \tilde{w}_0 \left( -\Omega - \omega_2 \right), -\omega_2 \right\} e^{i\omega_2 t} \, d\omega_2 \]

(59)

where \( \omega_1 = \Omega - kV \) and \( \omega_2 = -\Omega - kV \). In this case, the Eq. (53) can be applied to the functions
The functions (62) and (63) can be approximated by using the Eq. (53) in order to determine the displacement instead of \( \tilde{f}(\omega) \).

One should note that considering the features of the functions (60a) and (60b), and (61a) and (61b), and the Fourier integral form of the displacement formula (Eq. (59)), it is sufficient to investigate the properties of the functions

\[
\begin{align*}
  f_u(\omega) &= \begin{cases} 
    f_u^1(\omega) & \text{for } \omega \leq 0 \\
    f_u^2(\omega) & \text{for } \omega > 0 
  \end{cases} \\
  f_w(\omega) &= \begin{cases} 
    f_w^1(\omega) & \text{for } \omega \leq 0 \\
    f_w^2(\omega) & \text{for } \omega > 0 
  \end{cases}
\end{align*}
\]  

(62)

and

instead of the four functions (Eqs. (60a) and (60b), and (61a) and (61b)). This is acceptable because the functions (60a), (61a) and (60b), (61b) are equal to zero for the variables smaller and bigger than zero, respectively. The functions (62) and (63) can be approximated by using the Eq. (53) in order to determine the displacement amplitude.

The numerical calculations in this section have been carried out for the following system of parameters which was previously used in the papers (Krylov, 1995; Metrikine and Vrouwenvelder, 2000):

- Young’s moduli: \( E_1 = c_E E_2 \), \( E_2 = 3 \times 10^7 \text{ N/m}^2 \); the mass densities \( \rho_1 = c_\rho \rho_2 \), \( \rho_2 = 1700 \text{ kg/m}^3 \); \( \mu^*_1 = \mu^*_2 = \lambda^* = \lambda^*_2 = 3 \times 10^8 \text{ kg/s} \) and Poisson’s ratio \( v = v_1 = v_2 = 1/3 \) for the layer above and under the beam, respectively.

- Two parameters have been introduced to discuss the influence of the layers properties on the amplitude of vibrations at the surface: \( c_E = E_1/E_2 \) and \( c_\rho = \rho_1/\rho_2 \). It is assumed also (with the exception of Section 7) that the layers above and under the beam have the same properties: \( c_\rho = c_E = 1 \). The thickness of the upper layer is assumed to be \( h = 12 \text{ m} \).

The beam is characterised by: the mass density \( \rho_0/A = 3 \times 10^4 \text{ kg/m}^2 \); the bending stiffness \( El/A = 10^9 \text{ Nm} \); the characteristic length associated with the structure in \( y \) direction \( a = 4 \text{ m} \); the vertical point load \( P_0 = 4 \times 10^4 \text{ N} \); the velocity of the load \( V = 30 \text{ m/s} \) and the load frequency \( f_\omega = \Omega/2\pi = 1 \text{ Hz} \) for the harmonic load.

Fig. 3(a) and (b) show the variation of the displacement function \( f_u(\omega) \) in the transform domain and allow the determination of a proper range of summation in Eq. (53) for the harmonic load analysed in this paper. One can observe that the minimal appropriate interval of \( \omega \) must be taken as \([-30, 30] \), in order to investigate all points having a significant influence on the surface response. Using a bigger interval does not change significantly the characteristics of solution.

