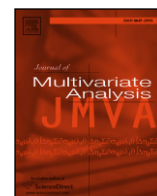




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journal homepage: www.elsevier.com/locate/jmvaOn the sensitivity of the one-sided t test to covariance misspecificationHuaizhen Qin^{a,b}, Alan T.K. Wan^{c,*}, Guohua Zou^d^a Department of Biostatistics and Medical Informatics, University of Wisconsin, Madison, WI 53706, USA^b Department of Statistics, University of Wisconsin, Madison, WI 53706, USA^c Department of Management Sciences, City University of Hong Kong, Kowloon, Hong Kong^d Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

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ABSTRACT

Sensitivity analysis stands in contrast to diagnostic testing in that sensitivity analysis aims to answer the question of whether it matters that a nuisance parameter is non-zero, whereas a diagnostic test ascertains explicitly if the nuisance parameter is different from zero. In this paper, we introduce and derive the finite sample properties of a sensitivity statistic measuring the sensitivity of the t statistic to covariance misspecification. Unlike the earlier work by Banerjee and Magnus [A. Banerjee, J.R. Magnus, On the sensitivity of the usual t - and F -tests to covariance misspecification, *Journal of Econometrics* 95 (2000) 157–176] on the sensitivity of the F statistic, the theorems derived in the current paper hold under both the null and alternative hypotheses. Also, in contrast to Banerjee and Magnus' [see the above cited reference] results on the F test, we find that the decision to accept the null using the OLS based one-sided t test is not necessarily robust against covariance misspecification and depends much on the underlying data matrix. Our results also indicate that autocorrelation does *not* necessarily weaken the power of the OLS based t test.

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1. Introduction

The traditional econometrics literature places a good deal of emphasis on the likely consequences of ignoring non-spherical errors on estimators and tests. For example, much has been written about the ordinary least squares (OLS) estimator being no longer best linear unbiased in the face of autocorrelated or heteroscedastic disturbances. In recent years, a large literature of diagnostic testing has been developed, and the idea that a model must be tested before it can be taken as an adequate basis for analysis has become widely accepted. Some econometricians, on the other hand, have contended that models that do not strictly fulfill the assumptions behind their validity are still useful if estimators of the parameters of interest are not sensitive to deviations from these assumptions. For example, in the presence of AR(1) disturbances, it occurs frequently that after fitting the model by feasible generalized least squares, the coefficient estimates do not change much from the OLS estimates. In other words, the OLS estimators of the coefficients are robust against AR(1) disturbances. In practice, econometric models are invariably misspecified, and whether the estimates of parameters are sensitive to deviations from the truth appears to be of greater importance than whether the underlying assumptions are satisfied, even though traditional econometrics has placed much greater emphasis on the latter.

Defined in the most general terms, sensitivity analysis is an analysis of the effects of various parameters and initial value changes on system behaviour. Over the past twenty years, a variety of sensitivity analysis tools have been developed in the mathematical modeling and statistics literatures. These tools are typically optimized for their particular applications and

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there are ample examples of the applications of these sensitivity analysis tools across various disciplines. See [1,2] for a good overview of the various sensitivity methodologies that have been developed.

In the context that is of interest to us here, studies by Banerjee and Magnus [3,4] and Magnus and Vasnev [5] developed a theory of sensitivity analysis for the linear model. Banerjee and Magnus [3] proposed a sensitivity statistic for the OLS estimator. They found that the OLS coefficient estimator is not very sensitive to covariance misspecification. In a limited Monte Carlo study they also found that the Durbin–Watson test statistic and the sensitivity statistic of the OLS coefficient estimator are nearly orthogonal. That is, information contained in the Durbin–Watson test is almost irrelevant for the sensitivity of the OLS coefficient estimator. This finding was later confirmed by theoretical results derived in [6] who also extended Banerjee and Magnus' sensitivity analysis to a restricted linear model allowing for the possibility of incorrect restrictions. The second paper by Banerjee and Magnus [4] discussed the sensitivity of the usual F and two-sided t tests in the linear model to covariance misspecification. They observed that the usual F test based on OLS residuals is generally sensitive to covariance misspecification, and the true size of the usual F test exceeds the stated size in the cases of AR(1), MA(1) and ARMA(1,1) disturbances. They then concluded that if the null is accepted using the usual F test it will also be accepted by the F test based on generalized least square (GLS) residuals, and hence accepting the null hypothesis using the OLS based F test is a robust decision.

It is worth noting that [4] findings depend crucially on *the null hypothesis being correct*. Thus, one cannot ascertain, for example, the question of whether rejecting the null using the OLS based F test is a robust decision. This shortcoming calls for further exploration of the problem and a new set of theoretical tools by which sensitivity of the test statistic may be examined under both the null and alternative hypotheses. The objective of the present paper is to show that an analysis of the test statistic's sensitivity under the alternative hypothesis is also well within the reach. Instead of focusing on the F and two-sided t tests as in [4], our main interest is in the one-sided t test but the theorems developed are in fact relevant to both one- and two-sided t tests and can be readily extended to consider the F statistic's robustness. Related studies by Smith [7], Magnus [8] and Qin and Wan [9] have examined the sensitivity of the t statistic to situations such as non-normal errors or dependence in the numerator and denominator of the t ratio.

The balance of the paper will begin with a discussion on the model set-up and the sensitivity measure for the decision based on the t statistic in the next section. Section 3 presents analytical results on the finite sample moments and limiting behaviour of the sensitivity statistic near the unit-root in the case of AR(1) disturbances. In Section 4, we conduct a comprehensive numerical study on the behaviour of the sensitivity statistic under AR(1) and MA(1) errors. Section 5 reports results of a comparison of the size and power of the one-sided t test based on OLS residuals with the corresponding test based on GLS residuals, while Section 6 discusses a rule of thumb as a practical guideline for the use of the sensitivity statistic. Section 7 concludes. Proofs of theorems are contained in [Appendices A and B](#).

2. Model set-up and sensitivity statistic

Consider the classical linear regression model

$$y = X\beta + u; \quad u \sim N(0, \sigma^2 \Omega(\theta)), \quad (2.1)$$

where y is an $n \times 1$ vector of observations on the dependent variable, X is an $n \times k$ non-stochastic matrix of full column rank containing values of k explanatory variables, β is a $k \times 1$ unknown coefficient vector, u is an $n \times 1$ vector of disturbances, $\sigma^2 > 0$ is a scalar and $\Omega(\theta)$ is an $n \times n$ matrix function of a nuisance parameter θ , positive definite and differentiable at least in a neighbourhood of $\theta = 0$. We assume for simplicity that θ is a scalar. The t statistic for testing

$$H_0 : R\beta = r \quad \text{vs.} \quad H_1 : R\beta < r, \quad (2.2)$$

where R is a known $1 \times k$ vector and r is a known scalar, is given by

$$t(\theta) = \frac{R\hat{\beta}(\theta) - r}{\sqrt{\hat{\sigma}^2(\theta)RS^{-1}(\theta)R}}, \quad (2.3)$$

where $\hat{\beta}(\theta) = S^{-1}(\theta)X'\Omega^{-1}(\theta)y$ is the GLS estimator of β , $S(\theta) = X'\Omega^{-1}(\theta)X$ and $\hat{\sigma}^2(\theta) = (y - X\hat{\beta}(\theta))'\Omega^{-1}(\theta)(y - X\hat{\beta}(\theta))/(n - k)$. Without loss of generality, we assume $\Omega(0) = I_n$. The familiar OLS estimator of β is $\hat{\beta}(0) = (X'X)^{-1}X'y$.

Notice that even if diagnostic tests suggest that $\theta \neq 0$, $t(0)$ may still be close to $t(\theta)$. So, one wants to find out if it is still legitimate to use the OLS based t statistic $t(0)$ instead of $t(\theta)$ when θ is not close to 0, and this is precisely what sensitivity analysis in the present context is about. If $t(\theta)$ is close to $t(0)$ even when θ is far from zero, then we say that $t(\theta)$ is insensitive to changes in θ . At issue here is whether the decision to accept/reject the null based on $t(0)$ is robust when θ is not close to 0. Now, consider the Taylor series expansion

$$t(\theta) \approx t(0) + \theta\tau, \quad (2.4)$$

where $\tau = \left. \frac{dt(\theta)}{d\theta} \right|_{\theta=0}$. Note that if $\theta\tau > 0$, then $t(0) < t(\theta)$. Under this case if H_0 is accepted using $t(0)$ it will also be accepted using $t(\theta)$ and accepting H_0 using $t(0)$ is said to be a robust decision. On the other hand, if $\theta\tau < 0$, then $t(0) > t(\theta)$ and rejecting H_0 based on $t(0)$ is a robust decision. Thus, by considering whether $E_\theta(\theta\tau) > 0$ (or equivalently, $E_\theta(\tau) > 0$

assuming θ is positive) or whether $\Pr_{\theta}(\theta\tau > 0) > 1/2$ (or equivalently, $\Pr_{\theta}(\tau > 0) > 1/2$ assuming θ is positive) one can gain insights into the robustness of the decisions based on $t(0)$. In the following, we first investigate the properties of the sensitivity statistic τ .

Theorem 2.1. *The sensitivity statistic τ has the stochastic representation*

$$\tau = -\frac{t(0)}{2} \left\{ -\frac{v'MAMv}{v'Mv} + \frac{H'AH}{H'H} \right\} - \sqrt{\frac{n-k}{H'H}} \frac{H'AMv}{\sqrt{v'Mv}}, \tag{2.5}$$

where $t(0) = \sqrt{n-k}(H'v - \delta) / \sqrt{H'H(v'Mv)}$, $H = XS^{-1}R'$, $v = u/\sigma$, $\delta = (r - R\beta)/\sigma$, $S = X'X$, $M = I_n - XS^{-1}X'$ is a symmetric idempotent matrix of rank $n - k$, and $A = \partial\Omega(\theta)/\partial\theta|_{\theta=0}$.

Proof. See Appendix A. \square

Eq. (2.5) is helpful for analyzing the exact finite sample moments of τ and the behaviour of τ in certain extreme cases (e.g., near the unit-root). We also observe from Eq. (2.5) that τ depends firstly on the data, and secondly on the regression parameters β and σ^2 through δ . If the null is incorrect then $\delta > 0$. For a given value of δ , both $t(0)$ and τ are distributional invariant with respect to the regression parameters.

