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The Burau matrix and Fiedler’s invariant for a closed braid

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Abstract

It is shown how Fiedler’s ‘small state-sum’ invariant for a braid β can be calculated from the 2-variable Alexander polynomial of the link which consists of the closed braid $\hat{\beta}$ together with the braid axis A . © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a recent paper [1] Fiedler introduced a simple invariant for a knot K in a line bundle over a surface F by means of a ‘small’ state-sum, which keeps a count of features of the links resulting from smoothing each crossing of the projection of K on F . The invariant takes values in a quotient of the integer group ring of $H_1(F)$. Fiedler gives a number of applications of his general construction. In particular, where K is a closed braid, and can thus be regarded as a knot in a solid torus V , his method gives an invariant of a braid $\beta \in B_n$ in $\mathbb{Z}[H_1(V)] = \mathbb{Z}[x^{\pm 1}]$ modulo the relation $x^n = 1$. This invariant depends only on the closure of the braid in V and hence gives an invariant of β up to conjugacy in B_n . Its behaviour under Birman and Menasco’s exchange moves has been used to help in detecting when two braids may be related by such a move.

The purpose of this paper is to show how Fiedler’s invariant for a closed braid $\hat{\beta}$ can be found in terms of the Burau representation of β , and hence from the 2-variable Alexander polynomial of the link $\hat{\beta} \cup A$ consisting of the closed braid $\hat{\beta}$ and its axis A . Its construction

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here from the Alexander polynomial can be compared with methods which yield Vassiliev invariants of degree 1 in other contexts, and suggests possible interpretations of Fiedler’s invariants as Vassiliev invariants of degree 1 in the line bundle.

Having seen how the special case of Fiedler’s invariant is related to an Alexander polynomial I finish the paper with a suggestion of extracting similar invariants from the 2-variable Alexander polynomial of a more general 2-component link. These might be regarded as degree 1 Vassiliev invariants of one component of the link when considered as a knot in the complement of the other component. It would be interesting to know if there was any similar state sum interpretation of these invariants in the more general setting.

2. Burau matrices

I make use of the fact that the 2-variable Alexander polynomial $\Delta_{\hat{\beta} \cup A}(t, x)$ of a closed braid and its axis can be calculated as the characteristic polynomial, $\det(I - x\overline{B}(t))$, of the reduced $(n - 1) \times (n - 1)$ Burau matrix $\overline{B}(t)$ of the braid β , [2]. Since the full $n \times n$ Burau matrix $B(t)$ is conjugate to $\begin{pmatrix} \overline{B}(t) & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix}$ we can write

$$(1 - x)\Delta_{\hat{\beta} \cup A}(t, x) = \det(I - xB(t)).$$

Put $t = e^h$ in $\det(I - xB(t)) = 1 + b_1(t)x + \dots + b_n(t)x^n$, and expand this as a power series in h to give

$$\det(I - xB(e^h)) = \sum_{i=0}^{\infty} a_i(x)h^i,$$

where each coefficient $a_i(x)$ is a polynomial in x of degree at most n .

When we set $h = 0$, and thus $t = 1$, we must get $\Delta_A(x) \times (1 - x^n)$ by the Torres–Fox formula, since the two components A and $\hat{\beta}$ of the link have linking number n . Hence $a_0(x) = 1 - x^n$. Setting $x = 0$ shows also that $a_1(x) = f_1x + f_2x^2 + \dots + f_nx^n$ for some integers f_1, \dots, f_n . We know that the determinant of the Burau matrix is $(-t)^{w(\beta)}$, where $w(\beta)$ is the writhe of the braid, and so $b_n(t) = (-1)^n(-t)^{w(\beta)}$. Now $w(\beta) = n - 1 \pmod 2$ since β closes to a single component. Hence $b_n(e^h) = -1 - w(\beta)h + O(h^2)$, giving $f_n = -w(\beta)$. We shall relate the remaining coefficients f_1, \dots, f_{n-1} directly to Fiedler’s invariant.