Fig. 4(a)–(e) and 5(a)–(e) show the first five terms \( n = 0,1,2,3,4 \) of approximating sequence (52) applied to the functions \( f_u(\omega) \) (Eq. (62)) and \( f_w(\omega) \) (Eq. (63)) for the horizontal component and the vertical component, respectively. As seen in Figs. 4d and 6 for the horizontal displacement, and Figs. 5d and 7 for the vertical displacement, the fourth term \( (n = 3) \) almost coincides with the numerical solution and it is almost identical with the fifth term \( (n = 4) \). It means that the fourth term \( (u_3 \text{ and } w_3) \) is sufficient for good approximation of solution \( u \text{ and } w \) with considered system of parameters. It should be noted that, according to the wavelet theory (Wang et al., 2003), the \( n \)th term of the sequence (53) should coincide approximately with the original
function \( f(x) \) within the frequency interval \([-2\pi, 2\pi]\), which means that the components \( u_3 \) and \( w_3 \) presented on the Figs. 4d and 5d represent a good approximation for the horizontal and vertical displacements in the frequency interval \([-8\pi, 8\pi]\) with the error less than 0.3% for the system of the filter coefficients (Table 1) used in this approximation.

When compared to the exact solution (Figs. 6 and 7), the small differences between the displacement patterns for the wavelet approximation \( u_3(t) \) and \( w_3(t) \) appearing only for \( t < -5 \) in Figs. 4d and 5d, suggest the need for a higher order approximation to be used for a better estimation of the solution in this interval. These differences diminish when analysing the fifth order approximation \( u_4(t) \) (Fig. 4(e)) and \( w_4(t) \) (Fig. 5(e)). The approximation \( u_3(t), w_3(t) \), and the numerical solution, are almost the same for a time interval sufficiently large to be used for the analysis of the dynamic response. For some systems of parameters, the numerical approach leads to instabilities, as in the case of the vertical displacement for relatively high (or low) frequency loads, and the wavelet approximation should be used as an alternative method of solution.

7. Parametrical analysis of surface vibration

The wavelet approximation method applied in this work allows a parametrical analysis of the governed system with the possibility of calculation of the displacement for different load frequencies.

In numerical calculations, the horizontal component \( u \) and the vertical component \( w \) were estimated by using the wavelet approach and in the following discussion the terms \( u_3 \) and \( w_3 \) (as shown previously – Eqs. (53) and (57)) of the approximating sequence were used. The magnitude of vibrations at the surface is investigated by using the two parameters \( c_q \) and \( c_E \) describing the properties of the solid. In order to analyse the behaviour of the system, three different velocities of the moving load are used in numerical calculations:

- 50 m/s \( \approx 0.7 \cdot c_{R2} \),
- 80 m/s \( \approx 1.05 \cdot c_{R2} \approx 0.98 \cdot c_{T2} \),
- 100 m/s \( \approx 1.3 \cdot c_{R2} \approx 1.2 \cdot c_{T2} \),

where \( c_{R2} \approx 76 \text{ m/s} \) and \( c_{T2} \approx 82 \text{ m/s} \) are the Rayleigh velocity and the velocity of the shear waves in the layer supporting the beam, respectively. In the following investigations, the velocities smaller than the Rayleigh velocity \( c_{R2} \) will be called sub-critical and the velocities which are bigger than the shear velocity \( c_{T2} \) for the supporting layer, will be named super-critical. The analysis was carried out for two depths of the tunnel \( (h = 10 \text{ m}, h = 20 \text{ m}) \) and two load frequencies \( (f_{\Omega} = 2 \text{ Hz}, f_{\Omega} = 4 \text{ Hz}) \). One should note that the load frequency \( f_{\Omega} = 2 \text{ Hz} \) is approximately equal to the eigenfrequency of the vertical vibrations of a wagon (Metrikine and Vrouwenvelder, 2000).

The Figs. 8–14 show the plots of the horizontal and vertical displacements for different values of the load velocity, load frequency, depth of the tunnel and different properties of the layers. The used set of parameters, allows to highlight the main features of the model and its relevance for the actual situations.
One can observe that the level of vibrations at the surface decreases and the vibrations vanish faster when the parameter $c_E$ is increasing. The displacement patterns become smoother, the sinusoidal character of the shapes vanishes and the magnitude of the displacement at the discussed point $(0,0,t)$ decreases, for both velocities: sub-critical velocity $50$ m/s, when $f_X = 2$ Hz, and super-critical velocity $100$ m/s for the load frequencies $f_X = 2$ Hz and $f_X = 4$ Hz. This feature changes in the case of sub-critical velocity when the depth $h$ increases together with the load frequency (Fig. 10(b) and (d), and 12(b) and (d)). The parametrical study shows that the vibrations at the surface can be effectively diminished by increasing the stiffness or the thickness of the upper layer for most of the considered variations of the parameters.

