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# A United Approach to Accelerating Trigonometric Expansions

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**Abstract**—The Fourier series of a smooth function on a compact interval usually has slow convergence due to the fact that the periodic extension of the function has jumps at the interval endpoints. For various symmetry conditions polynomial interpolation methods have been developed for performing a boundary correction. The resulting variants of Krylov approximants are a sum of a correction polynomial and a Fourier sum of the corrected function [1-8]. In this paper, we review these methods and derive estimates in the maximum norm. We further show that derivatives of the Krylov approximants are again Krylov approximants of derivatives of the considered function. This enables us to give a unified treatment of the problem of simultaneous approximation.

**Keywords**—Fourier, Sine, Cosine, Series expansion, Simultaneous approximation, Convergence acceleration.

## 1. INTRODUCTION

We start with a description of different forms of Fourier series of a $2\pi$-periodic sufficiently smooth real valued function $f$ depending on symmetry conditions [9].

The *trigonometric form* is given by

$$f = \frac{1}{2}\,a_0(f) + \sum_{n=1}^{\infty} \left(a_n(f)\,\cos nx + b_n(f)\,\sin nx\right).$$

The real Fourier coefficients are determined by

$$a_n(f) := \frac{1}{\pi}\int_0^{2\pi} f(y)\,\cos ny\,dy, \qquad n = 0, 1, \dots,$$
$$b_n(f) := \frac{1}{\pi}\int_0^{2\pi} f(y)\,\sin ny\,dy, \qquad n = 1, 2, \dots .$$

The *complex exponential form* is given by

$$f = \sum_{n=-\infty}^{\infty} c_n(f)\,e_n, \qquad e_n(x) := e^{inx}$$

We have been unable to communicate with the author(s) with respect to galley proof corrections. Hence, this work is published without the benefit of such corrections. (Ed.)

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with complex Fourier coefficients

$$c_n(f) := \frac{1}{2\pi} \int_0^{2\pi} f(y)\, e_{-n}(y)\, dy, \qquad n = 0, \pm 1, \dots .$$

We have

$$a_0(f) = 2c_0(f), \quad a_n(f) - ib_n(f) = 2c_n(f), \quad a_n(f) + ib_n(f) = 2c_{-n}(f), \quad n = 1, 2, \dots .$$

A Fourier series without any further symmetry conditions is called a *waveform*. As special cases, two types of symmetries can occur:

(I') $f(-x) = f(x)$

Then the Fourier series is called an *even waveform*. Its Fourier series is given by the *Fourier cosine series*

$$f(x) = \frac{1}{2}\, a_0(f) + \sum_{n=1}^{\infty} a_n(f) \cos nx$$

with the simplified Fourier coefficients

$$a_n(f) = \frac{2}{\pi} \int_0^{\pi} f(y) \cos ny\, dy, \qquad n = 0, 1, \dots .$$

(I'') $f(-x) = -f(x)$

Then the Fourier series is called an *odd waveform*. Its Fourier series is given by the *Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n(f) \sin nx$$

with simplified Fourier coefficients

$$b_n(f) = \frac{2}{\pi} \int_0^{\pi} f(y) \sin ny\, dy, \qquad n = 1, 2, \dots .$$

A periodic function $f$ is called *antiperiodic* if it satisfies the symmetry condition

(II) $f(x + \pi) = -f(x)$.

In this case the Fourier series is called a *half-waveform*. It possesses the form

$$f = \sum_{n=-\infty}^{\infty} c_{2n-1}(f)\, e_{2n-1}$$

with simplified Fourier coefficients

$$c_{2n-1}(f) = \frac{1}{\pi} \int_0^{\pi} f(y)\, e_{-2n+1}(y)\, dy, \qquad n = 0, \pm 1, \dots .$$

As in the general waveform case, the half-waveform can have two types of symmetries.

(II') $f(-x) = f(x), \quad f(x + \pi) = -f(x)$.

Then the Fourier series is called an *even half-waveform*. Its Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} a_{2n-1}(f) \cos(2n-1)x$$

with the simplified Fourier coefficients

$$a_{2n-1}(f) = \frac{4}{\pi} \int_0^{\pi/2} f(y) \cos(2n-1)y\, dy, \qquad n = 1, 2, \dots .$$

(II″) $f(-x) = -f(x), \quad f(x+\pi) = -f(x)$.

Then the Fourier series is called an *odd half-waveform*. Its Fourier series is given by the *Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_{2n-1}(f) \sin(2n-1)x$$

with simplified Fourier coefficients

$$b_{2n-1}(f) = \frac{4}{\pi} \int_0^{\pi/2} f(y) \sin(2n-1)y)\, dy, \qquad n = 1, 2, \dots .$$

In general, these Fourier series converge slowly for smooth (sufficiently differentiable) functions due to the fact that certain boundary conditions related to the symmetry assumptions are violated. There are different methods to accelerate the convergence of these series by polynomial correction [1–8].

The aim of this paper is to present a unified derivation of these accelerating methods by using only the Euler representation of a smooth function. In this way we obtain explicit integral remainder representations in terms of the Bernoulli functions for all types of acceleration methods. Moreover, using the concept of Korobov spaces we will derive simultaneous error estimates for the acceleration methods.

## 2. WAVEFORMS

The first Bernoulli function $B_1$ is given by

$$B_1(x) := \pi - x, \quad 0 < x < 2\pi, \quad B_1(0) = B_1(2\pi) := 0, \quad B_1(x) = B_1(x + 2\pi).$$

It possesses the Fourier series

$$B_1(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (in)^{-1} e_n.$$

The $q^{\text{th}}$ Bernoulli function $B_q$, $q = 2, 3, \dots$, is defined by

$$B_q(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (in)^{-q} e_n.$$

For $q = 2, 3, \dots$ we have

$$DB_q = B_{q-1}. \tag{2.1}$$

where $D$ denotes the derivative. $B_{2r}$ is even, $B_{2r-1}$ is odd. Note that $B_q$ restricted to $(0, 2\pi)$ coincides with the $q^{\text{th}}$ Bernoulli polynomial.

A $2\pi$-periodic square integrable function $f$ is said to be an element of the Korobov space $\mathcal{E}^q$, $q \geq 1$ if

$$|c_n(f)| = \mathcal{O}(|n|^{-q}), \qquad |n| \to \infty.$$

Thus the function $B_q$ is an element of $\mathcal{E}^q$.