3. Finite sample moments and behaviour near the unit-root

To gain insight into the sensitivity of the decision based on $t(0)$, we derive the first two moments of τ :

Theorem 3.1. *Let the distribution of y be evaluated at $\theta = 0$. Then we have*

$$E_0(\tau) = \frac{\Gamma\left(\frac{n-k-1}{2}\right) [(n-k)H'AH/H'H - \text{tr}(AM)] \delta}{2\sqrt{2(n-k)H'H} \Gamma\left(\frac{n-k}{2}\right)} \tag{3.1}$$

and

$$E_0(\tau^2) = \frac{H'AMAH}{H'H} + \frac{(n-k)\Delta}{4(n-k-2)} \left(1 + \frac{\delta^2}{H'H} \right), \tag{3.2}$$

where

$$\Delta = \left(\frac{H'AH}{H'H} \right)^2 - \frac{2\text{tr}(AM)}{n-k} \frac{H'AH}{H'H} + \frac{2\text{tr}(AM)^2 + \text{tr}^2(AM)}{(n-k)(n-k+2)}. \tag{3.3}$$

Proof. See Appendix A. \square

Unlike the corresponding theorem given in [4] (which holds only under the null), **Theorem 3.1** holds for all values of δ irrespective of whether $R\beta = r$ is valid. When the distribution of y is evaluated at values of θ other than 0, a corresponding result has also been obtained and is available on request from the authors. The next theorem presents results on the limiting behaviour of τ near the unit-root.

Theorem 3.2. *Let u_t be generated by the stationary AR(1) process $u_t = \phi_1 u_{t-1} + \varepsilon_t$, where $0 \leq \phi_1 < 1$ and ε_t 's are i.i.d. $N(0, \sigma^2)$ such that*

$$\Omega(\phi_1) = (\omega_{ij}(\phi_1)), \quad \text{where } \omega_{ij}(\phi_1) = \begin{cases} 1/(1 - \phi_1^2) & \text{if } i = j, \\ \phi_1^{|i-j|}/(1 - \phi_1^2) & \text{if } i \neq j, \end{cases} \tag{3.4}$$

i be an $n \times 1$ vector of ones, and $T^{(1)} = (t_{ij})$ be the symmetric Toeplitz matrix such that $t_{ij} = 1$ if $|i - j| = 1$ and $t_{ij} = 0$ otherwise. Note that when u_t follows an AR(1) process, $A = \partial\Omega(\phi_1)/\partial\phi_1|_{\phi_1=0} = T^{(1)}$. We have the following cases:

(i) If $Mi \neq 0$, $H'i \neq 0$ and $H'T^{(1)}Mi = 0$, then for any real $\tau_0 \neq \bar{\tau}_1(0)$,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \begin{cases} 0 & \text{if } \tau_0 < -|\bar{\tau}_1(0)|, \\ \frac{1}{2} & \text{if } -|\bar{\tau}_1(0)| < \tau_0 < |\bar{\tau}_1(0)|, \\ 1 & \text{if } \tau_0 > |\bar{\tau}_1(0)|, \end{cases} \tag{3.5}$$

where $\bar{\tau}_1(0) = -(\bar{t}(0)/2) (d1 + H'T^{(1)}H/H'H)$, $\bar{t}(0) = \sqrt{(n-k)/H'H} i/\sqrt{i'Mi} \neq 0$, and $d1 = -i'MT^{(1)}Mi/i'Mi$.

(ii) If $Mi \neq 0, H'i \neq 0$ and $H'T^{(1)}Mi \neq 0$, then for any real $\tau_0 \neq \bar{\tau}(0)$,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \begin{cases} 0 & \text{if } \tau_0 < -|\bar{\tau}(0)|, \\ \frac{1}{2} & \text{if } -|\bar{\tau}(0)| < \tau_0 < |\bar{\tau}(0)|, \\ 1 & \text{if } \tau_0 > |\bar{\tau}(0)|, \end{cases} \tag{3.6}$$

where $\bar{\tau}(0) = \bar{\tau}_1(0) - \bar{\gamma}(0)$ with $\bar{\gamma}(0) = \sqrt{(n-k)/H'H}H'T^{(1)}Mi/\sqrt{i'Mi}$.

(iii) If $Mi \neq 0, H'i = 0$ and $H'T^{(1)}Mi \neq 0$, then for any real $\tau_0 \neq \bar{\gamma}(0)$,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \begin{cases} 0 & \text{if } \tau_0 < -|\bar{\gamma}(0)|, \\ \frac{1}{2} & \text{if } -|\bar{\gamma}(0)| < \tau_0 < |\bar{\gamma}(0)|, \\ 1 & \text{if } \tau_0 > |\bar{\gamma}(0)|. \end{cases} \tag{3.7}$$

(iv) If $Mi \neq 0, H'i = 0$ and $H'T^{(1)}Mi = 0$, then for any real τ_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \begin{cases} 0 & \text{if } \tau_0 < 0, \\ \Pr\{H'[d^*I_n + T^{(1)}M]\bar{P}\eta \geq d^*\delta\} & \text{if } \tau_0 = 0, \\ 1 & \text{if } \tau_0 > 0, \end{cases} \tag{3.8}$$

where $d^* = \frac{1}{2}(d1 + H'T^{(1)}H/H'H)$, $\eta \sim N(0, I_{n-1})$, $\bar{P} = \bar{J}P, \bar{J}$ is an $n \times (n - 1)$ matrix such that $\bar{J}' = [0|I_{n-1}]$, and P is an $(n - 1) \times (n - 1)$ lower triangular matrix with ones on and below the diagonal and zeros elsewhere.

(v) If $Mi = 0$ and $H'i \neq 0$, then for any real τ_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \frac{1}{2}. \tag{3.9}$$

(vi) If $Mi = 0, H'i = 0$ and $H'T^{(1)}M\bar{J} = 0$, then for any real τ_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \Pr(\tau_1^{(1)}(\eta) \leq \tau_0), \tag{3.10}$$

where $\tau_1^{(1)}(\eta) = -(t^{(1)}(\eta)/2)(D^{(1)}(\eta) + H'T^{(1)}H/H'H)$, $D^{(1)}(\eta) = -\eta'\bar{P}MT^{(1)}M\bar{P}\eta/\eta'\bar{P}M\bar{P}\eta$, and $t^{(1)}(\eta) = \sqrt{(n-k)/H'H}(H'\bar{P}\eta - \delta)/\sqrt{\eta'\bar{P}M\bar{P}\eta}$.

(vii) If $Mi = 0, H'i = 0$ and $H'T^{(1)}M\bar{J} \neq 0$, then for any real τ_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(\tau \leq \tau_0) = \Pr(\tau^{(1)}(\eta) \leq \tau_0), \tag{3.11}$$

where $\tau^{(1)}(\eta) = \tau_1^{(1)}(\eta) - \gamma^{(1)}(\eta)$ with $\gamma^{(1)}(\eta) = \sqrt{(n-k)/H'H}H'T^{(1)}M\bar{P}\eta/\sqrt{\eta'\bar{P}M\bar{P}\eta}$.

Proof. See Appendix A. □

Different models are implied under the different cases of Theorem 3.2. First, when $Mi \neq 0$ (cases i–iv), the model contains no intercept. Second, when $Mi = 0$ and $H'i \neq 0$ (case v), the model has an intercept and the constraint under H_0 involves the intercept. Third, when $Mi = 0$ and $H'i = 0$ (cases vi and vii), the model has an intercept but the intercept is not part of the constraint implied by H_0 . The form of the regressors determines the differences among the cases within these three broad scenarios. Thus, in the case of AR(1) disturbances, the behaviour of τ near the unit-root can be vastly different depending on the form of the regressor matrix and whether the intercept is part of the null hypothesis. For example, the results of (3.10) and (3.11) indicate that whether $H'T^{(1)}M\bar{J}$ is zero or not (which in turn depends on the data matrix) can result in very different limiting behaviour of τ , even though both (3.10) and (3.11) are associated with models with an intercept that is not part of the restriction; similarly, depending on the underlying data matrix, sensitivity statistics in models that contain no intercept do not necessarily have the same limiting behaviour, as shown in (3.5)–(3.8). Unlike the results of Banerjee and Magnus [4], Theorem 3.2 holds under both H_0 and H_1 .

4. Numerical analysis

This section reports results of a comprehensive numerical study on the properties of τ . Our numerical study considers AR(1) and MA(1) disturbances. In the latter case, $u_t = \psi_1 \varepsilon_{t-1} + \varepsilon_t$, and so $\Omega(\psi_1) = (1 + \psi_1^2)I_n + \psi_1 T^{(1)}$. Under both AR(1) and MA(1) disturbances, $A = \partial\Omega(\phi_1)/\partial\phi_1|_{\phi_1=0} = \partial\Omega(\psi_1)/\partial\psi_1|_{\psi_1=0} = T^{(1)}$. Following the previous study of Wan et al. [6], our numerical analysis is based on design matrices formed by columns or linear combinations of columns from the following two data sets: the first comprises the eigenvectors t_1, t_2, \dots, t_n that correspond to the eigenvalues of the $n \times n$ Toeplitz matrix $T^{(1)}$ in ascending order; in the second data set, the regressors are $s_1 = i_n/\sqrt{n}$ representing an intercept term, where i_p is a $p \times 1$ vector of ones, and $s_p = (i'_{p-1}, 1 - p, 0_{1 \times (n-p)})'/\sqrt{p(p-1)}$, $2 \leq p \leq n$. We set $n = 15, 50$ and $100, k = 4$, and $R = [1, 0, 0, 0]$. Table 1 presents the design matrices on which the numerical investigations are based. In

Table 1
Regression models for numerical analysis.

n	Model	X_1	X_2	$i'Mi$	$H'i$	$H'T^{(1)}Mi$	ℓ
15	1	t_3	$[t_{12}, t_{13} + t_{14}, t_{14} + t_{15}]$	7.7345	-0.8035	-1.1480	1.2818
15	2	s_{15}	$[s_2, s_4, s_{14}]$	15	0	0.8971	0.2323
15	3	s_1	$[s_{10} + s_{13}, s_{11} + s_{14}, s_9 + s_{15}]$	0	3.8730	0	0.1956
15	4	s_6	$[s_1, s_2, s_3]$	0	0	0	1.3025
50	1	t_{38}	$[t_{47}, t_{48} + t_{49}, t_{49} + t_{50}]$	49.568	-0.3111	-0.7767	1.2438
50	2	s_{50}	$[s_{37}, s_{39}, s_{49}]$	50	0	0.9698	0.1386
50	3	s_1	$[s_{10} + s_{13}, s_{11} + s_{14}, s_9 + s_{15}]$	0	7.0711	0	0.1443
50	4	s_{41}	$[s_1, s_2, s_3]$	0	0	0	1.4124
100	1	t_{88}	$[t_{97}, t_{98} + t_{99}, t_{99} + t_{100}]$	99.844	-0.3275	-1.1178	1.1668
100	2	s_{100}	$[s_{87}, s_{89}, s_{99}]$	100	0	0.9849	0.0990
100	3	s_1	$[s_{10} + s_{13}, s_{11} + s_{14}, s_9 + s_{15}]$	0	10.0000	0	0.1022
100	4	s_{91}	$[s_1, s_2, s_3]$	0	0	0	1.4138

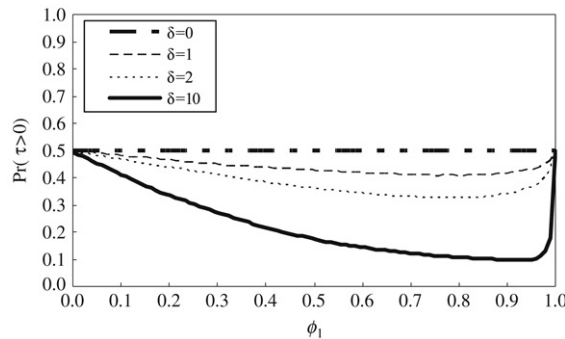


Fig. 1a. $\Pr(\tau > 0)$ under AR(1) errors—Model 1 ($n = 15$).

