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Oscillation Criteria for Second-Order Nonlinear Difference Equations with Damped Term

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Abstract—Several oscillation criteria are established for the second-order damped nonlinear difference equation

$$\Delta[a_{n-1}(\Delta y_{n-1})^{\sigma}] + p_{n-1}(\Delta y_{n-1})^{\sigma} + q_n f(y_n) = 0, \quad n \geq n_0 > 0,$$

where $\sigma > 0$ is any quotient of odd integers, $\{p_n\}$ and $\{q_n\}$ are real sequences, and $f \in C(R, R)$ such that $xf(x) > 0$ for $x \neq 0$. Several examples which dwell upon the importance of our results are also included.

Keywords—Damped, Difference equations, Oscillation, Nonoscillation.

1. INTRODUCTION

Consider the second-order damped nonlinear difference equation

$$\Delta\left[a_{n-1}(\Delta y_{n-1})^{\sigma}\right] + p_{n-1}(\Delta y_{n-1})^{\sigma} + q_n f(y_n) = 0, \qquad n \geq n_0 > 0, \tag{1.1}$$

where σ is a positive quotient of odd integers, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is an eventually positive real sequence, $\{p_n\}$ and $\{q_n\}$ are real sequences, and f is a real-valued continuous function on the real line R.

A number of dynamical behaviors of solutions of second-order difference equations are possible. Here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory. Our concern is motivated by several recent papers, especially those by Thandapani, Györi and Lalli [1], Wong and Agarwal [2–4], as well as Zhang and Chen [5]. In [1,5], the authors obtained oscillation criteria for a special case of (1.1)

$$\Delta^2 y_{n-1} + q_n f(y_n) = 0, \qquad n \geq n_0. \tag{1.2}$$

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In [4], the authors employed techniques similar to those in [1] and obtained oscillation criteria (see Theorems 3.3, 3.4, and 3.5(a)) for the equation

$$\Delta(a_{n-1}(\Delta y_{n-1})^\sigma) + q_n f(y_n) = 0, \qquad n \ge n_0. \tag{1.3}$$

Unfortunately, the oscillation criteria in [4] are theoretical in nature since additional assumptions have to be imposed on the unknown solutions. For example,

$$|\Delta y_n| \begin{cases} \ge L^{1/(\sigma-1)}, & \sigma < 1, \\ \le \infty, & \sigma = 1, \\ \le L^{1/\sigma - 1}, & \sigma > 1, \end{cases} \qquad \text{for } L > 0,$$

where $\{y_n\}$ is a solution of (1.3). Since solutions are unknown in general, these assumptions are, if not impossible, difficult to verify.

Very recently, Wong and Agarwal [2] obtained two oscillation criteria for (1.1), but the coefficient $\{q_n\}$ and the damping $\{p_n\}$ are required to satisfy $q_n \ge 0$ and $p_n \ge 0$, for all $n \ge n_0$ (see [2, Corollary 3.5]).

We will bypass the above-mentioned difficulties for our equation (1.1) by means of techniques similar to those in Thandapani, Gyori and Lalli [1], Wong and Agarwal [4], and Zhang and Chen [5]. In the same time, we will also be able to extend and improve several results in [1–4, 6–14].

By a solution of (1.1), we mean a nontrivial sequence $\{y_n\}$ satisfying (1.1) for $n \ge n_0$. A solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor negative, and nonoscillatory, otherwise. We shall denote $N_\beta = \{\beta, \beta+1, \dots\}$ and $N_\beta^\alpha = \{\beta, \beta+1, \dots, \alpha\}$.

Throughout, we shall assume the following.

(a) $uf(u) > 0$, for all $u \ne 0$.

(b) $f(u) - f(v) = g(u,v)(u-v)^\delta$, for all $u, v \ne 0$, where g is a nonnegative function, δ is a positive quotient of odd integers. This means that any quantity raised to the $(\delta + \sigma)$ power is positive. Also, if $u \ge v$, then $f(u) \ge f(v)$.

2. SEVERAL LEMMAS

LEMMA 2.1. *(See [4, Lemma 2.1].) Let the function $K(n,s,y) : N_{n_0} \times N_{n_0} \times R \to R$ be such that for each fixed n and s, the function $K(n,s,y)$ is nondecreasing. Furthermore, let $\{h_n\}$ be a given sequence and $\{u_n\}, \{v_n\}$ be sequences satisfying, for $n \in N_{n_0}$,*

$$u_n \ge (\le) h_n + \sum_{s=n_0}^{n-1} K(n,s,u_s) \tag{2.1}$$

and

$$v_n = h_n + \sum_{s=n_0}^{n-1} K(n,s,v_s). \tag{2.2}$$

Then, $u_n \ge (\le) v_n$, for all $n \in N_{n_0}$.

LEMMA 2.2. *Suppose that $\{y_n\}$ is positive (negative) solution of (1.1) for $n \in N_{n_0-1}^\alpha (1 \le n_0 < \alpha)$, and there exist a positive sequence $\{\rho_n\}$, $n_1 \in N_{n_0}^\alpha$, and $m > 0$ such that*

$$\begin{aligned} -\frac{a_{n_0-1}(\Delta y_{n_0-1})^\sigma \rho_{n_0-1}}{f(y_{n_0})} &+ \sum_{s=n_0}^{n}\left[q_s\rho_s + \frac{p_{s-1}\rho_s(\Delta y_{s-1})^\sigma}{f(y_s)} - \frac{a_{s-1}(\Delta y_{s-1})^\sigma \Delta\rho_{s-1}}{f(y_s)}\right] \\ &+ \sum_{s=n_0}^{n_1-1} \frac{a_s(\Delta y_s)^{\sigma+\delta}\rho_s g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})} \ge m, \end{aligned} \tag{2.3}$$

for all $n \in N^{\alpha}_{n_1}$. *Then,*

$$a_n \rho_n (\Delta y_n)^{\sigma} \leq (\geq) - m f(y_{n_1}), \qquad n \in N^{\alpha}_{n_1}. \tag{2.4}$$

PROOF. Set

$$w_n = a_n (\Delta y_n)^{\sigma} \rho_n,$$

then

$$\Delta w_n = \Delta (a_n (\Delta y_n)^{\sigma}) \rho_{n+1} + a_n (\Delta y_n)^{\sigma} \Delta \rho_n,$$

so that, in view of (1.1), we have

$$\frac{\Delta w_{n-1}}{f(y_n)} = -q_n \rho_n - \frac{p_{n-1} \rho_n (\Delta y_{n-1})^{\sigma}}{f(y_n)} + \frac{a_{n-1} (\Delta y_{n-1})^{\sigma} \Delta \rho_{n-1}}{f(y_n)}.$$

