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On the Upper Bound of Eigenvalues for Elliptic Equations with Higher Orders*

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Let Ω be a bounded domain in R^m with piecewise smooth boundary. We consider the upper bound of the $(n+1)$th eigenvalue λ_{n+1} for the two problems

$$\begin{cases} (-\Delta)^l u = \lambda(-\Delta)^r u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1}}{\partial \nu^{l-1}} = 0, & x \in \partial\Omega \end{cases}$$

and

$$\begin{cases} P(-\Delta) u = \lambda(-\Delta)^r u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial v} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & x \in \partial\Omega, \end{cases}$$

where l and r are positive integers with $l > r$, $\mathbf{v}$ is the unit outward normal to $\partial\Omega$, and $P(t) = a_{l-r} t^l + a_{l-r-1} t^{l-1} + \cdots + a_1 t^{r+1}$ with the constant coefficients $a_{l-r} = 1$, $a_i \geq 0$ for $i = 1, 2, ..., l-r-1$. The bounds of λ_{n+1} are expressed in terms of the preceding eigenvalues. This generalizes the inequalities obtained by Payne, Polya, Weinberger, Protter, Hile, and Yeh. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $\Omega \subset R^m$ $(m \geqslant 2)$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. We consider the eigenvalue problems

$$\begin{cases} (-\Delta)^l u = \mu u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} (-\Delta)^l u = \lambda(-\Delta)^r u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots \dfrac{\partial^{l-1} u}{\partial v^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where $\mathbf{v}$ is the unit outward normal to $\partial\Omega$, l and r are positive integers with $l > r$. Let

$$0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots \leqslant \mu_n \leqslant \mu_{n+1} \leqslant \cdots,$$

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \lambda_{n+1} \leqslant \cdots$$

denote the successive eigenvalues for (1.1) and (1.2), respectively. There is a series of works on problem (1.1) (See [1–5].). But little information is known for problem (1.2). Hile and Yeh discussed the problem (1.2) in 1984 only for the special case where $l=2$, $r=1$, and $n=1$. They obtained the upper bound of λ_2 in terms of λ_1 (see [3]). We now consider problem (1.2) for the case where l and r are any positive integers with $l > r$. The upper bound of λ_{n+1} is formulated by the preceding eigenvalues. We also consider the same problem for the general case where

$$\begin{cases} P(-\Delta)\, u = \lambda(-\Delta)^r\, u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial \mathbf{v}} = \cdots = \dfrac{\partial^{l-1} u}{\partial \mathbf{v}^{l-1}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $P(t) = a_{l-r} t^l + a_{l-r-1} t^{l-1} + \cdots + a_1 t^{r+1}$, and $a_{l-r} = 1$, $a_i \geqslant 0$, $i = 1, 2, ..., l-r-1$, are constants. The method used here is somewhat different from [1–3]. This kind of problem is interesting and significant both in the theory of partial differential equations and in the application or potential application to mechanics and physics (see [6]).

Our main result are as follows:

THEOREM 1. *Let $m \geqslant 2$ and λ_1, λ_2 be the eigenvalues of (1.2) with $\lambda_1 \leqslant \lambda_2$. Then*

$$\lambda_2 \leqslant \frac{(m+2r)^2 + 4l(2l+m-2) - 4r(2r+m-2)}{(m+2r)^2}\, \lambda_1. \tag{1.4}$$

Remark 1. Choose $l=2$, $r=1$ in (1.4). We then obtain

$$\lambda_2 \leqslant \frac{m^2 + 8m + 20}{(m+2)^2}\, \lambda_1.$$

This is just Theorem 6 of [3]. Therefore, one might say that this paper is a proper generalization of [3].

THEOREM 2. *Let $m > 2 + 2r$, and λ_i $(i = 1, 2, ..., n+1)$ be the eigenvalues of (1.2). If $1 \leqslant n \leqslant 1 + (m-2)/2r$, then*

$$\lambda_{n+1} \leqslant \lambda_n + \frac{\begin{pmatrix} 4(2r+m-2)^2\, [l(2l+m-2+nr) \\ + r(2nr+nl-2r-m+2)] \end{pmatrix}}{n^2(m+2r)^2\, (2r+m-2nr-2)^2}$$

$$\cdot \left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)} \right) \left(\sum_{i=1}^{n} \lambda_i^{1-1/(l-r)} \right), \qquad \text{when} \quad n < \frac{2r+m-2}{2r+l}, \tag{1.5}$$

$$\lambda_{n+1} \leqslant \lambda_n + \frac{\begin{pmatrix} 4l(2r+m-2)^2\,[(2l+m-2) \\ \times(2r+m-2nr-2)+ln^2r] \end{pmatrix}}{n^2(m+2r)^2\,(2r+m-2nr-2)^3}$$

$$\cdot\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)\left(\sum_{i=1}^{n} \lambda_i^{1-1/(l-r)}\right), \qquad when \quad n \geqslant \frac{2r+m-2}{2r+l}. \tag{1.5$'$}$$