We denote by $C_{2\pi}^q$ the set of all $2\pi$-periodic functions $f$ on $\mathbb{R}$ such that all derivatives $D^j f$, $j = 0, \dots, q$ are continuous. Let $\mathcal{P}C_{2\pi}^q$ be the set of all $2\pi$-periodic functions with piecewise continuous $D^j f$, $j = 0, \dots, q$. Let $\Pi_q$ be the set of all polynomials of degree $\leq q$. Any $g \in \mathcal{C}^1[0, 2\pi]$ has a unique periodic extension which is in $\mathcal{E}^1$ in view of the relation $c_n(Dg) = c_0(Dg) + in\, c_n(g)$. More general, if $g \in \mathcal{P}C_{2\pi}^1$ then $g \in \mathcal{E}^1$.

Moreover, it is shown in [10, p. 115] that $g \in C^q_{2\pi} \cap C^{q+2}[0,2\pi]$, $q = 0,1,\ldots$ implies $g \in \mathcal{E}^{q+2}$. Since $\mathcal{E}^q$ is shift-invariant any function $g \in C^q_{2\pi} \cap \mathcal{P}C^{q+2}_{2\pi}$ is also in $\mathcal{E}^{q+2}$. For more information on Korobov spaces, see [10, pp. 107–115].

Let $L^1$ be the Banach space of all $2\pi$-periodic functions which are absolutely integrable on $[0,2\pi]$. The $2\pi$-periodic (or waveform) convolution of $f,g \in L^1$ is defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)\,g(y)\,dy.$$

We have $f * g \in L^1$. By the periodic convolution property, we have

$$c_n(f * g) = c_n(f)\,c_n(g), \qquad n = 0, \pm 1, \ldots. \tag{2.2}$$

Note that the convolution is antiperiodic if at least one factor is antiperiodic. If both factors are even or odd, then the convolution is even. If one factor is even and the other is odd, then the convolution is odd.

For $f \in C^1_{2\pi}, g \in L^1$ it follows that

$$D(f * g) = (Df) * g.$$

Let $f \in C^q[0,2\pi]$, $q = 1,2,\ldots$ be given. We set

$$F_q(f)(x) := (B_q * D^q f)(x) = \frac{1}{2\pi}\int_0^{2\pi} B_q(x-y) D^q f(y)\,dy. \tag{2.3}$$

Partial integration yields the following proposition.

PROPOSITION 2.1. *Let $f \in C^q[0,2\pi]$, $q = 1,2,\ldots$ be given. Then we have the recursion formula*

$$F_q(f) = \frac{D^{q-1}f(2\pi) - D^{q-1}f(0)}{2\pi}\,B_q + F_{q-1}(f), \qquad q = 2,3,\ldots \tag{2.4}$$
$$F_1(f) = \frac{f(2\pi) - f(0)}{2\pi}\,B_1 + f - c_0(f)$$

*and the Euler decomposition* [11, p.108f]

$$f = M_q(f) + F_q(f)$$

*with the correcting polynomial (restricted on $(0,2\pi)$)*

$$\begin{aligned} M_q(f)(x) &:= -\sum_{j=0}^{q} c_0(D^j f)\,B_j(x) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(y)\,dy - \sum_{j=0}^{q-1} \frac{D^j f(2\pi) - D^j f(0)}{2\pi}\,B_j(x) \in \Pi_q \quad (0 < x < 2\pi) \end{aligned} \tag{2.5}$$

*(where we set $B_0(x) := -1$) and the corrected function $F_q(f)$ of the form* (2.3).

The *Krylov method* for accelerating the convergence of the Fourier series of a given function $f \in C^q[0,2\pi]$ is based on approximating the corrected function $F_q(f)$ by its $n^{\text{th}}$ Fourier sum. The $n^{\text{th}}$ Fourier sum of the function $f \in C^q[0,2\pi]$ is denoted by

$$G_n(f) := \sum_{k=-n}^{n} c_k(f)\,e_k.$$

The sequence $G_n(f)$, $n = 1,2,\ldots$ converges in the square mean to $f$.

The *Krylov approximant* of $f \in \mathcal{C}^q[0,2\pi]$ is defined by

$$K_{q,n}(f) := M_q(f) + G_n(F_q(f)). \tag{2.6}$$

Note that by (2.5), (2.6) the remainder satisfies

$$f - K_{q,n}(f) = F_q(f) - G_n(F_q(f)).$$

$G_n(F_q(f))$ is the $n^{\text{th}}$ partial sum of the rapidly converging Fourier series of the corrected function $F_q(f)$.

PROPOSITION 2.2. *Let $f \in \mathcal{C}^{q+1}[0,2\pi]$, $q = 1,2,\dots$ be given. Then*

$$\|f - K_{q,n}(f)\|_\infty = \mathcal{O}(n^{-q}), \quad n \to \infty.$$

PROOF. It follows from the definition (2.3) of $F_q(f)$ and from the convolution property (2.2) that

$$c_k(F_q(f)) = c_k(B_q)\, c_k(D^q f) = \begin{cases} (ik)^{-q} c_k(D^q f), & k = \pm 1, \pm 2, \dots, \\ 0, & k = 0. \end{cases}$$

Since $D^q f \in \mathcal{C}^1[0,2\pi]$, we have $D^q f \in \mathcal{E}^1$, and therefore,

$$c_k(D^q f) = \mathcal{O}(k^{-1}), \qquad |k| \to \infty.$$

This implies

$$c_k(F_q(f)) = \mathcal{O}(k^{-q-1}), \qquad |k| \to \infty.$$

Hence,

$$\|F_q(f) - G_n(F_q(f))\|_\infty = \mathcal{O}\left(\sum_{k=n+1}^{\infty} k^{-q-1}\right) = \mathcal{O}(n^{-q}).$$

Since $f - K_{q,n}(f) = F_q(f) - G_n(F_q(f))$, the proof if complete. ∎

Again let $f \in \mathcal{C}^q[0,2\pi]$, $q = 1,2,\dots$ be given. Our next purpose is to investigate *simultaneous approximation properties* of the Krylov operator $K_{q,n}(f)$ given by (2.6), i.e., we are interested in the asymptotic behaviour of

$$\|D^j f - D^j K_{q,n}(f)\|_\infty = \|D^j F_q(f) - D^j G_n(F_q(f))\|_\infty, \qquad 0 \le j < q.$$

By (2.1), differentiation of the correcting polynomial $M_q(f)$ yields the polynomial

$$DM_q(f) = -\sum_{j=0}^{q-1} c_0(D^{j+1} f)\, B_j,$$

i.e., $DM_q(f)$ is the correcting polynomial of degree $q-1$ of $Df$:

$$DM_q(f) = M_{q-1}(Df).$$

Similarly, we get

$$DF_q(f) = F_{q-1}(Df),$$

i.e., $DF_q(f)$ is the corrected function of $Df$. Thus the Euler decomposition of $Df$ is given by

$$Df = M_{q-1}(Df) + F_{q-1}(Df).$$

Proceeding in this way we obtain

$$D^j f = M_{q-j}(D^j f) + F_{q-j}(D^j f), \qquad j = 0, \dots, q-1.$$

Now we note that the $n^{\text{th}}$ Fourier sum projector $G_n$ and the operator of differentiation commute on $\mathcal{C}^1_{2\pi}$. Since the Krylov approximant $K_{q,n}(f)$ of $f$ is given by (2.6), we obtain

$$D^j K_{q,n}(f) = M_{q-j}(D^j f) + G_n(F_{q-j}(D^j f)), \qquad 0 \le j < q.$$

Thus an iterated application of Proposition 2.2 yields the simultaneous approximation properties of the Krylov approximant.

