Homogenization of the Peierls–Nabarro model for dislocation dynamics

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A R T I C L E   I N F O

A B S T R A C T

This paper is concerned with a result of homogenization of an integro-differential equation describing dislocation dynamics. Our model involves both an anisotropic Lévy operator of order 1 and a potential depending periodically on u/ϵ. The limit equation is a non-local Hamilton–Jacobi equation, which is an effective plastic law for densities of dislocations moving in a single slip plane.

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1. Introduction

In this paper we are interested in homogenization of the Peierls–Nabarro model, which is a phase field model describing dislocations. In this model a dislocation is described by a phase transition. Dislocations are moving defects in crystals that can be described at several scales by different models:

- atomic scale (Frenkel–Kontorova model),
- microscopic scale (Peierls–Nabarro model),
- mesoscopic scale (discrete dislocation dynamics),
- macroscopic scale (elasto-visco-plasticity with density of dislocations).

Several changes of scales already exist in the literature: see for instance [12] for a presentation of rigorous passages from atomic scale to microscopic scale, from microscopic scale to mesoscopic scale and from mesoscopic scale to macroscopic scale. Notice that the passage from Peierls–Nabarro model to the Discrete dislocation dynamics is only done in dimension 1 (see [12] and [19]). On the contrary in

higher dimensions, the large scale limit of a single phase transition described by the Peierls–Nabarro model shows that the line tension effect is the much stronger term. The limit model appears to be the mean curvature motion (see [25]).

Our goal in this paper is to understand the large scale limit of the Peierls–Nabarro model in the case of a large number of phase transitions (i.e. of dislocations), recovering at the limit a model with evolution of dislocation densities. In other words, we want to *perform a direct passage in any dimensions from the microscopic scale* (*Peierls–Nabarro model*) *to the macroscopic scale* (*elasto-visco-plasticity with density of dislocations*). In physics and mechanics, it is a great challenge to try to predict macroscopic elasto-visco-plasticity properties of materials (like metals), based on microscopic properties like dislocations. In our work, we try to tackle this question in a very simplified geometry where all the dislocations are contained in the same slip plane with the same Burgers vector. For a physical introduction to the Peierls–Nabarro model, see for instance [20]; for a recent reference, see [38]; we also refer the reader to the paper of Nabarro [35] which presents an historical tour on the Peierls–Nabarro model. See also Section 2 for a more physical presentation of the Peierls–Nabarro model and an interpretation of our results.

1.1. Setting of the problem

The Peierls–Nabarro model has been originally introduced as a variational (stationary) model (see [35]). The time evolution Peierls–Nabarro model as a gradient flow dynamics has only been introduced quite recently, see for instance [33] and [10]. In the present paper we consider such a time evolution Peierls–Nabarro model that can be written at the microscopic scale for the parameter $\epsilon = 1$ as the following equation

$$
\begin{cases}
\partial_t u^\epsilon = \mathcal{I}_1 \big[u^\epsilon(t, \cdot) \big] - W' \left(\dfrac{u^\epsilon}{\epsilon} \right) + \sigma \left(\dfrac{t}{\epsilon}, \dfrac{x}{\epsilon} \right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\
u^\epsilon(0, x) = u_0(x) & \text{on } \mathbb{R}^N.
\end{cases}
\tag{1.1}
$$

For the physical application that we have in mind, we consider a three-dimensional crystal which contains a crystallographic plane $\mathbb{R}^N$ with $N = 2$. This plane contains the dislocations that are represented by transitions of the phase function u^ϵ. Here u^ϵ solves the non-local (and non-linear) heat equation (1.1). Indeed $\mathcal{I}_1$ stands here for an anisotropic half Laplacian (whose expression will be precised below). Here the anisotropy comes both from the possible anisotropy of the elasticity of the crystal and from the fact that the Burgers vector is assumed to be contained in the slip plane $\mathbb{R}^N$ which creates a preferable direction. The dynamics is assumed to be fully overdamped and then the right hand side of the equation is the sum of three force terms: $\mathcal{I}_1[u^\epsilon]$ is the elastic stress created by the dislocation themselves, $-W'$ is the force deriving from the potential W describing the misfit between the two half crystals separated by the plane $\mathbb{R}^N$, and σ is a stress created by the obstacles in the crystal or/and an applied exterior stress. For simplicity σ is assumed to be periodic in order to analyze by homogenization the effect on the dynamics of periodic obstacles everywhere in the crystal. We consider time periodicity for two reasons: one in order to take into account exterior periodic loads, and the second for generality. Indeed, if $\sigma(t/\varepsilon, x/\varepsilon)$ is replaced by an oscillation at a different scale like $\sigma(t/\varepsilon^\gamma, x/\varepsilon^\gamma)$ with $\gamma \neq 1$, then we expect (but it is not proven) that there is a two-scales homogenization effect. If $\gamma > 1$, then we expect that there is first homogenization of σ, where only its mean value will be taken into account at the microscopic scale, and in a second step, we get the macroscopic model by homogenization of the Peierls–Nabarro model with constant σ. If $\gamma < 1$, we expect first to freeze σ and get the macroscopic model by homogenization of the Peierls–Nabarro model for constant σ, and in a second step we remind us that σ is slowly oscillating, and there is a second homogenization of the macroscopic model.

Here ϵ describes the ratio between the microscopic scale and the macroscopic scale, and then is a small parameter. After a suitable rescaling at the macroscopic scale, the Peierls–Nabarro model becomes (1.1). In this paper we investigate the limit as $\epsilon \to 0$ of the viscosity solution u^ϵ of (1.1).

We give the precise definitions and assumptions on the terms involved in (1.1). Here $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1, defined on bounded C^2-functions for $r > 0$ by

$$\mathcal{I}_1[U](x) = \int\limits_{|z| \leqslant r} \left(U(x+z) - U(x) - \nabla U(x) \cdot z \right) \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|} \right) dz$$

$$+ \int\limits_{|z| > r} \left(U(x+z) - U(x) \right) \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|} \right) dz, \tag{1.2}$$

where the function g satisfies

(H1) $g \in C(\mathbf{S}^{N-1})$, $g > 0$, g even.

On the functions W, σ and u_0 we assume:

(H2) $W \in C^{1,1}(\mathbb{R})$ and $W(v + 1) = W(v)$ for any $v \in \mathbb{R}$;
(H3) $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\sigma(t+1, x) = \sigma(t, x)$, $\sigma(t, x+k) = \sigma(t, x)$ for any $k \in \mathbb{Z}^N$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$;
(H4) $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

When $g \equiv C_N$, with C_N a suitable constant depending on the dimension N, then (1.2) is the integral representation of $-(-\Delta)^{\frac{1}{2}}$ for bounded real smooth functions defined on $\mathbb{R}^N$ (see Theorem 1 in [11]). We recall that $(-\Delta)^{\frac{1}{2}}$ is the fractional operator defined for instance on the Schwartz class $S(\mathbb{R}^N)$ by

$$\widehat{(-\Delta)^{\frac{1}{2}} v}(\xi) = |\xi| \widehat{v}(\xi), \tag{1.3}$$

where $\widehat{w}$ is the Fourier transform of w.

We prove that the limit u^0 of u^ϵ as $\epsilon \to 0$ exists and is the unique solution of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}\left(\nabla_x u, \mathcal{I}_1[u(t, \cdot)] \right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{1.4}$$

for some continuous function $\overline{H}$ usually called *effective Hamiltonian*. The function u^0 will be interpreted later as a macroscopic plastic strain satisfying the macroscopic plastic flow rule (1.4). Moreover $\mathcal{I}_1[u^0]$ will be the stress created by the macroscopic density of dislocations.

1.2. Main results

As usual in periodic homogenization, the limit equation is determined by a *cell problem*. In our case, such a problem is for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the following:

$$\begin{cases} \lambda + \partial_\tau v = \mathcal{I}_1[v(\tau, \cdot)] + L - W'(v + \lambda\tau + p \cdot y) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ v(0, y) = 0 & \text{on } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $\lambda = \lambda(p, L)$ is the unique number for which there exists a solution v of (1.5) which is bounded on $\mathbb{R}^+ \times \mathbb{R}^N$. In order to solve (1.5), we show for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the existence of a unique solution of

$$\begin{cases} \partial_\tau w = \mathcal{I}_1\big[w(\tau, \cdot)\big] + L - W'(w + p \cdot y) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ w(0, y) = 0 & \text{on } \mathbb{R}^N, \end{cases} \tag{1.6}$$

and we look for some $\lambda \in \mathbb{R}$ for which $w - \lambda \tau$ is bounded. Precisely we have:

Theorem 1.1 (Ergodicity). *Assume* (H1)–(H4). *For $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique viscosity solution $w \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ of* (1.6) *and there exists a unique $\lambda \in \mathbb{R}$ such that w satisfies: $\frac{w(\tau, y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}(p, L)$. The function $\overline{H}(p, L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.*

Unfortunately, we cannot directly use the bounded solution of (1.5), usually called *corrector*, in order to prove the convergence of the sequence u^ϵ to the solution of (1.4). Nevertheless we have the following result:

Theorem 1.2 (Convergence). *Assume* (H1)–(H4). *The solution u^ϵ of* (1.1) *converges towards the solution u^0 of* (1.4) *locally uniformly in (t, x), where $\overline{H}$ is defined in Theorem* 1.1.

Let us mention that in a companion paper [32], we show that we can recover Orowan's law in dimension $N = 1$ for $\sigma = 0$, i.e.

$$\overline{H}(\delta p, \delta L) \simeq c_0 \delta^2 |p| L \quad \text{as } \delta \to 0$$

i.e. the plastic strain velocity is asymptotically proportional to the product of dislocation density $|p|$ by the effective stress L.

1.3. Brief review of the literature

This non-local equation (1.1) is related to the local equation

$$\begin{cases} \partial_t u^\epsilon = F\left(\dfrac{x}{\epsilon}, \dfrac{u^\epsilon}{\epsilon}, \nabla u^\epsilon\right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u^\epsilon(0, x) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{1.7}$$

that was studied in [23] under the assumption that $F(x, u, p)$ is periodic in (x, u) and coercive in p. The homogenization problem (1.7) when F does not depend on u, has been completely solved by Lions Papanicolaou and Varadhan [31]. After this seminal paper, homogenization of Hamilton–Jacobi equations for coercive Hamiltonians has been treated for a wider class of periodic situations, cf. Ishii [27], for problems set on bounded domains, cf. Alvarez [1], Horie and Ishii [21], for equations with different structures, cf. Alvarez and Ishii [4], for deterministic control problems in L^∞, cf. Alvarez and Barron [2], for almost periodic Hamiltonians, cf. Ishii [26], and for Hamiltonians with stochastic dependence, cf. Souganidis [37]. More recently, inspired by [23], Barles [6] gave an homogenization result for non-coercive Hamiltonians and, as a by-product, obtained a simpler proof of the results [23] of Imbert and Monneau but under slightly more restrictive assumptions on the Hamiltonians. We can also mention the work of Imbert, Monneau and Rouy [24] where the authors studied homogenization of certain integro-differential equations depending explicitly on u^ϵ / ϵ. Notice that in the present paper, the operator $\mathcal{I}_1$ involves a singular kernel which creates some additional difficulties that were not present for instance in [24].