Remark 2. Choose $r=0$ in (1.5) and (1.5)′ for $m \geqslant 2$, $n \geqslant 1$. We obtain

$$\lambda_{n+1} \leqslant \lambda_n + \frac{4}{m^2n^2}\, l(2l+m-2)\left(\sum_{i=1}^{n} \lambda_i^{1/l}\right)\left(\sum_{i=1}^{n} \lambda_i^{1-1/l}\right), \tag{1.6}$$

Furthermore, replacing λ_i in (1.6) by λ_n yields

$$\lambda_{n+1} \leqslant \frac{1}{m^2}\,[m^2 + 4l(2l+m-2)]\,\lambda_n. \tag{1.7}$$

Inequalities (1.6) and (1.7) are just the results of Theorems 1 and 2 in [5]. Because [5] is the generalization of [1–4], we say therefore that this article is a generalization of [1–5].

THEOREM 3. *Let $m \geqslant 2$, and λ_1, λ_2 be the eigenvalues of* (1.3). *Then*

$$\lambda_2 \leqslant \left[1 + \frac{4}{(m+2r)^2} \cdot\left(\sum_{t=r+1}^{l} t(2t+m-2)\, a_{t-r} \lambda_1^{(t-1)/(l-r)} - r(2r+m-2)\right)\right]\lambda_1. \tag{1.8}$$

Remark 3. Taking $a_i = 0$ for $i = 1, 2, ..., l-1$ in (1.8) yields (1.4). Hence, this theorem is a generalization of Theorem 1.

THEOREM 4. *Let $m > 2r+2$, and λ_i $(i = 1, 2, ..., n+1)$ be the eigenvalues of* (1.3). *Then*

$$\lambda_{n+1} \leqslant \lambda_n + \frac{4(2r+m-2)^2}{n^2(m+2r)^2\,(2r+m-2nr-2)^2}\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)$$

$$\cdot\left[\sum_{t=r+1}^{l}\sum_{i=1}^{n} ta_{t-r}(2t+m-2+nr\lambda_i^{(t-l)/(l-r)})\,\lambda_i^{(t-r-1)/(l-r)}\right.$$

$$\left. + r\left(2nr-2r-m+2+n\sum_{t=r+1}^{l} ta_{t-r}\right)\left(\sum_{i=1}^{n} \lambda_i^{1-1/(l-r)}\right)\right],$$

$$when \quad n < \frac{2r+m-2}{2r+\sum_{t=r+1}^{l} ta_{t-r}}, \tag{1.9}$$

$$\lambda_{n+1} \leqslant \lambda_n + \frac{4(2r+m-2)^2}{n^2(m+2r)^2\,(2r+m-2nr-2)^3}\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)$$

$$\cdot \sum_{t=r+1}^{l} \sum_{i=1}^{n} t a_{t-r} \lambda_i^{(t-r-1)/(l-r)} \left[(2t+m-2)(2r+m-2nr-2)\right.$$

$$\left. + n^2 r \sum_{t=r+1}^{l} t a_{t-r} \lambda_i^{(t-l)/(l-r)}\right],$$

$$\text{when} \quad n \geqslant \frac{2r+m-2}{2r+\sum_{t=r+1}^{l} t a_{t-r}}. \tag{1.9$'$}$$

Remark 4. Taking $a_i = 0$ for $i = 1, 2, ..., l-1$ in (1.9) and (1.9)′, one can get (1.5) and (1.5)′, respectively. Choosing $r = 0$ in (1.8) and (1.9) (or (1.9)′), then (17) of [5] follows. Therefore, Theorem 3 and Theorem 4 are further generalizations of Theorem 1 and Theorem 2 respectively. Moreover, all the results in [1–5] can be obtained from Theorem 3 and Theorem 4 by choosing the proper parameters in these two theorems.

The proof of our main results is based on the variational formula. First of all, we construct some test functions and then use the Rayleigh theorem to obtain a basic inequality. Secondly, for clearness, we divide the proof into four lemmas. At last, the main results turn out immediately.

