The bipartite edge frustration of composite graphs

Z. Yarahmadi\textsuperscript{a}, T. Došlić\textsuperscript{b,\textdagger}, A.R. Ashrafi\textsuperscript{a,\textdagger}

\textsuperscript{a} Department of Mathematics, Faculty of Science, University of Kashan, Kashan 87317-51167, Islamic Republic of Iran
\textsuperscript{b} Faculty of Civil Engineering, University of Zagreb, Kačićeva 26, 10000 Zagreb, Croatia
\textsuperscript{c} School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Islamic Republic of Iran

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\section*{ABSTRACT}

The smallest number of edges that have to be deleted from a graph to obtain a bipartite spanning subgraph is called the bipartite edge frustration of G and denoted by \( \varphi(G) \). In this paper we determine the bipartite edge frustration of some classes of composite graphs.

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\section*{1. Introduction}

The problem of finding large bipartite spanning subgraphs of a given non-bipartite graph has a long and rich history. The first results were obtained by Erdős \cite{6} and Edwards \cite{5}, who showed that every graph \( G \) on \( |V(G)| \) vertices and \( |E(G)| \) edges contains a bipartite subgraph with at least \( \frac{|E(G)|}{2} + \frac{|V(G)| - 1}{4} \) edges. Those bounds were further improved for various classes of graphs; for example, the lower bound of \( \frac{4}{5}|E(G)| \) was established for cubic triangle-free graphs \cite{8} and also for sub-cubic triangle-free graphs \cite{1}. The best currently known \cite{2} lower bound for cubic, planar and triangle-free graphs is \( \frac{39}{32}|V(G)| - \frac{9}{16} \).

Instead of looking for large bipartite subgraphs of a given graph \( G \), it is sometimes more convenient to look at the equivalent problem of finding a smallest set of edges that must be deleted from \( G \) in order to make the remaining graph bipartite. Borrowing from the terminology of the anti-ferromagnetic Ising model, the cardinality of any such set is then called the bipartite edge frustration of a graph. More formally, let \( G \) be a graph with the vertex and edge sets \( V(G) \) and \( E(G) \) respectively. The \textbf{bipartite edge frustration} of \( G \) is then defined as the minimum number of edges that have to be deleted from \( G \) to obtain a bipartite spanning subgraph. We denote it by \( \varphi(G) \).

Clearly, if \( G \) is bipartite then \( \varphi(G) = 0 \). It can be easily shown that \( \varphi(G) \leq \frac{|E(G)|}{2} \) and that the complete graph on \( n \) vertices has the maximum possible bipartite edge frustration among all graphs on \( n \) vertices. Hence, the bipartite edge frustration has properties that make it useful as a measure of non-bipartivity of a given graph.

The quantity \( \varphi(G) \) is, in general, difficult to compute; it is NP-hard for general graphs. Hence, it makes sense to search for classes of graphs that allow its efficient computation. Some results in this direction are reported in \cite{4} for fullerene and other polyhedral graphs and in \cite{7} for some classes of nanotubes. It is also worthwhile to investigate how the bipartite edge frustration of some composite graphs that arise \textit{via} graph products is related to the bipartite edge frustrations of their components. Elucidating those relationships for some classes of composite graphs is the main goal of the present paper.

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\textsuperscript{\textdagger} Corresponding author. Tel.: +385 1 2393903.
\textit{E-mail} addresses: doслиc@grad.hr, doслиc@math.hr (T. Došlić).

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In particular, we will present explicit formulas for the bipartite edge frustration for Cartesian product, chain, bridge and extended bridge graphs. We will also compute the bipartite edge frustration of join, corona, suspension and composition of bipartite graphs. Finally, some inequalities of the Nordhaus–Gaddum type will be presented.

The notation we use is mostly standard and taken from standard graph theory textbooks such as, e.g., [12].

2. Definitions and preliminaries

In this section we introduce the composite graphs that will be considered here and recall their basic properties relevant for our goal. We start by composite graphs that arise by splicing, i.e., by identifying certain vertices.