THEOREM 2.1. *Let $f \in \mathcal{C}^{q+1}[0, 2\pi]$, $q = 1, 2, \dots$ be given. Let*

$$M_q(f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y)\, dy - \sum_{j=0}^{q-1} \frac{D^j f(2\pi) - D^j f(0)}{2\pi} B_j(x)$$

*be the correcting polynomial on $[0, 2\pi]$ and $F_q(f) = f - M_q(f)$ be the corrected function with $n^{\text{th}}$ Fourier sum*

$$G_n(F_q(f))(x) = \sum_{k=-n}^{n} c_k(F_q(f))\, e^{ikx}, \quad c_k(g) = \frac{1}{2\pi} \int_0^{2\pi} g(y)\, e^{-iky}\, dy.$$

*Then we have for the derivatives of order $j = 0, \dots, q-1$ of the Krylov approximant $K_{q,n}(f) = M_q(f) + G_n(F_q(f))$ the estimates:*

$$\|D^j f - D^j K_{q,n}(f)\|_\infty = \mathcal{O}(n^{-q+j}), \qquad n \to \infty.$$

## 3. EVEN AND ODD WAVEFORMS

Now let $f \in \mathcal{C}^{2r}[0, \pi]$, $r = 1, 2, \dots$ be given. Let $g \in \mathcal{C}_{2\pi}$ be the even $2\pi$-periodic extension of the function $D^{2r} f \in \mathcal{C}[0, \pi]$. If in addition $f \in \mathcal{C}^{2r+1}[0, \pi]$, then the Fourier cosine series of $g$ converges uniformly and absolutely to $g$:

$$g(x) = \frac{1}{2} a_0(f) + \sum_{k=1}^{\infty} a_k(g) \cos kx.$$

Let $L^1_0 \subset L^1$ be the subspace of even functions. The periodic convolution of $f, g \in L^1_0$ reads now as follows

$$\begin{aligned}(f * g)(x) &= \frac{1}{2\pi} \int_0^{\pi} f(x-y)\, g(y)\, dy + \frac{1}{2\pi} \int_0^{\pi} f\left(x - (2\pi - y)\right)\, g(2\pi - y)\, dy \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} \left(f(x-y) + f(x+y)\right)\, g(y)\, dy.\end{aligned}$$

From the periodic convolution property (2.2), it follows that

$$a_n(f * g) = 2\, a_n(f)\, a_n(g), \qquad n = 0, 1, \dots.$$

For even $f \in \mathcal{C}^2_{2\pi}$, $g \in L^1_0$, we have

$$D^2(f * g) = (D^2 f) * g.$$

Note that the Bernoulli functions $B_{2r}$, $r = 1, 2, \ldots$ are even. Then we obtain the corrected function of $f \in C^{2r}[0, \pi]$:

$$F_{2r}(f)(x) = (B_{2r} * g)(x) = \frac{1}{\pi} \int_0^\pi \frac{1}{2} \left(B_{2r}(x - y) + B_{2r}(x + y)\right) D^{2r} f(y)\, dy. \tag{3.1}$$

Note that the properties of the Bernoulli function $B_{2r}$ imply that $F_{2r}(f)$ is even. Thus the corrected function $F_{2r}(f)$ defined by (3.1) on $\mathbb{R}$ is an *even waveform*.

Now partial integration yields the following.

PROPOSITION 3.1. *Let $f \in C^{2r}[0, \pi]$, $r = 1, 2, \ldots$ be given. Then we have the recursion formula*

$$F_{2r}(f)(x) = \frac{1}{\pi} B_{2r}(x - \pi) (D^{2r-1} f)(\pi) - \frac{1}{\pi} B_{2r}(x) (D^{2r-1} f)(0) + F_{2r-2}(f)(x), \quad r = 2, 3, \ldots,$$
$$F_2(f)(x) = \frac{1}{\pi} B_2(x - \pi) (Df)(\pi) - \frac{1}{\pi} B_2(x) (Df)(0) + f(x) - \frac{1}{2} a_0(f).$$

*We further have the Jones-Hardy decomposition of $f$:*

$$f = M_{2r}(f) + F_{2r}(f) \tag{3.2}$$

*with the correcting polynomial (restricted on $(0, \pi)$)*

$$M_{2r}(f)(x) := \sum_{j=1}^{r} \left( -\frac{1}{\pi} B_{2j}(x - \pi) (D^{2j-1} f)(\pi) + \frac{1}{\pi} B_{2j}(x) (D^{2j-1} f)(0) \right) + \frac{1}{2} a_0(f) \in \Pi_{2r} \quad (0 < x < \pi) \tag{3.3}$$

*and with the corrected function $F_{2r}(f)$ of the form (3.1).*

REMARK 3.1. *The correcting polynomial $M_{2r}(f)$ was first introduced by Jones and Hardy [2], without the constant term $a_0(f)/2$, however. We have shown that the Jones-Hardy decomposition is an immediate consequence of the Euler decomposition, which takes into account the specific symmetry properties of $D^{2r} f$.*

*The Jones-Hardy approximant of $f \in C^{2r}[0, \pi]$, $r = 1, 2, \ldots$, is defined by*

$$K_{2r,n}(f) := M_{2r}(f) + G_n(F_{2r}(f)). \tag{3.4}$$

PROPOSITION 3.2. *Let $f \in C^{2r+2}[0, \pi]$, $r = 1, 2, \ldots$. Then*

$$\|f - K_{2r,n}(f)\|_\infty = \mathcal{O}(n^{-2r-1}), \quad n \to \infty.$$

PROOF. If we extend $D^{2r} f$ evenly and $2\pi$-periodically to the function $g \in C_{2\pi} \cap \mathcal{P}C^2_{2\pi}$ then

$$F_{2r}(f)(x) = \frac{1}{\pi} \int_0^\pi \frac{1}{2} \left((B_{2r}(x - y) + B_{2r}(x + y)\right) D^{2r} f(y)\, dy = (B_{2r} * g)(x),$$

which implies by the periodic convolution property (2.2)

$$c_k(F_{2r}(f)) = c_k(B_{2r})\, c_k(D^q g) = \begin{cases} (ik)^{-2r} c_k(g), & k = \pm 1, \pm 2, \ldots, \\ 0, & k = 0. \end{cases}$$

