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Positive and Copositive Spline Approximation in $\mathbf{L}_p[0,\ 1]$

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Abstract—The order of positive and copositive spline approximation in the $\mathbf{L}_p$-norm, $1 \le p < \infty$, is studied; the main results are

(1) the error of positive approximation by splines is bounded by $C\omega_2(f, 1/n)_p$ if f has a nonnegative extension;
(2) the order deteriorates to ω_1 if f does not have such an extension;
(3) the error of copositive spline approximation is bounded by $C\omega(f, 1/n)_p$;
(4) if f is also continuous, the error in (3) can be estimated in terms of the third τ-modulus $\tau_3(f, 1/n)_p$.

All constants in the error bounds are absolute.

Keywords—Degree of copositive approximation, Constrained approximation in $\mathbf{L}_p$ space, Spline approximation, Polynomial approximation.

1. INTRODUCTION AND MAIN RESULTS

Positive approximation of a positive function $f \in \mathbf{C}[0,\ 1]$ has the same order as that of nonconstrained approximation. For example, positive approximation by splines of order r with $n-1$ equally spaced knots has the order $\omega_r(f, 1/n)_\infty$. When f changes its sign in $(0,\ 1)$, following the sign of f is no so easy and the order of approximation deteriorates. S. P. Zhou [1], Y.-K. Hu, D. Leviatan and X. M. Yu [2,3] and Hu and Yu [4] proved that $\omega_3(f, 1/n)_\infty$ is the best order of copositive approximation by polynomials of degree $\le n$ or splines of any order with $n-1$ equally spaced knots, in the sense that one can not replace ω_3 by ω_4. In $\mathbf{L}_p[0,\ 1]$, $1 \le p < \infty$, things become more complicated and even positive approximation is no longer trivial. Zhou proved in [1] that for positive polynomial approximation in $\mathbf{L}_p$, it is impossible to reach ω_3 if $1 < p < \infty$ or ω_4 if $p = 1$, and that in the copositive case, it is impossible to reach ω_2 if $1 < p < \infty$ or ω_3 if $p = 1$. He also conjectures that the case of $p = 1$ is no better than that of $1 < p < \infty$, and that even ω_2 is impossible to reach in positive polynomial approximation for all $1 \le p < \infty$. Little is known for the spline case, the only relevant result we know is one on one-sided approximation by A. Andreev, V. A. Popov and B. Sendov [5]:

THEOREM A. *Let f have integrable bounded k^{th} derivative $f^{(k)}$ on the interval* $[0,\ 1]$, *and let* $\mathbf{T}$ *be a given partition of* $[0,\ 1]$. *Then there exist splines S and s of degree k on the knot sequence*

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T *such that*

$$S(x) \geq f(x) \geq s(x), \qquad x \in [0,\, 1],$$
$$\|S - s\|_p \leq 2(k+1)!(\Delta_{\mathbf{T}})^k \tau(f^{(k)}, \Delta_{\mathbf{T}})_p, \tag{1.1}$$

where $1 \leq p \leq \infty$, $\Delta_{\mathbf{T}}$ *is the mesh size of* $\mathbf{T}$, *and* τ *is the average modulus of smoothness (see the definition below).*

It is then the purpose of this paper to thoroughly investigate the order of positive and copositive spline approximation of functions in $\mathbf{L}_p$, $1 \leq p < \infty$. We should point out here that the technique used in this paper applies to the case of $0 < p < 1$ with almost no changes, but the constant C will then depend on p. Some changes can be made to cover the case of unequal spacing, which will be described in a forthcoming paper by Hu, K. A. Kopotun and Yu [6].

We first introduce the notation we will use. We denote by $\mathbf{P}_m[a,\, b]$ the space of all polynomials of degree $\leq m$ on $[a,\, b]$, and by $\mathbf{W}_p^k[a,\, b]$ the Sobolev space, the space of functions whose $(k-1)^{\text{st}}$ derivative is absolutely continuous and whose k^{th} derivative lies in $\mathbf{L}_p$, $1 \leq p \leq \infty$. We say that $f(x)$ changes sign at $y \in (0,1)$ if: (1) there exists an $\varepsilon_1 > 0$ such that $\eta f(x) \geq 0$ for any $x \in [y - \varepsilon_1,\, y)$, where $\eta = \pm 1$; (2) there exists an $\varepsilon_2 > 0$ such that $\eta f(x) \leq 0$, for any $x \in (y,\, y + \varepsilon_2]$; (3) the inequalities hold strictly at least at one x in each of the neighborhoods above, that is, f has a true sign change at y. Such a y is called a point of sign change of f. We assume f only has $k < \infty$ sign changes at $0 < y_1 < y_2 < \cdots < y_k < 1$, and denote $y_0 := 0$ and $y_{k+1} := 1$. A function g is said to be copositive with f if it has the same sign with f on each interval $[y_i,\, y_{i+1}]$ and changes its sign exactly at each y_i, $1 \leq i \leq k$. We denote the m^{th} modulus of smoothness of $f \in \mathbf{L}_p[a,\, b]$ by