3. Fiedler’s braid invariant

Fiedler’s invariant F_β for an n -braid β which closes to a single curve is a symmetric Laurent polynomial, which is even or odd depending on the parity of n . Suppose that the braid

$$\beta = \prod_{r=1}^k \sigma_{i_r}^{\varepsilon_r}$$

has been given in terms of the Artin generators σ_i , where $\varepsilon_r = \pm 1$. Suppose that the product reads from top to bottom in the braid and the strings are oriented downwards. For the

r th crossing define a positive integer $m(r)$ by smoothing the crossing and following the ‘ascending string’ at the smoothed crossing around the closed braid until it closes again after $m(r)$ turns around the axis. Here the ascending string means the string which starts from the end of the overcrossing, and is thus string i_r for a positive crossing and string $i_r + 1$ for a negative crossing. Fiedler’s polynomial $F_\beta(X)$ is defined as a sum over the k crossings of β by

$$F_\beta(X) = \sum_{r=1}^k \varepsilon_r X^{2m(r)-n}.$$

For a given m we can then write the coefficient of X^{2m-n} as $\sum_{m(r)=m} \varepsilon_r$.

Theorem 1. *Let the n -string braid β have Burau matrix $B(t)$, and write $\det(I - xB(e^h)) = a_0(x) + a_1(x)h + O(h^2)$. Fiedler’s polynomial for β satisfies*

$$F_\beta(x^{1/2}) = (f_1x + \dots + f_{n-1}x^{n-1})x^{-(n/2)},$$

where $a_1(x) = f_1x + \dots + f_{n-1}x^{n-1} + f_nx^n$.

Proof. Use the classical trace formula for the characteristic polynomial of a matrix B . Suppose that B has eigenvalues $\lambda_1, \dots, \lambda_n$. Then B^m has eigenvalues $\lambda_1^m, \dots, \lambda_n^m$ and $\det(I - xB) = \prod_{i=1}^n (1 - x\lambda_i)$. Hence

$$\begin{aligned} \ln(\det(I - xB)) &= \sum_{i=1}^n \ln(I - x\lambda_i) = - \sum_{m=1}^{\infty} \sum_{i=1}^n \frac{1}{m} x^m \lambda_i^m \\ &= - \sum_{m=1}^{\infty} \frac{x^m}{m} \operatorname{tr}(B^m), \end{aligned}$$

as power series in x .

Now expand $\ln(a_0(x) + a_1(x)h + \dots)$ as a power series in h , only as far as the term in h . We get

$$\begin{aligned} \ln(a_0(x) + a_1(x)h + \dots) &= \ln a_0(x) + \ln\left(1 + \frac{a_1(x)}{a_0(x)}h + O(h^2)\right) \\ &= \ln a_0(x) + \frac{a_1(x)}{a_0(x)}h + O(h^2) \\ &= -x^n - x^{2n}/2 - \dots + h(f_1x + f_2x^2 + \dots + f_nx^n) \\ &\quad \times (1 + x^n + x^{2n} + \dots) + O(h^2). \end{aligned}$$

The trace formula above applied to $B(e^h)$ shows at once that $\operatorname{tr}((B(e^h))^m) = -mf_mh + O(h^2)$ for $1 \leq m < n$.

The proof will be completed by relating the term in h in the trace of this matrix to the appropriate coefficient of Fiedler’s polynomial. It is thus enough to show that

$$\operatorname{tr}((B(e^h))^m) = -m \left(\sum_{m(r)=m} \varepsilon_r \right) h + O(h^2) \quad \text{for } 1 \leq m < n.$$

The Burau representation $\rho : B_n \rightarrow GL(n, \mathbb{Z}[t^{\pm 1}])$ is the group homomorphism defined on the elementary braid σ_i by

$$\rho(\sigma_i) = B_i = \left(\begin{array}{c|cc} I_{i-1} & 0 & 0 \\ \hline 0 & 1-t & t \\ & 1 & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array} \right).$$