Since $g \in C_{2\pi} \cap \mathcal{P}C^2_{2\pi}$ it follows that $g \in \mathcal{E}^2$, and hence,

$$c_k(g) = \mathcal{O}(k^{-2}), \quad |k| \to \infty.$$

Consequently,

$$c_k(F_{2r}(f)) = \mathcal{O}(k^{-2k-2}), \qquad |k| \to \infty.$$

Hence,

$$\|F_{2r}(f) - G_n(F_{2r}(f))\|_\infty = \mathcal{O}\left(\sum_{k=n+1}^{\infty} k^{-2r-2}\right) = \mathcal{O}(n^{-2r-1}).$$

Since $f - K_{2r,n}(f) = F_{2r}(f) - G_n(F_{2r}(f))$ the proof is complete. ∎

In view of deriving simultaneous approximation properties of the Jones-Hardy approximant, we have to investigate the derivative of the Jones-Hardy decomposition. In contrast to the Euler decomposition we obtain a new type of decomposition, which corresponds to Fourier expansions of odd periodic functions. Recall that for $f \in \mathcal{C}^{2r}[0,\pi]$, we have $f = M_{2r}(f) + F_{2r}(f)$ with (3.1) and (3.3). Differentiating $M_{2r}(f)$ and $F_{2r}(f)$, we get by (2.1) that

$$DM_{2r}(f)(x) = \sum_{j=1}^{r}\left(-\frac{1}{\pi}B_{2j-1}(x-\pi)\,(D^{2j-1}f)(\pi) + \frac{1}{\pi}B_{2j-1}(x)\,(D^{2j-1}f)(0)\right),$$

$$DF_{2r}(f)(x) = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\left((B_{2r-1}(x-y) + B_{2r-1}(x+y)\right) D^{2r}f(y)\,dy.$$

Setting $h := Df$, we obtain the Lanczos decomposition [1] of $h \in \mathcal{C}^{2r-1}[0,\pi]$, $r = 1,2,\ldots$:

$$h = N_{2r-1}(h) + H_{2r-1}(h) \tag{3.5}$$

with the correcting polynomial

$$N_{2r-1}(h)(x) := \sum_{j=1}^{r}\left(-\frac{1}{\pi}B_{2j-1}(x-\pi)\,(D^{2j-2}h)(\pi) + \frac{1}{\pi}B_{2j-1}(x)\,(D^{2j-2}h)(0)\right) \in \Pi_{2r-1} \tag{3.6}$$

and with the corrected function

$$H_{2r-1}(h)(x) = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\left((B_{2r-1}(x-y) + B_{2r-1}(x+y)\right) D^{2r-1}h(y)\,dy. \tag{3.7}$$

Note that properties of the Bernoulli functions imply that $H_{2r-1}(h)$ as defined by (3.7) on $\mathbb{R}$ is an *odd waveform*.

Thus, the Jones-Hardy decomposition yields an even waveform as polynomially corrected function with a rapidly converging Fourier series while the Lanczos decomposition gives an odd waveform as polynomially corrected function, again with a rapidly converging Fourier series. The Lanczos decomposition may be obtained by differentiating the Jones-Hardy representation.

The *Lanczos approximant* of $h \in \mathcal{C}^{2r-1}[0,\pi]$, $r = 1,2,\ldots$ is now defined by

$$L_{2r-1,n}(h) := N_{2r-1}(h) + G_n(H_{2r-1}(h)). \tag{3.8}$$

PROPOSITION 3.3. *Let $h \in \mathcal{C}^{2r+1}[0,\pi]$, $r = 1,2,\ldots$. Then*

$$\|h - L_{2r-1,n}(h)\|_\infty = \mathcal{O}(n^{-2r}), \qquad n \to \infty. \tag{3.9}$$

PROOF. Note first that (3.5) and (3.8) imply

$$h - L_{2r-1,n}(h) = H_{2r-1}(h) - G_n\left(H_{2r-1}(h)\right).$$

If we extend $D^{2r-1}h$ evenly and periodically to the function $g \in \mathcal{C}_{2\pi}$ then we have by (3.7)

$$H_{2r-1}(h) = B_{2r-1} * g. \tag{3.10}$$

As in the proof of Proposition 3.2, it follows that $g \in \mathcal{E}^2$. This implies

$$c_k(g) = \mathcal{O}(k^{-2}), \qquad |k| \to \infty.$$

Since by (2.2) and (3.10), we have

$$c_k(H_{2r-1}(h)) = c_k(g)\,(ik)^{1-2r}, \qquad k \neq 0,$$

we can conclude

$$c_k(H_{2r-1}(h)) = \mathcal{O}(k^{-2r-1}), \quad |k| \to \infty.$$

Hence,

$$\|H_{2r-1}(h) - G_n(H_{2r-1}(h))\|_\infty = \mathcal{O}\left(\sum_{k=n+1}^{\infty} k^{-2r-1}\right) = \mathcal{O}(n^{-2r}).$$

By (3.9) the proof is complete. ∎

We will now apply the preceding results to derive simultaneous approximation properties of both the Jones-Hardy and Lanczos approximants. Recall that for $f \in \mathcal{C}^{2r}[0,\pi]$, $h \in \mathcal{C}^{2r-1}[0,\pi]$, $r = 1, 2, \ldots$, we have

$$f = M_{2r}(f) + F_{2r}(f) \quad \text{(Jones-Hardy)},$$
$$h = N_{2r-1}(h) + H_{2r-1}(h) \quad \text{(Lanczos)}.$$

In particular,

$$Df = N_{2r-1}(Df) + H_{2r-1}(Df).$$

Next we differentiate the Lanczos decomposition for $h \in \mathcal{C}^{2r-1}[0,\pi]$, $r = 2, 3, \ldots$. From (3.7) and (2.1), it follows that

$$DH_{2r-1}(h)(x) = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\left((B_{2r-2}(x-y) + B_{2r-2}(x+y)\right) D^{2r-1}h(y)\,dy,$$

i.e., we have

$$DH_{2r-1}(h) = F_{2r-2}(Dh).$$

Similarly, we get by (3.6) and (2.1):

$$\begin{aligned} DN_{2r-1}(h)(x) &= \sum_{j=2}^{r}\left(-\frac{1}{\pi}B_{2j-2}(x-\pi)\,(D^{2j-2}h)(\pi) + \frac{1}{\pi}B_{2j-2}(x)\,(D^{2j-2}h)(0)\right) \\ &\quad + \left(-\frac{1}{\pi}(-1)\,h(\pi) + \frac{1}{\pi}(-1)\,h(0)\right) \\ &= \sum_{k=1}^{r-1}\left(-\frac{1}{\pi}B_{2k}(x-\pi)\,(D^{2k-1}Dh)(\pi) + \frac{1}{\pi}B_{2k-1}(x)\,(D^{2k-1}Dh)(0)\right) + \frac{1}{2}a_0(Dh), \end{aligned}$$