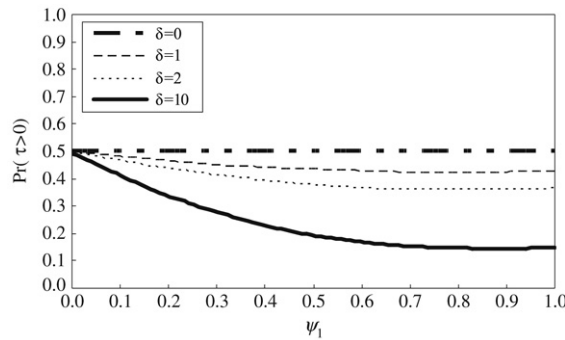


Fig. 1b. $\Pr(\tau > 0)$ under MA(1) errors—Model 1 ($n = 15$).

each case the design matrix is $X = [X_1|X_2]$ and the null hypothesis of interest is $H_0 : \beta_1 = r$, where β_1 is the first element of β . Of the four models considered, only Models 3 and 4 contain an intercept term, and only the null hypothesis of Model 3 involves the intercept. In Table 1, $\ell = \sqrt{H'T^{(1)}M\bar{J}\bar{J}'MT^{(1)}H}$ is the length of $H'T^{(1)}M\bar{J}$. For all models we set $\sigma = 1$ and the values of δ are varied at 0, 1, 2 and 10.

The robustness of the decision to accept/reject H_0 based on the $t(0)$ statistic is assessed by the magnitudes of $\Pr_\theta(\tau > 0)$ and $E_\theta(\tau)$ (where $\theta = \phi_1$ or ψ_1). If $\Pr_\theta(\tau > 0) > 1/2$ or $E_\theta(\tau) > 0$, then typically $t(0) < t(\theta)$ and in this case, accepting H_0 based on $t(0)$ is a robust decision. On the other hand, if $\Pr_\theta(\tau > 0) < 1/2$ or $E_\theta(\tau) < 0$, then typically $t(0) > t(\theta)$ and the decision to reject H_0 based on $t(0)$ is robust. The results on $\Pr_\theta(\tau > 0)$ and $E_\theta(\tau)$ under the four model settings based on $n = 15$ appear in Figs. 1a, 1b, 2a, 2b, 3a, 3b, 4a, 4b, 5a, 5b, 6a, 6b, 7a, 7b, 8a and 8b. We observe, first, that the limiting behaviour of $\Pr_\theta(\tau > 0)$ portrayed in Figs. 1a, 1b, 2a, 2b, 3a, 3b, 4a and 4b under AR(1) errors concurs with the theoretical results presented in Theorem 3.2. For example, Fig. 1a shows that under AR(1) errors, $\Pr_\theta(\tau > 0)$ for all δ 's approach 0.5 as ϕ_1 approaches 1. This is precisely the result expected as in the case of Model 1 ($Mi \neq 0, H'i \neq 0, H'T^{(1)}Mi \neq 0$, and $\bar{\tau}(0) = 0.8056$ for $\delta = 0, 1, 2$ and 10), Eq. (3.6) shows that the limiting probability is 0.5. Figs. 1a, 1b, 2a, 2b, 3a, 3b, 4a and 4b also show that both the form of the data matrix and the specification of the model have a large impact on the results. For all cases depicted in Figs. 1a, 1b, 2a, 2b, 4a and 4b, $\Pr_\theta(\tau > 0) < 1/2$ under both AR(1) and MA(1) errors, but exactly the opposite is observed in Figs. 3a and 3b. So under the model settings of Figs. 1a, 1b, 2a, 2b, 4a and 4b, if we reject the null

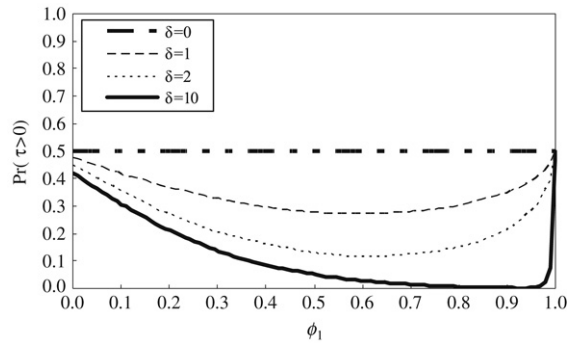


Fig. 2a. $\Pr(\tau > 0)$ under AR(1) errors—Model 2 ($n = 15$).

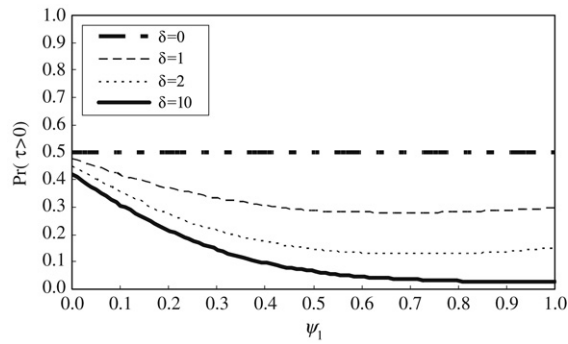


Fig. 2b. $\Pr(\tau > 0)$ under MA(1) errors—Model 2 ($n = 15$).

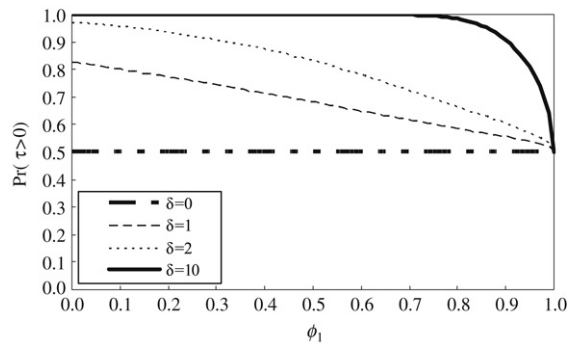


Fig. 3a. $\Pr(\tau > 0)$ under AR(1) errors—Model 3 ($n = 15$).

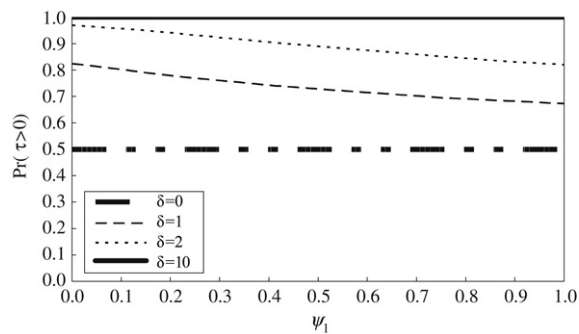


Fig. 3b. $\Pr(\tau > 0)$ under MA(1) errors—Model 3 ($n = 15$).

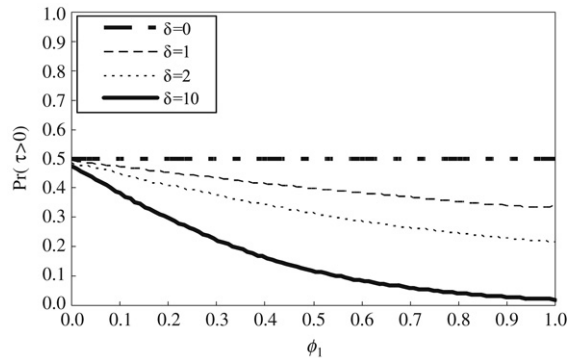


Fig. 4a. $\Pr(\tau > 0)$ under AR(1) errors—Model 4 ($n = 15$).

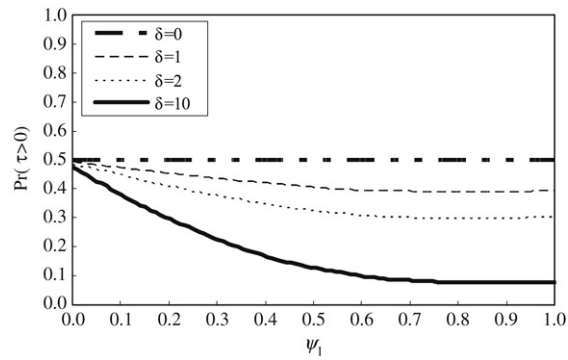


Fig. 4b. $\Pr(\tau > 0)$ under MA(1) errors—Model 4 ($n = 15$).

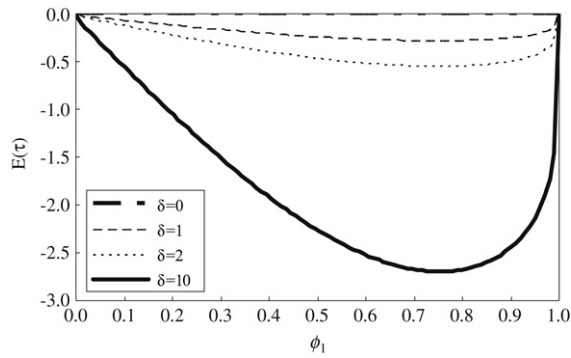


Fig. 5a. $E(\tau)$ under AR(1) errors—Model 1 ($n = 15$).

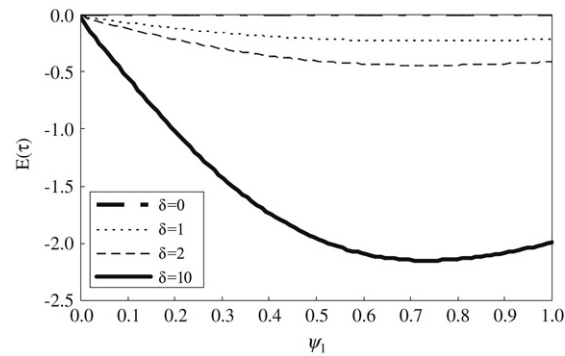


Fig. 5b. $E(\tau)$ under MA(1) errors—Model 1 ($n = 15$).