By summing from n_0 to n, where $n \in N^{\alpha}_{n_0}$, we have

$$\begin{aligned} -\frac{w_n}{f(y_{n+1})} = -\frac{w_{n_0 - 1}}{f(y_{n_0})} &+ \sum_{s=n_0}^{n} \left[q_s \rho_s + \frac{p_{s-1} \rho_s (\Delta y_{s-1})^{\sigma}}{f(y_s)} - \frac{a_{s-1} (\Delta y_{s-1})^{\sigma} \Delta \rho_s}{f(y_s)} \right] \\ &+ \sum_{s=n_0}^{n} \frac{a_s (\Delta y_s)^{\sigma+\delta} \rho_s g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}. \end{aligned} \tag{2.5}$$

In view of (2.3), we see further that

$$-w_n \geq m f(y_{n+1}) + \sum_{s=n_1}^{n} \frac{f(y_{n+1}) a_s (\Delta y_s)^{\sigma+\delta} \rho_s g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}, \tag{2.6}$$

where $n \in N^{\alpha}_{n_1}$.

CASE 1. Suppose that $\{y_n\}$ is positive. Then, (2.6) implies $-w_n > 0$, or equivalently $\Delta y_n < 0$, $n \in N^{\alpha}_{n_1}$. Let $u_n = -w_n = -a_n \rho_n (\Delta y_n)^{\sigma}$. Then, (2.6) becomes

$$u_n \geq m f(y_{n+1}) + \sum_{s=n_1}^{n} \frac{f(y_{n+1}) (-\Delta y_s)^{\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} u_s. \tag{2.7}$$

Define

$$K(n, s, x) = \frac{f(y_{n+1}) (-\Delta y_s)^{\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} x, \qquad n, s \in N^{\alpha}_{n_1}, \quad x \in R^{+}. \tag{2.8}$$

Since $\Delta y_n < 0$, $n \in N^{\alpha}_{n_1}$, we observe that for fixed n and s, $K(n, s, x)$ is nondecreasing in x. With $h_n = m f(y_{n+1})$, we apply Lemma 2.1 to get

$$u_n \geq v_n, \qquad n \in N^{\alpha}_{n_1}, \tag{2.9}$$

where v_n satisfies

$$v_n = m f(y_{n+1}) + \sum_{s=n_1}^{n} \frac{f(y_{n+1}) (-\Delta y_s)^{\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} v_s, \tag{2.10}$$

provided $v_n \in R^{+}$ for $n \in N^{\alpha}_{n_1}$. From (2.10), we find

$$\begin{aligned} \Delta \left[\frac{v_n}{f(y_{n+1})} \right] &= \Delta \left[m + \sum_{s=n_1}^{n} \frac{(-\Delta y_s)^{\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} v_s \right] \\ &= \frac{(-\Delta y_{n+1})^{\delta} g(y_{n+2}, y_{n+1})}{f(y_{n+1}) f(y_{n+2})} v_{n+1}. \end{aligned} \tag{2.11}$$

On the other hand,

$$\Delta\left[\frac{v_n}{f(y_{n+1})}\right]=\frac{\Delta v_n}{f(y_{n+1})}-\frac{v_{n+1}g(y_{n+2},y_{n+1})(\Delta y_{n+1})^\delta}{f(y_{n+1})f(y_{n+2})}. \tag{2.12}$$

Equating (2.11) and (2.12), we obtain $\Delta v_n = 0$ and so $v_n = v_{n_1} = mf(y_{n_1})$, $n \in N_{n_1}^\alpha$. The inequality (2.4) is now immediate from (2.9).

CASE 2. Suppose that $\{y_n\}$ is negative. Then, (2.6) gives $w_n > 0$, or equivalently $\Delta y_n > 0$, $n \in N_{n_1}^\alpha$. Let $u_n = w_n = a_n\rho_n(\Delta y_n)^\sigma$. It follows from (2.6) that

$$u_n \ge -mf(y_{n+1})+\sum_{s=n_1}^{n}\frac{[-f(y_{n+1})](\Delta y_s)^\delta g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}u_s.$$

With $K(n,s,x)$ defined as (2.8), we note that for fixed n and $s, K(n,s,x)$ is nondecreasing in x. Applying Lemma 2.1 with $h_n = -mf(y_{n+1})$, we get (2.9) where v_n satisfies

$$v_n = -mf(y_{n+1})+\sum_{s=n_1}^{n}\frac{[-f(y_{n+1})](\Delta y_s)^\delta g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}v_s.$$

As in Case 1, $\Delta v_n = 0$ and hence $v_n = v_{n_1} = mf(y_{n_1})$, $n \in N_{n_1}^\alpha$. Then inequality (2.9) immediately reduces to from (2.4). The proof is complete.

COROLLARY 2.1. *Let $\{y_n\}$, $n = n_1-1, n_1, \dots$, be a positive solution of (1.1) and*

$$\rho_n=\prod_{i=n_0}^{n-1}\left(1-\frac{p_i}{a_i}\right)^{-1}, \qquad a_n-p_n>0, \quad n\ge n_0. \tag{2.13}$$

If

$$\liminf_{n\to\infty}\sum_{s=n_1}^{n} q_s\rho_s > -\infty, \tag{2.14}$$

and

$$\sum_{s=n_1}^{\infty}\frac{1}{(\rho_s a_s)^{1/\sigma}}=\infty, \tag{2.15}$$

then

$$\sum_{s=n_1}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+\delta}\rho_s g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}<\infty. \tag{2.16}$$

PROOF. Otherwise,

$$\sum_{s=n_1}^{\infty}\frac{a_s(\Delta y_s)^{\sigma+\delta}\rho_s g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}=\infty.$$

Hence there exists $n_1^* \ge n_1$ such that

$$\frac{a_{n_0-1}(\Delta y_{n_0-1})^\sigma\rho_{n_0-1}}{f(y_{n_0})}+\sum_{s=n_0}^{n}q_s\rho_s+\sum_{s=n_0}^{n_1^*-1}\frac{a_s(\Delta y_s)^{\sigma+\delta}\rho_s g(y_{s+1},y_s)}{f(y_s)f(y_{s+1})}\ge m,$$

where $m > 0$ is a constant. Since our choice of ρ_n makes $p_s\rho_{s+1} - a_s\Delta\rho_s = 0$, Lemma 2.2 implies that

$$a_n(\Delta y_n)^\sigma\rho_n \le -mf(y_{n_1}^*), \qquad \text{for } n\ge n_1^*. \tag{2.17}$$

Since σ =odd/odd, by (2.17) we have

$$\Delta y_n \le -(mf(y_{n_1}^*))^{1/\sigma}\frac{1}{(\rho_n a_n)^{1/\sigma}}, \qquad \text{for } n\ge n_1^*. \tag{2.18}$$

In view of (2.15), relation (2.18) implies that $\{y_n\}$ is negative eventually, which is a contradiction. The proof is complete.