i.e., we have

$$DN_{2r-1}(h) = M_{2r-2}(Dh).$$

This shows

$$Dh = M_{2r-2}(Dh) + F_{2r-2}(Dh).$$

We recollect the different decompositions and their differentiated forms for $f \in \mathcal{C}^{2r}[0,\pi]$, $r = 1, 2, \ldots$ and $h \in \mathcal{C}^{2r-1}[0,\pi]$, $r = 2, 3, \ldots$:

$$\begin{aligned} f &= M_{2r}(f) + F_{2r}(f) \quad \text{(Jones-Hardy)}, \\ h &= N_{2r-1}(h) + H_{2r-1}(h) \quad \text{(Lanczos)}, \\ Df &= N_{2r-1}(Df) + H_{2r-1}(Df) \quad \text{(Lanczos)}, \\ Dh &= M_{2r-2}(Dh) + F_{2r-2}(Dh) \quad \text{(Jones-Hardy)}. \end{aligned}$$

A repeated application of Propositions 3.2 and 3.3 yields the following.

THEOREM 3.1.

(a) *Let* $f \in C^{2r+2}[0,\pi]$, $r = 1,2,\dots$ *be given. Let*

$$M_{2r}(f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)\,dy + \sum_{j=1}^{r}\left(-\frac{1}{\pi}B_{2j}(x-\pi)\,(D^{2j-1}f)(\pi) + \frac{1}{\pi}B_{2j}(x)\,(D^{2j-1}f)(0)\right)$$

*be the correcting polynomial on* $[0,\pi]$ *and* $F_{2r}(f) = f - M_{2r}(f)$ *be the corrected function with* $n^{\text{th}}$ *Fourier sum*

$$G_n(F_{2r}(f))(x) = \frac{1}{2}a_0(F_{2r}(f)) + \sum_{k=1}^{n} a_k(F_{2r}(f))\cos kx, \quad a_k(g) = \frac{2}{\pi}\int_0^{\pi} g(y)\cos ky\,dy.$$

*Then we have for the derivatives of order* $j = 0,\dots,2r$ *of the Jones-Hardy approximant* $K_{2r,n}(f) = M_{2r}(f) + G_n(F_{2r}(f))$ *the estimates:*

$$\|D^j f - D^j K_{2r,n}(f)\|_\infty = \mathcal{O}(n^{-2r-1+j}), \qquad n \to \infty.$$

(b) *Let* $h \in C^{2r+1}[0,\pi]$, $r = 1,2,\dots$ *be given. Let*

$$N_{2r-1}(h)(x) = \sum_{j=1}^{r}\left(-\frac{1}{\pi}B_{2j-1}(x-\pi)\,(D^{2j-2}h)(\pi) + \frac{1}{\pi}B_{2j-1}(x)\,(D^{2j-2}h)(0)\right)$$

*be the correcting polynomial on* $[0,\pi]$ *and* $H_{2r-1}(h) = h - N_{2r-1}(h)$ *be the corrected function with* $n^{\text{th}}$ *Fourier sum*

$$G_n(H_{2r-1}(h))(x) = \sum_{k=1}^{n} b_k(H_{2r-1}(h))\sin kx, \quad b_k(g) = \frac{2}{\pi}\int_0^{\pi} g(y)\sin ky\,dy.$$

*Then we have for the derivatives of order* $j = 0,\dots,2r-1$ *of the Lanczos approximant* $L_{2r-1,n}(h) = N_{2r-1}(h) + G_n(H_{2r-1}(h))$ *the estimates:*

$$\|D^j h - D^j L_{2r-1,n}(h)\|_\infty = \mathcal{O}(n^{-2r+j}), \qquad n \to \infty.$$

## 4. HALF-WAVEFORMS

Now let $f \in C^q[0,\pi]$, $q = 1,2,\dots$ be given. Let $g \in C_{2\pi}$ be the antiperiodic extension of $D^q f \in C[0,\pi]$. Then the corrected function $F_q(f)$ reads as follows

$$F_q(f)(x) = (B_q * g)(x) = \frac{1}{\pi}\int_0^{\pi} E_{q-1}(x-y)\,D^q f(y)\,dy \tag{4.1}$$

with the $(q-1)^{\text{th}}$ Euler function

$$E_{q-1}(x) := \frac{1}{2}\left(B_q(x) - B_q(x+\pi)\right), \qquad q = 1,2,\dots.$$

Recall that

$$DE_{q-1} = E_{q-2}, \qquad q = 2,3,\dots. \tag{4.2}$$

We obtain the following proposition.

PROPOSITION 4.1. *Let* $f \in C^q[0,\pi]$, $q = 1,2,\dots$ *be given. Then we have the recursion formula*

$$F_q(f) = -\frac{D^{q-1}f(\pi) + D^{q-1}f(0)}{\pi}E_{q-1} + F_{q-2}(f), \qquad q = 2,3,\dots$$

$$F_1(f) = -\frac{f(\pi) + f(0)}{\pi}E_0 + f$$

*and the Boole decomposition* [11, p. 110f]

$$f = M_q(f) + F_q(f) \tag{4.3}$$

*with the correcting polynomial*

$$M_q(f) := \sum_{j=0}^{q-1} \frac{D^j f(\pi) + D^j f(0)}{\pi} E_j \in \Pi_{q-1}$$

*and the corrected function $F_q(f)$ of the form* (4.1).

Note that the properties of the Euler function $E_{q-1}$ imply that $F_q(f)$ is antiperiodic, thus the corrected function $F_q(f)$ is a *half-waveform*.

The *Krylov method* for accelerating the convergence of the half-waveform of $f$ is based on approximating the corrected half-waveform $F_q(f)$ by its $(2n-1)^{\text{th}}$ Fourier sum. The antiperiodic $(2n-1)^{\text{th}}$ Fourier sum of $f \in C^q[0,\pi]$ is given by

$$G_{2n-1}(f) := \sum_{k=-n+1}^{n} c_{2k-1}(f)\, e_{2k-1}$$

with the simplified Fourier coefficients

$$c_{2k-1}(f) := \frac{1}{\pi} \int_0^{\pi} f(y)\, e_{1-2n}(y)\, dy, \qquad n = 0, \pm 1, \dots .$$

The sequence $G_{2n-1}(f)$ converges in the square mean to $f$ for $n \to \infty$.