2. Proof of Theorems

We only give the proof of Theorem 1 and Theorem 2. For Theorem 3 and Theorem 4, the proofs are similar but rather complicated in calculation. We take the mn trial functions such that

$$\varphi_{ik} = x_k u_i - \sum_{j=1}^{n} a_{ij}^k u_j, \tag{2.1}$$

where $i = 1, 2, ..., n$, $k = 1, 2, ..., m$, $x = (x_1, x_2, ..., x_n) \in \Omega$, and the constants a_{ij}^k are defined by

$$a_{ij}^k = \int x_k u_i (-\Delta)^r u_j \, dx, \qquad i, j = 1, 2, ..., n, \quad k = 1, 2, ..., m.$$

Here and throughout the notation $\int$ is used for $\int_\Omega$. Suppose that the eigenfunction u_i $(i = 1, 2, ...)$ of (1.2) corresponding to the eigenvalue λ_i is weighted orthogonal to u_j such that

$$\int u_i (-\Delta)^r u_j \, dx = \delta_{ij}, \qquad i, j = 1, 2, ...$$

It is easy to see that

$$\int u_i(-\Delta)^r u_j\,dx = \int \nabla^r u_i \cdot \nabla^r u_j\,dx = \int u_j(-\Delta)^r u_i\,dx$$
$$= \delta_{ij}, \qquad i, j = 1, 2, \ldots. \tag{2.2}$$

From (2.1) and (2.2), we know that φ_{ik} is weighted orthogonal to u_j $(j = 1, 2, \ldots, n)$, i.e.,

$$\int \varphi_{ik}(-\Delta)^r u_j\,dx = \int u_j(-\Delta)^r \varphi_{ik}\,dx = 0.$$

Moreover, $\varphi_{ik} = \partial\varphi_{ik}/\partial\nu = \cdots = \partial^{l-1}\varphi_{ik}/\partial\nu^{l-1} = 0$ on $\partial\Omega$. Hence, we can use the well-known Rayleigh theorem to obtain

$$\lambda_{n+1} \leqslant \int \varphi_{ik}(-\Delta)^l \varphi_{ik}\,dx \bigg/ \int |\nabla^r \varphi_{ik}|^2\,dx. \tag{2.3}$$

To obtain a more explicit inequality between λ_{n+1} and λ_n from (2.3), we give a further calculation to the right-hand side of (2.3) as

$$a_{ij}^k = \int x_k u_i(-\Delta)^r u_j\,dx = \int x_k \nabla^r u_i \cdot \nabla^r u_j\,dx + r\int \nabla^r u_j \cdot \nabla^{r-2} u_{i,x_k}\,dx$$
$$= \int x_k \nabla^r u_i \cdot \nabla^r u_j\,dx + r\int \nabla^{r-1} u_i \cdot \nabla^{r-1} u_{j,x_k}\,dx. \tag{2.4}$$

Here and throughout the notation u_{i,x_k} denotes $\partial u_i/\partial x_k$. By a straightforward calculation, it yields

$$(-\Delta)^l \varphi_{ik} = (-\Delta)^l \left(x_k u_i - \sum_{j=1}^n a_{ij}^k u_j\right)$$
$$= \lambda_i x_k (-\Delta)^r u_i - 2l(-\Delta)^{l-1} u_{i,x_k}$$
$$- \sum_{j=1}^n a_{ij}^k \lambda_j (-\Delta)^r u_j.$$

It is easy to see that

$$\int |\nabla^r \varphi_{ik}|^2\,dx = \int \varphi_{ik}(-\Delta)^r \varphi_{ik}\,dx$$
$$= \int x_k \varphi_{ik}(-\Delta)^r u_i\,dx - 2r\int \varphi_{ik}(-\Delta)^{r-1} u_{i,x_k}\,dx.$$

Noticing the weighted orthogonality of φ_{ik} with u_j, $j=1, 2, ..., n$, we obtain

$$\int \varphi_{ik}(-\Delta)^l \varphi_{ik}\, dx = \lambda_i \int |\nabla^r \varphi_{ik}|^2\, dx + 2r\lambda_i \int \varphi_{ik}(-\Delta)^{r-1} u_{i, x_k}\, dx - 2l \int \varphi_{ik}(-\Delta)^{l-1} u_{i, x_k}\, dx. \tag{2.5}$$