COROLLARY 2.2. *Assume that (2.15) holds. If*

$$\sum_{s=n_0}^{\infty} \rho_s q_s = \infty, \tag{2.19}$$

where $\{\rho_n\}$ is defined by (2.13), then every solution of (1.1) is oscillatory.

REMARK 1. Corollary 2.2 extends Corollary 3.4 of Wong and Agarwal [2]. In particular, when $p_n \equiv 0$ and $\delta = 1$, Corollary 2.2 reduces to Theorems 3.5(a) and 2.1 of [2,4], respectively, and also extends Corollary 2.2 of [5].

We now consider the case that $\lim_{n\to\infty} \sum_{s=n_0}^{n} q_s \rho_s$ exists.

LEMMA 2.3. *Let (2.13) be satisfied. Suppose that*

(i) $\lim_{|y|\to\infty} |f(y)| = \infty$;
(ii) $\lim_{n\to\infty} \sum_{s=n_0}^{n} q_s \rho_s$ *exists.*

Let $\{y_n\}$ be a nonoscillatory solution of (1.1), then

$$\sum_{s=n_0}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} < \infty, \tag{2.20}$$

$$\lim_{n\to\infty} \frac{a_n \rho_n (\Delta y_n)^{\sigma}}{f(y_{n+1})} = 0, \tag{2.21}$$

and

$$\frac{a_n \rho_n (\Delta y_n)^{\sigma}}{f(y_{n+1})} = \sum_{s=n+1}^{\infty} q_s \rho_s + \sum_{s=n+1}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}, \tag{2.22}$$

for sufficiently large n, where $\{\rho_n\}$ is defined by (2.13).

PROOF. Let $\{y_n\}$ be a nonosillatory solution of (1.1). Without loss of generality, assume $y_n > 0$ for $n \geq n_0$. By Corollary 2.1 it follows that (2.20) holds. Similar to the proof of Lemma 2.2, it follows that (2.5) holds. We rewrite (2.5) as

$$\begin{aligned} \frac{w_n}{f(y_{n+1})} &= \frac{w_{n_0-1}}{f(y_{n_0})} - \sum_{s=n_0}^{n} \left[q_s \rho_s + \frac{p_{s-1} \rho_s (\Delta y_{s-1})^{\sigma}}{f(y_s)} - \frac{a_{s-1} (\Delta y_{s-1})^{\sigma} \Delta \rho_{s-1}}{f(y_s)} \right] \\ &\quad - \sum_{s=n_0}^{n} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} \\ &= \frac{w_{n_0-1}}{f(y_{n_0})} - \sum_{s=n_0}^{\infty} q_s \rho_s - \sum_{s=n_0}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} \\ &\quad + \sum_{s=n+1}^{\infty} q_s \rho_s + \sum_{s=n+1}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} \\ &= \beta + \sum_{s=n+1}^{\infty} q_s \rho_s + \sum_{s=n+1}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}, \end{aligned} \tag{2.23}$$

where

$$\beta = \frac{w_{n_0-1}}{f(y_{n_0})} - \sum_{s=n_0}^{\infty} q_s \rho_s - \sum_{s=n_0}^{\infty} \frac{a_s \rho_s (\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})}. \tag{2.24}$$

We claim that $\beta = 0$.

If $\beta < 0$, we choose n_2 so large that

$$\left| \sum_{s=n_2}^{n} q_s \rho_s \right| \leq -\frac{\beta}{4}, \qquad n \geq n_2$$

and

$$\sum_{s=n_2}^{\infty} \frac{a_s\rho_s(\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})} < -\frac{\beta}{4}.$$

We take $n_0 = n_1 = n_2$ in Lemma 2.2, so that all assumptions of Lemma 2.2 hold. From Lemma 2.2 and (2.18), we obtain

$$\Delta y_n \le -(mf(y_{n_2}))^{1/\sigma} \frac{1}{(a_n\rho_n)^{1/\sigma}}, \qquad \text{for } n \ge n_2,$$

which contradicts the positivity of $\{y_n\}$ since (2.15) holds.

If $\beta > 0$, from (2.23) we have

$$\lim_{n\to\infty} \frac{a_n\rho_n(\Delta y_n)^\sigma}{f(y_{n+1})} = \beta > 0,$$

which implies that $\Delta y_n > 0$, eventually. Hence there exists $n_1 \ge n_0$ such that

$$\frac{a_n\rho_n(\Delta y_n)^\sigma}{f(y_{n+1})} \ge \frac{\beta}{2}, \qquad n \ge n_1. \tag{2.25}$$

Therefore,

$$\sum_{s=n_1}^{\infty} \frac{a_s\rho_s(\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})} \ge \frac{\beta}{2} \sum_{s=n_1}^{\infty} \frac{(\Delta y_s)^\sigma g(y_{s+1}, y_s)}{f(y_s)} = \frac{\beta}{2} \sum_{s=n_1}^{\infty} \frac{\Delta f(y_s)}{f(y_s)}. \tag{2.26}$$

Define

$$r(t) = f(y_n) + (t-n)\Delta f(y_n), \qquad n \le t \le n+1.$$

Note that since $\Delta y_n > 0$, Assumption (b) implies that $\Delta f(y_n) > 0$. It is easy to see that $r'(t) = \Delta f(y_n)$ and $f(y_n) \le r(t) \le f(y_{n+1})$ for $n \le t \le n+1$. Hence

$$\frac{\Delta f(y_n)}{f(y_n)} = \int_n^{n+1} \frac{\Delta f(y_n)}{f(y_n)}\, dt = \int_n^{n+1} \frac{r'(t)}{f(y_n)} dt \ge \int_n^{n+1} \frac{r'(t)}{r(t)}\, dt.$$