The case of the velocity $V = 80$ m/s (Fig. 14(a) and (b)) shows significant changes in the shapes of the plots suggesting that this value might be placed very close to the critical velocity for this system of parameters. One can observe that the vibrations of the surface last for a relatively long period of time, when compared to the sub-critical cases (Figs. 6, 7, 8(a) and (b), 9(a) and (b), 10 and 12). However, the increase of the parameter $c_E$ influences the magnitude of the vibrations in a similar way to the cases with sub-critical and super-critical velocities (Figs. 8–13). The numerical simulations show that the strongest suppression of vibrations appears for a fixed value of $c_\rho$ and increasing $c_E$ whereas the magnitude increases very slowly with constant $c_E$ and increasing $c_\rho$. 

![Graphs showing horizontal displacement $u$ for different values of $n$.](image-url)

Fig. 4. The approximation $u_n$ of the horizontal displacement $u$ for: (a) $n = 0$, (b) $n = 1$, (c) $n = 2$, (d) $n = 3$, (e) $n = 4$. 
Figs. 6–13 show the Doppler effect appearing in the moving load system for the sub-critical velocities. The wave patterns have a smaller amplitude but higher frequency for $t < 0$, which means that the wave has a higher frequency when the load is moving towards the observation point and the frequency becomes smaller when the...
distance between the excitation and the observation point grows. This effect vanishes when the thickness of the upper layer becomes smaller and the load frequency increases (Fig. 10(a) and (c), 11(a), 12(a), (c), 13(a)) or the thickness $h$ increases along with decreasing the load frequency $X$ in the case of the super-critical velocity 100 m/s (Figs. 8d and 9d).

The pattern of vibration for the horizontal displacement (Figs. 6, 8, 10 and 11) is more complex than for the vertical one (Figs. 7, 9, 12 and 13) however the vertical vibrations are more important due to their amplitude level. One should note that, according to the model formulation, the beam vibrates only vertically and therefore the extraction of the energy by the vibrating beam is not significant in the horizontal plane.

One can see (Figs. 8–13) that a higher load frequency $f$ produces a more complicated pattern of the response but the amplitude of the vibrations is smaller. This fact is the consequence of the significant part of the spectrum being placed at higher frequencies.

The analysis of the system’s behaviour by using the wavelet approximation can be extended further in order to carry out a complex parametrical analysis for the validation of results obtained with other methods.
Fig. 9. The vertical displacement in the case of the load frequency $\Omega = 4\pi$ and different load velocities for $c_p = 1, c_E = 0.75$ (dashed) and $c_p = 1, c_E = 1.3$ (solid): (a) and (c) $h = 10$ m, (b) and (d) $h = 20$ m.

Fig. 10. The horizontal displacement in the case of the load frequency $\Omega = 8\pi$ and sub-critical velocity: (a) $c_p = 1, c_E = 0.75$ and $h = 10$ m, (b) $c_p = 1, c_E = 0.75$ and $h = 20$ m, (c) $c_p = 1, c_E = 1.3$ and $h = 10$ m, (d) $c_p = 1, c_E = 1.3$ and $h = 20$ m.
8. Critical velocity