Notice also that the model studied in [24] was introduced to approximate a level set model like in [14]. The phase field model in [24] was therefore closer in the spirit to a model for discrete dislocation dynamics at the mesoscopic scale. On the contrary, the Peierls–Nabarro model (1.1) is a well-established physical model which is really devoted to the description of dislocations at the microscopic scale.

1.4. Organization of the paper

The paper is organized as follows. In Section 2, we give more details about the Peierls–Nabarro model yielding to the study of (1.1) and the mechanical interpretation of the homogenization results. In Section 3 we present briefly the strategies of the main proofs. In Section 4, we state various comparison principles, existence and regularity results for solutions of non-local Hamilton–Jacobi equations. In Section 5, we prove the convergence result (Theorem 1.2) by assuming the existence of smooth approximate sub and supercorrectors (Proposition 3.1). In order to show their existence, in Section 6, we first construct Lipschitz continuous sub and supercorrectors (Proposition 6.1). As a byproduct, we prove the ergodicity of the problem (Theorem 1.1) and some properties of the effective Hamiltonian (Proposition 5.4). Proposition 3.1 is then proved in Section 7. The proofs of Lemma 4.7 and of Proposition 6.2 are done in Appendix A.

1.5. Notations

We denote by $B_r(x)$ the ball of radius r centered at x. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted by $Q_{\tau,r}(t,x)$.

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and the ceil integer parts of a real number x.

It is convenient to introduce the singular measure defined on $\mathbb{R}^N \setminus \{0\}$ by

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz = \mu_0(z)\, dz,$$

and to denote

$$\mathcal{I}_1^{1,r}[U, x] = \int\limits_{|z| \leqslant r} \left(U(x+z) - U(x) - \nabla U(x) \cdot z\right) \mu(dz),$$

$$\mathcal{I}_1^{2,r}[U, x] = \int\limits_{|z| > r} \left(U(x+z) - U(x)\right) \mu(dz).$$

Sometimes when $r = 1$ we will omit r and we will write simply $\mathcal{I}_1^1$ and $\mathcal{I}_1^2$.

For a function u defined on $(0, T) \times \mathbb{R}^N$, $0 < T \leqslant +\infty$, for $0 < \alpha < 1$ we denote by $\langle u \rangle_x^\alpha$ the seminorm defined by

$$\langle u \rangle_x^\alpha := \sup_{\substack{(t,x),(t,x') \in (0,T) \times \mathbb{R}^N \\ x \neq x'}} \frac{|u(t,x) - u(t,x')|}{|x - x'|^\alpha}$$

and by $C_x^\alpha((0, T) \times \mathbb{R}^N)$ the space of continuous functions defined on $(0, T) \times \mathbb{R}^N$ that are bounded and with bounded seminorm $\langle u \rangle_x^\alpha$.

Finally, we denote by $USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) the set of upper (resp., lower) semi-continuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0, T) \times \mathbb{R}^N$ for any $T > 0$ and we set $C_b(\mathbb{R}^+ \times \mathbb{R}^N) := USC_b(\mathbb{R}^+ \times \mathbb{R}^N) \cap LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$.

2. Physical modeling and mechanical interpretation of the homogenization results

2.1. The Peierls–Nabarro model

Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in

the crystallographic planes and their dynamics is one of the main explanation of the plastic behavior of metals.

The Peierls–Nabarro model is a phase field model for dislocation dynamics incorporating atomic features into continuum framework. In a phase field approach, the dislocations are represented by transition of a continuous field.

We briefly review the model (see [20] for a detailed presentation). As an example, consider an edge dislocation in a crystal with simple cubic lattice. In a Cartesian system of coordinates $x_1 x_2 x_3$, we assume that the dislocation is located in the slip plane $x_1 x_2$ (where the dislocation can move) and that the Burgers' vector (i.e. a fixed vector associated to the dislocation) is in the direction of the x_1 axis. We write this Burgers' vector as $b e_1$ for a real b. The disregistry of the upper half crystal $\{x_3 > 0\}$ relative to the lower half $\{x_3 < 0\}$ in the direction of the Burgers' vector is $\phi(x_1, x_2)$, where ϕ is a phase parameter between 0 and b. Then the dislocation loop can be for instance localized by the level set $\phi = b/2$. For a closed loop, we expect to have $\phi \simeq b$ inside the loop and $\phi \simeq 0$ far outside the loop.

In the Peierls–Nabarro model, the total energy is given by

$$\mathcal{E} = \mathcal{E}^{el} + \mathcal{E}^{mis}. \tag{2.1}$$

In (2.1), $\mathcal{E}^{mis}$ is the so called *misfit energy* due to the non-linear atomic interaction across the slip plane

$$\mathcal{E}^{mis}(\phi) = \int_{\mathbb{R}^2} W\big(\phi(x)\big)\, dx \quad \text{with} \quad x = (x_1, x_2),$$

where $W(\phi)$ is the interplanar potential. In the classical Peierls–Nabarro model [36,34], $W(\phi)$ is approximated by the sinusoidal potential

$$W(\phi) = \frac{\mu b^2}{4\pi^2 d}\left(1 - \cos\left(\frac{2\pi\phi}{b}\right)\right),$$

where d is the lattice spacing perpendicular to the slip plane.

The elastic energy $\mathcal{E}^{el}$ induced by the dislocation is (for $X = (x, x_3)$ with $x = (x_1, x_2)$)

$$\mathcal{E}^{el}(\phi, U) = \frac{1}{2}\int_{\mathbb{R}^3} e : \Lambda : e\, dX \quad \text{with} \quad e = e(U) - \phi(x)\delta_0(x_3)e^0 \quad \text{and}$$

$$\begin{cases} e(U) = \dfrac{1}{2}\big(\nabla U + (\nabla U)^T\big), \\ e^0 = \dfrac{1}{2}(e_1 \otimes e_3 + e_3 \otimes e_1), \end{cases}$$

where $U : \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement and $\Lambda = \{\Lambda_{ijkl}\}$ are the elastic coefficients.

Given the field ϕ, we minimize the energy $\mathcal{E}^{el}(\phi, U)$ with respect to the displacement U and define

$$\mathcal{E}^{el}(\phi) = \inf_U \mathcal{E}^{el}(\phi, U).$$

Following the proof of Proposition 6.1 (iii) in [3], we can see that (at least formally)

$$\mathcal{E}^{el}(\phi) = -\frac{1}{2}\int_{\mathbb{R}^2} (c_0 \star \phi)\phi$$

where c_0 is a certain kernel. In the case of isotropic elasticity, we have

$$\Lambda_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

where λ, μ are the Lamé coefficients. Then the kernel c_0 can be written (see Proposition 6.2 in [3], translated in our framework):

$$c_0(x) = \frac{\mu}{4\pi} \left(\partial_{22} \frac{1}{|x|} + \gamma \partial_{11} \frac{1}{|x|} \right) \quad \text{with} \quad \gamma = \frac{1}{1 - \nu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

where $\nu \in (-1, 1/2)$ is called the Poisson ratio.

The equilibrium configuration of straight dislocations is obtained by minimizing the total energy with respect to ϕ, under the constraint that far from the dislocation core, the function ϕ tends to 0 in one half plane and to b in the other half plane. In particular, the phase transition ϕ is then solution of the following equation

$$\mathcal{I}_1[\phi] = W'(\phi) \quad \text{on } \mathbb{R}^2, \tag{2.2}$$

where formally $\mathcal{I}_1[\phi] = c_0 \star \phi$, which is the anisotropic Lévy operator defined in (1.2) for $N = 2$ and $g(z_1, z_2) = \frac{\mu}{4\pi}((2\gamma - 1)z_1^2 + (2 - \gamma)z_2^2)$. Let us now recall the expression of the kernel after a Fourier transform (see Paragraph 6.2.2.2 in [3])

$$\widehat{c_0}(\xi) = -\frac{\mu}{2|\xi|} \left(\xi_2^2 + \gamma \xi_1^2 \right).$$

Then for $\gamma = 1$ and $\mu = 2$, we see that $\mathcal{I}_1 = -(-\Delta)^{\frac{1}{2}}$. In that special case, we recall that the solution ϕ of (2.2) satisfies $\phi(x) = \tilde{\phi}(x, 0)$ where $\tilde{\phi}(X)$ is the solution of (see [30,19])

$$\begin{cases} \Delta \tilde{\phi} = 0 & \text{in } \{x_3 > 0\}, \\ \dfrac{\partial \tilde{\phi}}{\partial x_3} = W'(\tilde{\phi}) & \text{on } \{x_3 = 0\}. \end{cases}$$

Moreover, we have in particular an explicit solution for $b = 1$, $d = 2$ (with $W'(\tilde{\phi}) = \frac{1}{2\pi} \sin(2\pi \tilde{\phi})$)

$$\tilde{\phi}(X) = \frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{x_1}{x_3 + 1} \right).$$

Then by rescaling, it is easy to check that we can recover the explicit solution found in Nabarro [34]

$$\begin{cases} \phi(x) = \dfrac{b}{2} + \dfrac{b}{\pi} \arctan \left(\dfrac{2(1 - \nu)x_1}{d} \right) & \text{(edge dislocation)}, \\ \phi(x) = \dfrac{b}{2} + \dfrac{b}{\pi} \arctan \left(\dfrac{2x_2}{d} \right) & \text{(screw dislocation)}. \end{cases}$$

In a more general model, one can consider a potential W satisfying

(i) $W(v + b) = W(u)$ for all $v \in \mathbb{R}$;
(ii) $W(b\mathbb{Z}) = 0 < W(a)$ for all $a \in \mathbb{R} \setminus b\mathbb{Z}$.

The periodicity of W reflects the periodicity of the crystal, while the minimum property is consistent with the fact that the perfect crystal is assumed to minimize the energy.