Let us define

$$I_{ik} = 2r\lambda_i \int \varphi_{ik}(-\Delta)^{r-1} u_{i, x_k}\, dx, \qquad J_{ik} = -2l \int \varphi_{ik}(-\Delta)^{l-1} u_{i, x_k}\, dx,$$

$$I = \sum_{k=l}^{m} \sum_{i=1}^{n} I_{ik}, \qquad J = \sum_{k=1}^{m} \sum_{i=1}^{n} J_{ik}.$$

By (2.5), we find

$$\sum_{k=1}^{m} \sum_{i=1}^{n} \int \varphi_{ik}(-\Delta)^l \varphi_{ik}\, dx = \sum_{k=1}^{m} \sum_{i=1}^{n} \lambda_i \int |\nabla^r \varphi_{ik}|^2\, dx + I + J.$$

Denoting $S = \sum_{k=1}^{m} \sum_{i=1}^{n} \int |\nabla^r \varphi_{ik}|^2\, dx$, then by (2.3) it holds that

$$\lambda_{n+1} S \leqslant \sum_{k=1}^{m} \sum_{i=1}^{n} \lambda_i \int |\nabla^r \varphi_{ik}|^2\, dx + I + J. \tag{2.6}$$

Replacing λ_i in (2.6) by λ_n yields

$$(\lambda_{n+1} - \lambda_n)\, S \leqslant I + J, \tag{2.7}$$

which is the basic inequality we need. Now, it is sufficient to estimate S, I, and J for obtaining the main results. We would like to divide the following proof, for clearness, into four lemmas.

LEMMA 1. *Suppose that u_i $(i = 1, 2, ...)$ is the eigenfunction of* (1.2) *corresponding to the eigenvalue λ_i. Then*

(a) $\int |\nabla^{r+p} u_i|^2\, dx \leqslant \lambda_i^{p/(l-r)}$, $p = 1, 2, ..., l-r$,

(b) $\int |\nabla^{r-1} u_i|^2\, dx \geqslant \lambda_i^{1/(r-l)}$.

Proof. By induction, it is easy to show that

$$\int |\nabla^{r+p} u_i|^2\, dx \leqslant \left(\int |\nabla^{r+p+1} u_i|^2\, dx \right)^{p/(p+1)}, \quad p = 1, 2, ..., l-r-1. \tag{2.8}$$

Using (2.8) inductively yields

$$\int |\nabla^{r+p} u_i|^2\,dx \leqslant \left(\int |\nabla^{r+p+1} u_i|^2\right)^{p/(p+1)} \leqslant \cdots \leqslant \left(\int |\nabla^l u_i|^2\,dx\right)^{p/(l-r)}.$$

From (1.2) we know that

$$\lambda_i = \int u_i(-\Delta)^l u_i\,dx = \int |\nabla^l u_i|^2\,dx, \qquad i = 1, 2, \ldots. \tag{2.9}$$

Therefore, $\int |\nabla^{r+p} u_i|^2\,dx \leqslant \lambda_i^{p/(l-r)}$.

(b) We have

$$1 = \left(\int |\nabla^r u_i|^2\,dx\right)^2 \leqslant \left(\int |\nabla^{r-1} u_i|^2\,dx\right)\left(\int |\nabla^{r+1} u_i|^2\,dx\right).$$

By (a), $\int |\nabla^{r+1} u_i|^2\,dx \leqslant \lambda_i^{1/(l-r)}$, which by combining with the above inequality gives $\int |\nabla^{r-1} u_i|^2\,dx \geqslant \lambda_i^{1/(r-l)}$. Q.E.D.

LEMMA 2. *Let u_i $(i = 1, 2, \ldots)$ be an eigenfunction of* (1.2) *corresponding to the eigenvalue λ_i. Then*

(a) $\sum_{k=1}^m \int |\nabla^t u_{i,x_k}|^2\,dx = \int |\nabla^{t+1} u_i|^2\,dx$, $t = 0, 1, 2, \ldots, l-1$;

(b) $-\sum_{k=1}^m \int x_k u_i(-\Delta)^t u_{i,x_k}\,dx = (1/2)(2t+m)\int |\nabla^t u_i|^2\,dx$, $t = 0, 1, 2, \ldots, l-1$.