Let \( \{G_i\}_{i=1}^n \) be a set of finite pairwise disjoint graphs with \( v_i, w_i \in V(G_i) \). The **chain graph** \( C(G_1, G_2, \ldots, G_n, v_1, w_1, \ldots, v_n, w_n) \) of \( \{G_i\}_{i=1}^n \) with respect to the vertices \( \{v_i, w_i\}_{i=1}^n \) is the graph obtained from graphs \( G_1, G_2, \ldots, G_n \) by identifying the vertex \( w_i \) with \( v_{i+1} \), for all \( i = 1, 2, \ldots, n-1 \), as shown in Fig. 1. We abbreviate the notation to \( C(G_1, G_2, \ldots, G_n) \) when the vertices \( v_i \) and \( w_i \) are clear from context.

Let \( \{G_i\}_{i=1}^n \) be a set of finite pairwise disjoint graphs with \( v_i \in V(G_i) \). The **bridge graph** \( B(G_1, G_2, \ldots, G_n, v_1, \ldots, v_n) \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_n \) by connecting the vertices \( v_i \) and \( v_{i+1} \) by an edge, for all \( i = 1, 2, \ldots, n-1 \), as shown in Fig. 2. Again, the dependence on \( v_1, \ldots, v_n \) will be often omitted in notation.

Now we define an extension of bridge graph. Let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( \{H_i\}_{i=1}^n \). We also assume that \( H_i, 1 \leq n \), are finite connected pairwise disjoint graphs such that \( V(G) \cap V(H_i) = \{v_i\} \). The **extended bridge graph** \( EB(G; H_1, \ldots, H_n, v_1, \ldots, v_n) \) of \( G \) and \( \{H_i\}_{i=1}^n \) with respect to \( \{v_i\}_{i=1}^n \) is constructed by identifying the vertex \( v_i \) in \( G \) and \( H_i \), for all \( i = 1, 2, \ldots, n \). An example is shown in Fig. 3.

The above classes of graphs were considered in [10]. The splices and links considered in [3] could be viewed as their special cases.

The **Cartesian product** \( G \square H \) of graphs \( G \) and \( H \) has the vertex set \( V(G \square H) = V(G) \times V(H) \) and \( (a, b)(c, d) \) is an edge of \( G \square H \) if either \((a = c \text{ and } bd \in E(H))\), or \((ac \in E(G) \text{ and } b = d)\). If \( G_1, G_2, \ldots, G_s \) are graphs then we denote \( G_1 \square G_2 \square \cdots \square G_s \) by \( \Pi_{i=1}^s G_i \). If \( G_1 = G_2 = \cdots = G_s = G \), we have the \( s \)-th Cartesian power of \( G \) and denote it by \( G^s \). A Cartesian product is bipartite if and only if all of its components are bipartite.

The **join** \( G + H \) of graphs \( G \) and \( H \) with disjoint vertex sets \( V(G) \) and \( V(H) \) and edge sets \( E(G) \) and \( E(H) \) is the graph union \( G \cup H \) together with all the edges joining vertices of \( V(G) \) and \( V(H) \). If \( G = H + \cdots + H \), we denote \( G \) by \( nH \). The graph \( \nabla G \) is obtained from \( G \) by adding a new vertex and making it adjacent to all vertices of \( G \). The graph \( \nabla G \) is called **suspension** of
G. Obviously, $\nabla G = G + K_1$. A join of two graphs is bipartite if and only if both graphs are empty, i.e., without edges. Hence $\varphi(G + H) > 0$ if at least one of components contains an edge.

Both Cartesian product and join are standard graph operations. We refer the reader to monograph [9] for more information on those products.

Let $G$ and $H$ be two graphs. Their **corona product** $G \circ H$ is defined as the graph obtained by taking one copy of $G$ and joining the $i$-th vertex of $G$ to every vertex in $i$-th copy of $H$. An example is shown in Fig. 4.

The **composition** of two graphs $G$ and $H$, $G[H]$, is defined as the graph obtained by replacing every vertex of $G$ with a copy of $H$ and inserting all possible edges among the copies of $H$ that correspond to vertices adjacent in $G$. As in the case of corona, the roles played by the components of a composition are distinctly different. The outer graph provides a skeleton or a scaffold for multiple copies of the inner graph. Both operations can be expressed in terms of joins.