From (2.26), we obtain

$$\begin{aligned}
\infty > \sum_{s=n_1}^{\infty} \frac{a_s\rho_s(\Delta y_s)^{\sigma+\delta} g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})} &= \sum_{s=n_1}^{\infty} \frac{a_s\rho_s(\Delta y_s)^\sigma \Delta f(y_s)}{f(y_s)f(y_{s+1})} \\
&\ge \frac{\beta}{2} \sum_{s=n_1}^{\infty} \frac{\Delta f(y_s)}{f(y_s)} = \frac{\beta}{2} \sum_{s=n_1}^{\infty} \frac{r'(t)}{f(y_s)} \\
&\ge \frac{\beta}{2} \sum_{s=n_1}^{\infty} \int_s^{s+1} \frac{r'(t)}{r(t)}\, dt = \frac{\beta}{2} \lim_{n\to\infty} \ln\left(\frac{r(n)}{r(n_1)}\right).
\end{aligned}$$

Hence $\ln(r(t)) < \infty$, which implies that $f(y_n) < \infty$, as $n \to \infty$. Due to Condition (i) and the fact that $\{y_n\}$ is increasing eventually, if $\{y_n\}$ is unbounded, then $\lim_{n\to\infty} y_n = \infty$. Hence $f(y_n) \to \infty$ as $n \to \infty$, which is a contradiction. Therefore, $\{y_n\}$ is bounded.

On the other hand, from (2.25) and the monotonicity of f, we get

$$a_n\rho_n(\Delta y_n)^\sigma \ge \frac{\beta}{2} f(y_{n+1}) \ge \frac{\beta}{2} f(y_{n_1+1}),$$

and so

$$\Delta y_n \ge \left[\frac{\beta}{2} f(y_{n_1+1})\right]^{1/\sigma} \frac{1}{(a_n\rho_n)^{1/\sigma}}, \qquad n \ge n_1.$$

By (2.15), it follows that $\lim_{n\to\infty} y_n = \infty$, which contradicts the boundedness of $\{y_n\}$. The proof is complete.

3. MAIN RESULTS

In this section, we will give several new sufficient conditions for oscillation of all solutions of (1.1).

THEOREM 3.1. *Let (2.15) be satisfied. Suppose that*

(H1) $0 < \int_\epsilon^\infty \frac{dy}{f(y)^{1/\sigma}} < \infty$, $\int_{-\epsilon}^{-\infty} \frac{dy}{f(y)^{1/\sigma}} < \infty$, *for every* $\epsilon > 0$;

(H2) $\sum_{s=n_0}^\infty q_s \rho_s$ *exists and*

$$\lim_{n\to\infty} \sum_{s=n_0}^{n} \frac{1}{(a_s\rho_s)^{1/\sigma}} \left(\sum_{i=s+1}^{\infty} q_i \rho_s \right)^{1/\sigma} = \infty, \tag{3.1}$$

where $\{\rho_n\}$ *is defined by (2.13).*

Then every solution of (1.1) is oscillatory.

PROOF. Suppose to the contrary. Without loss of generality, we assume that (1.1) has an eventually positive solution. Under our assumptions Lemma 2.3 is true. Let $\{y_n\}$ be an eventually positive solution of (1.1). Then (2.16) holds. Since f is nondecreasing and $(\Delta y_n)^{\sigma+\delta} \geq 0$ by Assumption (b), the second sum in (2.22) is nonnegative. Hence

$$\frac{\Delta y_n}{f(y_{n+1})^{1/\sigma}} \geq \frac{1}{(a_n\rho_n)^{1/\sigma}} \left(\sum_{s=n+1}^{\infty} q_s \rho_s \right)^{1/\sigma}.$$

Summing it from n_0 to n, we obtain

$$\sum_{s=n_0}^{n} \frac{\Delta y_s}{f(y_{s+1})^{1/\sigma}} \geq \sum_{s=n_0}^{n} \frac{1}{(a_s\rho_s)^{1/\sigma}} \left(\sum_{i=s+1}^{\infty} q_i \rho_s \right)^{1/\sigma}. \tag{3.2}$$

We define $r(t) = y_n + (t-n)\Delta y_n, n \leq t \leq n+1$. If $\Delta y_n \geq 0$, then $y_n \leq r(t) \leq y_{n+1}$ and

$$\frac{\Delta y_n}{f(y_{n+1})^{1/\sigma}} \leq \frac{r(t)}{f(r(t))^{1/\sigma}} \leq \frac{\Delta y_n}{f(y_n)^{1/\sigma}}. \tag{3.3}$$

If $\Delta y_n < 0$, then $y_{n+1} \leq r(t) \leq y_n$ and (3.3) also holds.

From (3.2) and (3.3) we obtain

$$\begin{aligned} \int_{r(n_0)}^{\infty} \frac{dy}{f(y)^{1/\sigma}} &\geq \int_{n_0}^{n+1} \frac{dr(t)}{f(r(t))^{1/\sigma}} \geq \sum_{s=n_0}^{n} \frac{\Delta y_s}{f(y_{s+1})^{1/\sigma}} \\ &\geq \sum_{s=n_0}^{n} \frac{1}{(a_s\rho_s)^{1/\sigma}} \left(\sum_{i=s}^{\infty} q_{i+1}\rho_{s+1} \right)^{1/\sigma}. \end{aligned} \tag{3.4}$$

Let

$$G(y) = \int_y^\infty \frac{dr}{f(r)^{1/\sigma}},$$

then (3.4) implies that

$$G(y(n_0)) \geq \sum_{s=n_0}^{n} \frac{1}{(a_s\rho_s)^{1/\sigma}} \left(\sum_{i=s+1}^{\infty} q_i \rho_s \right)^{1/\sigma}, \tag{3.5}$$

which contradicts (3.1). Similarly, we can prove that (1.1) does not posses an eventually negative solution. The proof is complete.

REMARK 2. Theorem 3.1 improves Corollary 3.5 of Wong and Agarwal [2] by dropping the two conditions:

(i) $\{p_n\}$ is eventually nonnegative; and
(ii) $\{a_n\}$ is eventually nondecreasing.

In particular, when $p_n \equiv 0$ and $\delta = 1$, Theorem 3.1 concludes Theorem 2.2 of [2] and Theorem 3.1 of [5].