Proof. (a) Integrating by parts, we find

$$\begin{aligned}\sum_{k=1}^m \int |\nabla^t u_{i,x_k}|^2\,dx &= \sum_{k=1}^m \int u_{i,x_k}(-\Delta)^t u_{i,x_k}\,dx \\ &= -\sum_{k=1}^m \int u_{i,x_k x_k}(-\Delta)^t u_i\,dx = -\int \Delta u_i(-\Delta)^t u_i\,dx \\ &= \int |\nabla^{t+1} u_i|^2\,dx.\end{aligned}$$

(b) Similarly, we have

$$\begin{aligned}-\int x_k u_i(-\Delta)^t u_{i,x_k}\,dx &= \int u_i(-\Delta)^t u_i\,dx + \int x_k u_{i,x_k}(-\Delta)^t u_i\,dx \\ &= \int |\nabla^t u_i|^2\,dx + \int x_k u_i(-\Delta)^t u_{i,x_k}\,dx \\ &\quad - 2t\int u_i(-\Delta)^{t-1} u_{i,x_k x_k}\,dx.\end{aligned}$$

Then,

$$-\int x_k u_i(-\Delta)^t u_{i,x_k}\,dx=\tfrac{1}{2}\int|\nabla^t u_i|^2\,dx+t\int|\nabla^{t-1}u_{i,x_k}|^2\,dx. \quad (2.10)$$

Using (a), we obtain

$$-\sum_{k=1}^{m}\int x_k u_i(-\Delta)^t u_{i,x_k}\,dx=\frac{1}{2}(2t+m)\int|\nabla^t u_i|^2\,dx. \qquad \text{Q.E.D.}$$

Lemma 3. *Let λ_1 be the first eigenvalue of* (1.2). *We then have*

(a) $I+J\leqslant[l(2l+m-2)-r(2r+m-2)]\,\lambda_1^{1-1/(l-r)}$.

(b) $S\geqslant((m+2)^2/4)\,\lambda_1^{1/(r-l)}$, *where* $S=\sum_{k=1}^{m}\int|\nabla^r\varphi_k|^2\,dx$.

Proof. (a) Choose $\varphi_k=x_k u_1$ $(k=1,2,\ldots,m)$ and take a coordinate transformation such that

$$\int x_k\,|\nabla^r u_1|^2\,dx=0, \qquad \text{for} \quad k=1,2,\ldots,m;$$

then, φ_k is weighted orthogonal to u_1 with

$$\varphi_k=\frac{\partial\varphi_k}{\partial\nu}=\cdots=\frac{\partial^{l-1}\varphi_k}{\partial\nu^{l-1}}=0 \qquad \text{on } \partial\Omega.$$

Hence, by the Rayleigh theorem, $\lambda_2\leqslant\int\varphi_k(-\Delta)^l\varphi_k\,dx/\int|\nabla^r\varphi_k|^2\,dx$ and then

$$I=2r\lambda_1\sum_{k=1}^{m}\int\varphi_k(-\Delta)^{r-1}u_{1,x_k}\,dx, \qquad J=-2l\sum_{k=1}^{m}\int\varphi_k(-\Delta)^{l-1}u_{1,x_k}\,dx.$$

By the definition of φ_k, Lemma 2(b), we find

$$I=-r(2r+m-2)\lambda_1\int|\nabla^{r-1}u_1|^2\,dx, \qquad J=l(2l+m-2)\int|\nabla^{l-1}u_1|^2\,dx.$$

Using Lemma 1(a) for I and Lemma 1(b) for J, we get

$$I\leqslant-r(2r+m-2)\,\lambda_1^{1-1/(l-r)}, \quad (2.11)$$

$$J\leqslant l(2l+m-2)\,\lambda_1^{1-1/(l-r)}. \quad (2.12)$$

Adding (2.11) to (2.12), we see that Lemma 3(a) follows.

(b) Using (2.10), we have

$$\int\nabla^r\varphi_k\cdot\nabla^r u_{1,x_k}\,dx=\int x_k u_1(-\Delta)^r u_{1,x_k}\,dx$$

$$=-r\int|\nabla^{r-1}u_{1,x_k}|^2\,dx-\tfrac{1}{2}\int|\nabla^r u_1|^2\,dx.$$

Then, by Lemma 2(a) the following holds

$$\sum_{k=1}^{m} \int \nabla^r \varphi_k \cdot \nabla^r u_{1,x_k}\, dx = -\frac{1}{2}(2r+m) \int |\nabla^r u_1|^2\, dx = -\frac{m+2r}{2}.$$

By the Schwarz inequality, Lemma 2(a) and Lemma 1(a), we obtain

$$\frac{(m+2r)^2}{4} \leqslant \left(\sum_{k=1}^{m} \int |\nabla^r \varphi_k|^2\, dx\right)\left(\sum_{k=1}^{m} \int |\nabla^r u_{1,x_k}|^2\, dx\right)$$

$$= S \int |\nabla^{r+1} u_1|^2\, dx \leqslant S \lambda_1^{1/(l-r)},$$

i.e, $S \leqslant ((m+2r)^2/4)\, \lambda_1^{1/(r-l)}$. Q.E.D.