The complement $\overline{G}$ of graph $G$ has $V(G)$ as its vertex set, and two vertices are adjacent in $G$ if and only if they are not adjacent in $G$.

It is obvious from the definition that the bipartite edge frustration of a disconnected graph is equal to the sum of bipartite edge frustration of its components. Hence, it suffices to consider connected graphs. The following observation shows that this type of additive behavior extends also to the graphs with cut-vertices. We will find it useful when dealing with some classes of composite graphs introduced above.

**Lemma 0.** Let $v \in V(G)$ be a cut-vertex of a graph $G$ and $G_i, i = 1, \ldots, s$ be the components of $G - \{v\}$. Then $\varphi(G) = \sum_{i=1}^{s} \varphi(G_i \cup \{v\})$. Here $G_i \cup \{v\}$ denotes the graph induced in $G$ by $V(G_i) \cup \{v\}$.  

The notation we used for a graph induced by a certain set of vertices should not be confused with a similar notation used for composition; here in the square brackets is a set of vertices, while in the composition case there is a whole graph.

We close the section by formulas for the bipartite edge frustration of cycles and complete graphs.

$$\varphi(C_n) = \frac{1 - (-1)^n}{2}, \quad \varphi(K_n) = \left\lceil \frac{n - 1}{2} \right\rceil \left\lfloor \frac{n - 1}{2} \right\rfloor.$$

3. The bipartite edge frustration of some composite graphs

3.1. Chain, bridge and extended bridge

The three classes of graphs considered in this subsection share a certain number of similarities that enable their synoptic treatment. In both chain and bridge graphs their building blocks are so well isolated from each other that their bipartite edge frustrations can be computed separately and then added in order to obtain the bipartite edge frustration of the whole graph. All interaction between components of a bridge graph is via its path backbone, which is itself bipartite. If the backbone is replaced by a non-bipartite scaffold, as in the case of extended bridges, the only additional complication is to compute the bipartite edge frustration of the scaffold graph. This results in (at most) one additional term in the formula for the total bipartite edge frustration.

**Theorem 1.** (i) Let $G = C(G_1, G_2, \ldots, G_n)$ be a chain graph. Then $\varphi(G) = \sum_{i=1}^{n} \varphi(G_i)$.

(ii) Let $G = B(G_1, G_2, \ldots, G_n)$ be a bridge graph. Then $\varphi(G) = \sum_{i=1}^{n} \varphi(G_i)$.

(iii) Let $K = EB(G, H_1, \ldots, H_n)$ be an extended bridge graph. Then $\varphi(K) = \varphi(G) + \sum_{i=1}^{n} \varphi(H_i)$.  

The results of this subsection can be specialized in a straightforward way to the cases where all building blocks are identical, yielding the explicit formulas for the bipartite edge frustrations of rooted products of two graphs. Similarly, the results for chain graphs remain valid without any modifications also for splices of two or more graphs and for generalized cactus graphs. The results and proofs follow directly from Lemma 0 and we leave their formulation and proofs to the reader.
Let $G$ be a non-bipartite graph on $n$ vertices. Then $\phi(G) ≤ \phi(1) + \phi(2)$.

Proof. Let us first look at a special case of Cartesian product when there are only two graphs and one of them is $K_2$. The graph $G = G_1 \sqcap G_2$ consists of two copies of $G_1$, and each edge of $G_1$ has two copies in $G$, connected by a pair of "parallel" edges. We may say that each edge of $G_1$ has been expanded into a cycle of length four. The only sources of non-bipartivity in $G$ are those already present in $G_1$. Hence the operation $G_1 \sqcap G_2$ neither introduced any new non-bipartivity, nor destroyed any that was present. Clearly, the number of edges to be deleted from a copy of $G_1$ in order to make it bipartite was not affected by the product operation, and hence $\phi(G_1 \sqcap G_2) = 2\phi(G_1)$.