EXAMPLE 1. Consider the superlinear difference equation

$$\Delta\left[(n-1)^2 \Delta y_{n-1}\right] - 3(n-1)\Delta y_{n-1} + |y_n|^\gamma sgn y_n = 0, \qquad n \geq 3, \tag{3.6}$$

where $\gamma > 1$ is constant. Then $\rho_n = \prod_{i=1}^{n-1}(1-(p_i/a_i))^{-1} = 6/n(n+1)(n+2)$,

$$\sum_{s=2}^{\infty} q_s\rho_s = \sum_{s=2}^{\infty} \frac{6}{(s-1)s(s+1)} < \infty, \qquad \sum_{s=2}^{\infty} \frac{1}{a_s\rho_s} = \sum_{s=1}^{\infty} \frac{(s+1)(s+2)}{6s} = \infty,$$

and

$$\sum_{s=2}^{\infty} \frac{1}{a_s\rho_s} \sum_{i=s+1}^{\infty} q_i\rho_s = \sum_{s=2}^{\infty} \frac{(s+1)(s+2)}{s} \sum_{i=s+1}^{\infty} \frac{1}{(s-1)s(s+1)} = \infty.$$

By Theorem 3.1, this equation is oscillatory. But the results of [2, Corollaries 3.4 and 3.5] are not applicable to the equation since $p_n = -n < 0$ and $\sum_{s=2}^{\infty} q_s\rho_s < \infty$.

In Corollary 2.2 and Theorem 3.1, we require that

$$a_n \Delta\rho_n - p_n\rho_{n+1} = 0$$

for $n \geq n_0$. Such a condition is not required in the following results.

THEOREM 3.2. *Let $\sigma = \delta$. Suppose that $g(u,v) \geq \mu > 0$ for $u \neq v$ and that there exists a positive sequence $\{\rho_n\}$ such that (2.15) and (2.19) hold and*

$$a_n \Delta\rho_n \geq p_n\rho_{n+1}, \qquad n \geq n_0, \tag{3.7}$$

$$\sum_{n=n_0}^{\infty} \frac{\rho_{n+1}p_n^2}{a_n} < \infty, \tag{3.8}$$

$$\sum_{n=n_0}^{\infty} \frac{a_n(\Delta\rho_n)^2}{\rho_{n+1}} < \infty. \tag{3.9}$$

Then every solution of (1.1) is oscillatory.

PROOF. To the contrary, let $\{y_n\}$ be a nonoscillatory solution of (1.1) which may (and do) assume to be eventually positive, i.e., $y_n > 0$ for $n \geq n_0 - 1$. For the sake of convenience, let $w_n = a_n(\Delta y_n)^\sigma \rho_n$ for $n \geq n_0$. Then $w_n \Delta y_n = a_n\rho_n(\Delta y_n)^{\sigma+1} > 0$ for $n \geq n_0$, and

$$\Delta w_n = \Delta[a_n(\Delta y_n)^\sigma]\rho_{n+1} + a_n(\Delta y_n)^\sigma \Delta\rho_n, \tag{3.10}$$

so that, in view of (1.1), we have

$$\frac{\Delta w_n}{f(y_{n+1})} = -q_{n+1}\rho_{n+1} - \frac{p_n(\Delta y_n)^\sigma \rho_{n+1}}{f(y_{n+1})} + \frac{a_n(\Delta y_n)^\sigma \Delta\rho_n}{f(y_{n+1})}.$$

Since

$$\Delta\left(\frac{w_n}{f(y_n)}\right) = \frac{f(y_n)\Delta w_n - w_n g(y_{n+1}, y_n)(\Delta y_n)^\sigma}{f(y_n)f(y_{n+1})},$$

therefore, by summing from n_0 to $n-1$, we have

$$\begin{aligned} &\frac{a_n(\Delta y_n)^\sigma \rho_n}{f(y_n)} + \sum_{s=n_0}^{n-1}\left[q_{s+1}\rho_{s+1} + \frac{p_s(\Delta y_s)^\sigma \rho_{s+1}}{f(y_{s+1})} - \frac{a_s(\Delta y_s)^\sigma \Delta\rho_s}{f(y_{s+1})}\right] \\ &\quad + \sum_{s=n_0}^{n-1} \frac{a_s\rho_s g(y_{s+1}, y_s)(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})} = \frac{a_{n_0}(\Delta y_{n_0})^\sigma \rho_{n_0}}{f(y_{n_0})}. \end{aligned} \tag{3.11}$$

Using Schwartz's inequality, we have

$$\left(\sum_{s=n_0}^{n-1} \frac{p_s(\Delta y_s)^\sigma \rho_{s+1}}{f(y_{s+1})}\right)^2 \le K_1^2 \sum_{s=n_0}^{n-1} \frac{a_s \rho_{s+1}(\Delta y_s)^{2\sigma}}{[f(y_{s+1})]^2}, \tag{3.12}$$

and

$$\left(\sum_{s=n_0}^{n-1} \frac{a_s \Delta\rho_s(\Delta y_s)^\sigma}{f(y_{s+1})}\right)^2 \le K_2^2 \sum_{s=n_0}^{n-1} \frac{a_s \rho_{s+1}(\Delta y_s)^\sigma 2}{[f(y_{s+1})]^2}, \tag{3.13}$$

where

$$K_1^2 = \sum_{s=n_0}^{\infty} \frac{\rho_{s+1} p_s^2}{a_s}, \quad K_1 > 0, \qquad K_2^2 = \sum_{s=n_0}^{\infty} \frac{a_s(\Delta\rho_s)^2}{\rho_{s+1}}, \quad K_2 > 0.$$

Now, we use $g(u,v) \ge \mu > 0$ for $u \ne v$, (3.12) and (3.13) in (3.11) to get

$$\begin{aligned} &\frac{a_n(\Delta y_n)^\sigma \rho_n}{f(y_n)} + \sum_{s=n_0}^{n-1} q_{s+1}\rho_{s+1} + \mu \sum_{s=n_0}^{n-1} \frac{a_s\rho_s(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})} \\ &-(K_1+K_2)\left(\sum_{s=n_0}^{n-1} \frac{a_s\rho_{s+1}(\Delta y_s)^{2\sigma}}{[f(y_{s+1})]^2}\right)^{1/2} \le \frac{a_{n_0}(\Delta y_{n_0})^\sigma \rho_{n_0}}{f(y_{n_0})}. \end{aligned} \tag{3.14}$$