As in the waveform case the *Krylov approximant* of $f \in C^q[0,\pi]$, $q = 1, 2, \dots$ is defined by

$$K_{q,2n-1}(f) := M_q(f) + G_{2n-1}(F_q(f)). \tag{4.4}$$

Again by (4.3), (4.4) the remainder satisfies

$$f - K_{q,2n-1}(f) = F_q(f) - G_{2n-1}(F_q(f)).$$

$G_{2n-1}(F_q(f))$ is the $(2n-1)^{\text{th}}$ partial sum of the rapidly converging Fourier series of the corrected half-waveform $F_q(f)$.

PROPOSITION 4.2. *Let $f \in C^{q+1}[0,\pi]$, $q = 1, 2, \dots$ be given. Then*

$$\|f - K_{q,2n-1}(f)\|_\infty = \mathcal{O}(n^{-q}), \qquad n \to \infty.$$

PROOF. We extend $D^q f \in C^1[0,\pi]$ antiperiodically to $g \in \mathcal{PC}^1_{2\pi}$. Hence, it follows that

$$F_q(f)(x) = \frac{1}{\pi} \int_0^{\pi} E_{q-1}(x-y)\, D^q f(y)\, dy = (B_q * g)(x). \tag{4.5}$$

Since $g \in \mathcal{E}^1$, the proof of Proposition 2.2 can be applied to the present situation. ∎

Assume that $f \in C^{q+1}[0,\pi]$, $q = 1, 2, \dots$. Our next purpose is to investigate *simultaneous approximation properties* of the Krylov operator $K_{q,2n-1}(f)$ for half-waveforms, given by (4.4). Taking into account the properties (4.2) of the Euler functions, differentiation of the correcting polynomial $M_q(f)$ yields the polynomial

$$DM_q(f) := \sum_{j=0}^{q-2} \frac{D^j(Df)(\pi) + D^j(Df)(0)}{\pi} E_j \in \Pi_{q-2},$$

i.e., $DM_q(f)$ is the correcting polynomial of $Df$:

$$DM_q(f) = M_{q-1}(Df). \tag{4.6}$$

Similarly, we get

$$DF_q(f) = F_{q-1}(Df). \tag{4.7}$$

Thus the Euler decomposition of $Df$ is given by

$$Df = M_{q-1}(Df) + F_{q-1}(Df).$$

Proceeding this way, we obtain

$$D^j f = M_{q-j}(D^j f) + F_{q-j}(D^j f), \qquad j = 0, \dots, q-1.$$

Again the Fourier sum projector and the operator of differentiation commute on $\mathcal{C}^1_{2\pi}$:

$$DG_{2n-1} = G_{2n-1}D. \tag{4.8}$$

Then (4.4) and (4.6)–(4.8) imply that

$$D^j K_{q,2n-1}(f) = M_{q-j}(D^j f) + G_{2n-1}(F_{q-j}(D^j f)), \qquad 0 \le j < q.$$

Thus an iterated application of Proposition 4.2 yields the simultaneous approximation properties of the Krylov approximant.

THEOREM 4.1. *Let $f \in \mathcal{C}^{q+1}[0,\pi]$, $q = 1, 2, \dots$ be given. Let*

$$M_q(f)(x) = \sum_{j=0}^{q-1} \frac{D^j f(\pi) + D^j f(0)}{\pi} E_j(x)$$

*be the correcting polynomial on $[0,\pi]$ and $F_q(f) = f - M_q(f)$ be the corrected function with $(2n-1)^{\text{st}}$ Fourier sum*

$$G_{2n-1}(F_q(f))(x) = \sum_{k=-n+1}^{n} c_{2k-1}(F_q(f))\, e^{i(2k-1)x}, \quad c_{2k-1}(g) = \frac{1}{\pi} \int_0^\pi g(y)\, e^{-i(2k-1)y}\, dy.$$

*Then we have for the derivatives of order $j = 0, \dots, q-1$ of the Krylov approximant $K_{q,2n-1}(f) = M_q(f) + G_{2n-1}(F_q(f))$ the estimates:*

$$\|D^j f - D^j K_{q,2n-1}(f)\|_\infty = \mathcal{O}(n^{-q+j}), \qquad n \to \infty.$$

## 5. EVEN AND ODD HALF-WAVEFORMS

Now we consider $f \in \mathcal{C}^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$. Let $g$ be the even antiperiodization of $D^{2r} f$, that means

$$g(x) = D^{2r} f(x), \qquad 0 \le x < \frac{\pi}{2}, \qquad g\left(\frac{\pi}{2}\right) = 0, \qquad g(-x) = g(x), \qquad g(x+\pi) = -g(x).$$

Then we have $g \in \mathcal{P}C_{2\pi}$ and

$$g\left(\frac{\pi}{2} - x\right) = -g\left(\frac{\pi}{2} + x\right).$$

As in (4.1) the correctecd function $F_{2r}(f)$ of $f$ reads as follows

$$\begin{aligned} F_{2r}(f)(x) = (B_{2r} * g)(x) &= \frac{1}{\pi} \int_0^\pi E_{2r-1}(x-y)\, g(y)\, dy \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{2} \left(E_{2r-1}(x-y) + E_{2r-1}(x+y)\right) (D^{2r} f)(y)\, dy. \end{aligned} \tag{5.1}$$

Then we obtain with integration by parts:

PROPOSITION 5.1. *Let $f \in C^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$. Then we have the recursion formula*

$$\begin{aligned} F_{2r}(f)(x) =& \frac{2}{\pi} E_{2r-2}\left(x - \frac{\pi}{2}\right) \left(D^{2r-2} f\right)\left(\frac{\pi}{2}\right) \\ &- \frac{2}{\pi} E_{2r-1}(x) \left(D^{2r-1} f\right)(0) + F_{2r-2}(f)(x), \qquad r = 2, 3, \dots, \\ F_2(f)(x) =& \frac{2}{\pi} E_0\left(x - \frac{\pi}{2}\right) (f)\left(\frac{\pi}{2}\right) - \frac{2}{\pi} E_1(x)\, (Df)(0) + f(x) \end{aligned}$$

*and the Boole decomposition of $f$:*

$$f = M_{2r}(f) + F_{2r}(f) \tag{5.2}$$

*with the correcting polynomial (restricted on $\left(0, \frac{\pi}{2}\right)$)*

$$\begin{aligned} M_{2r}(f)(x) := \frac{2}{\pi} \sum_{j=1}^{r} \left(E_{2j-1}(x)\, (D^{2j-1} f)(0) - E_{2j-2}\left(x - \frac{\pi}{2}\right) \left(D^{2j-2} f\right)\left(\frac{\pi}{2}\right)\right) \\ \in \Pi_{2r-1} \quad \left(0 < x < \frac{\pi}{2}\right) \end{aligned} \tag{5.3}$$

*and with the corrected function $F_{2r}(f)$ given by* (5.1).