Let us now look at $G_1 \sqcap G_2$ for general graphs $G_1$ and $G_2$ with given $\phi(G_1)$ and $\phi(G_2)$. Let $G = G_1 \sqcap G_2$. The two copies of $G_1$ indexed by $a$ and $b$ look locally as $G_1 \sqcap G_2$. And $2\phi(G_1)$ edges must be deleted to make this part of $G_1 \sqcap G_2$ bipartite. By summing over all edges of $G_2$ we see that $|V(G_2)||\phi(G_1)|$ edges must be deleted to account for the non-bipartivity of $G_1 \sqcap G_2$ inherited from $G_1$. By symmetry of the Cartesian product it follows that another $|V(G_1)||\phi(G_2)|$ edges must be deleted to get rid of the non-bipartivity inherited from $G_2$. Hence,

$$\phi(G_1 \sqcap G_2) = |V(G_2)||\phi(G_1)| + |V(G_1)||\phi(G_2)| = |V(G_1)||V(G_2)| \left( \frac{\phi(G_1)}{|V(G_1)|} + \frac{\phi(G_2)}{|V(G_2)|} \right).$$

The general case now follows by a straightforward inductive argument and we omit the details. \qed

Corollary 3. Let $B$ be a bipartite graph on $n$ vertices. Then for any graph $G$ we have $\phi(B \sqcap G) = n\phi(G)$. \qed

This corollary covers the case of linear polymers $P_n \sqcap G$ induced by an arbitrary graph $G$.

Corollary 4. Let $G$ be a non-bipartite graph on $n$ vertices. Then $\phi(G) = n^{-1}\phi(G)$. \qed

We close the subsection by presenting explicit formulas for the bipartite edge frustration of $C_4$ nanotubes and nanotori. We omit the trivial case of bipartite structures.

Corollary 5. (a) $\phi(P_n \sqcap C_{2m+1}) = n$.
(a) $\phi(C_{2n} \sqcap C_{2m+1}) = 2n$.
(c) $\phi(C_{2m+1} \sqcap C_{2m+1}) = 2(m + n + 1)$. \qed

3.3. Join and suspension

We have already mentioned that $G + H$ is non-bipartite as soon as any of its component contains an edge. It is intuitively clear that joins are "very much" non-bipartite, and our findings confirm this feeling.

Theorem 6. Let $G_1$ and $G_2$ be two connected bipartite graphs with bipartitions $(A_1, B_1)$ and $(A_2, B_2)$, respectively. Let us denote $a_i = |A_i|$, $b_i = |B_i|$, $i = 1, 2$, and let $a_i ≤ b_i$ for $i = 1, 2$. Then

$$\phi(G_1 + G_2) ≤ \min\{a_1a_2 + b_1b_2, a_1b_2 + b_1a_2, a_1|V(G_2)| + |E(G_2)|, a_2|V(G_1)| + |E(G_1)|, |E(G_1)| + |E(G_2)|\}.$$

Proof. The graph $G_1 + G_2$ is schematically shown in Fig. 5.

It is clear that deleting all edges of $G_1$ and of $G_2$ will make $G_1 + G_2$ bipartite. Hence $\phi(G_1 + G_2) ≤ |E(G_1)| + |E(G_2)|$. Similarly, deleting all edges between $A_1$ and $G_2$ together with all edges of $G_2$ will result in a bipartite graph. (The same will happen if we delete all edges between $B_1$ and $G_2$ together with all edges of $G_2$, but $b_1 ≥ a_1$ implies that this set cannot be of smaller cardinality than the previous one.) By symmetry, the graph can be bipartitized by deleting all edges between $A_2$ and $G_1$ along with all edges of $G_1$. (Again, we need not consider deleting the edges between $B_2$ and $G_1$.) Also, $G_1 + G_2$ can be bipartitized by deleting all edges between $A_1$ and $A_2$ and between $B_1$ and $B_2$. Finally, deleting all edges between $A_1$ and $B_2$ and between $A_2$ and $B_1$ will also yield a bipartite graph. \qed
Now, the logical thing to do would be to proceed and show that the above upper bound is always achieved. The next example shows that this cannot be done in all cases.