$$\omega_m(f,t)_p := \sup_{0 \leq h \leq t} \|\Delta_h^m(f,\cdot)\|_{p,[a,b-mh]},$$

where $t \geq 0$ and Δ_h^m is the usual m^{th} forward difference operator, and denote the m^{th} average modulus, or τ-modulus, of smoothness by

$$\tau_m(f,t)_p := \|\omega_m(f,\cdot,t)\|_{p,[a,\, b]},$$

where

$$\omega_m(f,x,t) := \sup\left\{|\Delta_h^m(f,y)| : y, y+mh \in \left[x - \frac{mt}{2},\, x + \frac{mt}{2}\right] \cap [a,\, b]\right\}$$

is the m^{th} local modulus of smoothness of f. If there is any possible confusion about the interval over which the modulus is taken, we will indicate the interval in the notation such as $\omega_m(f, 1/n, [0,\, 1])_p$ or $\tau_m(f,t,[x_0,\, x_3])_p$. From the definitions, it is easy to see (Sendov and Popov [7, Theorem 1.4]) that if $f \in \mathbf{C}[a,\, b]$, then

$$\omega_m(f,t)_p \leq \tau_m(f,t)_p \leq (b-a)^{1/p}\omega_m(f,t)_\infty,$$
$$\tau_m(f,t)_\infty = \omega_m(f,t)_\infty, \tag{1.2}$$

and that if t is large, τ_m can be quite large. As an extreme, we have

$$\omega_m(f,x,b-a) = \omega_m(f,b-a)_\infty, \quad \forall x \in [a,\, b],$$
$$\tau_m(f,b-a)_p = (b-a)^{1/p}\omega_m(f,b-a)_\infty. \tag{1.3}$$

We now state our main results; all the proofs will be given later in Section 3. The first two of them deal with the order of positive spline approximation in $\mathbf{L}_p$. Note that an analogue of Theorem 1 for polynomials can be readily obtained by using a positive linear operator, but then f has to have a nonnegative extension on a much larger interval.

THEOREM 1. *Let $f \in \mathbf{L}_p[0, 1]$, $1 \le p < \infty$, be nonnegative, and let $n > 0$ be an integer. If f has a nonnegative extension F on $[-(1/2n), 1 + (1/2n)]$ with*

$$\omega_2\left(F, \frac{1}{n}, \left[-\frac{1}{2n}, 1 + \frac{1}{2n}\right]\right)_p \le C_0\omega_2\left(f, \frac{1}{n}, [0, 1]\right)_p, \tag{1.4}$$

then there exists a nonnegative $\mathbf{C}^1$ quadratic spline s on $[0, 1]$ with the interior knots $\{(2i-1)/(2n)\}_{i=1}^n$ such that

$$\|f - s\|_p \le CC_0\omega_2\left(f, \frac{1}{n}, [0, 1]\right)_p, \tag{1.5}$$

where C is an absolute constant.

If f fails to have a nonnegative extension satisfying (1.4), then the order deteriorates at the ends of the interval, and that leads to the order of ω_1.

THEOREM 2. *Let $f \in \mathbf{L}_p[0, 1]$, $1 \le p < \infty$, be nonnegative, and let $n > 0$ be an integer. Then there exists a nonnegative $\mathbf{C}^1$ quadratic spline s on $[0, 1]$ with the interior knots $\{(2i-1)/(2n)\}_{i=1}^n$ such that*

$$\|f - s\|_p \le C\omega\left(f, \frac{1}{n}\right)_p, \tag{1.6}$$

where C is an absolute constant.

Because positive approximation is a special case of copositive approximation, it should not be surprising that the theorem below does not give an order higher than that in (1.6). We suspect that all these are the highest possible orders, especially for $1 < p < \infty$, just as the case of polynomials, since as far as we know in all known cases, splines with equally spaced knots have the same approximation power as polynomials, except that the latter approximate better near the ends of the interval; see [1–4,8] and the references therein.