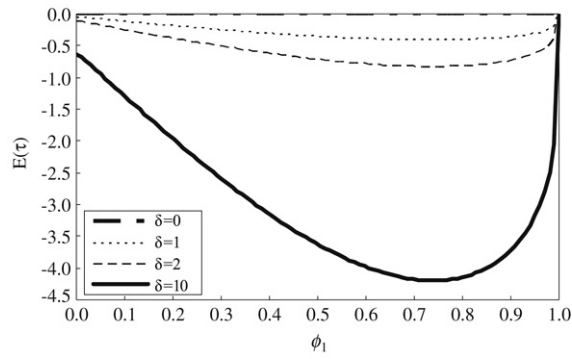


Fig. 6a. $E(\tau)$ under AR(1) errors—Model 2 ($n = 15$).

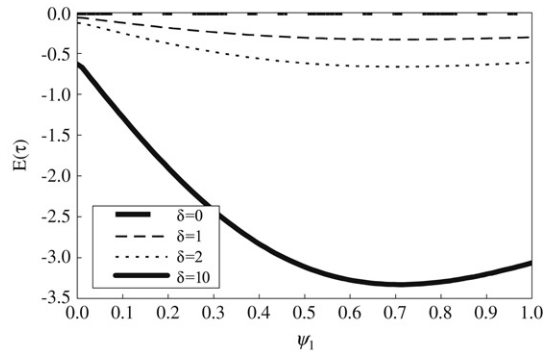


Fig. 6b. $E(\tau)$ under MA(1) errors—Model 2 ($n = 15$).

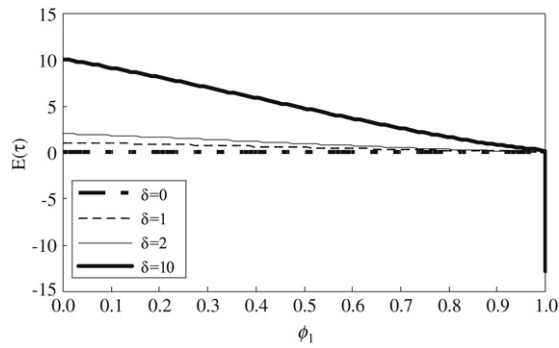


Fig. 7a. $E(\tau)$ under AR(1) errors—Model 3 ($n = 15$).

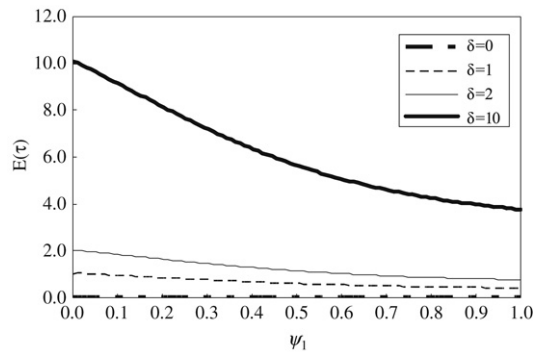


Fig. 7b. $E(\tau)$ under MA(1) errors—Model 3 ($n = 15$).

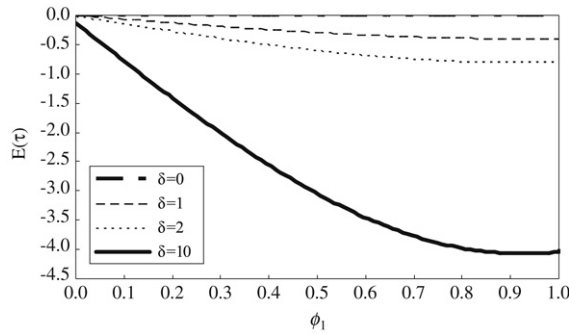


Fig. 8a. $E(\tau)$ under AR(1) errors—Model 4 ($n = 15$).

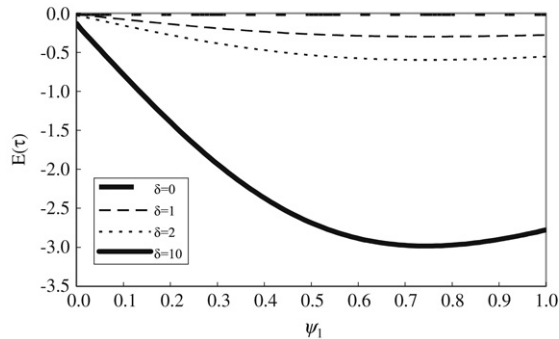


Fig. 8b. $E(\tau)$ under MA(1) errors—Model 4 ($n = 15$).

using $t(0)$, we will continue to do so using $t(\theta)$, that is, the decision to reject H_0 based on $t(0)$ is robust. In contrast, in the case of Figs. 3a and 3b, the decision to accept H_0 based on $t(0)$ is a robust decision. Qualitatively, these results are consistent with those observed based on the analysis of $E_\theta(\tau)$. In Figs. 5a, 5b, 6a, 6b, 8a and 8b, $E_\theta(\tau) < 0$ for all cases under both types of error processes under examination, implying that rejecting H_0 based on $t(0)$ is a robust decision under Models 1, 2 and 4; in Figs. 7a and 7b, however, $E_\theta(\tau) > 0$ for all cases, implying that accepting the null based on $t(0)$ is a robust decision in the case of Model 3. Interestingly, these results contrast with the findings of Banerjee and Magnus [4], who show that in the cases of the OLS based F and two-sided t tests, accepting the null is a robust decision even though the test statistics are sensitive to covariance misspecification. Our results have shown that for the one-sided t test, this conclusion does not generally hold true and depends much on the underlying regression matrix. While these commentaries are based on results for $n = 15$, the results obtained for $n = 50$ and 100 are qualitatively very similar and are available on request from the authors. In particular, we have found that even with a sample size as large as 100 , accepting the null based on $t(0)$ is not always a robust decision and again depends much on the underlying data matrix.

5. Direct power comparisons

The preceding discussion is based on local sensitivity analysis. In this section we conduct a direct comparison of rejection probabilities between $t(0)$ and $t(\theta)$ for a range of ϕ_1 and ψ_1 values based on the design matrices of Section 4. Given the findings of the last section, the size and power of $t(0)$ are expected to be smaller than those of $t(\theta)$ for Models 1, 2 and 4 but larger for Model 3 when $\theta \neq 0$. Our aim here is to obtain some idea of the possible magnitude of power as well as size distortions when $t(0)$ is used in place of $t(\theta)$ when $\theta \neq 0$. We first derive some theoretical results concerning the limiting size and power of $t(0)$:

Theorem 5.1. Let u_i be generated by the stationary AR(1) process, then we have the following cases:

(i) If $M_i \neq 0$, then for any real t_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(t(0) \leq t_0) = \begin{cases} 0 & \text{if } t_0 < -|\bar{t}(0)| \text{ and } H'i \neq 0, \\ 1 & \text{if } -|\bar{t}(0)| < t_0 < |\bar{t}(0)| \text{ and } H'i \neq 0, \\ 2 & \text{if } t_0 > |\bar{t}(0)| \text{ and } H'i \neq 0, \\ 1 & \text{if } t_0 < 0 \text{ and } H'i = 0, \\ 0 & \text{if } t_0 = 0 \text{ and } H'i = 0, \\ \Pr(H'\bar{P}\eta \leq \delta) & \text{if } t_0 = 0 \text{ and } H'i = 0, \\ 1 & \text{if } t_0 > 0 \text{ and } H'i = 0. \end{cases} \quad (5.1)$$

Table 2a

Comparison between $t(0)$ and $t(\theta)$ under Model 3 (Nominal $\alpha = 0.05$; $n = 15$; critical value $t_0 = -1.7959$).

AR(1)	Pr($t(0) \leq t_0$)				Pr($t(\phi_1) \leq t_0$)			
	δ							
ϕ_1	0	1	2	10	0	1	2	10
0.000	0.050	0.240	0.591	1.000	0.050	0.240	0.591	1.000
0.100	0.071	0.257	0.579	1.000	0.050	0.214	0.522	1.000
0.200	0.093	0.276	0.570	1.000	0.050	0.189	0.452	1.000
0.300	0.118	0.294	0.558	1.000	0.050	0.166	0.384	1.000
0.400	0.148	0.312	0.543	1.000	0.050	0.145	0.319	1.000
0.500	0.176	0.330	0.528	1.000	0.050	0.126	0.259	1.000
0.600	0.214	0.350	0.514	1.000	0.050	0.109	0.205	0.993
0.700	0.261	0.371	0.503	0.995	0.050	0.093	0.158	0.940
0.800	0.314	0.400	0.495	0.967	0.050	0.079	0.119	0.748
0.900	0.377	0.433	0.491	0.869	0.050	0.066	0.086	0.399
1.000	0.500	0.500	0.500	0.500	0.050	0.050	0.050	0.050

MA(1)	Pr($t(0) \leq t_0$)				Pr($t(\psi_1) \leq t_0$)			
	δ							
ψ_1	0	1	2	10	0	1	2	10
0.000	0.050	0.240	0.591	1.000	0.050	0.240	0.591	1.000
0.100	0.069	0.255	0.579	1.000	0.050	0.216	0.528	1.000
0.200	0.085	0.268	0.570	1.000	0.050	0.197	0.474	1.000
0.300	0.099	0.276	0.558	1.000	0.050	0.181	0.428	1.000
0.400	0.109	0.279	0.542	1.000	0.050	0.168	0.390	1.000
0.500	0.118	0.280	0.524	1.000	0.050	0.157	0.356	1.000
0.600	0.125	0.277	0.507	1.000	0.050	0.148	0.328	1.000
0.700	0.128	0.272	0.489	1.000	0.050	0.140	0.304	1.000
0.800	0.131	0.267	0.469	1.000	0.050	0.134	0.283	1.000
0.900	0.132	0.260	0.449	1.000	0.050	0.128	0.264	1.000
1.000	0.132	0.253	0.428	1.000	0.050	0.123	0.248	0.999

(ii) If $M_i = 0$ and $H'i \neq 0$, then for any real t_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(t(0) \leq t_0) = \frac{1}{2}. \tag{5.2}$$

(iii) If $M_i = 0$ and $H'i = 0$, then for any real t_0 ,

$$\lim_{\phi_1 \rightarrow 1} \Pr(t(0) \leq t_0) = \Pr(t^{(1)}(\eta) \leq t_0), \tag{5.3}$$

where $t^{(1)}(\eta) = \sqrt{(n-k)/H'H} (H'\bar{P}\eta - \delta) / \sqrt{\eta'\bar{P}'M\bar{P}\eta}$ as in (3.10).