Take $K_{5,50}$ and attach a path of length 4 to its smaller class by identifying one of its end-vertices with any vertex of the smaller class. Denote the obtained bipartite graph by $G_1$, and its bipartition by $(A_1, B_1)$. Obviously, $a_1 = 7$, $b_1 = 52$, and $|E(G_1)| = 254$. Call the vertices from the path exceptional. Take $K_{8,9}$ and call it $G_2$. Now consider $G = G_1 + G_2$ as shown in Fig. 6. By computing all terms of the right-hand side of the inequality of Theorem 6 it follows that the minimum is achieved for $|V(G_1)| + |E(G_2)| = 191$. Hence $G$ can be made bipartite by deleting 119 edge between $A_1$ and $G_2$ and 72 edges of $G_2$.

Let us denote so obtained bipartite graph by $G^\prime$. Now take any two vertices of $A_2$ and connect them to the two exceptional vertices of $A_1$ by all 4 possible edges. The new edges are shown by dashed lines in Fig. 6. The resulting graph is not bipartite, but it can be made bipartite by removing the 3 edges connecting the exceptional vertices of $A_1$ with the exceptional vertices in $B_1$. The total result is a bipartite spanning subgraph of $G_1 + G_2$ obtained by deleting 190 edges, a strictly smaller number than the minimum of the right-hand side of the inequality of Theorem 6. With some care the number of vertices in the example could be made smaller, but this is not essential for our conclusion.

The inequality of Theorem 6 can be converted to equality when the minimum of the right-hand side is equal to $|E(G_1)| + |E(G_2)|$.

**Theorem 7.** Let $G_1$ and $G_2$ be two connected bipartite graphs as in Theorem 6. If the minimum of the right-hand side of the inequality of Theorem 6 is achieved for $|E(G_1)| + |E(G_2)|$, then $\varphi(G_1 + G_2) = |E(G_1)| + |E(G_2)|$.

**Proof.** Obviously $\varphi(G_1 + G_2)$ cannot exceed $|E(G_1)| + |E(G_2)|$. Let us suppose that it is strictly smaller, i.e., $\varphi(G_1 + G_2) < |E(G_1)| + |E(G_2)|$. Let $G^\prime$ be the bipartite graph obtained from $G_1 + G_2$ by deleting $\varphi(G_1 + G_2)$ edges. Since the number of deleted edges is smaller than $|E(G_1)| + |E(G_2)|$, there must be some edges from $E(G_1) \cup E(G_2)$ in $E(G^\prime)$. Those edges span two bipartite graphs, $M_{pq} \subseteq G_1$ and $M_{rs} \subseteq G_2$. Additional edges, at most $\max\{pr + qs, ps + rq\}$ of them, can be added to
We immediately obtain

\[ |V(G_1)||V(G_2)| - (p + q)(a_2 + b_2) - (r + s)(a_1 + b_1) + pq + rs + \max\{pr + qs, ps + rq\}. \]

Consider the expression \((p + q)(a_2 + b_2) + (r + s)(a_1 + b_1) - pq - rs - pr - qs \geq \min\{p, s\}(a_1 + a_2) + rb_1 + qa_2 > 0.\)

Similarly,

\[(p + q)(a_2 + b_2) + (r + s)(a_1 + b_1) - pq - rs - ps - rq \geq \min\{q, s\}(b_1 + b_2) + pa_2 + ra_1 > 0.\]

Hence the number of edges gained by admitting some edges from \(|E(G_1)| \cup |E(G_2)|\) in \(E(G^0)\) is more than offset by the number of edges between \(G_1\) and \(G_2\) that must be deleted in order to preserve the bipartiteness. From there it follows that, within the conditions of Theorem 7, we cannot bipartize \(G_1 \cup G_2\) by deleting fewer than \(|E(G_1)| + |E(G_2)|\) edges. \(\square\)

Two important special cases of join admit exact determination of \(\varphi(G_1 + G_2):\) the suspension \(\nabla G\) and the join of two copies of the same graph \(G + G.\)