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane. In the present paper we are interested in describing the effective dynamics for a collection of dislocations curves with the same Burgers' vector and all contained in a single slip plane $x_1 x_2$, and moving in a landscape with periodic obstacles (that can be for instance precipitates in the material). These dislocations are represented by a single phase parameter $u(t, x_1, x_2)$ defined on the slip plane $x_1 x_2$. The dynamic of dislocations is then described by the evolutive version of the Peierls–Nabarro model (see for instance [33] and [10]):

$$\partial_t u = \mathcal{I}_1\big[u(t, \cdot)\big] - W'(u) + \sigma_{13}^{\text{obst}}(t, x) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N \tag{2.3}$$

for $x \in \mathbb{R}^N$ with the physical dimension $N = 2$. In the model, the component $\sigma_{13}^{\text{obst}}$ of the stress (evaluated on the slip plane) has been introduced to take into account the shear stress not created by the dislocations themselves. This shear stress is created by the presence of the periodic *obstacles* and the possible external applied stress on the material.

We want to identify at *large scale* an evolution model for the dynamics of a density of dislocations. We consider the following rescaling

$$u^\epsilon(t, x) = \epsilon u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right),$$

where ϵ is the ratio between the typical length scale for dislocation (of the order of the micrometer) and the typical macroscopic length scale in mechanics (millimeter or centimeter). With such a rescaling, we see that the number of dislocations is typically of the order of $1/\epsilon$ per unit of macroscopic scale. Moreover, assuming suitable initial data

$$u(0, x) = \frac{1}{\epsilon} u_0(\epsilon x) \quad \text{on } \mathbb{R}^N \tag{2.4}$$

(where u_0 is a regular bounded function), we see that the functions u^ϵ are solutions of (1.1). This indicates that at the limit $\epsilon \to 0$, we will recover a model for the dynamics of (renormalized) densities of dislocations.

Remark 2.1. Fractional reaction–diffusion equations of the form

$$\partial_t u = \mathcal{I}_1[u] + f(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N \tag{2.5}$$

where $N \geqslant 2$ and f is a bistable nonlinearity have been studied by Imbert and Souganidis [25]. In this paper the authors show that solutions of (2.5), after properly rescaling them, exhibit the limit evolution of an interface by (anisotropic) mean curvature motion.

Other results have been obtained by González and Monneau [19] for a rescaling of the evolutive Peierls–Nabarro model in dimension $N = 1$. In the one-dimensional space, the limit moving interfaces are points particles interacting with forces as $1/x$. The dynamics of these particles corresponds to the classical discrete dislocation dynamics, in the particular case of parallel straight edge dislocation lines in the same slip plane with the same Burgers' vector. In [14], considering another rescaling of the model of particles obtained in [19], the authors identify at large scale an evolution model for the dynamics of a density of dislocations, that is analogous to (1.4). In the present paper, we directly deduce the model (1.4) at larger scale from the Peierls–Nabarro model at smaller scale in any dimension $N \geqslant 1$. That way we remove the limitation to the dimension $N = 1$ that appears in [19].

Finally, let us mention that in [17] and [18] Garroni and Muller study a variational model for dislocations that is the variational formulation of the stationary Peierls–Nabarro equation, where they derive a line tension model.

2.2. Mechanical interpretation of the homogenization

Let us briefly explain the meaning of the homogenization result. In the macroscopic model, the function $u^0(t, x)$ can be interpreted as the plastic strain (localized in the slip plane $\{x_3 = 0\}$). Then the three-dimensional displacement $U(t, X)$ is obtained as a minimizer of the elastic energy

$$U(t, \cdot) = \arg\min_{\widetilde{U}} \mathcal{E}^{el}\big(u^0(t, \cdot), \widetilde{U}\big)$$

and the stress is

$$\sigma = \Lambda : e \quad \text{with} \quad e = e(U) - u^0(t, x)\delta_0(x_3)e^0.$$

Then the resolved shear stress is

$$\mathcal{I}_1\big[u^0\big] = \sigma_{13}^{obst}$$

The homogenized equation (1.4), i.e.

$$\partial_t u^0 = \overline{H}\big(\nabla_x u^0, \mathcal{I}_1\big[u^0(t, \cdot)\big]\big)$$

which is the evolution equation for u^0, can be interpreted as the plastic flow rule in a model for macroscopic crystal plasticity. This is the law giving the plastic strain velocity $\partial_t u^0$ as a function of the resolved shear stress σ_{13}^{obst} and the dislocation density ∇u^0.

The typical example of such a plastic flow rule is the Orowan's law:

$$\overline{H}(p, L) \simeq |p|L.$$

This is also the law that we recover in dimension $N = 1$ in a forthcoming paper [32] in the case where there are no obstacles (i.e. $\sigma_{13}^{obst} \equiv 0$) and for small stress L and small density $|p|$. When $\sigma_{13}^{obst} \not\equiv 0$ with zero mean value (i.e. $\langle \sigma_{13}^{obst} \rangle = 0$), we expect a threshold phenomenon as in [24] (see also Norton's law with threshold in [16]), i.e.

$$\overline{H}(p, L) = 0 \quad \text{if } |L| \text{ is small enough.}$$

This means more generally that our homogenization procedure describes correctly the mechanical behavior of the stress at large scales, but keeps the memory of the microstructure in the plastic law with possible threshold effects.

3. Strategies of the main proofs

3.1. Strategy for the proof of convergence

3.1.1. The general approach

It has been already noticed that for problems periodic in u^ϵ/ϵ, we have to introduce twisted correctors (see for instance [23]). It is also known that if we can claim that the limit function satisfies

$$\partial_t u^0 \neq 0 \quad \text{or} \quad \nabla_x u^0 \neq 0 \tag{3.1}$$

then we do not have to introduce an additional dimension to perform the proof of convergence. The idea (see [23]) is that we can twist the corrector either dividing by $p_i := \partial_{x_i} u^0$ for some index i, or by $\lambda := \partial_t u^0$ like considering the ansatz:

$$u^\varepsilon(t, x) \simeq u^0(t, x) + \varepsilon v\left(\frac{u^0(t, x) - p \cdot x}{\varepsilon \lambda}, \frac{x}{\varepsilon}\right).$$

On the contrary, we do not know how to deal with the case where both quantities in (3.1) vanish, except adding a dimension and considering twisted correctors in higher dimension. Here we have to face a similar difficulty in the much more involved framework of non-local equations. Notice also that it does not seem possible to apply the approach of Barles [6]. Therefore following the idea in [23], we consider the solution U^ϵ of

$$\begin{cases} \partial_t U^\epsilon = \mathcal{I}_1\big[U^\epsilon(t, \cdot, x_{N+1})\big] - W'\left(\frac{U^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ U^\epsilon(0, x, x_{N+1}) = u_0(x) + p_{N+1} x_{N+1} & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{3.2}$$

where $p_{N+1} \neq 0$. We then consider the following ansatz:

$$U^\epsilon(t, x, x_{N+1}) \simeq U^0(t, x, x_{N+1}) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^0(t, x, x_{N+1}) - \lambda t - p \cdot x}{\epsilon p_{N+1}}\right)$$

where $U^0(t, x, x_{N+1}) = u_0(x) + p_{N+1} x_{N+1}$. This ansatz turns out to be the good one, and plugging this expression of U^ϵ into (3.2), we find formally with $\tau = \frac{t}{\epsilon}$, $y = \frac{x}{\epsilon}$, $y_{N+1} = \frac{U^0(t,x,x_{N+1}) - \lambda t - p \cdot x}{p_{N+1} \epsilon}$:

$$\lambda + \partial_\tau V = L + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] - W'(V + p \cdot y + p_{N+1} y_{N+1} + \lambda \tau) + \sigma(\tau, y), \tag{3.3}$$

where

$$\lambda = \partial_t U^0(t, x, x_{N+1}) = \partial_t u^0(t, x), \qquad p = \nabla_x U^0(t, x, x_{N+1}) = \nabla_x u^0(t, x)$$

and

$$L = \mathcal{I}_1\big[U^0(t, \cdot, x_{N+1})\big] = \mathcal{I}_1\big[u^0(t, \cdot)\big].$$

Then, we expect u^0 to be solution of (1.4) with $\overline{H}(p, L) = \lambda(p, L)$. This heuristic computation, that permits first of all to identify the cell problem in the higher dimensional space, can be made rigorous through the perturbed test function method by Evans [13].

3.1.2. Additional difficulty

Let us enter a bit more in the details of the proof. Fix $P_0 = (t_0, x_0, x_{N+1}^0) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$ and define

$$\widetilde{U}^\epsilon(t, x, x_{N+1}) = U^0(t, x, x_{N+1}) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^0(t, x, x_{N+1}) - \lambda t - p \cdot x}{\epsilon p_{N+1}}\right), \tag{3.4}$$

where V is solution of (3.3) with $\lambda = \partial_t U^0(P_0)$, $p = \nabla_x U^0(P_0)$ and $L = \mathcal{I}_1[U^0(t_0, \cdot, x_{N+1}^0), x_0]$. Let us call $F(t, x, x_{N+1}) = \frac{U^0(t,x,x_{N+1}) - \lambda t - p \cdot x}{p_{N+1}}$. Here we assume for simplicity that U^0 and V are smooth. The proof of convergence consists in showing that $\widetilde{U}^\epsilon$ is a solution of (3.2) in a cylinder $(t_0 - r, t_0 + r) \times B_r(x_0, x_{N+1}^0)$ for $r > 0$ small enough, up to an error that goes to 0 as $r \to 0^+$. This will allow us to

compare U^ϵ with $\widetilde{U}^\epsilon$ and, thanks to the boundedness of V, to conclude that U^ϵ converges to U^0 as $\epsilon \to 0$.

When we plug $\widetilde{U}^\epsilon$ into (3.2), we find the equation

$$\lambda + \partial_\tau V = L + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] - W'(V + p \cdot y + p_{N+1}y_{N+1} + \lambda\tau) + \sigma(\tau, y) + o_r(1) + \theta_r,$$

with $\tau = \frac{t}{\epsilon}$, $y = \frac{x}{\epsilon}$, $y_{N+1} = \frac{F(t,x,x_{N+1})}{\epsilon}$, where

$$\theta_r = \big(\partial_t U^0(P_0) - \partial_t U^0(t, x, x_{N+1})\big)\partial_{y_{N+1}}V(\tau, y, y_{N+1})$$

$$+ \mathcal{I}_1\left[V\left(\tau, \cdot, \frac{F(\epsilon\tau, \epsilon\cdot, \epsilon y_{N+1})}{\epsilon}\right)\right] - \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big].$$

Then, $\widetilde{U}^\epsilon$ will be a solution of (3.2) up to a small error if $\theta_r = o_r(1)$ as $r \to 0+$. This last property holds true if the corrector V satisfies: $|V|, |\partial_{y_{N+1}}V| \leqslant C$ in $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ for some $C > 0$, and

$$\partial_{y_{N+1}}V(\tau, \cdot, \cdot) \quad \text{is Hölder continuous, uniformly in time.} \tag{3.5}$$

In the case of the local first order equation (1.7) considered in [23], or non-local equations considered in [24], approximate correctors were only required to be Lipschitz continuous in the additional variable. Here the additional regularity (3.5) is required because we deal with an operator $\mathcal{I}_1$ whose kernel is singular.