Note that

$$\mu \sum_{s=n_0}^{n-1} \frac{a_s\rho_s(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})} - (K_1+K_2)\left(\sum_{s=n_0}^{n-1} \frac{a_s\rho_{s+1}(\Delta y_s)^{2\sigma}}{[f(y_{s+1})]^2}\right)^{1/2}$$

remains bounded below as $n \to \infty$. Thus, taking (2.19) into account, we observe from (3.14) that

$$\frac{a_n(\Delta y_n)^\sigma \rho_n}{f(y_n)} \to -\infty$$

as $n \to \infty$. Hence, there exists an integer $M \ge n_0$ such that

$$\Delta y_n < 0, \qquad n \ge M. \tag{3.15}$$

We rewrite (3.11) as

$$\begin{aligned} &\frac{a_n(\Delta y_n)^\sigma \rho_n}{f(y_n)} + \sum_{s=M}^{n-1} \frac{a_s\rho_s g(y_{s+1},y_s)(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})} \\ &= \frac{a_{n_0}(\Delta y_{n_0})^\sigma \Delta\rho_{n_0}}{f(y_{n_0})} + \sum_{s=n_0}^{M-1} \frac{(a_s\Delta\rho_s - p_s\rho_{s+1})(\Delta y_s)^\sigma}{f(y_{s+1})} - \sum_{s=n_0}^{n-1} q_{s+1}\rho_{s+1} \\ &+ \sum_{s=M}^{n-1} \frac{(a_s\Delta\rho_s - p_s\rho_{s+1})(\Delta y_s)^\sigma}{f(y_{s+1})} - \sum_{s=n_0}^{M-1} \frac{a_s\rho_s g(y_{s+1},y_s)(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})}, \end{aligned} \tag{3.16}$$

and use (3.7), (3.11), and (3.15) to find an integer $M_1 \ge M$ such that

$$\frac{a_n(\Delta y_n)^\sigma \rho_n}{f(y_n)} + \sum_{s=M}^{n-1} \frac{a_s\rho_s g(y_{s+1},y_s)(\Delta y_s)^{2\sigma}}{f(y_s)f(y_{s+1})} \le -m, \qquad n \ge M_1,$$

where m is a positive constant. Hence,

$$u_n \ge -mf(y_n) + \sum_{s=M}^{n-1} \frac{f(y_n)g(y_{s+1},y_s)(-(\Delta y_s)^\sigma)}{f(y_s)f(y_{s+1})} u_s, \qquad n \ge M_1,$$

where $u_n = -a_n \rho_n (\Delta y_n)^\sigma$. By Lemma 2.1, we obtain

$$u_n \geq v_n,$$

where v_n satisfies

$$v_n = -mf(y_n) + \sum_{s=M}^{n-1} \frac{f(y_n) g(y_{s+1}, y_s)(-\Delta y_s)}{f(y_s) f(y_{s+1})} v_s \tag{3.17}$$

provided that $v_s \in R^+$, for all $s \geq M_1$. Dividing (3.17) first by $f(y_n)$ and then applying the difference operator Δ, it is easy to verify that $\Delta v_n \equiv 0$. Therefore,

$$u_n \geq v_n = -C = -mf(y_{M_1}), \qquad n \geq M_1,$$

and

$$\Delta y_n \leq -[mf(y_{M_1})]^{1/\sigma} \left(\frac{1}{a_n \rho_n} \right)^{1/\sigma}, \qquad n \geq M_1.$$

Summing the last inequality from M_1 to $n-1$, we have

$$y_n \leq y_{M_1} - [mf(y_{M_1})]^{1/\sigma} \sum_{s=M_1}^{n-1} \frac{1}{(a_s \rho_s)^{1/\sigma}}. \tag{3.18}$$

By (2.15), we have $y_n \to \infty$, which yields a contradiction to the fact that $\{y_n\}$ is eventually positive. The proof is similar to the case when $\{y_n\}$ is eventually negative. This completes the proof.

From Theorem 3.2, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$.

Let $\{\rho_n\}$ be defined by (2.13). Then we can obtain Corollary 2.2 in Section 2.

If we let

$$\rho_n = n, n \geq n_0,$$

then we have the following oscillation criterion, which extends Theorem 1 of Thandapani and Lalli [13], $a_n \equiv 1$, $\sigma = 1$.

COROLLARY 3.1. *Let $\sigma = \delta$. Suppose that (2.15), (2.19), (3.7), (3.8), and (H_1) in Theorem 3.1 hold for $\rho_n = n$. Suppose further that $g(u, v) \geq \mu > 0$ for $u \neq v$. Then every solution of (1.1) is oscillatory.*

Define

$$\rho_n = n^\alpha, \qquad n \geq n_0,$$

where α is a constant such that $0 \leq \alpha < 1$, then the following result extends Theorem 2 of Thandapani and Lalli [13], $a_n \equiv 1$, $\sigma = 1$.

COROLLARY 3.2. *Let $\sigma = \delta$ and $\rho_n = n^\alpha (0 \leq \alpha < 1)$. Suppose that (2.15), (2.19), (3.7)–(3.9) hold and that $g(u, v) \geq \mu > 0$ for $u \neq v$. Then every solution of (1.1) is oscillatory.*

THEOREM 3.3. *Let $\sigma = \delta$. Suppose that $g(u, v) \geq \mu > 0$ for $u \neq v$ and that there exists a positive sequence $\{\rho_n\}$ such that (2.15), (2.19), and (3.7) hold and*

$$\sum_{n=n_0}^{\infty} \frac{(\rho_{n+1} p_n - a_n \Delta \rho_n)^2}{a_n \rho_n} < \infty. \tag{3.19}$$

Then every solution of (1.1) is oscillatory.

PROOF. We proceed as in the proof of Theorem 3.2 and obtain (3.11). Then, we use Schwartz's inequality to get

$$\left(\sum_{s=n_0}^{n-1} \frac{(p_s \rho_{s+1} - a_s \Delta \rho_s)(\Delta y_s)^\sigma}{f(y_{s+1})} \right)^2 \leq K^2 \sum_{s=n_0}^{n-1} \frac{a_s \rho_s (\Delta y_s)^{2\sigma}}{[f(y_{s+1})]^2},$$

where

$$K^2 = \sum_{s=n_0}^{\infty} \frac{(\rho_{s+1}p_s - a_s\Delta\rho_s)^2}{a_s\rho_s}.$$

Hence, an inequality similar to (3.14) holds. The rest of the proof follows that of Theorem 3.2. The proof is complete.

We remark that, by Theorem 3.3, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. As space is limited, we omit it.