**Corollary 8.** Let \(G\) be a connected bipartite graph with bipartition \((A, B)\), and let \(a = |A| \leq |B| = b.\) Then \(\varphi(\nabla G) = a.\)

**Proof.** By setting \(a_1 = 0, b_1 = 1, a_2 = a\) and \(b_2 = b\) in Theorem 6 we immediately obtain \(\varphi(\nabla G) \leq \min\{a, |E(G)|\}.\) The right-hand side is always minimized by \(a\), since no connected bipartite graph can have more elements in the smaller class of bipartition than there are edges incident with them. \(\square\)

If we allow that a bipartite graph \(G\) is not connected, the bipartite edge frustration of its suspension can be computed as \(\varphi(\nabla G) = \min\{a, b, |E(G)|\}\), where \(a\) and \(b\) denote the sizes of classes of bipartition of \(G.\)

**Corollary 9.** Let \(G\) be a connected bipartite graph with bipartition \((A, B)\). Then

\[ \varphi(G + G) = 2|E(G)|. \]

**Proof.** By Theorem 6, \(\varphi(G + G) \leq \min\{a^2 + ab + |E(G)|, 2|E(G)|\}.\) Since \(|E(G)| \leq ab\), the expression above is always minimized by \(2|E(G)|\), and the claim now follows from Theorem 7. \(\square\)

When the components of a join are not bipartite, we have an obvious upper bound

\[ \varphi(G_1 + G_2) \leq \varphi(G_1) + \varphi(G_2) + \varphi(G_1^0 + G_2^0), \]

where \(G_1^0\) and \(G_2^0\) are the bipartite graphs obtained from \(G_1\) and \(G_2\) by deleting \(\varphi(G_1)\) and \(\varphi(G_2)\) edges, respectively.

The following results can be easily verified by direct computation.

(a) \(\varphi(\nabla S_n) = 1,\)

(b) \(\varphi(\nabla W_n) = n + 1,\)

(c) \(\varphi(\nabla P_n) = \left\lfloor \frac{n}{2} \right\rfloor.\)

(d) \(\varphi(\nabla C_n) = \left\lfloor \frac{n}{2} \right\rfloor.\)

We proceed by showing that the suspensions of stars and paths have the smallest and the largest bipartite edge frustrations, respectively, among the suspensions of all bipartite graphs on the same number of vertices.

**Theorem 10.** Let \(G\) be a connected bipartite graph on \(n\) vertices. Then \(\varphi(\nabla S_n) \leq \varphi(\nabla G) \leq \varphi(\nabla P_n).\)

**Proof.** The minimum possible size of a class in the bipartition of a graph is one, and this is exactly the size of the smaller class in a star on any number of vertices. Moreover, the star is the only graph with this property. Hence for all other bipartite graphs on the same number \(n\) of vertices we have \(\varphi(\nabla G) > \varphi(\nabla S_n) = 1.\) Similarly, no bipartite graph on \(n\) vertices can have the smaller class in the bipartition with more than \(\left\lfloor \frac{n}{2} \right\rfloor\) vertices. Hence, by Corollary 8, the bipartite edge frustration of its suspension cannot exceed the size of its smaller class, i.e., \(\varphi(\nabla G) \leq \left\lfloor \frac{n}{2} \right\rfloor.\) But the right-hand side is exactly the bipartite edge frustration of \(\nabla (P_n),\) and the claim follows. Notice that \(\varphi(\nabla S_n) = \varphi(\nabla G)\) if and only if \(G \cong S_n,\) but there exist \(n\)-vertex bipartite graphs \(G\) such that \(G \not\cong P_n\) and \(\varphi(\nabla P_n) = \varphi(\nabla G);\) any even cycle is an example. \(\square\)
3.4. Corona

The bipartite edge frustration of corona products can be neatly expressed when the non-scaffold graph is bipartite. Again, the result crucially depends on the formula for the bipartite edge frustration of a suspension.