THEOREM 3. *Let $f \in \mathbf{L}_p[0, 1]$, $1 \le p < \infty$, change its sign $k < \infty$ times at $0 < y_1 < y_2 < \cdots < y_k < 1$. Denote $\delta := \min_{j=0}^k(y_{j+1} - y_j)$, where $y_0 := 0$ and $y_{k+1} := 1$. Then for every $n > \delta^{-1}$ there exists a $\mathbf{C}^1$ quadratic spline s with at most $2n + 16k$ interior knots that is copositive with f and satisfies*

$$\|f - s\|_p \le C\omega\left(f, \frac{1}{n}\right)_p, \tag{1.7}$$

where C is an absolute constant.

Using the average modulus τ_m, which is more suitable than ω_m in this context, one can get a better estimate than (1.7), as stated in Theorem 4 below. In view of (1.2), this result is consistent with those for copositive approximation in $\mathbf{C}$, which say the best order in that case is ω_3; see the beginning of this section.

THEOREM 4. *Let $f \in \mathbf{C}[0, 1]$ change its sign $k < \infty$ times at $0 < y_1 < y_2 < \cdots < y_k < 1$. Denote $\delta := \min_{j=0}^k(y_{j+1} - y_j)$, where $y_0 := 0$ and $y_{k+1} := 1$. Then, for every $n > \delta^{-1}$ there exists a $\mathbf{C}^1$ quadratic spline s with at most $16n$ knots that is copositive with f and satisfies*

$$\|f - s\|_p \le C\tau_3\left(f, \frac{1}{n}\right)_p, \tag{1.8}$$

where C is an absolute constant.

REMARK. The requirement of $f \in \mathbf{C}$ is justified by the facts (1.3), with $[a, b]$ viewed as any subinterval of $[0, 1]$. Also see Theorem 5 in Section 2.

The corollary below follows directly from the inequality (Sendov and Popov, [7, Theorem 1.5])

$$\tau_m(f, t)_p \le C_m t\omega_{m-1}(f', t)_p, \qquad t \ge 0. \tag{1.9}$$

COROLLARY 1. *If the function f in Theorem 4 is also in $\mathbf{W}_p^1[0, 1]$, then*

$$\|f - s\|_p \le C n^{-1} \omega_2 \left(f', \frac{1}{n}\right)_p, \tag{1.10}$$

where C is an absolute constant.

2. PRELIMINARIES

In the proofs, we will make repeated use of the following relationship between the uniform and $\mathbf{L}_p$ norms of polynomials as described in [9].

LEMMA 1. *If $k \ge 0$, $q > 0$, there is a constant $C > 0$ depending at most on q, k and n such that for each $q \le p \le \infty$, each polynomial $P \in \mathbf{P}_k$ and each n-cube $Q \subset \mathbb{R}^n$,*

$$\left(\frac{1}{|Q|}\int_Q |P|^q\right)^{1/q} \le \left(\frac{1}{|Q|}\int_Q |P|^p\right)^{1/p} \le C\left(\frac{1}{|Q|}\int_Q |P|^q\right)^{1/q}. \tag{2.1}$$

When either q or $p = \infty$ the corresponding expression is replaced by $\|P\|_{\infty,Q}$.

REMARK. If $q \ge 1$, the constant C above can be chosen so that it is independent of q.

The auxiliary lemma below allows us to blend local overlapping polynomials into a smooth spline with the same approximation order.

LEMMA 2 (BEATSON, [10, LEMMA 3.2]). *Let $r \ge 2$ be an integer and $d = 2(r-1)^2$. Let $\mathbf{T} = \{t_i\}_{i=-\infty}^{\infty}$ be a strictly increasing knot sequence with $t_0 = a$ and $t_d = b$. Let g_1, g_2 be two polynomials of degree $< r$. Then there exists a spline g of order r with knot sequence $\mathbf{T}$ such that*

(1) *$g(x)$ is a number between $g_1(x)$ and $g_2(x)$ for each $x \in [a, b]$,*
(2) *$g = g_1$ on $(-\infty, a]$ and $g = g_2$ on $[b, \infty)$.*

Although simple and well-known, this fact is useful in our constructions:

PROPOSITION 1. *Let L be a bounded linear operator on $\mathbf{L}_p[a, b]$, $1 \le p \le \infty$. Then,*

$$\|f - Lf\|_p \le C\omega_r(f, b-a)_p \quad \text{for some } C = C(r) > 0, \tag{2.2}$$

if and only if L reproduces all polynomials of degree $< r$.