The Burau matrix for the given braid β is then

$$B(t) = \rho(\beta) = \prod_{r=1}^k B_{i_r}^{\varepsilon_r}.$$

Now

$$\begin{aligned} B_i(e^h) &= \left(\begin{array}{c|cc} I_{i-1} & 0 & 0 \\ \hline 0 & 0 & 1 \\ & 1 & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array} \right) + h \left(\begin{array}{c|cc} 0_{i-1} & 0 & 0 \\ \hline 0 & -1 & 1 \\ & 0 & 0 \\ \hline 0 & 0 & 0_{n-i-1} \end{array} \right) + O(h^2) \\ &= T_i + hP_i^+ + O(h^2), \text{ say.} \end{aligned}$$

We can similarly write $B_i^{-1} = T_i + hP_i^- + O(h^2)$ where

$$P_i^- = \left(\begin{array}{c|cc} 0_{i-1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ & -1 & 1 \\ \hline 0 & 0 & 0_{n-i-1} \end{array} \right).$$

Then

$$(B(e^h))^m = \left(\prod_{r=1}^k (T_{i_r} + hP_{i_r}^{\pm}) \right)^m + O(h^2).$$

We can write a matrix of the form $M = \prod_{r=1}^l (C_r + hD_r)$ as

$$\begin{aligned} M &= C_1 C_2 \cdots C_l + h(D_1 C_2 \cdots C_l + C_1 D_2 C_3 \cdots C_l + \cdots \\ &\quad + C_1 C_2 \cdots C_{l-1} D_l) + O(h^2), \end{aligned}$$

and then

$$\text{tr } M = \text{tr}(C_1 C_2 \cdots C_l) + h(\text{tr}(D_1 C_2 \cdots C_l) + \text{tr}(C_1 D_2 C_3 \cdots C_l) + \cdots) + O(h^2).$$

The term in h can be rewritten as

$$\text{tr}(C_2 \cdots C_l D_1) + \text{tr}(C_3 \cdots C_l C_1 D_2) + \cdots + \text{tr}(C_1 C_2 \cdots C_{l-1} D_l)$$

by cycling the matrices so that the r th product ends with the matrix D_r .

Apply this to find the term in h in $\text{tr}((B(e^h))^m)$ as the sum of mk terms, each of which is the trace of the product of mk matrices of the form $T_{i_{r+1}} \cdots T_{i_{r-1}} P_{i_r}^\pm$ with sign \pm according to the sign of ε_r . For each of the k crossings of the original braid the matrix $T_{i_{r+1}} \cdots T_{i_{r-1}} P_{i_r}^\pm$ occurs m times in the sum. Thus

$$f_m = - \sum_{r=1}^k \text{tr}(T_{i_{r+1}} \cdots T_{i_{r-1}} P_{i_r}^\pm).$$

The proof of Theorem 1 will be completed by showing that

$$\text{tr}(T_{i_{r+1}} \cdots T_{i_{r-1}} P_{i_r}^\pm) = \begin{cases} -\varepsilon_r & \text{if } m(r) = m, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix T_i is the permutation matrix for the transposition $(i \ i + 1)$. Hence a product of these matrices is also a permutation matrix, T say, whose permutation is the product π of the corresponding transpositions. Then the entries in T satisfy

$$T_{ij} = \begin{cases} 1 & \text{if } i = \pi(j), \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $T = T_{i_{r+1}} \cdots T_{i_{r-1}}$ above is thus a permutation matrix with permutation $\pi_r^{(m)}$, say. Notice that the permutation corresponding to the product TT_{i_r} is conjugate to the permutation of the braid β^m . Under the assumption that β closes to a single curve this will be the m th power of an n -cycle, and will hence not fix any number when $1 \leq m < n$. Hence $\pi_r^{(m)}$ cannot carry i_r to $i_r + 1$ or vice versa, in this range.

If the r th crossing is smoothed and the strings i_r and $i_r + 1$ are followed upwards around the braid m times, with $m < n$, they will not pass through the smoothed crossing. They then become the strings $\pi_r^{(m)}(i_r)$ and $\pi_r^{(m)}(i_r + 1)$, respectively when they return to the level of the bottom of the r th crossing. Now when $\varepsilon_r = +1$ the ascending string at the r th crossing, which is string i_r , returns to position i_r after the permutation $\pi_r^{(m)}$ if and only if $m = m(r)$. Similarly when $\varepsilon_r = -1$ the ascending string, in this case string $i_r + 1$ returns to position $i_r + 1$ exactly when $m = m(r)$.