In the following theorem, we are not require assumption (3.8) and (3.9) or (3.19), but we require the condition

$$q_n > 0, \qquad a_{n-1} - p_{n-1} > 0, \qquad n \geq n_0. \tag{3.20}$$

THEOREM 3.4. *Let $\sigma = \delta$. Suppose that $g(u,v) \geq \mu > 0$, for all $u \neq v$. Suppose further that there exists a positive sequence $\{\rho_n\}$ such that (2.15), (3.7), and (3.20) hold and that*

$$\sum_{s=n_0}^{\infty} \left[q_s\rho_s - \frac{(a_{s-1}\Delta\rho_{s-1} - p_{s-1}\rho_s)^2}{4\mu a_{s-1}\rho_{s-1}} \right] = \infty. \tag{3.21}$$

Then every solution of (1.1) is oscillatory.

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of (1.1) without loss of generality, and assume that $y_n > 0$ for $n \geq n_0 - 1$. We assert that $\Delta y_n \geq 0$, for all large n.

CASE 1. Assume that $\{\Delta y_n\}$ is oscillatory.

(a) Suppose that there exists $n_1 \geq n_0$ such that $y_{n_1-1} < 0$. Let $n = n_1$ in (1.1), and then multiply the resulting equation (1.1) by Δy_{n_1-1}, to obtain

$$\begin{aligned} \Delta(a_{n_1-1}(\Delta y_{n_1-1})^\sigma)\Delta y_{n_1-1} &= -p_{n_1-1}(\Delta y_{n_1-1})^{\sigma+1} - q_{n_1}f(y_{n_1})\Delta y_{n_1-1} \\ &\geq -p_{n_1-1}(\Delta y_{n_1-1})^{\sigma+1}. \end{aligned}$$

Hence, on using $a_n - p_n > 0$ for $n \geq n_0$, we get

$$a_{n_1}(\Delta y_{n_1})^\sigma \Delta y_{n_1-1} > (a_{n_1-1} - p_{n_1-1})(\Delta y_{n_1-1})^{\sigma+1} > 0,$$

which implies that

$$\Delta y_{n_1} < 0.$$

By induction, we obtain $\Delta y_n < 0$, for all large $n \geq n_1 - 1$, contradicting the assumption that $\{\Delta y_n\}$ oscillates.

(b) Suppose that there exists n_1 such that $\Delta y_{n_1-1} = 0$. Then, letting $n = n_1$ in (1.1) leads to

$$a_{n_1}(\Delta y_{n_1})^\sigma = -q_{n_1+1}f(y_{n_1+1}) < 0,$$

which implies that $\Delta y_{n_1} < 0$, i.e., Case 1(a). We have seen that this contradicts the assumption that $\{\Delta y_n\}$ is oscillatory.

CASE 2. Assume that $\Delta y_n < 0$ for $n \geq n_0$. Then we have (3.11) holds. By (3.21) we have

$$\sum_{s=n_0}^{n-1} \left[q_s\rho_s - \frac{(a_{s-1}\Delta\rho_{s-1} - p_{s-1}\rho_s)^2}{4\mu a_{s-1}\rho_{s-1}} \right] \leq \sum_{s=n_0}^{n-1} q_s\rho_s \to \infty$$

as $n \to \infty$. Thus, relation (3.16) and Lemma 2.1 imply that (3.17) holds, and hence, we have (3.18) holds. By (2.15), we have $y_n \to -\infty$ as $n \to \infty$, which yields a contradiction to the fact that $\{y_n\}$ is eventually positive. Thus, we prove that $\Delta y_n \geq 0$, for all large n. Furthermore,

$$\begin{aligned}
\Delta\left(\frac{a_{n-1}\rho_{n-1}(\Delta y_{n-1})^\sigma}{f(y_{n-1})}\right) &= \Delta\left(a_{n-1}(\Delta y_{n-1})^\sigma\right)\frac{\rho_n}{f(y_n)} + a_{n-1}(\Delta y_{n-1})^\sigma \Delta\left(\frac{\rho_{n-1}}{f(y_{n-1})}\right) \\
&= -q_n\rho_n - \frac{p_{n-1}\rho_n(\Delta y_{n-1})^\sigma}{f(y_n)} + a_{n-1}(\Delta y_{n-1})^\sigma \frac{\Delta \rho_{n-1}}{f(y_n)} \\
&\quad - \frac{a_{n-1}\rho_{n-1}(\Delta y_{n-1})^{2\sigma} g(y_n, y_{n-1})}{f(y_{n-1})f(y_n)} \\
&\leq -q_n\rho_n + \frac{(a_{n-1}\Delta\rho_{n-1} - p_{n-1}\rho_n)(\Delta y_{n-1})^\sigma}{f(y_n)} - \frac{\mu a_n \rho_{n-1}(\Delta y_{n-1})^{2\sigma}}{f(y_{n-1})f(y_n)} \\
&= -\left[q_n\rho_n - \frac{f(y_{n-1})(a_{n-1}\Delta\rho_{n-1} - p_{n-1}\rho_n)^2}{4a_n\mu\rho_n f(y_n)}\right] - I^2 \\
&\leq -\left[q_n\rho_n - \frac{f(y_{n-1})(a_{n-1}\Delta\rho_{n-1} - p_{n-1}\rho_n)^2}{4a_n\mu\rho_{n-1} f(y_n)}\right] \\
&\leq -\left[q_n\rho_n - \frac{(a_{n-1}\Delta\rho_{n-1} - p_{n-1}\rho_n)^2}{4a_n\mu\rho_{n-1}}\right],
\end{aligned}$$

for all large n, where the last inequality holds since f is nondecreasing and $(\Delta y_n)^\sigma \geq 0$ and

$$I^2 = \left[\left(\frac{\mu f(y_{n-1})}{a_{n-1}\rho_{n-1} f(y_n)}\right)^{1/2} \frac{a_{n-1}\rho_{n-1}(\Delta y_{n-1})^\sigma}{f(y_{n-1})} - \frac{1}{2}\left(\frac{f(y_{n-1})}{\mu a_n \rho_{n-1} f(y_n)}\right)^{1/2}(a_{n-1}\Delta\rho_{n-1} - p_{n-1}\rho_n)\right]^2.$$

Summing the above inequality from a sufficiently large integer N to n, we obtain

$$\sum_{s=N}^{n}\left[q_s\rho_s - \frac{(a_{s-1}\Delta\rho_{s-1} - p_{s-1}\rho_s)^2}{4a_s\mu\rho_{s-1}}\right] \leq \frac{a_{N-1}\rho_{N-1}(\Delta y_{N-1})^\sigma}{f(y_{N-1})} - \frac{a_n\rho_n(\Delta y_n)^\sigma}{f(y_n)} \leq \frac{a_{N-1}\rho_{N-1}(\Delta y_{N-1})^\sigma}{f(y_{N-1})},$$

which contradicts (3.8). The case where $\{y_n\}$ is eventually negative is similarly proved. The proof is complete.