Since in (3.3), the quantity $\mathcal{I}_1[V(\tau, \cdot, y_{N+1})]$ is computed only in the y variable, we cannot expect this kind of regularity for the correctors. Nevertheless, we are able to construct regular approximated sub and supercorrectors, i.e., sub and supersolutions of approximate $N + 1$-dimensional cell problems, and this is enough to conclude. Finally, this construction works for any $p_{N+1} \neq 0$ and to simplify the presentation we take $p_{N+1} = 1$.

3.2. Strategy for the construction of smooth approximate correctors

As explained in the previous subsection, in the proof of convergence we will need smooth approximate sub and super-correctors on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, i.e., for $P = (p, 1) \in \mathbb{R}^{N+1}$ and $L \in \mathbb{R}$, sub and supersolutions of

$$\begin{cases} \lambda + \partial_\tau V = L + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] - W'(V + P \cdot Y + \lambda\tau) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{3.6}$$

Here and in what follows, we denote $Y = (y, y_{N+1})$. More precisely, we will prove the following proposition.

Proposition 3.1 (*Smooth approximate correctors*). *Let λ be the constant defined by Theorem 1.1. For any fixed $p \in \mathbb{R}^N$, $P = (p, 1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda_\eta^+(p, L)$, $\lambda_\eta^-(p, L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors V_η^+, V_η^-, i.e. respectively a super and a subsolution of*

$$\begin{cases} \lambda_\eta^\pm + \partial_\tau V_\eta^\pm = L + \mathcal{I}_1\big[V_\eta^\pm(\tau, \cdot, y_{N+1})\big] \\ \qquad\qquad - W'\big(V_\eta^\pm + P \cdot Y + \lambda_\eta^\pm\tau\big) + \sigma(\tau, y) \mp o_\eta(1) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V_\eta^\pm(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{3.7}$$

where $0 \leqslant o_\eta(1) \to 0$ as $\eta \to 0^+$, such that

$$\lim_{\eta \to 0^+} \lambda_\eta^+(p, L) = \lim_{\eta \to 0^+} \lambda_\eta^-(p, L) = \lambda(p, L), \tag{3.8}$$

locally uniformly in (p, L), $\lambda_\eta^\pm$ *satisfy* (i) *and* (ii) *of Proposition* 5.4 *and for any* $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$

$$\left| V_\eta^\pm(\tau, Y) \right| \leqslant C. \tag{3.9}$$

Moreover $V_\eta^\pm$ *are of class* C^2 *w.r.t.* y_{N+1}, *and for any* $0 < \alpha < 1$

$$-1 \leqslant \partial_{y_{N+1}} V_\eta^\pm \leqslant \frac{\|W''\|_\infty}{\eta}, \tag{3.10}$$

$$\left\| \partial^2_{y_{N+1} y_{N+1}} V_\eta^\pm \right\|_\infty \leqslant C_\eta, \qquad \langle \partial_{y_{N+1}} V_\eta^\pm \rangle_y^\alpha \leqslant C_{\eta, \alpha}. \tag{3.11}$$

Here in order to build Lipschitz sub/super correctors, it does not seem easy to apply a kind of truncation of the Hamiltonian like in [23] or [24]. Therefore we use a different method to build such approximate correctors (similar to the one in [15]).

The proof of Proposition 3.1 is mainly performed in two steps:

Step 1. *Constructions of Lipschitz correctors.* Using the modified Cauchy problem

$$
\begin{cases}
\partial_\tau U = L + \mathcal{I}_1\big[U(\tau, \cdot, y_{N+1})\big] - W'(U + P \cdot Y) + \sigma(\tau, y) \\
\qquad + \eta \left\{ a_0 + \inf_{Y'} U(\tau, Y') - U(\tau, Y) \right\} |\partial_{y_{N+1}} U + 1| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\
U(0, Y) = 0 & \text{on } \mathbb{R}^{N+1},
\end{cases}
$$

we construct Lipschitz correctors. The Lipschitz bound comes formally from the equation satisfied by $w = \partial_{y_{N+1}} U$:

$$
\begin{cases}
\partial_\tau w = \mathcal{I}_1\big[w(\tau, \cdot, y_{N+1})\big] - W''(U + P \cdot Y)(w + 1) - \eta w(\tau, Y)|w + 1| \\
\qquad + \eta \left\{ a_0 + \inf_{Y'} U(\tau, Y') - U(\tau, Y) \right\} \mathrm{sign}(\partial_{y_{N+1}} U + 1) \partial_{y_{N+1}} w & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\
w(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}
\end{cases}
$$

and the comparison principle implies that

$$-1 \leqslant w \leqslant \frac{|W''|_\infty}{\eta}. \tag{3.12}$$

On the other hand we are able to show (as in [24]) that $\inf_{Y'} U(\tau, Y') - U(\tau, Y)$ remains bounded independently on η. Then an appropriate choice of a_0 large enough (resp. negative enough) provides us bounded supercorrectors W_η^+ (resp. subcorrectors W_η^-). We also show using Proposition 4.7 and the bound (3.12) that we have the following Hölder estimate:

$$\langle W_\eta^\pm \rangle_y^\alpha \leqslant C_\alpha.$$

Step 2. *Constructions of smooth correctors.* We make a convolution with respect to y_{N+1} of the Lipschitz correctors built in Step 1, with a sequence $(\rho_\delta)_\delta$ of mollifiers:

$$V_{\eta, \delta}^\pm(t, y, y_{N+1}) := W_\eta^\pm(t, y, \cdot) \star \rho_\delta(\cdot).$$

Those functions are finally the smooth approximate sub/super correctors of Proposition 3.1 with some small error term $o_\eta(1)$ on the right hand side of the equation, for a suitable choice $\delta = \delta(\eta)$.

4. Results about viscosity solutions for non-local equations

The classical notion of viscosity solution can be adapted for Hamilton–Jacobi equations involving non-local operators, see for instance [5]. In this section we state comparison principles, existence and regularity results for viscosity solutions of (1.1) and (1.4), that will be used later in the proofs.

4.1. Definition of viscosity solution

We first recall the definition of viscosity solution for a general first order non-local equation with associated initial condition:

$$\begin{cases} u_t = F\big(t, x, u, Du, \mathcal{I}_1[u]\big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $F(t, x, u, p, L)$ is continuous and non-decreasing in L.

Definition 4.1 (*r-Viscosity solution*). A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is an r-viscosity subsolution (resp., supersolution) of (4.1) if $u(0, x) \leqslant (u_0)^*(x)$ (resp., $u(0, x) \geqslant (u_0)_*(x)$) and for any $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$, any $\tau \in (0, t_0)$ and any test function $\phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^N)$ such that $u - \phi$ attains a local maximum (resp., minimum) at the point (t_0, x_0) on $Q_{(\tau, r)}(t_0, x_0)$, then we have

$$\partial_t \phi(t_0, x_0) - F\big(t_0, x_0, u(t_0, x_0), \nabla_x \phi(t_0, x_0), \mathcal{I}_1^{1,r}[\phi(t_0, \cdot), x_0] + \mathcal{I}_1^{2,r}[u(t_0, \cdot), x_0]\big) \leqslant 0$$

(resp., $\geqslant 0$).

A function $u \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ is an r-viscosity solution of (4.1) if it is an r-viscosity sub and supersolution of (4.1).

It is classical that the maximum in the above definition can be supposed to be global and this will be used later. We have also the following property, see e.g. [5]:

Proposition 4.1 (*Equivalence of the definitions*). *Assume $F(t, x, u, p, L)$ continuous and non-decreasing in L. Let $r > 0$ and $r' > 0$. A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is an r-viscosity subsolution (resp., supersolution) of (4.1) if and only if it is an r'-viscosity subsolution (resp., supersolution) of (4.1).*

Because of this proposition, if we do not need to emphasize r, we will omit it when calling viscosity sub and supersolutions.

4.2. Comparison principle and existence results

In this subsection, we successively give comparison principles and existence results for (1.1) and (1.4). The following comparison theorem is shown in [29] for more general parabolic integro-PDEs.

Proposition 4.2 (*Comparison principle for (1.1)*). *Consider $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ subsolution and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ supersolution of (1.1), then $u \leqslant v$ on $\mathbb{R}^+ \times \mathbb{R}^N$.*

Following [29] it can also be proved the comparison principle for (1.1) in bounded domains. Since we deal with a non-local equation, we need to compare the sub and the supersolution everywhere outside the domain.

Proposition 4.3 *(Comparison principle on bounded domains for (1.1)). Let Ω be a bounded domain of $\mathbb{R}^+ \times \mathbb{R}^N$ and let $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ be respectively a sub and a supersolution of*

$$\partial_t u^\epsilon = \mathcal{I}_1\big[u^\epsilon(t,\cdot)\big] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$$

in Ω. If $u \leqslant v$ outside Ω, then $u \leqslant v$ in Ω.

Proposition 4.4 *(Existence for (1.1)). For $\epsilon > 0$ there exists $u^\epsilon \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (unique) viscosity solution of (1.1). Moreover, there exists a constant $C > 0$ independent of ϵ such that*

$$\big|u^\epsilon(t,x) - u_0(x)\big| \leqslant Ct. \tag{4.2}$$

Proof. Adapting the argument of [22], we can construct a solution by Perron's method if we construct sub and supersolutions of (1.1). Since $u_0 \in W^{2,\infty}$, the two functions $u^\pm(t,x) := u_0(x) \pm Ct$ are respectively a super and a subsolution of (1.1) for any $\epsilon > 0$, if

$$C \geqslant D_N \|u_0\|_{2,\infty} + \big\|W'\big\|_\infty + \|\sigma\|_\infty,$$

with D_N depending on the dimension N. By comparison we also get the estimate (4.2). □

We next recall the comparison and the existence results for (1.4).

