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Note

On the Dependence of Functions on Their Variables

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Let $f: A_1 \times \cdots \times A_n \to A$ be a function of n variables, where $n \geqslant 2$. We say that f depends on the variable x_i iff there exist two sequence $(a_1,..., a_n)$, $(a_1,..., a_{i-1}, a, a_{i+1},..., a_n) \in A_1 \times \cdots \times A_n$ such that $f(a_1,..., a_n) \neq f(a_1,..., a_{i-1}, a, a_{i+1},..., a_n)$.

THEOREM. ***If f depends on all its variables, then there exist i, j, $a \in A_i$, and $b \in A_j$ such that $i \neq j$ and the functions $f(x_1,..., x_{i-1}, a, x_{i+1},..., x_n)$ and $f(x_1,..., x_{j-1}, b, x_{j+1},..., x_n)$ depend on all their $n-1$ variables.***

Proof. We let $x_i \leqslant_a x_j$ iff $a \in A_i$ and $f(x_1,..., x_{i-1}, a, x_{i+1},..., x_n)$ does not depend on x_j. Let us show that

(A) if $i \neq j$, $x_i \leqslant_a x_j$, and $x_j \leqslant_b x_k$, then $x_i \leqslant_a x_k$.

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In fact, if $f(x_1,..., x_{i-1}$, a, $x_{i+1},...,$ x,) does not depend on x_j and j $\neq i$, then

$$f(x_1,..., x_{i-1}, a, x_{i+1},..., x_n)$$
$$= f(x_1,..., \text{Xi-}_1, a, x_{i+1},..., x_{j-1}, b, x_{j+1},..., x_n).$$

But the right side does not depend on x_k if $x_j \leqslant_b x_k$, so neither does the left side, i.e., $x_i \leqslant_a x_k$. Thus, (A) is true.

Now let $x_i \leqslant x_j$ iff $x_i \leqslant_a x_j$ for some a $\in$ A $_i$. Clearly, $x_i \leqslant_a x_i$ for all a $\in$ A $_i$ so $\leqslant$ is reflexive, and $\leqslant$ is transitive by (A). We say that x_i can be frozen iff there is some a $\in A_i$ such that $f(x_1,..., x_{i-1}$, a, $x_{i+1},...,$ x,) depends on all its $n-1$ variables. Next we prove

(B) if $x_i \leqslant x_j$, $i \neq j$, and $x_k \leqslant x_j$ whenever $x_i \leqslant x_k$, then x_i can be frozen.

For any i, j let S_{ij} = (a: $x_i \leqslant_a x_j\}$. Notice that x_i can be frozen iff $\bigcup_{k\neq i} S_{ik} \neq A_i$. Assume x_i, x_j satisfy the hypothesis of (B). Let $k \neq i$. If $x_i \nleqslant x_k$, then $S_{ik} = \varnothing$, and if $x_i \leqslant x_k$, then $x_k \leqslant x_j$, so by (A), $S_{ik} \subseteq S_{ij}$. Consequently, $\bigcup_{k\neq i} S_{ik} = S_{ij}$. If $S_{ij} = A_i$, then f does not depend on x_j, contrary to the hypothesis of the theorem. Thus, x_i can be frozen.

Say that x_i is <-maximal iff $x_j \leqslant x_i$ whenever $x_i \leqslant x_j$. Then we prove

(C) each <-maximal variable can be frozen.

Let x_i be <-maximal. If $x_i \leqslant x_j$ for some j $\neq i$, then x_j is also S-maximal and it follows from (B) that x_i can be frozen. If $x_i \nleqslant x_j$ for all $j \neq i$, then $\bigcup_{j\neq i} S_{ij} = \varnothing \neq A_i$, and again x_i can be frozen.

Now we finish the proof of the theorem. If there are two or more $\leqslant$-maximal variables, then they can be frozen, but if there is only one $\leqslant$-maximal variable, then all variables can be frozen, by (B) and (C).

This theorem implies an affirmative answer to problem ($\mathbf{P}_1$) of [2] and to similar problems about some functions φ and ψ defined in [2, p. 284]. The theorem does not generalize to all f: ${}^{\omega}\{0,1\} \to \{0,1\}$. It cannot be improved to conclude that more than two variables can be frozen. To see this, definef: ${}^{n}\{0, 1\} \to \{0, 1\}$ as

$$\begin{aligned} f(a_1,..., \text{a,}) &= a_1, && \text{if} \quad a_3 = \cdots = a_n = 0, \\ &= a_2, && \text{if} \quad a_3 = \cdots = a_n = 1, \\ &= \sum_{i=3}^{n} \text{a, (mod 2)}, && \text{otherwise.} \end{aligned}$$

It is easy to check that f depends on all its variables and $x_i \leqslant x_j$ iff either $i = j$, or else, $i \in (3 ,..., n\}$ and j $\in \{1, 2\}$. The only variables which can be frozen are x_1 and x_2.

R be a preorder over the variables x, ,..., x_n, i.e., a reflexive and transitive relation. Then there is a function f: $^n\{0, 1\} \to (0, 1\}$ such that $x_i\ R\ x_j$ iff $x_i \leqslant x_j$. If R is a partial order, such a function may be obtained as follows: For any sequence a = (a, ,..., a,), let D_a = (j: if $x_i\ R\ x_j$, then $a_i = 1$ } and let f(a) = 0 if the cardinality of D_a is even, otherwise $f(a) = 1$. Suppose $x_i\ R\ x_j$. Then $x_i \leqslant_0 x_j$, for if a is a sequence with $a_i = 0$, thenj $\notin D_a$ and the value of f(u) cannot depend on a_j. Now suppose $x_i\ R\ x_j$ fails. Define a, a' $\in {}^n$ (0, 1 } by $a_k = 1$ iff $x_k\ R\ x_j$, and $a'_k = 1$ iff $x_k\ R\ x_j$ and $k \neq$ j. Then $a_k = a'_k$ whenever $k \neq$ j, and D_a has one more element than $D_{a'}$, so f(u) $\neq f(a')$. Thus, f depends on x_j, and a, = $a'_i = 0$ since $x_i\ R\ x_j$ fails, so, in fact, $x_i \nleqslant_0 x_j$. Define b, $b' \in {}^n$ (0, 1 } by $b_k = 1$ iff $k = i$ or $x_k\ R\ x_j$, and $b'_k = 1$ iff $k = i$ or $x_k\ R\ x_j$ and $k \neq$ j. Then $b_i = b'_i = 1$, $f(b) \neq f(b')$, $b_k = b'_k$ whenever $k \neq$ j, and so $x_i \nleqslant_1 x_j$. Hence, $x_i \nleqslant x_j$ whenever $x_i\ R\ x_j$ fails. Clearly, $\leqslant$ and R coincide, so f has the desired property. Thus, any partial order is isomorphic to the relation $\leqslant$ of some function. The relation $\leqslant$ of the product function $\prod_{i=1}^n x_i$ is universal, i.e., $x_i \leqslant x_j$ for all i, j. By a straightforward combination of product functions with the construction above, it can be shown that any preorder is isomorphic to the relation $\leqslant$ of some function.

Added in proof. **We learned recently that the theorem (restricted to Boolean functions) was announced without proof in [1], and another example showing that 2 is maximal is given there.**

REFERENCES

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