We now construct linear operators L by interpolating integral averages of f as follows. Let $I_1, I_2, \dots, I_r$ be subintervals of $[a, b]$, and $c_1, c_2, \dots, c_r$ their midpoints. Let z_i be the integral average of f on I_i:

$$z_i := \frac{1}{|I_i|}\int_{I_i} f, \qquad i = 1, 2, \dots, r.$$

Define $L : \mathbf{L}_p \to \mathbf{P}_{r-1}$ by $Lf = P_f$, where P_f is the polynomial of degree $< r$ interpolating (c_i, z_i), $i = 1, 2, \dots, r$.

LEMMA 3. *Let L be the linear operator defined above. Then*

$$\|f - Lf\|_p \le C\omega_2(f, b-a)_p, \tag{2.3}$$

where the constant C depends on r, and the ratios $(b-a)/(c_{i+1} - c_i)$ and $(b-a)/|I_i|$.

PROOF. It is trivial to see that L is linear and (only) reproduces all linear polynomials. We only need to show $\|L\|_p$ is finite. Let $f \in \mathbf{L}_p[a, b]$ be arbitrary. Since $|||q||| := \max_i |q(c_i)|$ defines a norm $|||\cdot|||$ for the space $\mathbf{P}_{r-1}[a, b]$ which is equivalent to $\|\cdot\|_\infty$ (with the equivalence constants depending on r and the ratios $(b-a)/(c_{i+1}-c_i)$). By Lemma 1, we have

$$\frac{1}{(b-a)^{1/p}}\|Lf\|_p \le \|Lf\|_\infty \le C|||Lf||| = C\max_i |z_i| \le C\sum_i |z_i|$$
$$\le \sum_i \frac{C}{|I_i|}\int_{I_i}|f| \le \frac{C}{b-a}\sum_i \int_{I_i}|f| \le \frac{C}{b-a}\|f\|_1 \le \frac{C}{(b-a)^{1/p}}\|f\|_p,$$

where, and throughout this paper, C denotes a constant whose value does not depend on f but may vary from one occurrence to another, even in the same line. Therefore, $\|L\|_p$ is finite. ∎

We will also need the following trivial variation in the constructions.

LEMMA 4. *If at least one of the midpoints c_i is replaced by another point $d_i \in I_i$, then the resulting operator L satisfies*

$$\|f - Lf\|_p \le C\omega(f, b-a)_p, \tag{2.4}$$

with C depending on the same quantities.

V. H. Hristov and K. G. Ivanov estimated the error of one-sided approximation by polynomials

$$\widetilde{E}_m(f)_p := \inf\{\|P - Q\|_p : P, Q \in \mathbf{P}_m[a, b],\ P \ge f \ge Q\};$$

see [11,12] and the references therein. They also kindly confirmed to the author the validity of the following Whitney-type theorem and sketched the proof in their letter.

THEOREM 5. *Let $f \in \mathbf{C}[a, b]$. Then*

$$\widetilde{E}_{r-1}(f)_p \sim \tau_r(f, b-a)_p, \qquad r = 1, 2, \dots, \tag{2.5}$$

with the constants of equivalence depending only on r.

PROOF. Denote by $\overline{P}$, $\overline{Q}$, $\widetilde{P}$ and $\widetilde{Q}$ best one-sided approximations of f by polynomials of degree $< r$ from above and below in the spaces $\mathbf{L}_p$ and $\mathbf{C}$, respectively. Then by the definitions and Lemma 1

$$\widetilde{E}_{r-1}(f)_p = \|\overline{P} - \overline{Q}\|_p \le \|\widetilde{P} - \widetilde{Q}\|_p \le (b-a)^{1/p}\|\widetilde{P} - \widetilde{Q}\|_\infty = (b-a)^{1/p}\widetilde{E}_{r-1}(f)_\infty$$

and

$$(b-a)^{1/p}\|\widetilde{P} - \widetilde{Q}\|_\infty \le (b-a)^{1/p}\|\overline{P} - \overline{Q}\|_\infty \le C\|\overline{P} - \overline{Q}\|_p.$$

These give the equivalence of $\widetilde{E}_{r-1}(f)_p$ and $(b-a)^{1/p}\widetilde{E}_{r-1}(f)_\infty$. Now denote by P^* the best unconstrained polynomial approximant of f of degree $< r$ in $\mathbf{C}$, that is, $E := E_{r-1}(f)_\infty = \|f - P^*\|_\infty$. By the definitions and Lemma 1 again, we have

$$\widetilde{E}_{r-1}(f)_\infty = \|\widetilde{P} - \widetilde{Q}\|_\infty \ge \|f - \widetilde{P}\|_\infty \ge \|f - P^*\|_\infty = E$$

and

$$\widetilde{E}_{r-1}(f)_\infty = \|\widetilde{P} - \widetilde{Q}\|_\infty \le \|(P^* + E) - (P^* - E)\|_\infty = 2E.$$