Proposition 4.5. *(See [24], Proposition 3.) Let $\overline{H} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous with $\overline{H}(p, \cdot)$ non-decreasing on $\mathbb{R}$ for any $p \in \mathbb{R}^N$. If $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ are respectively a sub and a supersolution of (1.4), then $u \leqslant v$ on $\mathbb{R}^+ \times \mathbb{R}^N$. Moreover there exists a (unique) viscosity solution of (1.4).*

In the next sections, we will embed the problem in the higher dimensional space $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ by adding a new variable x_{N+1} in the equations. We will need the following proposition showing that sub and supersolutions of the higher dimensional problem are also sub and supersolutions of the lower dimensional one. This in particular implies that the comparison principle between sub and supersolutions remains true increasing the dimension.

Proposition 4.6. *Assume $F(t, x, x_{N+1}, U, p, L)$ continuous and non-decreasing in L. Suppose that $U \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ (resp., $U \in USC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$) is a viscosity supersolution (resp., subsolution) of*

$$U_t = F\big(t, x, x_{N+1}, U, D_x U, \mathcal{I}_1\big[U(t,\cdot,x_{N+1})\big]\big) \quad in \ \mathbb{R}^+ \times \mathbb{R}^{N+1}, \tag{4.3}$$

then, for any $x_{N+1} \in \mathbb{R}$, U is a viscosity supersolution (resp., subsolution) of

$$U_t = F\big(t, x, x_{N+1}, U, D_x U, \mathcal{I}_1\big[U(t,\cdot,x_{N+1})\big]\big) \quad in \ \mathbb{R}^+ \times \mathbb{R}^N.$$

Proof. Notice that in (4.3), there is no derivative with respect to x_{N+1} and no integral with respect to dx_{N+1}. Therefore x_{N+1} only appears as a parameter that can (at least formally) be frozen.

We now do the (rigorous) proof for supersolutions. Fix $x_{N+1}^0 \in \mathbb{R}$. Let us consider a point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ and a smooth function $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ such that

$$U\big(t, x, x_{N+1}^0\big) - \varphi(t,x) \geqslant U\big(t_0, x_0, x_{N+1}^0\big) - \varphi(t_0, x_0) = 0 \quad \text{for } (t,x) \in Q_{\tau,r}(t_0, x_0),$$

with $r = 1$. We have to show that

$$\partial_t \varphi(t_0, x_0) \geqslant F\big(t_0, x_0, x_{N+1}^0, U\big(t_0, x_0, x_{N+1}^0\big), D_x \varphi(t_0, x_0), \mathcal{I}_1^1\big[\varphi(t_0, \cdot), x_0\big]$$
$$+ \mathcal{I}_1^2\big[U\big(t_0, \cdot, x_{N+1}^0\big), x_0\big]\big).$$

Without loss of generality, we can assume that the minimum is strict. For $\epsilon > 0$ let $\varphi_\epsilon : \mathbb{R}^+ \times \mathbb{R}^{N+1} \to \mathbb{R}$ be defined by

$$\varphi_\epsilon(t, x, x_{N+1}) = \varphi(t, x) - \frac{1}{\epsilon}\big|x_{N+1} - x_{N+1}^0\big|^2.$$

Let $(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon)$ be a minimum point of $U - \varphi_\epsilon$ in $Q_{\tau,r}(t_0, x_0, x_{N+1}^0)$. Standard arguments show that $(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon) \to (t_0, x_0, x_{N+1}^0)$ as $\epsilon \to 0$ and that $\lim_{\epsilon \to 0} U(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon) = U(t_0, x_0, x_{N+1}^0)$. In particular, $(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon)$ is internal to $Q_{\tau,r}(t_0, x_0, x_{N+1}^0)$ for ϵ small enough, then we get

$$\partial_t \varphi(t_\epsilon, x_\epsilon) \geqslant F\big(t_\epsilon, x_\epsilon, U\big(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon\big), D_x \varphi(t_\epsilon, x_\epsilon), \mathcal{I}_1^1\big[\varphi(t_\epsilon, \cdot), x_\epsilon\big]$$
$$+ \mathcal{I}_1^2\big[U\big(t_\epsilon, \cdot, x_{N+1}^\epsilon\big), x_\epsilon\big]\big). \tag{4.4}$$

By the Dominate Convergence Theorem $\lim_{\epsilon \to 0} \mathcal{I}_1^1[\varphi(t_\epsilon, \cdot), x_\epsilon] = \mathcal{I}_1^1[\varphi(t_0, \cdot), x_0]$; by Fatou's Lemma and the convergence of $U(t_\epsilon, x_\epsilon, x_{N+1}^\epsilon)$ to $U(t_0, x_0, x_{N+1}^0)$, we deduce that

$$\mathcal{I}_1^2\big[U\big(t_0, \cdot, x_{N+1}^0\big), x_0\big] \leqslant \liminf_{\epsilon \to 0} \mathcal{I}_1^2\big[U\big(t_\epsilon, \cdot, x_{N+1}^\epsilon\big), x_\epsilon\big].$$

Then, passing to the limit in (4.4) and using the continuity and monotonicity of F, we get the desired inequality. $\square$

4.3. Hölder regularity

In this subsection we state a regularity result for sub and supersolutions of semilinear non-local equations. The proof is postponed in Appendix A.

Proposition 4.7 (Hölder regularity). *Assume* (H1) *and let* $g_1, g_2 \in \mathbb{R}$. *Suppose that* $u \in C(\mathbb{R}^+ \times \mathbb{R}^N)$ *and bounded on* $\mathbb{R}^+ \times \mathbb{R}^N$ *is a viscosity subsolution of*

$$\begin{cases} \partial_t u = \mathcal{I}_1\big[u(t, \cdot)\big] + g_1 & in \ \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0, x) = 0 & on \ \mathbb{R}^N, \end{cases}$$

and a viscosity supersolution of

$$\begin{cases} \partial_t u = \mathcal{I}_1\big[u(t, \cdot)\big] + g_2 & in \ \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0, x) = 0 & on \ \mathbb{R}^N. \end{cases}$$

Then, for any $0 < \alpha < 1$, $u \in C_x^\alpha(\mathbb{R}^+ \times \mathbb{R}^N)$ *with* $\langle u \rangle_x^\alpha \leqslant C$, *where* C *depends on* $\|u\|_\infty$, g_1 *and* g_2.

Notice that this regularity result will be used to establish a bound on the Hölder regularity in y of $\partial_{y_{N+1}} V_\eta^\pm$ for smooth approximate correctors $V_\eta^\pm$ that will be used in Step 1.2 of the proof of Lemma 5.5 used in the proof of the convergence result (Theorem 1.2).

5. The proof of convergence

This section is dedicated to the proof of Theorem 1.2. As explained in Subsection 1.3, we imbed our problem in a higher dimensional one. We consider U^ϵ solution of

$$\begin{cases} \partial_t U^\epsilon = \mathcal{I}_1\big[U^\epsilon(t,\cdot,x_{N+1})\big] - W'\left(\dfrac{U^\epsilon}{\epsilon}\right) + \sigma\left(\dfrac{t}{\epsilon},\dfrac{x}{\epsilon}\right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ U^\epsilon(0,x,x_{N+1}) = u_0(x) + x_{N+1} & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{5.1}$$

By Proposition 4.6 and Proposition 4.2, the comparison principle holds true for (5.1). Then, as in the proof of Proposition 4.4, by Perron's method we have:

Proposition 5.1 *(Existence for (5.1))*. *For $\epsilon > 0$ there exists $U^\epsilon \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ (unique) viscosity solution of (5.1). Moreover, there exists a constant $C > 0$ independent of ϵ such that*

$$\big|U^\epsilon(t,x,x_{N+1}) - u_0(x) - x_{N+1}\big| \leqslant Ct. \tag{5.2}$$

Let us exhibit the link between the problem in $\mathbb{R}^N$ and the problem in $\mathbb{R}^{N+1}$.

Lemma 5.2 *(Link between the problems on $\mathbb{R}^N$ and on $\mathbb{R}^{N+1}$)*. *If u^ϵ and U^ϵ denote respectively the solution of (1.1) and (5.1), then we have*

$$\left|U^\epsilon(t,x,x_{N+1}) - u^\epsilon(t,x) - \epsilon\left\lfloor \frac{x_{N+1}}{\epsilon} \right\rfloor\right| \leqslant \epsilon,$$

$$U^\epsilon\left(t,x,x_{N+1} + \epsilon\left\lfloor \frac{a}{\epsilon} \right\rfloor\right) = U^\epsilon(t,x,x_{N+1}) + \epsilon\left\lfloor \frac{a}{\epsilon} \right\rfloor \quad \text{for any } a \in \mathbb{R}. \tag{5.3}$$

This lemma is a consequence of the comparison principle for (5.1), the invariance by ϵ-translations w.r.t. x_{N+1} and the monotonicity of U^ϵ w.r.t. x_{N+1}.

Let us now consider the problem

$$\begin{cases} \partial_t U = \overline{H}\big(\nabla_x U, \mathcal{I}_1[U(t,\cdot,x_{N+1})]\big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ U(0,x,x_{N+1}) = u_0(x) + x_{N+1} & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{5.4}$$

The link between problems (1.4) and (5.4) is given by the following lemma (analogue to Lemma 5.2).

Lemma 5.3. *Let u^0 and U^0 be respectively the solutions of (1.4) and (5.4). Then, we have*

$$U^0(t,x,x_{N+1}) = u^0(t,x) + x_{N+1}.$$

Lemma 5.3 is a consequence of the comparison principle for (5.4) and the invariance by translations w.r.t. x_{N+1}.

We need to make more precise the dependence of the real number λ given by Theorem 1.1 on its variables. The following properties will be shown in the next section.

Proposition 5.4 *(Properties of the effective Hamiltonian)*. *Let $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$. Let $\overline{H}(p,L)$ be the constant defined by Theorem 1.1, then $\overline{H} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the following properties:*

(i) $\overline{H}(p,L) \to \pm\infty$ as $L \to \pm\infty$ for any $p \in \mathbb{R}^N$.

(ii) $\overline{H}(p, \cdot)$ is non-decreasing on $\mathbb{R}$ for any $p \in \mathbb{R}^N$.

(iii) If $\sigma(\tau, y) = \sigma(\tau, -y)$ then

$$\overline{H}(p, L) = \overline{H}(-p, L).$$

(iv) If $W'(-s) = -W'(s)$ and $\sigma(\tau, -y) = -\sigma(\tau, y)$ then

$$\overline{H}(p, -L) = -\overline{H}(p, L).$$

5.1. Proof of Theorem 1.2

Step 1. *The classical approach.* By (5.2), we know that the family of functions $\{U^\epsilon\}_{\epsilon>0}$ is locally bounded, then $U^+ := \limsup^*_{\epsilon \to 0} U^\epsilon$ is everywhere finite. Classically we prove that U^+ is a subsolution of (5.4).