Proof. See Appendix A. \square

The results of the size and power comparisons for $n = 15$ are summarized in Tables 2a and 2b. Since the qualitative findings are similar for Models 1, 2 and 4, Tables 2a and 2b only present the results based on Models 3 and 4. In all evaluations, the nominal size is set to 0.05. Recall that the null hypothesis is incorrect when $\delta > 0$. Hence in Tables 2a and 2b, the null rejection probabilities corresponding to $\delta = 0$ are the true sizes of the tests, while the rejection probabilities for $\delta > 0$ represent powers. For $t(\theta)$, the true size of the test equals the nominal size over the whole range of ϕ_1 or ψ_1 considered. For $t(0)$, the true size is the same as the stated size at $\phi_1 = \psi_1 = 0$. The powers of the two tests are also identical at $\phi_1 = \psi_1 = 0$. No size corrections are made to the power functions of $t(0)$. The reason for this is that the purpose of this study is to determine the effects of autocorrelations on the properties of the t test. As such we presume that the researcher ignores the possibility of autocorrelation in the process. In the case of Model 3 under both AR(1) and MA(1) errors, the true size of the $t(0)$ test increases and exceeds the nominal 0.05 level as ϕ_1 and ψ_1 increase. The size inflation is quite substantial in the case of AR(1) errors but relatively mild in the case of MA(1) errors. Table 2a also shows that the $t(0)$ test is more powerful than $t(\theta)$ in the entire region of the parameter space. The deviations in powers between the two tests can be substantial in the case of AR(1) errors. In view of these observations it is also clear that if one accepts the null using $t(0)$, one will continue to do so using $t(\theta)$. In other words, the decision to accept H_0 based on $t(0)$ is a robust decision, a finding consistent with that based on the local sensitivity analysis in the last section. In the case of Model 4, the use of $t(0)$ instead of $t(\theta)$ does not seem to result in any size distortion, as Table 2b illustrates. On the other hand, there is a striking evidence that the OLS based $t(0)$ test lacks power when compared to $t(\theta)$. The drop in power caused by the use of $t(0)$ is very substantial for large values of ϕ_1 and ψ_1 . But this would suggest that rejecting the null based on $t(0)$ should be a robust decision which is again consistent with the findings in the last section based on the local sensitivity analysis.

In the case of Model 3, the limiting rejection probabilities of $t(0)$ under AR(1) errors approach 0.5 as ϕ_1 approaches 1 irrespective of the value of δ . The finding is consistent with the theoretical results obtained in Part (ii) of Theorem 5.1. For

Table 2b
Comparison between $t(0)$ and $t(\theta)$ under Model 4 (Nominal $\alpha = 0.05$; $n = 15$; critical value $t_0 = -1.7959$).

AR(1)	Pr($t(0) \leq t_0$)				Pr($t(\phi_1) \leq t_0$)			
	δ							
ϕ_1	0	1	2	10	0	1	2	10
0.000	0.050	0.240	0.591	1.000	0.050	0.240	0.591	1.000
0.100	0.051	0.241	0.595	1.000	0.050	0.242	0.597	1.000
0.200	0.051	0.238	0.589	1.000	0.050	0.248	0.609	1.000
0.300	0.051	0.234	0.577	1.000	0.050	0.255	0.628	1.000
0.400	0.050	0.228	0.561	1.000	0.050	0.266	0.652	1.000
0.500	0.047	0.218	0.540	1.000	0.050	0.279	0.681	1.000
0.600	0.044	0.209	0.510	1.000	0.050	0.295	0.713	1.000
0.700	0.040	0.192	0.478	1.000	0.050	0.313	0.747	1.000
0.800	0.034	0.176	0.442	1.000	0.050	0.333	0.781	1.000
0.900	0.028	0.158	0.399	1.000	0.050	0.355	0.815	1.000
1.000	0.028	0.139	0.348	0.998	0.050	0.380	0.847	1.000

MA(1)	Pr($t(0) \leq t_0$)				Pr($t(\psi_1) \leq t_0$)			
	δ							
ψ_1	0	1	2	10	0	1	2	10
0.000	0.050	0.240	0.591	1.000	0.050	0.240	0.591	1.000
0.100	0.051	0.242	0.594	1.000	0.050	0.242	0.597	1.000
0.200	0.052	0.237	0.588	1.000	0.050	0.247	0.609	1.000
0.300	0.052	0.231	0.574	1.000	0.050	0.256	0.629	1.000
0.400	0.051	0.227	0.556	1.000	0.050	0.269	0.658	1.000
0.500	0.051	0.220	0.535	1.000	0.050	0.287	0.697	1.000
0.600	0.051	0.209	0.512	1.000	0.050	0.312	0.745	1.000
0.700	0.051	0.201	0.485	1.000	0.050	0.345	0.800	1.000
0.800	0.051	0.191	0.458	1.000	0.050	0.385	0.854	1.000
0.900	0.052	0.183	0.431	1.000	0.050	0.422	0.893	1.000
1.000	0.051	0.173	0.402	1.000	0.050	0.421	0.893	1.000

Model 4, the limiting null rejection probability is a constant between 0 and 1, and the precise value of the limiting probability depends on both the data and value of δ , as Part (iii) of Theorem 5.1 and Table 2b illustrate. Power comparisons in the cases of Models 1 and 2, which are not reported here, are similar to those observed under Model 4. That is, the use of the OLS based one-sided t test typically leads to no discernable difference in test size, but there is also clear evidence that the OLS based test is less powerful than the GLS based test, particularly for large values of ϕ_1 and ψ_1 . For certain data matrices such as that of Model 1, the limiting rejection probabilities of $t(0)$ can drop to zero as $\phi_1 \rightarrow 1$ in the case of AR(1) errors.

Again, the preceding discussion focuses on the results obtained for $n = 15$. The qualitative findings under $n = 50$ and $n = 100$ are in fact very similar to those under $n = 15$ and are available upon request from the authors.

6. A rule of thumb for practical application

The preceding sections have provided a considerable amount of information on the likely consequences of using the OLS based $t(0)$ statistic when θ is non-zero. This section discusses a practical guide for the use of the sensitivity statistic τ in practice. As is clear from (2.4), other things being equal, a large value of $|\tau|$ should be taken as an indication of sensitivity, and vice versa. The following theorem enables the derivation of a “rule of thumb” for sensitivity based on an observed value of τ .

Theorem 6.1. Suppose that y is evaluated at $\theta = 0$ such that $u \sim N(0, \sigma^2 I_n)$. Consider the following two cases:

(i) If $\delta^2/H'H$ is bounded, then $t(0) = \tilde{t}(0) + O_p(n^{-1/2})$, where

$$\tilde{t}(0) = \frac{H'v - \delta}{\sqrt{H'H}}, \tag{6.1}$$

and $v = u/\sigma \sim N(0, I_n)$.

(ii) If $\delta^2/H'H$ and the eigenvalues of A are bounded, then $\tau = \tilde{\tau} + O_p(n^{-1/2})$, where

$$\tilde{\tau} = -\frac{H'AMv}{\sqrt{H'H}} + \frac{1}{2} \left[\frac{\text{tr}(AM)}{n - k} - \frac{H'AH}{H'H} \right] \tilde{t}(0). \tag{6.2}$$

Proof. See Appendix A. \square

It is readily seen from [Theorem 6.1](#) that as $n \rightarrow \infty$, $t(0)$ has an approximate $N(-\delta/\sqrt{H'H}, 1)$ distribution with a convergence rate of $O_p(n^{-1/2})$, and τ has an approximate $N(a_\tau, \sigma_\tau^2)$ distribution with a convergence rate of $O_p(n^{-1/2})$, where

$$a_\tau = \frac{\delta}{2\sqrt{H'H}} \left[\frac{H'AH}{H'H} - \frac{\text{tr}(AM)}{n-k} \right], \quad (6.3)$$

and

$$\sigma_\tau^2 = \frac{H'AMAH}{H'H} + \frac{1}{4} \left[\frac{H'AH}{H'H} - \frac{\text{tr}(AM)}{n-k} \right]^2. \quad (6.4)$$

Note that in any given application, σ_τ^2 can be readily computed while a_τ depends on the knowledge of δ in addition to the data. Now, to assess the robustness of $t(0)$ when θ deviates from 0, consider the probability $\Pr(|\tau| > c_\tau(\alpha)) = \alpha$. Results of [Theorem 6.1](#) facilitate the approximation of $c_\tau(\alpha)$ for a given α by the $N(a_\tau, \sigma_\tau^2)$ distribution. Note that under H_0 : $\delta = 0$, $a_\tau = 0$, and $c_\tau(\alpha)$ can be approximated as $c_{\tilde{\tau}}(\alpha) = c_N(\alpha)\sigma_\tau$, where $c_N(\alpha)$ is the upper $\alpha/2$ -quantile of the $N(0, 1)$ distribution. A value of $|\tau|$ greater than $c_{\tilde{\tau}}(\alpha)$ can be taken to imply that $t(\theta)$ is sensitive to a change of θ from 0 (or equivalently, $t(0)$ is not robust when θ deviates from 0) and vice versa. Clearly, the choice of α has an impact on the ultimate conclusion; α should neither be too small nor too large if one wants to avoid being too optimistic or too pessimistic about the robustness of $t(0)$. In their evaluation of the F test, [\[4\]](#) suggested setting α to 0.5. Now, since $c_N(0.5) \approx 0.6745$, we obtain the following 'rule of thumb' on the robustness of the OLS based t statistic $t(0)$ when θ departs from 0:

Rule of thumb. The OLS based statistic $t(0)$ is sensitive to a departure of θ from zero if $|\tau| > 0.6745\sigma_\tau$.

This rule of thumb provides a practical guideline for the use of the sensitivity statistic by practitioners in a given application. With a given A matrix (e.g., $A = T^{(1)}$ under Models 1–4 in [Table 1](#)), one may compute τ from [\(A.3\)](#) and σ_τ from [\(6.4\)](#) and check whether $|\tau| > 0.6745\sigma_\tau$. One may also get some idea on how well the rule of thumb works to warn against the use of the OLS based test for Models 1–4 by contrasting the probability $\Pr(|\tau| > 0.6745\sigma_\tau)$ with 0.5. For example, under Model 3 with $\delta = 0$, it is found that $\Pr(|\tau| > 0.6745\sigma_\tau)$ increases quickly to 1 as ϕ approaches 1 under AR(1) disturbances, while it is steady around 0.5 for ϕ over $[0, 1]$ under MA(1) disturbances. Judging from these observations, it is likely that the rule of thumb would indicate that $t(0)$ is sensitive to AR(1) misspecification especially when ϕ is near 1 but insensitive to MA(1) misspecification. Under all four models, $\Pr(|\tau| > 0.6745\sigma_\tau)$ increases beyond 0.5 when δ increases from 0. In other words, other things being equal, the likelihood of the rule of thumb indicating sensitivity increases as the constraint becomes increasing misspecified. The above results are not shown here but are available upon request from the authors.