Note that $F_{2r}(f)$ is an even antiperiodic function. Thus the corrected function $F_{2r}(f)$ of $f$ is an even half-waveform.

The *Shaw method* [4] for accelerating the convergence of the related even half-waveform $F_{2r}(f)$ of $f \in C^{2r}\left[0, \frac{\pi}{2}\right]$ is based on approximating $F_{2r}(f)$ by its $(2n-1)^{\text{st}}$ Fourier sum

$$G_{2n-1}(F_{2r}(f))(x) = \sum_{k=1}^{n} a_{2k-1}\left(F_{2r}(f)\right) \cos(2k-1)x$$

with

$$a_{2k-1}\left(F_{2r}(f)\right) = \frac{4}{\pi} \int_0^{\pi/2} F_{2r}(f)(y)\, \cos(2k-1)y\, dy, \qquad k = 1, 2, \dots .$$

The *Shaw approximant* of $f \in C^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$, is now defined by

$$S_{2r,2n-1}(f) := M_{2r}(f) + G_{2n-1}(F_{2r}(f)) \tag{5.4}$$

PROPOSITION 5.2. *Let $f \in C^{2r+1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ be given. Then we have*

$$\|f - S_{2r,2n-1}(f)\|_\infty = \mathcal{O}(n^{-2r}), \qquad n \to \infty.$$

PROOF. Let $g \in \mathcal{PC}^1_{2\pi}$ be the even antiperiodization of $D^{2r} f$. Then

$$F_{2r}(f) = B_{2r} * g$$

is an even antiperiodic function such that $c_{2k}(F_{2r}(f)) = 0$, $k = 0, \pm 1, \ldots$. By (2.2), we have

$$c_{2k-1}(F_{2r}(f)) = c_{2k-1}(B_{2r})\, c_{2k-1}(g) = (i(2k-1))^{-2r}\, c_{2k-1}(g), \qquad k = 0, \pm 1, \ldots .$$

Since $g \in \mathcal{PC}^1_{2\pi}$, it follows that $g \in \mathcal{E}^1$, and hence,

$$c_{2k-1}(g) = \mathcal{O}((2k-1)^{-1}), \qquad |k| \to \infty.$$

Note that

$$a_{2k-1}(g) = 2c_{2k-1}(g) = 2c_{1-2k}(g), \qquad k = 1, 2, \ldots .$$

Consequently,

$$c_{2k-1}(F_{2r}(f))) = \mathcal{O}((2k-1)^{-2r-1}).$$

By (5.2) and (5.4), we obtain

$$F - S_{2r,2n-1}(f) = F_{2r}(f) - G_{2n-1}(F_{2r}(f)),$$

and hence, the estimate

$$\begin{aligned} \|f(x) - S_{2r,2n-1}(f)(x)\|_\infty &= \left\| \sum_{k=n+1}^{\infty} a_{2k-1}(F_{2r}(f)) \cos(2k-1)x \right\|_\infty \\ &\le \left\| \sum_{k=n+1}^{\infty} |a_{2k-1}(F_{2r}(f))| \right\| = \mathcal{O}(n^{-2r}), \qquad n \to \infty. \end{aligned}$$

This completes the proof. ∎

In order to derive simultaneous approximation properties of the Shaw approximant, we have to investigate the derivative of the decomposition formula (5.2). In contrast to (5.2) we obtain now a new type of decomposition which corresponds to Fourier expansions of odd antiperiodic functions.

As already stated, we have for $f \in \mathcal{C}^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \ldots$ the decomposition $f = M_{2r}(f) + F_{2r}(f)$ with (5.1) and (5.3). Differentiating $M_{2r}(f)$, we get by (4.2) that

$$\begin{aligned} DM_{2r}(f)(x) &= \frac{2}{\pi} \sum_{j=1}^{r} (D^{2j-1}f)(0)\, E_{2j-2}(x) - \frac{2}{\pi} \sum_{j=2}^{r} (D^{2j-2}f)\left(\frac{\pi}{2}\right) E_{2j-3}\left(x - \frac{\pi}{2}\right) \\ &= \frac{2}{\pi} \sum_{k=0}^{r-1} (D^{2k}Df)(0)\, E_{2k}(x) - \frac{2}{\pi} \sum_{k=1}^{r-1} (D^{2k-1}Df)\left(\frac{\pi}{2}\right) E_{2k-1}\left(x - \frac{\pi}{2}\right) \end{aligned}$$

for $r = 2, 3, \ldots$ and

$$DM_2(f) = \frac{2}{\pi} Df(0)\, E_0.$$

We set $h := Df$. Then $h \in \mathcal{C}^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \ldots$. Introducing

$$\begin{aligned} L_{2r-1}(h)(x) &:= \frac{2}{\pi} \sum_{k=0}^{r-1} (D^{2k}h)(0)\, E_{2k}(x) - \frac{2}{\pi} \sum_{k=1}^{r-1} (D^{2k-1}h)\left(\frac{\pi}{2}\right) E_{2k-1}\left(x - \frac{\pi}{2}\right), \quad r = 2, 3, \ldots, \\ L_1(h) &:= \frac{2}{\pi}\, h(0)\, E_0, \end{aligned}$$

we obtain

$$DM_{2r}(f) = L_{2r-1}(Df).$$

Note that $L_{2r-1}(h)$ restricted to $\left(0, \frac{\pi}{2}\right)$ is a polynomial of degree $\le 2r - 2$.

Differentiating $F_{2r}(f)$, we get by (4.2)

$$DF_{2r}(f)(x) = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{2}\left((E_{2r-2}(x-y) + E_{2r-2}(x+y)\right) D^{2r} f(y)\, dy.$$

Defining for $h \in C^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$

$$U_{2r-1}(h)(x) = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{2}\left((E_{2r-2}(x-y) + E_{2r-2}(x+y)\right) D^{2r-1} h(y)\, dy, \tag{5.5}$$

we obtain

$$DF_{2r}(f) = U_{2r-1}(Df)$$

for $f \in C^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$. We summarize:

PROPOSITION 5.3. *Let $h \in C^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ be given. Then we have the decomposition formula*

$$h = L_{2r-1}(h) + U_{2r-1}(h) \tag{5.6}$$

*with the (restricted to $\left(0, \frac{\pi}{2}\right)$) correcting polynomial $L_{2r-1}(h) \in \Pi_{2r-2}$ and the corrected function $U_{2r-1}(h)$.*

Note that by the properties of the Euler functions $U_{2r-1}(h)$ defined by (5.5) on $\mathbb{R}$ is an odd half-waveform.