Theorem 11. Let $G$ and $H$ be two connected graphs and let $H$ be bipartite. Then $\varphi(G \circ H) = \varphi(G) + a|V(G)|$, where $a$ is the size of the smaller class in the bipartition $(A, B)$ of $H$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. From the definition of corona product of graphs, one can see that $G \circ H = EB(G, \nabla H, \ldots, \nabla H)$, where we suppressed the dependence on the vertices. By Theorem 1 (iii), $\varphi(G \circ H) = \varphi(EB(G, \nabla H, \ldots, \nabla H)) = \varphi(G) + \sum_{i=1}^{\nabla(H)} \varphi(\nabla_i)$. Therefore $\varphi(G \circ H) = \varphi(G) + |V(G)||\varphi(\nabla_i)|$, and this is, by Corollary 8, equal to $\varphi(G) + a|V(G)|$. 

3.5. Composition

The bipartite edge frustration of composition of two graphs can be expressed explicitly if the inner graph is bipartite. If one thinks of composition $G[H]$ as of a graph $G$ in which each edge has been expanded to a copy of $H + H$, the following results becomes immediately clear.

Theorem 12. Let $G$ and $H$ be two connected graphs and let $H$ be bipartite. Then

$$\varphi(G[H]) = \varphi(G)|V(H)|^2 + |V(G)||E(H)|. \quad \Box$$

4. Results of the Nordhaus–Gaddum type

A Nordhaus–Gaddum-type result is a lower or upper bound on the sum or product of an invariant of a graph and its complement. It is named after a paper [11] in which Nordhaus and Gaddum gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results were obtained for many other invariants.

A trivial lower bound $\varphi(G) + \varphi(\overline{G}) \geq 1$ is valid for all graphs on more than five vertices. It follows from Ramsey’s theorem, since at least one of $G$ and $\overline{G}$ contains a triangle if $G$ has at least six vertices.

We start with following simple observation: Let $G$ be a graph and $e$ an edge not in $E(G)$. Then $\varphi(G + e) \leq \varphi(G) + 1$. In other words, adding an edge to a graph cannot increase its bipartite edge frustration by more than one. By the same reasoning one can establish that $\varphi(G + e_1 + \cdots + e_k) \leq \varphi(G) + k$ for any choice $e_1, \ldots, e_k$ of edges not in $E(G)$.

Now we tackle the reverse problem: what happens to $\varphi(G)$ if we delete edges from $G$? It is intuitively clear that the answer depends on the density of edges in $G$; for graphs rich in edges, each edge removal will affect the value of $\varphi(G)$, while for graphs with few edges the removal is less likely to have an effect.

Theorem 13. Let $G = K_n - \{e_1, \ldots, e_l\}$.

(i) If $l \leq \lfloor \frac{n+1}{4} \rfloor$, then $\varphi(G) = \varphi(K_n) - l$.

(ii) If $l > \lfloor \frac{n+1}{4} \rfloor$, then $\varphi(K_n) - l \leq \varphi(G) \leq \varphi(K_n) - \lfloor \frac{n+1}{4} \rfloor$.

Proof. (i) The $l$ deleted edges together can have at most $2l$ end-vertices. Divide the vertices of $K_n$ into two sets of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. Since $2l \leq \lfloor \frac{n}{2} \rfloor$, we can always choose those two sets so that all end-vertices of the $l$ considered edges lie in the larger set. By deleting all edges in those two sets we obtain a bipartite graph. The number of deleted edges is equal to $\varphi(K_n) - l$, hence $\varphi(G) \leq \varphi(K_n) - l$.

Consider now $G'$ obtained by deleting $\varphi(G)$ edges from $G$. By adding $l$ edges to $G$ we cannot increase $\varphi(G)$ by more than $l$. But adding $l$ edges to $G$ results in the complete graph $K_n$. Hence, $\varphi(K_n) \leq \varphi(G) - l$, and the claim follows.