The matrices $P_{i_r}^\pm$ have only two nonzero entries. Suppose first that $\varepsilon_r = +1$. Then $\text{tr}(T P_{i_r}^+)$ is the sum of two terms. The off-diagonal entry gives a contribution only if the permutation matrix T maps it onto the diagonal. This requires $\pi_r^{(m)}(i_r) = i_r + 1$, which was excluded above. The diagonal entry contributes -1 if and only if $\pi_r^{(m)}(i_r) = i_r$, which is the condition that $m = m(r)$. Thus when $\varepsilon_r = +1$ we get a contribution of $-\varepsilon_r$ to the trace if and only if $m = m(r)$, and zero otherwise.

A similar argument holds when $\varepsilon_r = -1$. Again the off-diagonal entry does not contribute to the trace, while the diagonal entry contributes $+1$ if and only if $\pi_r^{(m)}(i_r + 1) = i_r + 1$. This corresponds once more to the condition that $m = m(r)$, and so in each case we have a contribution of $-\varepsilon_r$ if and only if $m = m(r)$. The total coefficient of h in $\text{tr}((B(e^h))^m)$ is then $-m \sum_{m=m(r)} \varepsilon_r$, showing that $f_m = \sum_{m=m(r)} \varepsilon_r$ as claimed. This completes the proof of Theorem 1. \square

4. Determination from an Alexander polynomial

If we are given the Alexander polynomial of the closed braid $\hat{\beta}$ and its axis A as a 2-variable polynomial we can recover Fiedler's invariant for the braid. First multiply by $1 - x$, where x is the variable for the axis. This gives the characteristic polynomial of the Burau matrix for β , up to multiplication by a power of x and a power of t , and a sign. Put $t = e^h$ and expand as a power series in h with coefficients depending on x . Then multiply by a power of x and a sign to make the constant term $1 - x^n$. The result will be the characteristic polynomial used above, up to a power of $t = e^h$. Extract the coefficient $f_0 + f_1x + \cdots + f_{n-1}x^{n-1} + f_nx^n$ of h . This will contain the Fiedler polynomial as before in the terms $f_1x + \cdots + f_{n-1}x^{n-1}$, while the remaining terms will come from a factor of t^{f_0} and will satisfy $f_0 + f_n = -w(\beta)$.

A similar interpretation looks plausible for the coefficients of the linear terms in h_1, \dots, h_k when the Alexander polynomial of a closed braid with k components and its axis is expanded in terms of the meridian generator x for the axis and meridians $t_i = e^{h_i}$ for the components. This polynomial can again be written in terms of the characteristic polynomial of a suitable 'colored' Burau matrix. The eventual coefficient of h_i should then have contributions from the overcrossings of the corresponding component of the closed braid, as in the Fiedler polynomial above.

As a possible extension to the case of a general link L with two components X and T say, we might put $t = e^h$ in the Alexander polynomial $\Delta_{X \cup T}(x, t)$ of L and consider only the terms $a_0(x) + a_1(x)h$ up to degree 1 in h . The polynomial $a_0(x)$ is $\Delta_X(x)(1 - x^n)/(1 - x)$, where n is the linking number of X and T , and $\Delta_X(x)$ is the Alexander polynomial of X . Now consider $a_1(x)$ as a polynomial modulo the ideal generated by $a_0(x)$. This is an invariant of L as it is unaffected by any ambiguity of powers of x and t in the Alexander polynomial. This seems to me to be the nearest analogue to Fiedler's invariant for the link component T with meridian t when regarded as a knot in the complement of X ; in the case of a closed braid we take X as the braid axis and T as the closed braid. It looks likely to be a Vassiliev invariant of type 1 for knots in the complement of X . There is not, however, any obvious candidate for a state-sum construction of this invariant along Fiedler's lines when the component X is knotted.

References

- [1] T. Fiedler, A small state-sum invariant for knots, *Topology* 32 (1993) 281–294.
- [2] H.R. Morton, Infinitely many fibred knots having the same Alexander polynomial, *Topology* 17 (1978) 101–104.