These give the equivalence of $\widetilde{E}_{r-1}(f)_\infty$ and $E_{r-1}(f)_\infty$, and now (2.5) follows from Whitney's Theorem and (1.3). ∎

Let subintervals J_i form a partition of the interval J, then from the definition of the τ-modulus it is obvious that

$$\sum \tau_m(f, t, J_i)_p^p \le \tau_m(f, t, J)_p^p.$$

If J_i's intersect each other, but each $x \in J$ is contained in at most k such subintervals, then this still holds true with the constant k added to the right hand side:

$$\sum_i \tau_m(f,t,J_i)_p^p \le k\tau_m(f,t,J)_p^p. \tag{2.6}$$

Similarly, but not trivially, we have [13]

$$\sum_i \omega_m(f,t,J_i)_p^p \le C\omega_m(f,t,J)_p^p, \tag{2.7}$$

where C depends on m. Inequalities (2.6) and (2.7) will be used in estimating the error of spline approximant from the information about its polynomial pieces.

3. PROOFS OF MAIN THEOREMS

PROOF OF THEOREM 1. For the sake of simplicity, we still use the lower case letter f for the extension F. Fix $n > 0$. Let $x_i := (i/n)$, $t_i := (2i-1)/(2n)$ and

$$z_i := n\int_{t_i}^{t_{i+1}} f, \qquad i = 0, 1, \dots, n.$$

Let l_i be the line through the two points $P_i(x_i, z_i)$ and $P_{i+1}(x_{i+1}, z_{i+1})$, $i = 0, 1, \dots, n-1$, and $\tilde{s}$ the broken line with vertices P_i, $i = 0, 1, \dots, n$. By Lemma 3,

$$\|f-\tilde{s}\|_{p,[x_i,\,x_{i+1}]} = \|f-l_i\|_{p,[x_i,\,x_{i+1}]} \le \|f-l_i\|_{p,[t_i,\,t_{i+2}]} \le C\omega_2\left(f, \frac{1}{n}, [t_i,\, t_{i+2}]\right)_p. \tag{3.1}$$

By (2.7), we have

$$\begin{aligned} \|f-\tilde{s}\|_{p,[0,\,1]}^p &= \sum_{i=0}^{n-1} \|f-\tilde{s}\|_{p,[x_i,\,x_{i+1}]}^p \le C^p \sum_{i=0}^{n-1} \omega_2\left(f, \frac{1}{n}, [t_i,\, t_{i+2}]\right)_p^p \\ &\le C^p\omega_2\left(f, \frac{1}{n}, \left[-\frac{1}{2n},\, 1+\frac{1}{2n}\right]\right)_p^p \le C^pC_0^p\omega_2\left(f, \frac{1}{n}, [0,\, 1]\right)_p^p. \end{aligned} \tag{3.2}$$

That is,

$$\|f-\tilde{s}\|_{p,[0,\,1]} \le CC_0\omega_2\left(f, \frac{1}{n}, [0,\, 1]\right)_p. \tag{3.3}$$

The spline $\tilde{s}$ is nonnegative and has the desired order of approximation, but is merely in $\mathbf{C}^0$. We now smooth it into a $\mathbf{C}^1$ spline with the same order of approximation. Denote the midpoint $(t_i, (z_{i-1}+z_i)/2)$ of the line segment $\overline{P_{i-1}P_i}$ by Q_i, $i = 1, \dots, n$. It is readily seen that there exists a quadratic polynomial s_i on $[t_i,\, t_{i+1}]$ that is tangent to l_{i-1} and l_i at Q_i and Q_{i+1}, respectively, and has its graph inside the triangle $Q_iP_iQ_{i+1}$. This s_i cuts off the corner of the broken line $\overline{P_{i-1}P_iP_{i+1}}$ and stays nonnegative. Therefore, the spline s defined by

$$s(x) := \begin{cases} l_0(x), & x \in [0,\, t_1] \\ s_i(x), & x \in [t_i,\, t_{i+1}], \quad i = 1, \dots, n-1 \\ l_{n-1}(x), & x \in [t_n,\, 1] \end{cases} \tag{3.4}$$

has a continuous derivative and n simple knots t_i, $i = 1, \dots, n$.