The approximant $V_{2r-1,2n-1}(h)$ of $h \in C^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ is defined by

$$V_{2r-1,2n-1}(h) := L_{2r-1}(h) + G_{2n-1}(U_{2r-1}(h)). \tag{5.7}$$

Since $U_{2r-1}(h)$ defined by (5.5) on $\mathbb{R}$ is odd and antiperiodic, we have

$$G_{2n-1}(U_{2r-1}(h))(x) = \sum_{k=1}^{n} b_{2k-1}(U_{2r-1}(h)) \sin(2k-1)\, x$$

with

$$b_{2k-1}(U_{2r-1}(h)) = \frac{4}{\pi}\int_0^{\pi/2} U_{2r-1}(h)(y) \sin(2k-1)\, y\, dy.$$

PROPOSITION 5.4. *Let $h \in C^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ be given. Then*

$$\|h - V_{2r-1,2n-1}(h)\|_\infty = \mathcal{O}(n^{-2r+1}), \qquad n \to \infty.$$

PROOF. Let $g$ be the odd antiperiodization of $D^{2r-1}h$, that means

$$g(x) = D^{2r-1}h(x), \qquad 0 < x \le \frac{\pi}{2}, \qquad g(0) = 0, g(-x) = -g(x), \qquad g(x+\pi) = -g(x).$$

Then we have $g \in \mathcal{PC}^1_{2\pi}$ and

$$g\left(\frac{\pi}{2} - x\right) = g\left(\frac{\pi}{2} + x\right).$$

It is easy to see that

$$U_{2r-1}(h) = B_{2r-1} * g.$$

Then,

$$c_{2k}(U_{2r-1}(h)) = 0, \qquad k = 0, \pm 1, \dots.$$

From the periodic convolution property (1.2) it follows that

$$c_{2k-1}(U_{2r-1}(h)) = c_{2k-1}(B_{2r-1})\, c_{2k-1}(g) = \mathcal{O}((2k-1)^{-2r}), \qquad |k| \to \infty.$$

Thus $U_{2r-1}(h) \in \mathcal{E}^{2r}$. Further, we have

$$b_{2k-1}(U_{2r-1}(h)) = 2ic_{2k-1}(U_{2r-1}(h)), \qquad k = 1, 2, \dots .$$

Hence, we obtain

$$\|U_{2r-1}(h)(x) - G_{2n-1}(U_{2n-1}(h))(x)\|_\infty = \left\| \sum_{k=n+1}^{\infty} b_{2k-1}(U_{2r-1}(h)) \sin(2k-1)\,x \right\|_\infty = \mathcal{O}(n^{-2r+1}), \qquad n \to \infty.$$

Further, by (5.6) and (5.7), we have

$$h - V_{2r-1,2n-1}(h) = U_{2r-1}(h) - G_{2n-1}(U_{2r-1}(h)).$$

This completes the proof. ∎

Now we apply the preceding results to the simultaneous approximation of both approximants. Recall that for $f \in \mathcal{C}^{2r}\left[0, \frac{\pi}{2}\right]$, $h \in \mathcal{C}^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ we have the decomposition formulas

$$f = M_{2r}(f) + F_{2r}(f),$$
$$h = L_{2r-1}(h) + U_{2r-1}(h).$$

In particular, we have

$$Df = L_{2r-1}(Df) + U_{2r-1}(Df).$$

Differentiating $L_{2r-1}(h)$ and $U_{2r-1}(h)$ for $h \in \mathcal{C}^{2r-1}\left[0, \frac{\pi}{2}\right]$, $r = 2, 3, \dots$ we obtain after simple calculations:

$$DL_{2r-1}(h) = M_{2r-2}(Dh), DU_{2r-1}(h) = F_{2r-2}(Dh),$$

i.e.,

$$Dh = M_{2r-2}(Dh) + F_{2r-2}(Dh).$$

A repeated application of Propositions 5.2 and 5.4 yields the following

THEOREM 5.1.

(a) *Let* $f \in \mathcal{C}^{2r+1}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ *be given. Let*

$$M_{2r}(f)(x) = \frac{2}{\pi} \sum_{j=1}^{r} \left( E_{2j-1}(x)\,(D^{2j-1}f)(0) - E_{2j-2}\left(x - \frac{\pi}{2}\right)\,(D^{2j-2}f)\left(\frac{\pi}{2}\right)\right)$$

*be the correcting polynomial on* $\left[0, \frac{\pi}{2}\right]$ *and* $F_{2r}(f) = f - M_{2r}(f)$ *be the corrected function with* $(2n-1)^{\text{st}}$ *Fourier sum*

$$G_{2n-1}(F_{2r}(f))(x) = \sum_{k=1}^{n} a_{2k-1}\left(F_{2r}(f)\right) \cos(2k-1)x, \quad a_{2k-1}(g) = \frac{4}{\pi} \int_0^{\pi/2} g(y)\,\cos(2k-1)\,y\,dy.$$

*Then we have for the derivatives of order* $j = 0, \dots, 2r-1$ *of the Shaw approximant* $S_{2r,2n-1}(f) = M_{2r}(f) + G_{2n-1}(F_{2r}(f))$ *the estimates:*

$$\|D^j f - D^j S_{2r,2n-1}(f)\|_\infty = \mathcal{O}(n^{-2r+j}), \qquad n \to \infty.$$

(b) *Let* $h \in \mathcal{C}^{2r}\left[0, \frac{\pi}{2}\right]$, $r = 1, 2, \dots$ *be given. Let*

$$L_{2r-1}(h)(x) = \frac{2}{\pi} \sum_{k=0}^{r-1} (D^{2k}h)(0)\,E_{2k}(x) - \frac{2}{\pi} \sum_{k=1}^{r-1} (D^{2k-1}h)\left(\frac{\pi}{2}\right)\,E_{2k-1}\left(x - \frac{\pi}{2}\right)$$

*be the correcting polynomial on* $\left[0, \frac{\pi}{2}\right]$ *and* $U_{2r-1}(h) = h - L_{2r-1}(h)$ *be the corrected function with* $(2n-1)^{\text{st}}$ *Fourier sum*

$$G_{2n-1}(U_{2r-1}(h))(x) = \sum_{k=1}^{n} b_{2k-1}(U_{2r-1}(h)) \sin(2k-1)\,x,$$

$$b_{2k-1}(g) = \frac{4}{\pi} \int_0^{\pi/2} g(y) \sin(2k-1)\,y\,dy.$$

*Then we have for the derivatives of order* $j = 0, \dots, 2r-2$ *of the approximant* $V_{2r-1,2n-1}(h) = L_{2r-1}(h) + G_{2n-1}(L_{2r-1}(h))$ *the estimates:*

$$\|D^j h - D^j V_{2r-1,2n-1}(h)\|_\infty = \mathcal{O}(n^{-2r+j+1}), \qquad n \to \infty.$$

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