7. Conclusions

The main aim of this paper is to explore the consequences of using the OLS based t statistic in a regression model with non-spherical errors. A sensitivity statistic τ has been introduced for this purpose. In contrast to the earlier contribution of Banerjee and Magnus [\[4\]](#), all the theorems derived in the current paper hold under both the null and alternative hypotheses. With AR(1) and MA(1) errors, it seems clear from our results based on both local sensitivity analysis and direct size and power comparisons that rejecting the null hypothesis based on the one-sided t statistic can often be a robust decision to covariance misspecification. This contrasts with the findings on the F test (or two-sided t test) that the decision to accept the null is a robust one. Our results also indicate that autocorrelation does not necessarily weaken the power of the OLS based t test. Another notable feature of this study is the extent to which the regressor matrix affects the results – sensitivity depends on the data and the decision based on $t(0)$ can be robust in one application and not so in another application. We have also derived a rule of thumb as guideline for the use of the sensitivity statistic in practice. Finally, it should be mentioned that throughout the analysis we have assumed that θ is a scalar. In the more general context when there are several autocorrelation parameters the sensitivity statistic will be multivariate and more difficult to treat. The latter situation is an interesting point of departure that certainly warrants investigation.

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Appendix A. Proofs of theorems

Proof of Theorem 2.1. Using [\(2.3\)](#) and applying the chain rule in Calculus, we have

$$\tau(\theta) \triangleq \frac{dt(\theta)}{d\theta} = -\frac{t(\theta)}{2} \left\{ \frac{R\kappa(\theta)R'}{RS^{-1}(\theta)R'} + \frac{\lambda(\theta)}{\hat{\sigma}^2(\theta)} \right\} + \frac{R\zeta(\theta)}{\sqrt{\hat{\sigma}^2(\theta)RS^{-1}(\theta)R'}}, \quad (A.1)$$

where $\kappa(\theta) = dS^{-1}(\theta)/d\theta$, $\lambda(\theta) = d\hat{\sigma}^2(\theta)/d\theta$, and $\zeta(\theta) = d\hat{\beta}(\theta)/d\theta$. Using [10, Ch. 8, Theorem 3] we observe that

$$\kappa(\theta) = -S^{-1}(\theta)X' (d\Omega^{-1}(\theta)/d\theta)XS^{-1}(\theta). \tag{A.2}$$

Now that $S(0) = S$ and $d\Omega^{-1}(\theta)/d\theta|_{\theta=0} = -A$ (see [3], equation 2.5). Setting $\theta = 0$ in (A.2) we obtain $\kappa(0) = S^{-1}X'AXS^{-1}$. In addition, $\lambda(0) = \lambda = -y'MAMy/(n - k)$ and $\zeta(0) = -S^{-1}X'AMy$ (see [3], Theorems 2 and 3). Thus, setting $\theta = 0$ in (A.1) and writing $H = XS^{-1}R'$ we obtain

$$\tau = \tau(0) = -\frac{t(0)}{2} \left\{ \frac{H'AH}{H'H} + D \right\} + \tilde{\tau}, \tag{A.3}$$

where $t(0) = (R\hat{\beta}(0) - r)/\sqrt{\hat{\sigma}^2(0)RS^{-1}R'}$, $D = \lambda/\hat{\sigma}^2(0)$ and $\tilde{\tau} = -H'AMy/\sqrt{\hat{\sigma}^2(0)H'H}$.

Under model (2.1), we have $\hat{\beta}(0) = S^{-1}X'y = \beta + S^{-1}X'u$, $My = Mu$ and $\hat{\sigma}^2(0) = y'My/(n - k) = u'Mu/(n - k)$. Together with $v = u/\sigma$ we obtain

$$t(0) = \frac{\sqrt{n - k} [H'u - (r - R\beta)]}{\sqrt{u'MuH'H}} = \frac{\sqrt{n - k} (H'v - \delta)}{\sqrt{v'MvH'H}} \tag{A.4}$$

and

$$\tilde{\tau} = -\frac{\sqrt{n - k}H'AMu}{\sqrt{u'MuH'H}} = -\frac{\sqrt{n - k}H'AMv}{\sqrt{v'MvH'H}}. \tag{A.5}$$

In addition, $D = \lambda/\hat{\sigma}^2(0) = -y'MAMy/y'My$ (see [3], Theorem 3). Hence $D = -v'MAMv/v'Mv$. Substituting the expression of D and Eqs. (A.4) and (A.5) into (A.3) completes the proof of Theorem 2.1. \square

Proof of Theorem 3.1. Define $a = \sqrt{(n - k)/H'H}$, $b = H'AH/H'H$, $l_1 = aH'v$, $l_2 = aH'AMv$, $q_1 = v'Mv$ and $q_2 = v'MAMv$. Then (2.5) may be written as

$$\tau = \frac{1}{2} (a\delta - l_1) (b - q_1^{-1}q_2) q_1^{-1/2} - l_2q_1^{-1/2}. \tag{A.6}$$

At $\theta = 0$, $v = u/\sigma \sim N(0, I_n)$, and (3.1) may be obtained by observing the following. First, note that $E_0(l_1) = 0$ and l_1 and (l_2, q_1, q_2) are independent since $\text{cov}(l_1, Mv) = aH'M = 0$. Second, $q_1^{1/2}$ and $q_2q_1^{-1}$ are independent and $E_0(q_2q_1^{-1}) = E_0(q_2)/E_0(q_1)$. By the well-known Eckhart-Young (SVD) factorization, we have $M = Q'Q$, where Q is an $(n - k) \times n$ column orthonormal matrix. Define $v = Qv$, we have $v \sim N(0, I_{n-k})$, $q_1 = v'v \sim \chi_{n-k}^2$, and $q_1^{1/2}$ and $\vartheta = vq_1^{-1/2}$ are mutually independent. Since $q_2q_1^{-1} = \vartheta'QAQ'\vartheta$, $q_2q_1^{-1}$ is independent of $q_1^{1/2}$. It follows that $E_0(q_2) = E_0(q_2q_1^{-1}q_1) = E_0(q_2q_1^{-1})E_0(q_1)$. Third, note that $l_2q_1^{-1/2} = aH'AQ'\vartheta$ and $q_1^{1/2}$ are mutually independent. So, $E_0(l_2q_1^{-1/2}) = 0$, since $E_0(l_2) = 0 = E_0(l_2q_1^{-1/2})E_0(q_1^{1/2})$. On the basis of these observations, we have

$$E_0(\tau) = \frac{1}{2} a\delta (b - E_0(q_2)/E_0(q_1)) E_0(q_1^{-1/2}). \tag{A.7}$$

By $v \sim N(0, I_n)$ and $M^2 = M$, we obtain $E_0(q_2) = \text{tr}(AM)$. Also, as $q_1 \sim \chi_{n-k}^2$, it follows that $E_0(q_1) = n - k$ and $E_0(q_1^{-1/2}) = \frac{1}{\sqrt{2}} \Gamma(\frac{n-k-1}{2})/\Gamma(\frac{n-k}{2})$. Eq. (3.1) follows by substituting these expressions in (A.7).

Using the above results, we also obtain from (A.6) that

$$E_0(\tau^2) = E_{01} - a\delta E_{02} + \frac{1}{4} E_{03}, \tag{A.8}$$

where $E_{01} = E_0(l_2^2q_1^{-1})$, $E_{02} = E_0\{(b - q_1^{-1}q_2)q_1^{-1}l_2\}$ and $E_{03} = \{a^2\delta^2 + E_0(l_2^2)\}E_0(q_1^{-1})\{b^2 - 2b\text{tr}(AM)/(n - k) + E_0(q_1^{-2}q_2^2)\}$. Now, note that $E_0(l_2^2) = a^2H'AMAH$ and $l_2^2q_1^{-1}$ and q_1 are independent. It follows that $E_0(l_2^2) = E_0(l_2^2q_1^{-1})E_0(q_1) = E_{01}E_0(q_1)$. Thus we obtain

$$E_{01} = E_0(l_2^2)/E_0(q_1) = H'AMAH/H'H. \tag{A.9}$$

One can easily show that

$$E_{02} = E_0\{(b - q_1^{-1}q_2)q_1^{-1}l_2\} = 0. \tag{A.10}$$

In fact, writing $f(v) = (b - q_1^{-1}q_2)q_1^{-1}l_2$, we have $f(v) + f(-v) = 0$ by the definition of (l_2, q_1, q_2) . In addition, $E_0f(v) = E_0f(-v)$ as v and $-v$ both follow the $N(0, I_n)$ distribution. Thus, $2E_0f(v) = 0$ and (A.10) follows.