LEMMA 4. *Let λ_i, $i=1, 2, ..., n$, be the eigenvalues of* (1.2), *for* $2 \leqslant n < 1 + (m-2)/2r$; *we then have*

(a) $I + J \leqslant [l(2l + m - 2 + nr/\sigma) + r(2nr + nl\sigma - 2r - m + 2)] (\sum_{i=1}^{n} \lambda_i^{1-1(l-r)})$;

(b) $S \geqslant (n^2(2r+m-2nr-2)^2\, (m+r)^2/4(2r+m-2)^2)(\sum_{i=1}^{n} \lambda_i^{1/(l-r)})^{-1}$,

where σ is a any constant satisfying $0 < \sigma < (1/nl)(2r + m - 2nr - 2)$, *and* $S = \sum_{i=1}^{n} \sum_{k=1}^{m} \int |\nabla^r \varphi_{ik}|^2\, dx$.

Proof. (a) We already have

$$I_{ik} = 2r\lambda_i \int \varphi_{ik}(-\Delta)^{r-1}\, u_{i,x_k}\, dx$$

$$= 2r\lambda_i \int x_k u_i (-\Delta)^{r-1}\, u_{i,x_k}\, dx$$

$$-2r\lambda_i \sum_{j=1}^{n} a_{ij}^k \int u_j (-\Delta)^{r-1}\, u_{i,x_k}\, dx.$$

Let us define

$$b_{ij}^k = \int x_k \nabla^r u_i \cdot \nabla^r u_j\, dx, \qquad c_{ij}^k = r \int \nabla^{r-1} u_i \cdot \nabla^{r-1} u_{j,x_k}\, dx, \tag{2.13}$$

then, $b_{ij}^k = b_{ji}^k$, $c_{ij}^k = -c_{ji}^k$, and, by (2.4), $a_{ij}^k = b_{ij}^k + c_{ij}^k$, for $i, j = 1, 2, ..., n$; $k = 1, 2, ..., m$, which with the fact that $\int u_j \Delta^{r-1} u_{i,x_k}\, dx = -\int u_i \Delta^{r-1} u_{j,x_k}\, dx$ yields

$$\sum_{i,j=1}^{n} b_{ij}^k \int u_j \Delta^{r-1} u_{i,x_k}\, dx = 0, \tag{2.14}$$

and then

$$I=\sum_{k=1}^{m}\sum_{i=1}^{n} I_{ik}=2r\sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i\int x_k u_i(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$-2r\sum_{k=1}^{m}\sum_{i,j=1}^{n}\lambda_i c_{ij}^k\int u_j(-\Delta)^{r-1}u_{i,x_k}\,dx=I_1+I_2.$$

By Lemma 2(b),

$$I_1=2r\sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i\int x_k u_i(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$=-r(2r+m-2)\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_i|^2\,dx. \tag{2.15}$$

Using the definition of c_{ij}^k, the Schwarz inequality, and Lemma 2(a), we obtain

$$I_2=-2r\sum_{k=1}^{m}\sum_{i,j=1}^{n}\lambda_i c_{ij}^k\int u_j(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$=2r^2\sum_{i,j=1}^{n}\lambda_i\sum_{k=1}^{m}\left(\int\nabla^{r-1}u_i\cdot\nabla^{r-1}u_{j,x_k}\,dx\right)^2$$
$$\leqslant 2r^2\sum_{i,j=1}^{n}\lambda_i\sum_{k=1}^{m}\left(\int|\nabla^{r-1}u_i|^2\,dx\right)\left(\int|\nabla^{r-1}u_{j,x_k}|^2\,dx\right)$$
$$=2r^2\sum_{i,j=1}^{n}\lambda_i\left(\int|\nabla^{r-1}u_i|^2\,dx\right)\left(\int|\nabla^{r}u_j|^2\,dx\right)$$
$$=2nr^2\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_i|^2\,dx.$$

Combining (2.16) with (2.15) yields

$$I=I_1+I_2\leqslant r(2nr-2r-m+2)\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_j|^2\,dx. \tag{2.17}$$