Similarly, we can prove that $U^- = \liminf_{*\epsilon \to 0} U^\epsilon$ is a supersolution of (5.4). Moreover $U^+(0, x, x_{N+1}) = U^-(0, x, x_{N+1}) = u_0(x) + x_{N+1}$. The comparison principle for (5.4), which is an immediate consequence of Propositions 4.5 and 4.6, then implies that $U^+ \leqslant U^-$. Since the reverse inequality $U^- \leqslant U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique viscosity solution of (5.4).

By Lemmata 5.2 and 5.3, the convergence of U^ϵ to U^0 proves in particular that u^ϵ converges towards u^0 viscosity solution of (1.4).

To prove that U^+ is a subsolution of (5.4), we argue by contradiction. In what follows we will use the notation $X = (x, x_{N+1})$. We consider a test function ϕ such that $U^+ - \phi$ attains a zero maximum at (t_0, X_0) with $t_0 > 0$ and $X_0 = (x_0, x_{N+1}^0)$. Without loss of generality we may assume that the maximum is strict and global. Suppose that there exists $\theta > 0$ such that

$$\partial_t \phi(t_0, X_0) = \overline{H}\big(\nabla_x \phi(t_0, X_0), L_0\big) + \theta,$$

where

$$L_0 = \int\limits_{|x| \leqslant 1} \big(\phi\big(t_0, x_0 + x, x_{N+1}^0\big) - \phi(t_0, X_0) - \nabla_x \phi(t_0, X_0) \cdot x\big) \mu(dx)$$

$$+ \int\limits_{|x| > 1} \big(U^+\big(t_0, x_0 + x, x_{N+1}^0\big) - U^+(t_0, X_0)\big) \mu(dx). \tag{5.5}$$

Step 2. *Construction of ϕ^ϵ.* By Proposition 5.4, we know that there exists $L_1 > 0$ (that we take minimal) such that

$$\overline{H}\big(\nabla_x \phi(t_0, X_0), L_0\big) + \theta = \overline{H}\big(\nabla_x \phi(t_0, X_0), L_0 + L_1\big).$$

By Propositions 3.1 and 5.4, we can consider a sequence $L_\eta \to L_1$ as $\eta \to 0^+$, such that $\lambda_\eta^+(\nabla_x \phi(t_0, X_0), L_0 + L_\eta) = \lambda(\nabla_x \phi(t_0, X_0), L_0 + L_1)$. We choose η so small that $L_\eta - o_\eta(1) \geqslant L_1/2 > 0$, where $o_\eta(1)$ is defined in Proposition 3.1. Let V_η^+ be the approximate supercorrector given by Proposition 3.1 with

$$p = \nabla_x \phi(t_0, X_0), \qquad L = L_0 + L_\eta$$

and

$$\lambda_\eta^+ = \lambda_\eta^+(p, L_0 + L_\eta) = \partial_t \phi(t_0, X_0).$$

For simplicity of notations, in the following we denote $V = V_\eta^+$. We consider the function $F(t, X) = \phi(t, X) - p \cdot x - \lambda t$, and as in [23] and [24] we introduce the "x_{N+1}-twisted perturbed test function" ϕ^ϵ defined by:

$$\phi^\epsilon(t, X) := \begin{cases} \phi(t, X) + \epsilon V(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}) + \epsilon k_\epsilon & \text{in } (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0), \\ U^\epsilon(t, X) & \text{outside}, \end{cases} \tag{5.6}$$

where $k_\epsilon \in \mathbb{Z}$ will be chosen later.

Step 3. *Checking that ϕ^ϵ is a supersolution.*

Step 3.1. *Outside $Q_{r,r}(t_0, x_0)$.* We are going to prove that ϕ^ϵ is a supersolution of (5.1) in $Q_{r,r}(t_0, X_0)$ for some $r < \frac{1}{2}$ properly chosen and such that $Q_{r,r}(t_0, X_0) \subset (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0)$. First, remark that since $U^+ - \phi$ attains a strict maximum at (t_0, X_0) with $U^+ - \phi = 0$ at (t_0, X_0) and V is bounded, we can ensure that there exists $\epsilon_0 = \epsilon_0(r) > 0$ such that for $\epsilon \leqslant \epsilon_0$

$$U^\epsilon(t, X) \leqslant \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) - \gamma_r \quad \text{in } \left(\frac{t_0}{3}, 3t_0\right) \times B_1(x_0) \setminus Q_{r,r}(t_0, x_0) \tag{5.7}$$

for some $\gamma_r = o_r(1) > 0$. Hence choosing $k_\epsilon = \lceil \frac{-\gamma_r}{\epsilon} \rceil$ we get $U^\epsilon \leqslant \phi^\epsilon$ outside $Q_{r,r}(t_0, X_0)$.

Step 3.2. *Inside $Q_{r_0,r_0}(t_0, x_0)$: ϕ^ϵ tested by ψ.* Let us next study the equation. From (5.3), we deduce that $U^+(t, x, x_{N+1} + a) = U^+(t, x, x_{N+1}) + a$ for any $a \in \mathbb{R}$, from which we derive that $\partial_{x_{N+1}} F(t_0, X_0) = \partial_{x_{N+1}} \phi(t_0, X_0) = 1$. Then, there exists $r_0 > 0$ such that the map

$$Id \times F : Q_{r_0,r_0}(t_0, X_0) \longrightarrow \mathcal{U}_{r_0}$$
$$(t, x, x_{N+1}) \longmapsto (t, x, F(t, x, x_{N+1}))$$

is a C^1-diffeomorphism from $Q_{r_0,r_0}(t_0, X_0)$ onto its range $\mathcal{U}_{r_0}$. Let $G : \mathcal{U}_{r_0} \to \mathbb{R}$ be the map such that

$$Id \times G : \mathcal{U}_{r_0} \longrightarrow Q_{r_0,r_0}(t_0, X_0)$$
$$(t, x, \xi_{N+1}) \longmapsto (t, x, G(t, x, \xi_{N+1}))$$

is the inverse of $Id \times F$. Let us introduce the variables $\tau = t/\epsilon$, $Y = (y, y_{N+1})$ with $y = x/\epsilon$ and $y_{N+1} = F(t, X)/\epsilon$. Let us consider a test function ψ such that $\phi^\epsilon - \psi$ attains a global zero minimum at $(\bar{t}, \overline{X}) \in Q_{r_0,r_0}(t_0, X_0)$ and define

$$\Gamma^\epsilon(\tau, Y) = \frac{1}{\epsilon}\left[\psi\left(\epsilon\tau, \epsilon y, G(\epsilon\tau, \epsilon y, \epsilon y_{N+1})\right) - \phi\left(\epsilon\tau, \epsilon y, G(\epsilon\tau, \epsilon y, \epsilon y_{N+1})\right)\right] - k_\epsilon.$$

Then

$$\psi(t, X) = \phi(t, X) + \epsilon \Gamma^\epsilon\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$$

and Γ^ϵ is a test function for V:

$$\Gamma^\epsilon(\overline{\tau}, \overline{Y}) = V(\overline{\tau}, \overline{Y}) \quad \text{and} \quad \Gamma^\epsilon(\tau, Y) \leqslant V(\tau, Y) \quad \text{for all } (\epsilon\tau, \epsilon Y) \in Q_{r_0,r_0}(t_0, X_0), \tag{5.8}$$

where $\overline{\tau} = \bar{t}/\epsilon$, $\overline{y} = \bar{x}/\epsilon$, $\overline{y}_{N+1} = F(\bar{t}, \overline{X})/\epsilon$, $\overline{Y} = (\overline{y}, \overline{y}_{N+1})$. From Proposition 3.1, we know that V is Lipschitz continuous w.r.t. y_{N+1} with Lipschitz constant M_η depending on η. This implies that

$$\left|\partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right| \leqslant M_\eta. \tag{5.9}$$

Simple computations yield with $P = (p, 1) \in \mathbb{R}^{N+1}$:

$$\begin{cases} \lambda_\eta^+ + \partial_\tau \Gamma^\epsilon(\overline{\tau}, \overline{Y}) = \partial_t \psi(\overline{t}, \overline{X}) + \left(1 + \partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y})\right)\left(\partial_t \phi(t_0, X_0) - \partial_t \phi(\overline{t}, \overline{X})\right), \\ \lambda_\eta^+ \overline{\tau} + P \cdot \overline{Y} + V(\overline{\tau}, \overline{Y}) = \dfrac{\phi^\epsilon(\overline{t}, \overline{X})}{\epsilon} - k_\epsilon. \end{cases} \tag{5.10}$$

Using (5.10) and (5.9), Eq. (3.7) yields for any $\rho > 0$

$$\partial_t \psi(\overline{t}, \overline{X}) + o_r(1) \geqslant L_0 + L_\eta + \mathcal{I}_1^{1,\rho}\left[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\right] + \mathcal{I}_1^{2,\rho}\left[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\right]$$
$$- W'\left(\frac{\phi^\epsilon(\overline{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) - o_\eta(1). \tag{5.11}$$

With the following lemma (which will be proved in the next subsection), we make rigorous the heuristic computations done in Subsection 3.1.2 to estimate the error when plugging (3.4) in (3.2).

Lemma 5.5 (*Supersolution property for ϕ^ϵ*). *For $\epsilon \leqslant \epsilon_0(r) < r \leqslant r_0$, we have*

$$\partial_t \psi(\overline{t}, \overline{X}) \geqslant \mathcal{I}_1^{1,1}\left[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right]$$
$$- W'\left(\frac{\phi^\epsilon(\overline{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) - o_\eta(1) + o_r(1) + L_\eta.$$

Let $r \leqslant r_0$ be so small that $o_r(1) \geqslant -L_1/4$. Then, recalling that $L_\eta - o_\eta(1) \geqslant L_1/2$, for $\epsilon \leqslant \epsilon_0(r)$ we have

$$\partial_t \psi(\overline{t}, \overline{X}) \geqslant \mathcal{I}_1^{1,1}\left[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right]$$
$$- W'\left(\frac{\phi^\epsilon(\overline{t}, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}}{\epsilon}\right) + \frac{L_1}{4},$$

and therefore ϕ^ϵ is a supersolution of (5.1) in $Q_{r,r}(t_0, X_0)$.