(ii) We consider the graph $G' = K_n - \{e_{i+1}, \ldots, e_{\lfloor \frac{n+1}{4} \rfloor}\}$. Then $G \subseteq G'$ and $G' = G + \{e_\lfloor \frac{n+1}{4} \rfloor + 1, \ldots, e_l\}$. But $\varphi(G') = \varphi(K_n) - \lfloor \frac{n+1}{4} \rfloor$, by claim (i), and hence $\varphi(G') \leq \varphi(G) + l - \lfloor \frac{n+1}{4} \rfloor$. By plugging in the expression for $\varphi(G')$ we obtain $\varphi(G) \geq \varphi(K_n) - l$. On the other hand, since $G \subseteq G'$ we have $\varphi(G) \leq \varphi(G') = \varphi(K_n) - \lfloor \frac{n+1}{4} \rfloor$. 

Let us now consider $G = K_n - \{e_1, \ldots, e_l\}$. Then $G = K_n - \{e_{i+1}, \ldots, e_{\lfloor \frac{n}{2} \rfloor}\}$. We can relabel the edges of $G$ by subtracting $l$ from their labels, so that $G$ can be written as $G = K_n - \{e_1, \ldots, e_l\}$, where $s = \left(\frac{n}{2}\right) - l$. It is easy to see that at least one of the numbers $l$ and $s$ must exceed the critical value $\lfloor \frac{n+1}{4} \rfloor$. Depending on whether the other one also exceeds it or not, we have three different situations. Two of them are symmetric, hence it suffices to consider only one of them. We look at the case $l \leq \lfloor \frac{n+1}{4} \rfloor$ and $s > \lfloor \frac{n+1}{4} \rfloor$ first by Theorem 13 it immediately follows that

$$\varphi(G) + \varphi(G) \leq 2\varphi(K_n) - \left\lfloor \frac{n+1}{4} \right\rfloor - |E(G)|.$$
By symmetry, in the case of $s \leq \left\lfloor \frac{n+1}{4} \right\rfloor$ and $l >\left\lfloor \frac{n+1}{4} \right\rfloor$, we obtain
\[
\varphi(G) + \varphi(\overline{G}) \leq 2\varphi(K_n) - \left\lfloor \frac{n+1}{4} \right\rfloor - |E(G)|.
\]
Finally, when both $l, s > \left\lfloor \frac{n+1}{4} \right\rfloor$, we obtain
\[
\varphi(G) + \varphi(\overline{G}) \leq 2\varphi(K_n) - \left\lfloor \frac{n+1}{4} \right\rfloor.
\]
In general case, when nothing is known on the value of $|E(G)|$, we have an upper bound equal to the worst case, i.e., to the maximum of the above three expressions. By plugging in the formula for $\varphi(K_n)$ and rearranging the terms we obtain an upper bound valid for all graphs.

**Theorem 14.**
\[
\varphi(G) + \varphi(\overline{G}) \leq 2\left\lfloor \frac{n-1}{2} \right\rfloor\left\lceil \frac{n-1}{2} \right\rceil - \left\lfloor \frac{n+1}{4} \right\rfloor - \min\left\{ |E(G)|, |E(\overline{G})|, \left\lfloor \frac{n+1}{4} \right\rfloor \right\}.
\]

For a bipartite graph $G$, $\varphi(G) + \varphi(\overline{G}) = \varphi(\overline{G})$. By combining this fact with the above results we can determine the bipartite edge frustration of complements of some graphs with (relatively) few edges.

**Corollary 15.** Let $T_n$ be a tree on $n \geq 7$ vertices. Then
\[
\varphi(T_n) = \left\lfloor \frac{n-1}{2} \right\rfloor\left\lceil \frac{n-1}{2} \right\rceil - (n - 1). \quad \square
\]

**Corollary 16.** Let $C_n$ be an even cycle on $n \geq 10$ vertices. Then
\[
\varphi(C_n) = \left\lfloor \frac{n-1}{2} \right\rfloor\left\lceil \frac{n-1}{2} \right\rceil - n. \quad \square
\]

It is tempting to think of $\varphi(G)$ as of a measure of bipartivity of a given bipartite graph $G$: the more frustration a bipartite graph leaves to its complement, the more bipartite it is. Based on this idea, we could say that the trees are the “most bipartite” among all connected graphs on the same number of vertices.

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**References**