To show (1.5), we need an estimate of $\|s - \tilde{s}\|_{p,[0,1]}$. Denote by $\tilde{l}_i$ the straight line through the two points Q_i and Q_{i+1}. For $i = 1, \ldots, n-1$ we have

$$\begin{aligned}
&C^p \omega_2\left(f, \frac{1}{n}, [t_{i-1}, t_{i+2}]\right)_p^p \geq \omega_2\left(\tilde{s}, \frac{1}{n}, [x_{i-1}, t_{i+1}]\right)_p^p \\
&\geq \int_{x_{i-1}}^{t_i} \left|\tilde{s}\left(x + \frac{1}{n}\right) - 2\tilde{s}\left(x + \frac{1}{2n}\right) + \tilde{s}(x)\right|^p \\
&= \int_{x_{i-1}}^{t_i} \left|l_i\left(x + \frac{1}{n}\right) - l_{i-1}\left(x + \frac{1}{n}\right) + l_{i-1}\left(x + \frac{1}{n}\right) - 2l_{i-1}\left(x + \frac{1}{2n}\right) + l_{i-1}(x)\right|^p \\
&= \int_{x_{i-1}}^{t_i} \left|l_i\left(x + \frac{1}{n}\right) - l_{i-1}\left(x + \frac{1}{n}\right)\right|^p = \|l_i - l_{i-1}\|_{p,[x_i,\, t_{i+1}]}^p \sim \frac{1}{2n}\|l_i - l_{i-1}\|_{\infty,[x_i,\, t_{i+1}]}^p \\
&= \frac{1}{2n}|l_i(t_{i+1}) - l_{i-1}(t_{i+1})|^p = \frac{2^p}{2n}|l_i(x_i) - \tilde{l}_i(x_i)|^p,
\end{aligned}$$

where we have used (3.1) in the first step, the fact that $\Delta_h^2(l, x) \equiv 0$ for any linear function l and any h in the fourth step, Lemma 1 in the sixth and a property of similar triangles in the last. This is equivalent to

$$|l_i(x_i) - \tilde{l}_i(x_i)| \leq C(2n)^{1/p}\omega_2\left(f, \frac{1}{n}, [t_{i-1}, t_{i+2}]\right)_p. \tag{3.5}$$

Basic analytic geometry on linear and quadratic polynomials gives

$$\begin{aligned}
(2n)^{1/p}\|s - \tilde{s}\|_{p,[t_i,\, t_{i+1}]} &= (2n)^{1/p}\|s_i - \tilde{s}\|_{p,[t_i,\, t_{i+1}]} = (2 \cdot 2n)^{1/p}\|s_i - \tilde{s}\|_{p,[x_i,\, t_{i+1}]} \\
&= (2 \cdot 2n)^{1/p}\|s_i - l_i\|_{p,[x_i,\, t_{i+1}]} \sim \|s_i - l_i\|_{\infty,[x_i,\, t_{i+1}]} = |s_i(x_i) - l_i(x_i)| \\
&= \frac{1}{2}|\tilde{l}_i(x_i) - l_i(x_i)|.
\end{aligned}$$

Combining this and (3.5) yields

$$\|s - \tilde{s}\|_{p,[t_i,\, t_{i+1}]} \leq C\omega_2\left(f, \frac{1}{n}, [t_{i-1}, t_{i+2}]\right)_p.$$

The same argument as that in (3.2) produces

$$\|s - \tilde{s}\|_{p,[0,1]} \leq CC_0\omega_2\left(f, \frac{1}{n}, [0, 1]\right)_p,$$

and (1.5) now follows from this and (3.3). ∎

PROOF OF THEOREM 2. We change the definition of z_0 and z_n in the proof of Theorem 1 by

$$z_0 := 2n\int_0^{t_1} f \qquad \text{and} \qquad z_n := 2n\int_{t_n}^1 f,$$

and notice that

$$\omega_2(f, t, I)_p \leq 2\omega(f, t, I)_p,$$

then an almost identical proof gives (1.6). ∎

Because the proof of Theorem 3 is a modification of that of Theorem 4, we need to prove Theorem 4 first.

PROOF OF THEOREM 4. Fix $n > \delta^{-1}$. Let $x_i := i/2n$. We call the interval $I_i := [x_i, x_{i+1}]$ contaminated if $x_i < y_j \leq x_{i+1}$ for some point y_j of sign change of f, $1 \leq j \leq k$. Since $\delta > 1/n$, there is exactly one y_j in each of the contaminated intervals I_{m_j}, $j = 1, \dots, k$. For convenience we also denote $m_0 := -1$, and $m_{k+1} := 2n$, then

$$m_j < m_j + 2 \le m_{j+1}, \qquad j = 0, 1, \dots, k, \tag{3.6}$$

that is, between I_{m_j} and $I_{m_{j+1}}$ for any $0 \le j \le k$ there is at least one interval I_i that is not contaminated. Note that f does not change sign between I_{m_j} and $I_{m_{j+1}}$.