Step 4. *Conclusion.* Since $U^\epsilon \leqslant \phi^\epsilon$ outside $Q_{r,r}(t_0, X_0)$, by the comparison principle, Proposition 4.3, we conclude that $U^\epsilon(t, X) \leqslant \phi(t, X) + \epsilon V(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}) + \epsilon k_\epsilon$ in $Q_{r,r}(t_0, X_0)$ and we obtain the desired contradiction by passing to the upper limit as $\epsilon \to 0$ at (t_0, X_0) using the fact that $U^+(t_0, X_0) = \phi(t_0, X_0)$: $0 \leqslant -\gamma_r$.

This ends the proof of Theorem 1.2.

5.2. Proof of Lemma 5.5

The result will follow from (5.11) and the following inequality

$$L_0 + \mathcal{I}_1^{1,\rho}\left[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\right] + \mathcal{I}_1^{2,\rho}\left[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\right]$$
$$\geqslant \mathcal{I}_1^{1,1}\left[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + o_r(1). \tag{5.12}$$

To show the result, we proceed in several steps. In what follows, we denote by C various positive constants independent of ϵ. We start to call

$$L_0^1 = \int\limits_{|x| \leqslant 1} \big(\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) - \nabla\phi(t_0, X_0) \cdot x\big)\mu(dx),$$

$$L_0^2 = \int\limits_{|x| > 1} \big(U^+(t_0, x_0 + x, x_{N+1}^0) - U^+(t_0, X_0)\big)\mu(dx).$$

Then, recalling the definition (5.5) of L_0, we can write

$$L_0 = L_0^1 + L_0^2. \tag{5.13}$$

Keep in mind that $\overline{y}_{N+1} = \frac{F(\overline{t}, \overline{X})}{\epsilon}$. Since $\psi(t, X) = \phi(t, X) + \epsilon \Gamma^\epsilon(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t, X)}{\epsilon}) + \epsilon k_\epsilon$, we have

$$\mathcal{I}_1^{1,1}\big[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\big] = I_1 + I_2, \tag{5.14}$$

where

$$\begin{cases} I_1 = \int\limits_{|x| \leqslant 1} \epsilon \left(\begin{array}{c} \Gamma^\epsilon(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\overline{t}, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}) - \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \\ -\nabla_y \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \cdot \frac{x}{\epsilon} - \partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \nabla_x F(\overline{t}, \overline{X}) \cdot \frac{x}{\epsilon} \end{array} \right) \mu(dx), \\ I_2 = \int\limits_{|x| \leqslant 1} \big(\phi(\overline{t}, \overline{x}+x, \overline{x}_{N+1}) - \phi(\overline{t}, \overline{X}) - \nabla\phi(\overline{t}, \overline{X}) \cdot x\big)\mu(dx). \end{cases}$$

In order to show (5.12), we show successively in Steps 1, 2 and 3:

$$\begin{cases} I_1 \leqslant \mathcal{I}_1^{1,\rho}\big[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] + \mathcal{I}_1^{2,\rho}\big[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] + o_r(1) + C_\epsilon \rho, \\ I_2 \leqslant L_0^1 + o_r(1), \\ \mathcal{I}_1^{2,1}\big[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\big] \leqslant L_0^2 + o_r(1). \end{cases}$$

Because the expressions are non-linear and non-local and with a singular kernel, there is no simple computation and we have to carefully check those inequalities sometimes splitting terms in easier parts to estimate.

Step 1. We can choose ϵ_0 so small that for any $\epsilon \leqslant \epsilon_0$ and any $\rho > 0$ small enough

$$I_1 \leqslant \mathcal{I}_1^{1,\rho}\big[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] + \mathcal{I}_1^{2,\rho}\big[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] + o_r(1) + C_\epsilon \rho.$$

Take $\rho > 0$, $\delta > \rho$ small and $R > 0$ large and such that $\epsilon R < 1$. Since g is even, we can write

$$I_1 = I_1^0 + I_1^1 + I_1^2 + I_1^3,$$

where

$$I_1^0 = \int\limits_{|x| \leqslant \epsilon\rho} \epsilon \left(\Gamma^\epsilon\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\overline{t}, \overline{x}+x, \overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y}) - \nabla_y \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \cdot \frac{x}{\epsilon} \right.$$

$$\left. - \partial_{y_{N+1}} \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \nabla_x F(\overline{t}, \overline{X}) \cdot \frac{x}{\epsilon} \right) \mu(dx),$$

$$I_1^1 = \int\limits_{\epsilon\rho \leqslant |x| \leqslant \epsilon\delta} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar{t}}{\epsilon}, \frac{\bar{x}+x}{\epsilon}, \frac{F(\bar{t}, \bar{x}+x, \bar{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\bar{\tau}, \bar{Y}) \right)\mu(dx),$$

$$I_1^2 = \int\limits_{\epsilon\delta \leqslant |x| \leqslant \epsilon R} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar{t}}{\epsilon}, \frac{\bar{x}+x}{\epsilon}, \frac{F(\bar{t}, \bar{x}+x, \bar{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\bar{\tau}, \bar{Y}) \right)\mu(dx),$$

$$I_1^3 = \int\limits_{\epsilon R \leqslant |x| \leqslant 1} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar{t}}{\epsilon}, \frac{\bar{x}+x}{\epsilon}, \frac{F(\bar{t}, \bar{x}+x, \bar{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\bar{\tau}, \bar{Y}) \right)\mu(dx).$$

Moreover

$$\mathcal{I}_1^{2,\rho}\big[V(\bar{\tau}, \cdot, \bar{y}_{N+1}), \bar{y}\big] = J_1 + J_2 + J_3,$$

where

$$J_1 = \int\limits_{\rho < |z| \leqslant \delta} \big(V(\bar{\tau}, \bar{y}+z, \bar{y}_{N+1}) - V(\bar{\tau}, \bar{Y})\big)\mu(dz),$$

$$J_2 = \int\limits_{\delta < |z| \leqslant R} \big(V(\bar{\tau}, \bar{y}+z, \bar{y}_{N+1}) - V(\bar{\tau}, \bar{Y})\big)\mu(dz),$$

$$J_3 = \int\limits_{|z| > R} \big(V(\bar{\tau}, \bar{y}+z, \bar{y}_{N+1}) - V(\bar{\tau}, \bar{Y})\big)\mu(dz).$$

Step 1.1. *Estimate of* I_1^0 *and* $\mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\bar{\tau}, \cdot, \bar{y}_{N+1}), \bar{y}]$.
Since Γ^ϵ is of class C^2, we have

$$\big|I_1^0\big|, \big|\mathcal{I}_1^{1,\rho}\big[\Gamma^\epsilon(\bar{\tau}, \cdot, \bar{y}_{N+1}), \bar{y}\big]\big| \leqslant C_\epsilon \rho, \tag{5.15}$$

where C_ϵ depends on the second derivatives of Γ^ϵ. Remark that if we knew that V is smooth in y too, we could choose $\rho = 0$.
Step 1.2. *Estimate of* $I_1^1 - J_1$.
Using (5.8) and the fact that g is even, we can estimate $I_1^1 - J_1$ as follows

$$I_1^1 - J_1 \leqslant \int\limits_{\rho < |z| \leqslant \delta} \left[V\left(\bar{\tau}, \bar{y}+z, \frac{F(\bar{t}, \bar{x}+\epsilon z, \bar{x}_{N+1})}{\epsilon} \right) - V\left(\bar{\tau}, \bar{y}+z, \frac{F(\bar{t}, \bar{x})}{\epsilon} \right) \right]\mu(dz)$$

$$= \int\limits_{\rho < |z| \leqslant \delta} \left\{ \left[V\left(\bar{\tau}, \bar{y}+z, \frac{F(\bar{t}, \bar{x}+\epsilon z, \bar{x}_{N+1})}{\epsilon} \right) - V\left(\bar{\tau}, \bar{y}+z, \frac{F(\bar{t}, \bar{x})}{\epsilon} \right) \right. \right.$$

$$\left. - \partial_{y_{N+1}} V\left(\bar{\tau}, \bar{y}+z, \frac{F(\bar{t}, \bar{X})}{\epsilon} \right) \nabla_x F(\bar{t}, \bar{X}) \cdot z \right]$$

$$\left. + \big[\partial_{y_{N+1}} V(\bar{\tau}, \bar{y}+z, \bar{y}_{N+1}) - \partial_{y_{N+1}} V(\bar{\tau}, \bar{Y}) \big] \nabla_x F(\bar{t}, \bar{X}) \cdot z \right\}\mu(dz).$$

Next, using (3.10) and (3.11), we get

$$I_1^1 - J_1 \leqslant C \int\limits_{|z| \leqslant \delta} \left(|z|^2 + |z|^{1+\alpha} \right) \mu(dz) \leqslant C\delta^\alpha. \tag{5.16}$$

Step 1.3. *Estimate of* $I_1^2 - J_2$.
If M_η is the Lipschitz constant of V w.r.t. y_{N+1}, then

$$I_1^2 - J_2 \leqslant \int\limits_{\delta < |z| \leqslant R} \left(V\left(\overline{\tau}, \overline{y} + z, \frac{F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1})}{\epsilon} \right) - V\left(\overline{\tau}, \overline{y} + z, \frac{F(\overline{t}, \overline{X})}{\epsilon} \right) \right) \mu(dz)$$

$$\leqslant M_\eta \int\limits_{\delta < |z| \leqslant R} \left| \frac{F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1})}{\epsilon} - \frac{F(\overline{t}, \overline{X})}{\epsilon} \right| \mu(dz)$$

$$\leqslant M_\eta \int\limits_{\delta < |z| \leqslant R} \sup_{|z| \leqslant R} \left| \nabla_x F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1}) \right| |z| \mu(dz).$$

Then

$$I_1^2 - J_2 \leqslant C \sup_{|z| \leqslant R} \left| \nabla_x F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1}) \right| \log(R/\delta). \tag{5.17}$$

Step 1.4. *Estimate of* I_1^3 *and* J_3.
Since V is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, we have

$$I_1^3 \leqslant \int\limits_{R < |z| \leqslant \frac{1}{\epsilon}} \left(V\left(\overline{\tau}, \overline{y} + z, \frac{F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1})}{\epsilon} \right) - V(\overline{\tau}, \overline{Y}) \right) \mu(dz)$$

$$\leqslant \int\limits_{|z| > R} 2\|v\|_\infty \mu(dz) \leqslant \frac{C}{R}. \tag{5.18}$$

Similarly

$$|J_3| \leqslant \frac{C}{R}. \tag{5.19}$$

Now, from (5.15)–(5.19), we infer that

$$I_1 \leqslant \mathcal{I}_1^{1,\rho} \left[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y} \right] + \mathcal{I}_1^{2,\rho} \left[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y} \right] + 2C_\epsilon \rho + C\delta^\alpha$$

$$+ C \sup_{|z| \leqslant R} \left| \nabla_x F(\overline{t}, \overline{x} + \epsilon z, \overline{x}_{N+1}) \right| \log\left(\frac{R}{\delta} \right) + \frac{C}{R}.$$

We choose $R = R(r)$ such $R \to +\infty$ as $r \to 0^+$, $\epsilon_0 = \epsilon_0(r)$ such that $R\epsilon_0(r) \leqslant r$ and $\delta = \delta(r) > 0$ such that $\delta \to 0$ as $r \to 0^+$ and $r \log(R/\delta) \to 0$ as $r \to 0^+$. With this choice, for any $\epsilon \leqslant \epsilon_0$ and any $\rho < \delta$

$$C\delta^\alpha + C \sup_{|z| \leqslant R} \left| \nabla_x F(\bar{t}, \bar{x} + \epsilon z, \bar{x}_{N+1}) \right| \log\left(\frac{R}{\delta} \right) + \frac{C}{R} = o_r(1) \quad \text{as } r \to 0^+,$$

and Step 1 is proved.