From Theorem 3.4, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$.

Let

$$\rho_n = n^\lambda, \qquad n \geq n_0,$$

where $\lambda > 1$ is a constant. By Theorem 3.4, we have the following result.

COROLLARY 3.3. *Let $\sigma = \delta$. Suppose that $g(u,v) \geq \mu > 0$, for all $u \neq v$. Suppose further that there exists a positive sequence $\{\rho_n\}$ such that (2.15), (3.7), and (3.20) hold. Suppose further that*

$$\sum_{s=n_0}^{\infty}\left[q_s s^\lambda - \frac{\left\{a_{s-1}\left[s^\lambda - (s-1)^\lambda\right] - p_{s-1}s^\lambda\right\}^2}{4\mu a_{s-1}(s-1)^\lambda}\right] = \infty$$

for some $\lambda > 1$. Then every solution of (1.1) is oscillatory.

If we choose

$$\rho_n = A_n^\lambda, \qquad \lambda > 1, \quad n \ge 1,$$

where

$$A_n = \sum_{s=n_0}^{n} \frac{1}{a_s}, \qquad n \ge 1,$$

then we obtain the following corollary.

COROLLARY 3.4. *Let $\sigma = \delta$. Suppose that $g(u,v) \ge \mu > 0$, for all $u \ne v$. Suppose further that there exists a positive sequence $\{\rho_n\}$ such that (2.15), (3.7), and (3.20) hold. Suppose further that*

$$\sum_{s=n_0}^{\infty} \left[q_s A_s^\lambda - \frac{\left\{ a_{s-1}\left[A_s^\lambda - A_{s-1}^\lambda\right] - p_{s-1} A_s^\lambda \right\}^2}{4\mu a_{s-1} A_{s-1}^\lambda} \right] = \infty$$

for some $\lambda > 1$. Then every solution of (1.1) is oscillatory.

EXAMPLE 2. Consider the discrete Euler equation

$$\Delta^2 y_{n-1} + \frac{\gamma}{n^2} y_n = 0, \qquad n \ge 2, \tag{3.22}$$

where $\gamma > 1/4$. If we take $\rho_n = n$ for $n \ge n_0 > 0$, then

$$\begin{aligned}
\sum_{s=n_0}^{n} \left[q_s \rho_s - \frac{(a_{s-1}\Delta\rho_{s-1} - p_{s-1}\rho_s)^2}{4a_{s-1}\mu\rho_{s-1}} \right] &= \sum_{s=n_0}^{n} \left[\gamma s s^{-2} - \frac{1}{4(s-1)} \right] \\
&= \sum_{s=n_0}^{n} \frac{4\gamma(s-1) - (s-1) - 1}{4(s-1)s} \\
&= \sum_{s=n_0}^{n} \frac{4\gamma - 1}{4s} - \sum_{s=n_0}^{n} \frac{1}{4(s-1)s} \to \infty,
\end{aligned}$$

as $n \to \infty$. By Theorem 3.4, every solution of (3.22) is oscillatory. It is known [5] that when $\gamma \le 1/4$, (3.22) has a nonoscillatory solution. Hence, Theorem 3.4 is sharp.

EXAMPLE 3. Consider the damped nonlinear difference equation

$$\Delta\left(\frac{n-1}{n}\Delta y_{n-1}\right) - \frac{(\sqrt{2}-1)(n-1)}{n^2}\Delta y_{n-1} + \frac{\gamma}{n^2}\left(y_n + y_n^3\right) = 0, \qquad n \ge 1. \tag{3.23}$$

Then

$$g(u,v) = 1 + \left(u + \frac{v}{2}\right)^2 + \frac{3}{4}v^2 \ge 1 = \mu.$$

If we take $\rho_n = n$, then

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n \rho_n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \infty,$$

and

$$\begin{aligned}
\sum_{s=n_0}^{n} \left[q_s \rho_s - \frac{(a_{s-1}\Delta\rho_{s-1} - p_{s-1}\rho_s)^2}{4a_{s-1}\mu\rho_{s-1}} \right] &= \sum_{s=1}^{n} \left[\gamma s^{-1} - \frac{2}{4s} \right] \\
&= \sum_{s=1}^{n} \left[\frac{\gamma}{s} - \frac{1}{2s} \right] \to \infty, \quad \text{as } n \to \infty,
\end{aligned}$$

where $\gamma > 1/2$. Thus, Theorem 3.4 asserts that every solution of (3.23) is oscillatory. But the results in [4] fail to apply to equation (3.23) since $p_n = -(\sqrt{2}-1)n/(n+1)^2 < 0$ and $\sum_{s=n_0}^{\infty} q_s \rho_s \le \sum_{s=n_0}^{\infty} q_s = \sum_{s=n_0}^{\infty} \gamma/s^2 < \infty$, where $\rho_n = \prod_{i=1}^{n-1}(i+1)/(i+\sqrt{2}) < 1$ defined as in [2, Corollary 3.4].

4. REMARKS

REMARK 1. Our result (Theorem 3.1) deals with superlinear equation (1.1). It remains to analyze (1.1) in which the function $f(x)$ is the sublinear [4]. Such an analysis will be the subject of the forthcoming paper.

REMARK 2. Due to the fact that the nonlinearlity of the damping is taken as σ, our results cannot apply for instance the simple (but important) equation

$$\Delta[a_{n-1}(\Delta y_{n-1})^{\sigma}] + p_{n-1}(\Delta y_{n-1}) + q_n f(y_n) = 0, \qquad n \geq n_0 \geq 0.$$

It is interesting question to consider the more general equation

$$\Delta[a_{n-1}(\Delta y_{n-1})_1^{\sigma}] + p_{n-1}(\Delta y_{n-1})^{\sigma_2} + q_n f(y_n) = 0, \qquad n \geq n_0 \geq 0,$$

where $\sigma_1 \neq \sigma_2$. Perhaps, this is very difficult because of the fact that $\{q_n\}$ is allowed to be oscillatory.

REMARK 3. Theorem 3.4 should be niced without condition (3.20), i.e., without the information that $(\Delta y_n)^{\sigma} \geq 0$.

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