Similarly,

$$J=J_1+J_2\leqslant l\left(2l+m-2+\frac{nr}{\sigma}\right)\sum_{i=1}^{n}\lambda_i^{1-1/(l-r)}$$
$$+\,lnr\sigma\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_i|^2\,dx. \tag{2.18}$$

Therefore,

$$I+J \leqslant r(2nr-2r-m+2+ln\sigma)\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_i|^2\,dx$$
$$+l\left(2l+m-2+\frac{nr}{\sigma}\right)\sum_{i=1}^{n}\lambda_i^{1-1/(l-r)}. \tag{2.19}$$

Because $n<1+(m-2)/2r$, we choose $\sigma<(1/ln)(2r+m-2-2nr)$ and then

$$2nr-2r-m+2+ln\sigma<0.$$

By Lemma 1(b),

$$r(2nr-2r-m+2+ln\sigma)\sum_{i=1}^{n}\lambda_i\int|\nabla^{r-1}u_i|^2\,dx$$
$$\leqslant r(2nr-2r-m+2+ln\sigma)\sum_{i=1}^{n}\lambda_i^{1-1/(l-r)}. \tag{2.20}$$

Substituting (2.20) into (2.19), we see that Lemma 4(a) then follows.

(b) We have the calculation

$$\int\nabla^r\varphi_{ik}\cdot\nabla^{r-2}u_{i,x_k}\,dx=-\int\varphi_{ik}(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$=-\int x_k u_i(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$+\sum_{j=1}^{n}a_{ij}^k\int u_j(-\Delta)^{r-1}u_{i,x_k}\,dx.$$

Using (2.14), we obtain

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\int\nabla^r\varphi_{ik}\cdot\nabla^{r-2}u_{i,x_k}\,dx=-\sum_{k=1}^{m}\sum_{i=1}^{n}\int x_k u_i(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$+\sum_{k=1}^{m}\sum_{i=1}^{n}c_{ij}^k\int u_j(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$=Q_1+Q_2. \tag{2.21}$$

Using Lemma 2(b), one can get

$$Q_1=-\sum_{k=1}^{m}\sum_{i=1}^{n}\int x_k u_i(-\Delta)^{r-1}u_{i,x_k}\,dx$$
$$=\frac{2r+m-2}{2}\sum_{i=1}^{n}\int|\nabla^{r-1}u_i|^2\,dx. \tag{2.22}$$

With the help of (2.13), the Schwarz inequality, and Lemma 2(a), we find

$$\begin{aligned}
|Q_2| &= \left| \sum_{k=1}^{m} \sum_{i,j=1}^{n} c_{ij}^k \int u_j(-\Delta)^{r-1} u_{i,x_k}\, dx \right| \\
&= \left| \sum_{k=1}^{m} \sum_{i,j}^{n} c_{ij}^k \int \nabla^{r-1} u_j \cdot \nabla^{r-1} u_{i,x_k}\, dx \right| \\
&= r \sum_{i,j=1}^{n} \sum_{k=1}^{m} \left(\int \nabla^{r-1} u_i \cdot \nabla^{r-1} u_{j,x_k}\, dx \right)^2 \\
&\leqslant r \sum_{i,j=1}^{n} \sum_{k=1}^{m} \left(\int |\nabla^{r-1} u_i|^2\, dx \right) \left(\int |\nabla^{r-1} u_{j,x_k}|^2\, dx \right) \\
&= nr \sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\, dx.
\end{aligned} \tag{2.23}$$

Because $n < 1 + (m-2)/2r$, then $2r + m - 2 - 2nr > 0$. By (2.22), (2.23), and (2.21), one can find

$$\begin{aligned}
0 &< \frac{1}{2}(2r + m - 2nr - 2) \sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\, dx \\
&\leqslant Q_1 + Q_2 = \sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^{r-2} u_{i,x_k}\, dx.
\end{aligned} \tag{2.24}$$

Using the Schwarz inequality and Lemma 2(a) on the right-hand side of (2.24), we obtain

$$\begin{aligned}
&\frac{1}{4}(2r + m - 2nr - 2)^2 \left(\sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\, dx \right)^2 \\
&\quad\leqslant \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^{r-2} u_{i,x_k}\, dx \right)^2 \\
&\quad\leqslant \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \int |\nabla^r \varphi_{ik}|^2\, dx \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \int |\nabla^{r-2} u_{i,x_k}|^2\, dx \right) \\
&\quad\leqslant \left(\sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\, dx \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \int |\nabla^r \varphi_{ik}|^2\, dx \right).
\end{aligned}$$