One may alternatively prove (A.10) by noting the following. Since $q_2 l_2 q_1^{-3/2}$ and $q_1^{-1/2}$ are independent, $E_0(q_2 l_2 q_1^{-3/2}) = E_0(q_2 l_2)/E_0(q_1^{3/2})$. Using [11, Ch.3, Theorem 3.2d.2], we can show that $E_0(q_2 l_2) = 0$. So, $E_0(q_2 l_2 q_1^{-3/2}) = 0$. Also, recall that $E_0(l_2 q_1^{-1/2}) = 0$ and that $l_2 q_1^{-1/2}$ and $q_1^{1/2}$ are independent. Therefore,

$$\begin{aligned} E_{02} &= bE_0(l_2 q_1^{-1/2} q_1^{-1/2}) - E_0(q_2 l_2 q_1^{-3/2} q_1^{-1/2}) \\ &= bE_0(l_2 q_1^{-1/2})E_0(q_1^{-1/2}) - E_0(q_2 l_2 q_1^{-3/2})E_0(q_1^{-1/2}) = 0. \end{aligned}$$

Now, writing $l_1^2 = a^2 v' H H' v$, it is straightforward to see that $E_0(l_1^2) = a^2 H' H$. Recall that q_1 and $q_2 q_1^{-1}$ are independent. Therefore, $E_0(q_1^{-2} q_2^2) = E_0(q_2^2)/E_0(q_1^2)$. Clearly, $E_0(q_1^2) = (n - k)(n - k + 2)$ and $E_0(q_1^{-1}) = (n - k - 2)^{-1}$. Using [11, Ch.3, Theorem 3.2b.2], and recognizing that $v \sim N(0, I_n)$ and $M^2 = M$, we can show that $E_0(q_2^2) = 2\text{tr}((AM)^2) + (\text{tr}(AM))^2$. Thus, it follows that

$$\begin{aligned} E_{03} &= \frac{a^2(\delta^2 + H'H)}{n - k - 2} \left\{ b^2 - \frac{2b\text{tr}(AM)}{n - k} + \frac{2\text{tr}((AM)^2) + (\text{tr}(AM))^2}{(n - k)(n - k + 2)} \right\} \\ &= \frac{(n - k)\Delta}{n - k - 2} \left(1 + \frac{\delta^2}{H'H} \right). \end{aligned} \tag{A.11}$$

Substituting (A.9)–(A.11) into (A.8) yields (3.2) directly. □

Proof of Theorem 3.2. In (2.5), write $A = T^{(1)}$ and $\tau = \tau(0)$. Then we have

$$\tau(0) = -\frac{t(0)}{2} \left\{ D1 + \frac{H'T^{(1)}H}{H'H} \right\} - \gamma(0), \tag{A.12}$$

where $t(0)$ is defined as in Theorem 2.1, $\gamma(0) = \sqrt{(n - k)/H'HH'T^{(1)}} Mv/\sqrt{v'Mv}$, and $D1 = -v'MT^{(1)}Mv/v'Mv$. Observe from [3] the following result:

$$D1 \xrightarrow{p} \begin{cases} d1 & \text{if } Mi \neq 0, \\ D^{(1)}(\eta) & \text{if } Mi = 0. \end{cases} \tag{A.13}$$

Applying Theorem B.1 in Appendix B, we have

$$\gamma(0) \xrightarrow{p} \begin{cases} \bar{\gamma}(0)\xi/|\xi| & \text{if } Mi \neq 0 \text{ and } H'T^{(1)}Mi \neq 0, \\ 0 & \text{if } Mi \neq 0 \text{ and } H'T^{(1)}Mi = 0, \\ \gamma^{(1)}(\eta) & \text{if } Mi = 0 \text{ and } H'T^{(1)}M\bar{J} \neq 0, \\ 0 & \text{if } Mi = 0 \text{ and } H'T^{(1)}M\bar{J} = 0, \end{cases} \tag{A.14}$$

where $\xi \sim N(0, 1)$. It is easy to see from (A.12) through (A.14) and (A.19) (see the proof of Theorem 5.1) that

$$\tau(0) \xrightarrow{p} \begin{cases} \bar{\tau}(0)\xi/|\xi| & \text{if } Mi \neq 0, H'i \neq 0 \text{ and } H'T^{(1)}Mi \neq 0, \\ \bar{\tau}_1(0)\xi/|\xi| & \text{if } Mi \neq 0, H'i \neq 0 \text{ and } H'T^{(1)}Mi = 0, \\ -\bar{\gamma}(0)\xi/|\xi| & \text{if } Mi \neq 0, H'i = 0 \text{ and } H'T^{(1)}Mi \neq 0, \\ 0 & \text{if } Mi \neq 0, H'i = 0 \text{ and } H'T^{(1)}Mi = 0, \\ \tau^{(1)}(\eta) & \text{if } Mi = 0, H'i = 0 \text{ and } H'T^{(1)}M\bar{J} \neq 0, \\ \tau_1^{(1)}(\eta) & \text{if } Mi = 0, H'i = 0 \text{ and } H'T^{(1)}M\bar{J} = 0. \end{cases} \tag{A.15}$$

Except for the cases of (3.9) and $\tau_0 = 0$ in (3.8), Theorem 3.2 follows readily from (A.15). We observe from (A.12) that for any $\rho > 0$ and real τ_0 that

$$\begin{aligned} \Pr(\tau(0) \leq \tau_0) &= \Pr(\rho\tau(0) \leq \rho\tau_0) \\ &= \Pr\left(-\frac{1}{2}\sqrt{\frac{n - k}{H'H}} \frac{\rho H'v - \rho\delta}{\sqrt{v'Mv}} \left[D1 + \frac{H'T^{(1)}H}{H'H} \right] - \rho\gamma(0) \leq \rho\tau_0\right). \end{aligned} \tag{A.16}$$

Setting $\tau_0 = 0$ in (A.16) yields

$$\Pr(\tau(0) \leq 0) = \Pr\left(-\frac{H'v - \delta}{2} \left[D1 + \frac{H'T^{(1)}H}{H'H} \right] - H'T^{(1)}Mv \leq 0\right). \tag{A.17}$$

Note that $H'i = 0$. So we have $\rho H'v = \rho H'\bar{P}\eta + O_p(\rho^2)$ upon setting $\rho = \sqrt{1 - \phi_1^2}$ in (B.4) and (B.6), i.e., $H'v = H'\bar{P}\eta + O_p(\rho)$ as $\phi_1 \rightarrow 1$. Similarly, by the condition $H'T^{(1)}Mi = 0$ we have $H'T^{(1)}Mv = H'T^{(1)}M\bar{P}\eta + O_p(\rho)$ as $\phi_1 \rightarrow 1$. Moreover, note

that $D1 \xrightarrow{p} d1$ when $Mi \neq 0$ (see (A.13)). Therefore, $\Pr(\tau(0) \leq 0) \rightarrow \Pr(-d^*(H'\bar{P}\eta - \delta) - H'T^{(1)}M\bar{P}\eta \leq 0)$, which gives the second equation in (3.8).

To prove (3.9), note that $Mi = 0$ and thus by setting $V = M$ and $u = v/\sigma$ in (B.10), we have $v'Mv \xrightarrow{p} \eta'\bar{P}'M\bar{P}\eta$ as $\phi_1 \rightarrow 1$. Accordingly, we obtain by setting $a = H$ and $u = v/\sigma$ in (B.9) of Appendix B that $\rho H'v/\sqrt{v'Mv} \xrightarrow{p} H'i\xi/\sqrt{\eta'\bar{P}'M\bar{P}\eta}$. In addition, it is readily seen from the last two items in (A.14) that $\rho\gamma(0) = O_p(\rho)$ regardless of whether $H'T^{(1)}MJ = 0$. By these arguments and the second item in (A.13), we have from (A.16) that

$$\Pr(\tau(0) \leq \tau_0) \rightarrow \Pr\left(-\frac{1}{2}\sqrt{\frac{n-k}{H'H}} \frac{H'i\xi}{\sqrt{\eta'\bar{P}'M\bar{P}\eta}} \left[D^{(1)}(\eta) + \frac{H'T^{(1)}H}{H'H} \right] \leq 0\right). \tag{A.18}$$

The probability value on the right-hand side of (A.18) is 1/2 because $\xi \stackrel{d}{=} -\xi$. This proves (3.9) and completes the proof of the theorem. □

Proof of Theorem 5.1. The proof of Theorem 5.1 relies heavily on Theorem B.1 presented in Appendix B, which gives the limiting properties of the t statistic under ARMA(1,1) disturbances and $\phi_1 \rightarrow 1$. Now, if $Mi \neq 0$, then it can be shown that $1/\sqrt{v'Mv} \xrightarrow{p} 0$ when $\phi_1 \rightarrow 1$, where v is defined as in (2.5). In fact, it can be seen from (B.8) in Appendix B that $1/\sqrt{v'Mv} = O_p(\rho)$, where $\rho = \sqrt{1 - \phi_1^2}$. If $Mi = 0$, then it follows from (B.10) in Appendix B that $v'Mv = \eta'\bar{P}'M\bar{P}\eta + O_p(\rho)$. Note that $H'H > 0$, so $H'i = 0$ implies $H'j \neq 0$. Putting $\psi_1 = 0$, $a = \sqrt{(n-k)/H'H}$ and $V = M$ in Theorem B.1 of Appendix B, it follows from (B.2) that

$$t(0) \xrightarrow{p} \begin{cases} \tilde{t}(0)\xi/|\xi| & \text{if } Mi \neq 0 \text{ and } H'i \neq 0, \\ 0 & \text{if } Mi \neq 0 \text{ and } H'i = 0, \\ t^{(1)}(\eta) & \text{if } Mi = 0 \text{ and } H'i = 0. \end{cases} \tag{A.19}$$

With the exception of the fifth item in (5.1) and (5.2), other results in Theorem 5.1 follow readily from (A.19). To prove the fifth item in (5.1), note from Theorem 2.1 that $\Pr(t(0) \leq 0) = \Pr(H'v \leq \delta)$. Now, Appendix B shows that the condition $H'i = 0$ implies $H'v = H'\bar{P}\eta + O_p(\rho)$. Hence the fifth item in (5.1) follows. In order to show (5.2), note from Theorem 2.1 that $\Pr(t(0) \leq t_0) = \Pr(\sqrt{(n-k)/H'H} [\rho H'v - \rho\delta] \leq \rho t_0 \sqrt{v'Mv})$ for any $\rho > 0$. Given the conditions $Mi = 0$ and $H'i \neq 0$ in (5.2), we obtain from (B.9) and (B.10) in Appendix B that $\Pr(t(0) \leq t_0) \rightarrow \Pr(\sqrt{(n-k)/H'H} H'i\xi \leq 0) = \frac{1}{2}$ for any real t_0 . This verifies (5.2) and completes the proof of Theorem 5.1.

Proof of Theorem 6.1. The proof of Theorem 6.1 requires the following lemma from [12]:

Lemma 1. For arbitrary $a \in (0, \infty)$ and $x \in [0, 1]$,

$$a^{1-x} \leq \frac{\Gamma(a+1)}{\Gamma(a+x)} \leq (a+x)^{1-x}. \tag{A.20}$$

Lemma 1 gives bounds for the gamma ratio. These bounds are useful for proving Theorem 6.1.

(i) Denote $t_*(0) = t(0) - \tilde{t}(0)$. To verify Part (i) of Theorem 6.1, we only need to show $E_0[t_*^2(0)] = O(n^{-1})$. Clearly,

$$E_0[t_*^2(0)] = E_0[t^2(0)] + E_0[\tilde{t}^2(0)] - 2E_0[t(0)\tilde{t}(0)]. \tag{A.21}$$

Since $v \sim N(0, I_n)$, we have $H'v/\sqrt{H'H} \sim N(0, 1)$, $v'Mv \sim \chi_{n-k}^2$, and $H'v$ and $v'Mv$ are independent. It thus follows that

$$E_0[t_*^2(0)] = 2\varepsilon_n \left\{ 1 + \frac{\delta^2}{H'H} \right\}, \tag{A.22}$$

where

$$\varepsilon_n = 1 - \frac{\sqrt{n-k}\Gamma\left(\frac{n-k-1}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n-k}{2}\right)} + \frac{1}{n-k-2}. \tag{A.23}$$

By Lemma 1, we observe that $1/(n-k-2) \geq \varepsilon_n \geq 1/(n-k-2) + 1 - \sqrt{(n-k)/(n-k-2)} = o(1)/(n-k-2)$, namely $\varepsilon_n = O(n^{-1})$. Part (i) of Theorem 6.1 thus follows.