The analysis of the critical velocities is an important aspect for the moving load problems. Two important critical velocities appear when studying the problems of the loads moving at the surface: the velocity of Rayleigh surface wave in the ground and the minimal phase velocity of bending waves propagating in the track (Metrikine and Vrouwenvelder, 2000; Krylov, 2001). In the soft ground both of these velocities become relatively low and can be easily exceeded by the present operated trains. It has been shown (Krylov, 1995, 2001) that for the trains moving with the velocity higher than the Rayleigh velocity \( c_R \) of the supporting medium, one can observe a ground vibration boom and this phenomenon was confirmed experimentally (Madshus and Kaynia, 1998; Krylov, 2001). Different studies show that for some types of soil, an increase in the train speed from 140 to 180 km/h caused an increase of ten times in the ground vibrations, and large track deflections can be observed when the train exceeds the track critical velocity. This response can lead to further changes in the

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Fig. 11. The horizontal displacement in the case of the load frequency \( \Omega = 8\pi \) and super-critical velocity for \( c_p = 1, c_E = 0.75 \) (dashed) and \( c_p = 1, c_E = 1.3 \) (solid): (a) \( h = 10 \) m, (b) \( h = 20 \) m.

Fig. 12. The vertical displacement in the case of the load frequency \( \Omega = 8\pi \) and sub-critical velocity: (a) \( c_p = 1, c_E = 0.75 \) and \( h = 10 \) m, (b) \( c_p = 1, c_E = 0.75 \) and \( h = 20 \) m, (c) \( c_p = 1, c_E = 1.3 \) and \( h = 10 \) m, (d) \( c_p = 1, c_E = 1.3 \) and \( h = 20 \) m.
dynamic behaviour of the system which can be a very serious problem for the operating train systems and for the safety of passengers, reflecting the need for further studies.

In previous studies (Metrikine and Vrouwenvelder, 2000), the minimal phase velocity for a moving load on a beam with a finite thickness of 12 m for the supporting layer was determined to be approximately 68 m/s for the same parameters discussed in this work. It has been shown that for this critical velocity the maximum amplitude displacement is reached for both components: horizontal and vertical.

When a moving load with relatively high harmonic frequency reaches the Rayleigh velocity, the displacements show high variations which prevent a clear analysis of the critical velocities. An extension of this study could be carried out in the frequency domain. The dynamics near critical velocities, show the importance of the wavelet method, along with an appropriate order of the coiflet approximation which could capture the high variations of the response.

The variations of the displacements presented in this work, suggest that the critical velocities are close to the Rayleigh velocity of the supporting layer, which is in agreement with previously published results (Kononov and Wolfert, 2000; Metrikine and Vrouwenvelder, 2000; Krylov, 2001).

9. Conclusions

A two-dimensional theoretical model of the surface vibration due to a point load moving along a beam in the layer has been investigated in this paper. The infinitely long beam is placed inside the solid parallel to the surface and the lower layer is considered to be a half space.

A special approximation method based on the wavelet theory has been used for calculation of the horizontal and vertical components of the displacement vector. The use of the wavelet approximation allowed a complex parametrical dynamic analysis of the model for harmonic types of loads. Extensive simulations for
different sets of parameters show that the numerical integration is, in some cases, much more computational extensive than the analytical wavelet approach.

The analytical solutions in the integral form, for the horizontal and vertical components of displacement, as well as for the amplitude spectra of displacement, have been derived. The level of amplitude of vibrations at the surface has been analysed depending on the mass density and Young’s modulus of the solid space. Further extended parametrical analysis can be carried out by using the wavelet approximation in order to characterise a dynamic behaviour of the system.

The two-dimensional model, presented in this paper, allows to simplify the parametric analysis of the system, especially for the vertical displacement estimation. Possible extension of this work could be:

– more complex models of the tunnel together with more realistic assumptions for the properties of the layer and the beam;
– the introduction of a stochastic model of variation for the natural properties of the solid;
– the analysis in three dimensions, which would allow the study of the effects in the $y$ direction for more general applications (in this case, it might be possible to simplify the numerical calculations by using other systems of coordinates, e.g. the spherical coordinates).

The wavelet approach opens the possibility for investigation of more complicated systems and can be applied alternatively to numerical integration, especially due to the wavelets ability to represent more accurately the complex variations of the system behaviour.

References