Then

$$\sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\, dx \leqslant \frac{4S}{(2r + m - 2nr - 2)^2}. \tag{2.25}$$

Integrating by parts and using the definition of φ_{ik}, we get

$$\int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx = \int \varphi_{ik}(-\Delta)^r u_{i,x_k}\,dx$$
$$= \int x_k u_i(-\Delta)^r u_{i,x_k}\,dx - \sum_{j=1}^{n} a_{ij}^k \int u_j(-\Delta)^r u_{i,x_k}\,dx.$$

By virtue of (2.14), we have

$$\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx = \sum_{k=1}^{m} \sum_{i=1}^{n} \int x_k u_i(-\Delta)^r u_{i,x_k}\,dx$$
$$- \sum_{k=1}^{m} \sum_{i,j=1}^{n} c_{ij}^k \int u_j(-\Delta)^r u_{i,x_k}\,dx$$
$$= R_1 + R_2. \tag{2.26}$$

With the help of Lemma 2(b) and (2.2), one can get

$$R_1 = \sum_{k=1}^{m} \sum_{i=1}^{n} \int x_k u_i(-\Delta)^r u_{i,x_k}\,dx = -nr - \frac{mn}{2}. \tag{2.27}$$

Similar to the estimate for Q_2, we have

$$|R_2| \leqslant nr\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)^{1/2} \left(\sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\right)^{1/2}. \tag{2.28}$$

Combining (2.27), (2.28), and (2.26) yields

$$\left|\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx + \frac{mn}{2} + nr\right|$$
$$= |R_2| \leqslant nr\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)^{1/2} \left(\sum_{i=1}^{n} \int |\nabla^{r-1} u_i|^2\,dx\right)^{1/2}. \tag{2.29}$$

Substituting (2.25) into the right-hand side of (2.29) and using the inequality that $|a| \leqslant |b| + |a+b|$, we obtain

$$nr + \frac{mn}{2} \leqslant \left|\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx\right| + |R_2|$$
$$\leqslant \left|\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx\right| + \frac{2nr\,(\sum_{i=1}^{n} \lambda_i^{1/(l-r)})^{1/2}}{2r+m-2-2nr} S^{1/2}. \tag{2.30}$$

For the first term on the right-hand side of (2.30), using the Schwarz inequality, Lemma 2(a), and Lemma 1(a) we find

$$\left|\sum_{k=1}^{m} \sum_{i=1}^{n} \int \nabla^r \varphi_{ik} \cdot \nabla^r u_{i,x_k}\,dx\right| \leqslant S^{1/2}\left(\sum_{i=1}^{n} \lambda_i^{1/(l-r)}\right)^{1/2}.$$

Substituting the above inequality into (2.30) gives

$$nr+\frac{mn}{2}\leqslant\frac{2r+m-2}{2r+m-2-2nr}\left(\sum_{i=1}^{n}\lambda_i^{1/(l-r)}\right)^{1/2}S^{1/2},$$

i.e.,

$$S\geqslant\frac{n^2(2r+m-2-2nr)^2}{4(2r+m-2)^2}(m+2r)^2\left(\sum_{i=1}^{n}\lambda_i^{1/(l-r)}\right)^{-1}.\qquad \text{Q.E.D.}$$

Proof of Theorem 1. Substituting Lemma 3 into (2.7), then (1.4) follows. Q.E.D.

Proof of Theorem 2. Substituting Lemma 4 into (2.7) yields

$$\lambda_{n+1}\leqslant\lambda_n+\frac{\begin{pmatrix}4(2r+m-2)^2\,[l(2l+m-2+nr/\sigma)\\ +r(2nr+nl\sigma-2r-m+2)]\end{pmatrix}}{n^2(m+2r)^2\,(2r+m-2nr-2)^2}\cdot\left(\sum_{i=1}^{n}\lambda_i^{1/(l-r)}\right)\left(\sum_{i=1}^{n}\lambda_i^{1-(l-r)}\right),\qquad(2.31)$$

where $0<\sigma<(1/nl)(2r+m-2nr-2)$. Minimizing the right-hand side of (2.31) at $\sigma=1$ when $n<(2r+m-2)/(2r+l)$ and at $\sigma=(2r+m-2nr-2)/nl$ when $n\geqslant(2r+m-2)/(2r+l)$, then (1.5) and (1.5)′ follow, respectively. Q.E.D.

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