(ii) To prove Part (ii) of Theorem 6.1, it is sufficient to show that

$$\hat{\tau} \triangleq \frac{t(0)}{2} \frac{v'MAMv}{v'Mv} - \frac{\sqrt{n-k} H'AMv}{\sqrt{v'Mv} \sqrt{H'H}} = \frac{\tilde{t}(0)\text{tr}[AM]}{2(n-k)} - \frac{H'AMv}{\sqrt{H'H}} + O_p(n^{-1/2}). \tag{A.24}$$

It follows from $v'Mv \sim \chi_{n-k}^2$ that $E_0 \left[1 - \sqrt{n-k}/\sqrt{v'Mv} \right]^2 = 2\varepsilon_n = O(n^{-1})$. Accordingly, $\sqrt{n-k}/\sqrt{v'Mv} = 1 + O_p(n^{-1/2})$. Hence, to verify (A.24), it is sufficient to show that

$$\frac{v'MAMv}{v'Mv} = \frac{\text{tr}[AM]}{n-k} + O_p(n^{-1/2}). \tag{A.25}$$

Clearly,

$$E_0 \left\{ \frac{v'MAMv}{v'Mv} - \frac{\text{tr}[AM]}{n-k} \right\}^2 = E_0 \left\{ \frac{v'MAMv}{v'Mv} \right\}^2 - \frac{2\text{tr}[AM]}{n-k} E_0 \left\{ \frac{v'MAMv}{v'Mv} \right\} + \frac{\text{tr}^2[AM]}{(n-k)^2}. \tag{A.26}$$

From the proof of Theorem 3.1 we observe that

$$E_0 \left\{ \frac{v'MAMv}{v'Mv} \right\} = \frac{\text{tr}[AM]}{n-k} \tag{A.27}$$

and

$$E_0 \left\{ \frac{v'MAMv}{v'Mv} \right\}^2 = \frac{\text{tr}^2(AM) + 2\text{tr}(AM)^2}{(n-k)(n-k+2)}. \tag{A.28}$$

Substituting (A.27) and (A.28) into (A.26), we obtain

$$\begin{aligned} E_0 \left\{ \frac{v'MAMv}{v'Mv} - \frac{\text{tr}[AM]}{n-k} \right\}^2 &= \frac{\text{tr}^2(AM) + 2\text{tr}(AM)^2}{(n-k)(n-k+2)} - \frac{\text{tr}^2[AM]}{(n-k)^2} \\ &= \frac{2}{(n-k)(n-k+2)} \left[\text{tr}(AM)^2 - \frac{\text{tr}^2(AM)}{n-k} \right] \geq 0. \end{aligned} \tag{A.29}$$

Since the eigenvalues of A are bounded, it follows that $\text{tr}(AM)^2 \leq (n-k)\mu^2$ for some constant $\mu^2 < \infty$. Thus, we observe (A.25) from (A.27).

Appendix B. Some results on the limiting properties of the t statistic

In this Appendix we derive some results on the limiting properties of the t statistic under the case of an ARMA(1,1) process as $\phi_1 \rightarrow 1$. The results given here are vital to the proof of Theorem 5.1 but are also of independent interest. The similar conclusions were given by Banerjee and Magnus [3] for F statistic. Now, let u be an $n \times 1$ vector of disturbances with mean zero and covariance matrix $\sigma^2\Omega$, a be an arbitrary $n \times 1$ non-zero vector, and V be a positive semi-definite $n \times n$ symmetric matrix with non-zero rank. We define the t ratio as

$$\gamma = \frac{a'u}{\sqrt{u'Vu}}. \tag{B.1}$$

Theorem B.1. Suppose that the elements in u are generated by a stationary ARMA(1,1) process, i.e., $u_t = \phi_1 u_{t-1} + \psi_1 \varepsilon_{t-1} + \varepsilon_t$ where ε_t 's are iid $N(0, \sigma^2)$ variables, $|\phi_1| < 1$ and $|\psi_1| < 1$. When $\phi_1 \rightarrow 1$, we have the following results:

(i) If $V_i = 0$ implies $a'i = 0$, then

$$\gamma \xrightarrow{P} \begin{cases} \frac{a'i}{\sqrt{i'Vi}} \frac{\xi}{|\xi|} & \text{if } V_i \neq 0 \text{ and } a'i \neq 0, \\ 0 & \text{if } V_i \neq 0 \text{ and } a'i = 0, \\ \frac{a'\bar{P}\eta}{\sqrt{\eta'\bar{P}'V\bar{P}\eta}} & \text{if } V_i = 0 \text{ and } a'J \neq 0, \end{cases} \tag{B.2}$$

where, again, $\xi \sim N(0, 1)$, $(\xi, \eta)' \sim N(0, I_n)$, $\bar{P} = \bar{J}P$, $\bar{J}' = [0_{(n-1) \times 1} | I_{n-1}]$, and P is an $(n-1) \times (n-1)$ lower triangular matrix such that the (i, j) th element of PP' is $\min\{i, j\} - \psi_1(1 + \delta_{ij})/(1 + \psi_1)^2$, and δ_{ij} is the Kronecker sign.

(ii) If $V_i = 0$ but $a'i \neq 0$, then for any real γ_0 , both $\Pr(\gamma > \gamma_0)$ and $\Pr(\gamma \leq \gamma_0)$ tend to $\frac{1}{2}$.

Proof. According to the proof of Theorem B.1 in [3] the matrix $(1 - \phi_1^2)\Omega(\phi_1)$ can be expressed as $(1 - \phi_1^2)\Omega(\phi_1) = LL'$, where

$$L = L_0 + \rho L_1 - \frac{1}{2}\rho^2 L_2 + \rho^3 L_3 + O(\rho^4) \tag{B.3}$$

as $\phi_1 \rightarrow 1, \rho = \sqrt{1 - \phi_1^2}, L_0 = [i|0_{n \times (n-1)}], L_1 = \text{diag}(0, P), L_2 = [l|0_{n \times (n-1)}]$ with the components of l being $l_1 = \rho/(1 + \rho)^2, l_s = s - l_1 - 1$ for $2 \leq s \leq n, L_3 = \text{diag}(0, \bar{L}_3)$, and \bar{L}_3 is an $(n - 1) \times (n - 1)$ lower triangular matrix. It is readily seen that when $\phi_1 \rightarrow 1,$

$$a'L = \begin{cases} a'L_0 + O(\rho) & \text{if } a'i \neq 0, \\ (0, \rho a'\bar{P}) + O(\rho^2) & \text{if } a'i = 0 \text{ and } a'J \neq 0, \\ O(\rho^2) & \text{if } a'i = 0 \text{ and } a'J = 0, \end{cases} \tag{B.4}$$

and

$$L'VL = \begin{cases} \text{diag}(i'Vi, 0_{(n-1) \times (n-1)}) + O(\rho) & \text{if } Vi \neq 0, \\ \text{diag}(0, \rho^2 \bar{P}'V\bar{P}) + O(\rho^3) & \text{if } Vi = 0. \end{cases} \tag{B.5}$$

Clearly, $\gamma = a'v/\sqrt{v'Vv}$, where $v = u/\sigma$. Since $\rho v = L(\xi, \eta)'$ with $\xi \sim N(0, 1)$ and $(\xi, \eta)' \sim N(0, I_n)$, we have

$$\rho a'v = a'L(\xi, \eta)' \tag{B.6}$$

and

$$\rho^2 v'Vv = (\xi, \eta)'L'VL(\xi, \eta)' \tag{B.7}$$

Now, **Theorem B.1** can be verified using (B.4)–(B.7).

Case (i). Consider item 1 in (B.2). It follows from (B.7) and the first item in (B.5) that

$$\rho^2 v'Vv = i'Vi\xi^2 + O_p(\rho). \tag{B.8}$$

Additionally, it follows from (B.6) and the first item in (B.4) that

$$\rho a'v = a'i\xi + O_p(\rho). \tag{B.9}$$

Accordingly, the first item in (B.2) follows from (B.1), (B.8) and (B.9).

Next consider item 2 in (B.2). Given that $a'J \neq 0$, it follows from the second item in (B.4) that $\rho a'v = \rho a'\bar{P}\eta + O_p(\rho^2)$. So we obtain from (B.1) and (B.8) that $\gamma = O_p(\rho)$. In the case of $a'J = 0$, it follows from the third item in (B.4) that $\rho a'v = O_p(\rho^2)$. We therefore observe from (B.1) and (B.8) that $\gamma = O_p(\rho^2)$. This proves the second item in (B.2).

To consider item 3 in (B.2), note that by (B.7) and the second item in (B.5),

$$\rho^2 v'Vv = \rho^2 \eta' \bar{P}'V\bar{P}\eta + O_p(\rho^3). \tag{B.10}$$

Since $a'i = 0$ and $a'J \neq 0, \rho a'v = \rho a'\bar{P}\eta + O_p(\rho^2)$ (see the proof of item 2 in (B.2) above). Combining this with (B.10), the third item in (B.2) follows immediately.

Case (ii). Note that $\Pr(\gamma \leq \gamma_0) = \Pr(a'v \leq \gamma_0 \sqrt{v'Vv})$. It follows from (B.9) and (B.10) that $\Pr(\gamma \leq \gamma_0) = \Pr(a'i\xi + O_p(\rho) \leq \rho\gamma_0 \sqrt{\eta' \bar{P}'V\bar{P}\eta + O_p(\rho)}) \rightarrow \Pr(a'i\xi \leq 0) = \frac{1}{2}$ and $\Pr(\gamma > \gamma_0) = 1 - \Pr(\gamma \leq \gamma_0) \rightarrow \frac{1}{2}$. This completes the proof to **Theorem B.1**. □

Two aspects of **Theorem B.1** are of particular interest and deserve mention here. First, **Theorem B.1** covers all possible cases except for the trivial case of $Vi = 0, a'i = 0$ and $a'J = 0$. This case is trivial because it leads to $a = 0$. Second, the normality assumption is not necessary. Although the proof is not given here, it can be readily shown that **Theorem B.1** carries over to the wider disturbance term assumption of elliptical symmetry.

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