If $m_{j+1} > m_j + 2$, that is, there are at least two noncontaminated intervals between I_{m_j} and $I_{m_{j+1}}$, we apply Theorem 5 with $r = 3$ on each of the intervals $[x_i,\, x_{i+2}]$, $i = m_j + 1, m_j + 2, \dots, m_{j+1} - 2$ and obtain two quadratic polynomials P_i and Q_i such that

$$\begin{aligned} &P_i(x) \ge f(x) \ge Q_i(x), \qquad \forall x \in [x_i,\, x_{i+2}], \\ &\|P_i - Q_i\|_{p,[x_i,\, x_{i+2}]} \le C\tau_3\left(f, \frac{1}{n}, [x_i,\, x_{i+2}]\right)_p . \end{aligned} \tag{3.7}$$

Set $q_i := P_i$ if $f \ge 0$; $q_i := Q_i$ if $f \le 0$. Hence, q_i is copositive with f and satisfies

$$\|f - q_i\|_{p,[x_i,\, x_{i+2}]} \le \|P_i - Q_i\|_{p,[x_i,\, x_{i+2}]} \le C\tau_3\left(f, \frac{1}{n}, [x_i,\, x_{i+2}]\right)_p . \tag{3.8}$$

Near each point of sign change of f, we construct a local quadratic polynomial by interpolation. More precisely, on each $[x_{m_j-1},\, x_{m_j+2}]$, $j = 1, \dots, k$, we interpolate f at x_{m_j-1}, y_j and x_{m_j+2} by a quadratic polynomial q_{m_j-1}. It is obvious that this parabola is copositive with f on $[x_{m_j-1},\, x_{m_j+2}]$. For its approximation error, we consider the two quadratics P_{m_j-1} and Q_{m_j-1} guaranteed to exist by Theorem 5. Since

$$P_{m_j-1}(x) \ge f(x) \ge Q_{m_j-1}(x), \qquad \forall x \in [x_{m_j-1},\, x_{m_j+2}], \tag{3.9}$$

it holds true at x_{m_j-1}, y_j and x_{m_j+2} in particular. Because the distance between any two of these points is no less than $1/2n = (x_{m_j+2} - x_{m_j-1})/3$, the norm $|||\cdot|||$ for the space $\mathbf{P}_2[x_{m_j-1},\, x_{m_j+2}]$ defined by

$$|||q||| := \max\{|q(x_{m_j-1})|,\ |q(y_j)|,\ |q(x_{m_j+2})|\}$$

is equivalent to $\|\cdot\|_\infty$ with equivalence constants being absolute ones. Therefore, it is also equivalent to $(2n/3)^{1/p}\|\cdot\|_p$ by Lemma 1. From (3.9), we have

$$|||P_{m_j-1} - q_{m_j-1}||| \le |||P_{m_j-1} - Q_{m_j-1}|||,$$

hence,

$$\|P_{m_j-1} - q_{m_j-1}\|_p \le C\|P_{m_j-1} - Q_{m_j-1}\|_p \le C\tau_3\left(f, \frac{1}{n}, [x_{m_j-1},\, x_{m_j+2}]\right)_p ,$$

and finally

$$\begin{aligned} \|f - q_{m_j-1}\|_p &\le \|f - P_{m_j-1}\|_p + \|P_{m_j-1} - q_{m_j-1}\|_p \\ &\le C\tau_3\left(f, \frac{1}{n}, [x_{m_j-1},\, x_{m_j+2}]\right)_p . \end{aligned} \tag{3.10}$$

Having constructed the overlapping local quadratics which are copositive with f and have approximation order τ_3, we now blend them for a smooth spline approximant s with the same approximation order. If I_{i-1} is a noncontaminated interval and q_{i-1} and q_i overlap on I_i, then I_i must be noncontaminated, too, or there would be no q_i at all. We insert $d - 1 = 2(3-1)^2 - 1 = 7$ equally spaced interior knots in $(x_i,\, x_{i+1})$. By Lemma 2, there exists a spline s_i on I_i on these knots that connects with q_{i-1} and q_i in a $\mathbf{C}^1$ manner at x_i and x_{i+1}, respectively. Moreover, the graph of s_i lies between those of q_{i-1} and q_i, hence s_i is also copositive with f and satisfies

$$\int_{I_i} |f - s_i|^p \le \int_{I_i} |f - q_{i-1}|^p + \int_{I_i} |f - q_i|^p,$$

or

$$\|f - s_i\|_{p,I_i}^p \le C^p \left(\tau_3 \left(f, \frac{1}{n}, [x_{i-1}, x_{i+1}] \right)_p^p + \tau_3 \left(f, \frac{1}{n}, [x_i, x_{i+2}] \right)_p^p \right). \tag{3.11}$$

If I_{i-1} is a contaminated interval, then I_i is noncontaminated by (3.6), and this time it is q_{i-2} and q_i that overlap on I_i. We construct s_i the same way as above and, noticing that q_{i-2} is an approximation of f on $[x_{i-2}, x_{i+1}]$, have

$$\|f - s_i\|_{p,I_i}^p \le C^p \left(\tau_3 \left(f, \frac{1}{n}, [x_{i-2}, x_{i+1}] \right)_p^p + \tau_3 \left(f, \frac{1}{n}, [x_i, x_{i+2}] \right)_p^p \right). \tag{3.12}$$

In both (3.11) and (3.12), the interval in the second τ-modulus will be $[x_i, x_{i+3}]$ instead of $[x_i, x_{i+2}]$ if I_{i+1} is contaminated, but this makes no difference in the rest of the proof.

We define the final spline approximant s on each I_i as follows: if there is only one local quadratic polynomial over I_i, set s to this quadratic; if there are two quadratics overlapping on I_i, then there must be a blending local spline s_i, we set s to s_i. It is clear from its construction that s is copositive with f on the whole interval $[0, 1]$, that it lies in $\mathbf{C}^1[0, 1]$ and has no more than $d(2n-1) < 16n$ single interior knots. Using the fact that each s_i is determined by the behavior of f on no more than four (consecutive) intervals of the form I_i, we obtain from (3.8), (3.10)–(3.12) and (2.6)

$$\|f - s\|_{p,[0,1]}^p = \sum_{i=0}^{2n-1} \int_{I_i} |f - s|^p \le C^p \tau_3 \left(f, \frac{1}{n}, [0, 1] \right)_p^p, \tag{3.13}$$

which is (1.8). ∎

PROOF OF THEOREM 3. We use all the notation defined in the first paragraph in the proof of Theorem 4. If $m_{j+1} > m_j + 2$, we apply Theorem 2 with $[0, 1]$ replaced by $[x_{m_j+1}, x_{m_{j+1}}]$ and n by $m_{j+1} - m_j - 1$, and obtain a $\mathbf{C}^1$ quadratic spline s_{m_j} on this interval with interior knots $(2k-1)/(4n)$, $k = m_j + 2, m_j + 3, \dots, m_{j+1}$, and

$$\|f - s_{m_j}\|_{p,[x_{m_j+1}, x_{m_{j+1}}]} \le C\omega \left(f, \frac{1}{n}, [x_{m_j+1}, x_{m_{j+1}}] \right)_p. \tag{3.14}$$

Near each of the points y_j of sign change of f, we construct a quadratic polynomial q_{m_j} on $[x_{m_j} - 1/4n, x_{m_j+1} + 1/4n]$, $j = 1, \dots, k$, as follows. Suppose at y_j, f changes sign from nonpositive to nonnegative for certainty. Let $h(t) := 2n \int_{t-1/2n}^{t} f$, then h is continuous with $h(y_j + 1/2n) \ge 0$ and $h(y_j) \le 0$. Therefore, there exists a $t_j \in [y_j, y_j + 1/2n]$ such that $h(t_j) = 0$. Let $z_{j1} := h(x_{m_j}) \le 0$ and $z_{j3} := h(x_{m_j+2}) \ge 0$. Let q_{m_j} be the quadratic polynomial that interpolates $(x_{m_j} - 1/4n, z_{j1})$, $(y_j, 0)$ and $(x_{m_j+1} + 1/4n, z_{j3})$. Then it is copositive with f. Since the x-coordinates of these points are at least $1/4n$ away from one another, it has approximation order ω_1 by Lemma 4:

$$\|f - q_{m_j}\|_{p,[x_{m_j} - 1/4n,\, x_{m_j+1} + 1/4n]} \le C\omega \left(f, \frac{1}{n}, [x_{m_j-1}, x_{m_j+2}] \right)_p. \tag{3.15}$$

Inserting 7 equally spaced interior knots into each of the intervals $[x_{m_j} - 1/4n, x_{m_j}]$ and $[x_{m_j+1}, x_{m_j+1} + 1/4n]$, $j = 1, \dots, k$, we can blend all of s_{m_j} and q_{m_j} into a $\mathbf{C}^1$ quadratic spline s that is copositive with f and satisfies (1.7) in the same way as in the proof of Theorem 4. ∎

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