Step 2. $I_2 \leqslant L_0^1 + o_r(1)$. For $0 < \nu < 1$ we can split I_2 and L_0^1 as follows

$$I_2 = \int_{|x| \leqslant \nu} \left(\phi(\bar{t}, \bar{x} + x, \bar{x}_{N+1}) - \phi(\bar{t}, \overline{X}) - \nabla\phi(\bar{t}, \overline{X}) \cdot x \right) \mu(dx)$$

$$+ \int_{\nu \leqslant |x| \leqslant 1} \left(\phi(\bar{t}, \bar{x} + x, \bar{x}_{N+1}) - \phi(\bar{t}, \overline{X}) \right) \mu(dx) = I_2^1 + I_2^2,$$

$$L_0^1 = \int_{|x| \leqslant \nu} \left(\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) - \nabla\phi(t_0, X_0) \cdot x \right) \mu(dx)$$

$$+ \int_{\nu \leqslant |x| \leqslant 1} \left(\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) \right) \mu(dx) = T_1 + T_2.$$

Since ϕ is of class C^2 we have

$$I_2^1, T_1 \leqslant C\nu.$$

Using the Lipschitz continuity of ϕ we get

$$I_2^2 - T_2 = \int_{\nu < |x| \leqslant 1} Cr\mu(dx) \leqslant C\frac{r}{\nu}.$$

Hence, Step 2 follows choosing $\nu = \nu(r)$ such that $\nu \to 0$ and $r/\nu \to 0$ as $r \to 0^+$.

Step 3. $\mathcal{I}_1^{2,1}[\phi^\epsilon(\bar{t}, \cdot, \bar{x}_{N+1}), \bar{x}] \leqslant L_0^2 + o_r(1)$. Remark that

$$U^\epsilon(\bar{t}, \bar{x} + x, \bar{x}_{N+1}) - \phi(\bar{t}, \overline{X}) - \epsilon V(\overline{\tau}, \overline{Y}) - \epsilon k_\epsilon$$

$$\leqslant U^+(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, X_0) + o_\epsilon(1) + o_r(1).$$

Then, recalling that $\phi(t_0, X_0) = U^+(t_0, X_0)$, for $\epsilon \leqslant \epsilon_0$ we get

$$\mathcal{I}_1^{2,1}[\phi^\epsilon(\bar{t}, \cdot, \bar{x}_{N+1}), \bar{x}] - L_0^2 \leqslant o_r(1)$$

and Step 3 is proved.

Finally (5.13), (5.14), Steps 1, 2 and 3 give

$$\mathcal{I}_1^{1,1}\big[\psi(\bar{t},\cdot,\bar{x}_{N+1}),\bar{x}\big]+\mathcal{I}_1^{2,1}\big[\phi^\epsilon(\bar{t},\cdot,\bar{x}_{N+1}),\bar{x}\big]$$
$$\leqslant \mathcal{I}_1^{1,\rho}\big[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}\big]+\mathcal{I}_1^{2,\rho}\big[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}\big]+L_0+o_r(1)+C_\epsilon\rho,$$

from which, using inequality (5.11) and letting $\rho\to 0^+$, we get for $\epsilon\leqslant\epsilon_0$

$$\partial_t\psi(\bar{t},\overline{X})\geqslant\mathcal{I}_1^{1,1}\big[\psi(\bar{t},\cdot,\bar{x}_{N+1}),\bar{x}\big]+\mathcal{I}_1^{2,1}\big[\phi^\epsilon(\bar{t},\cdot,\bar{x}_{N+1}),\bar{x}\big]-W'\left(\frac{\phi^\epsilon(\bar{t},\overline{X})}{\epsilon}\right)+\sigma\left(\frac{\bar{t}}{\epsilon},\frac{\bar{x}}{\epsilon}\right)$$
$$-o_\eta(1)+o_r(1)+L_\eta$$

and this concludes the proof of the lemma. □

6. Building of Lipschitz sub and supercorrectors

In this section we construct bounded sub and supersolutions of (3.6) that are Lipschitz w.r.t. y_{N+1}. As a byproduct, we will prove Theorem 1.1 and Proposition 5.4.

Proposition 6.1 (*Lipschitz continuous sub and supercorrectors*). *Let λ be the quantity defined by Theorem 1.1. Then, for any fixed $p\in\mathbb{R}^N$, $P=(p,1)$, $L\in\mathbb{R}$ and $\eta>0$ small enough, there exist real numbers $\lambda_\eta^+(p,L)$, $\lambda_\eta^-(p,L)$, a constant $C>0$ (independent of η, p and L) and bounded super and subcorrectors W_η^+, W_η^- i.e. respectively a super and a subsolution of (3.6) (with respectively λ_η^+ and λ_η^- in place of λ) such that*

$$\lim_{\eta\to 0^+}\lambda_\eta^+(p,L)=\lim_{\eta\to 0^+}\lambda_\eta^-(p,L)=\lambda(p,L),$$

$\lambda_\eta^\pm$ satisfy (i) *and* (ii) *of Proposition 5.4 and for any $(\tau,Y)\in\mathbb{R}^+\times\mathbb{R}^{N+1}$*

$$\big|W_\eta^\pm(\tau,Y)\big|\leqslant C.\tag{6.1}$$

Moreover $W_\eta^\pm$ are Lipschitz continuous w.r.t. y_{N+1} and α-Hölder continuous w.r.t. y for any $0<\alpha<1$, with

$$-1\leqslant\partial_{y_{N+1}}W_\eta^\pm\leqslant\frac{\|W''\|_\infty}{\eta},\tag{6.2}$$

$$\big\langle W_\eta^\pm\big\rangle_y^\alpha\leqslant C_\eta.\tag{6.3}$$

In order to prove the proposition, for $\eta\geqslant 0$, a_0, $L\in\mathbb{R}$, $p\in\mathbb{R}^N$ and $P=(p,1)$, we introduce the problem

$$\begin{cases}\partial_\tau U=L+\mathcal{I}_1\big[U(\tau,\cdot,y_{N+1})\big]-W'(U+P\cdot Y)+\sigma(\tau,y)\\\qquad\quad+\eta\Big[a_0+\inf_{Y'}U(\tau,Y')-U(\tau,Y)\Big]|\partial_{y_{N+1}}U+1| & \text{in }\mathbb{R}^+\times\mathbb{R}^{N+1},\\U(0,Y)=0 & \text{on }\mathbb{R}^{N+1}.\end{cases}\tag{6.4}$$

We have the following result whose proof is postponed to Appendix A.

Proposition 6.2 (*Comparison principle for (6.4)*). *Let $U_1\in USC_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ and $U_2\in LSC_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ be respectively a viscosity subsolution and supersolution of (6.4), then $U_1\leqslant U_2$ on $\mathbb{R}^+\times\mathbb{R}^{N+1}$.*

6.1. Lipschitz regularity

Proposition 6.3 (*Lipschitz continuity in y_{N+1}*). *Suppose $\eta > 0$. Let $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ be the viscosity solution of (6.4). Then U_η is Lipschitz continuous w.r.t. y_{N+1} and for almost every $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$*

$$-1 \leqslant \partial_{y_{N+1}} U_\eta(\tau, Y) \leqslant \frac{\|W''\|_\infty}{\eta}. \tag{6.5}$$

For a formal argument, we refer the reader to Step 1 of Subsection 3.2.

Proof. Let us define $\widehat{U}(\tau, Y) = U(\tau, Y) + y_{N+1}$, then $\widehat{U}$ satisfies

$$\begin{cases} \partial_\tau \widehat{U} = L + \mathcal{I}_1\big[\widehat{U}(\tau, \cdot, y_{N+1})\big] - W'(\widehat{U} + p \cdot y) + \sigma(\tau, y) \\ \qquad + \eta\Big[a_0 + \inf_{Y'}\big(\widehat{U}(\tau, Y') - y'_{N+1}\big) - \big(\widehat{U}(\tau, Y) - y_{N+1}\big)\Big]|\partial_{y_{N+1}}\widehat{U}| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ \widehat{U}(0, Y) = y_{N+1} & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{6.6}$$

We are going to prove that $\widehat{U}$ is Lipschitz continuous w.r.t. y_{N+1} with

$$0 \leqslant \partial_{y_{N+1}} \widehat{U}(\tau, Y) \leqslant 1 + \frac{\|W''\|_\infty}{\eta}.$$

By comparison, $\widehat{U}(t, y, y_{N+1}) \leqslant \widehat{U}(t, y, y_{N+1} + h)$ for $h \geqslant 0$, from which immediately follows that $\partial_{y_{N+1}} \widehat{U} \geqslant 0$. In particular we can replace $|\partial_{y_{N+1}} \widehat{U}|$ by $\partial_{y_{N+1}} \widehat{U}$ in (6.6).

Let us now show that $\partial_{y_{N+1}} \widehat{U} \leqslant 1 + \frac{\|W''\|_\infty}{\eta}$. We argue by contradiction by assuming that for some $T > 0$ the supremum of the function $\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}|$ on $[0, T] \times \mathbb{R}^{N+1}$ is strictly positive as soon as $K > 1 + \frac{\|W''\|_\infty}{\eta}$. Then for $\delta, \beta > 0$ small enough, M defined by

$$M = \max_{\substack{(\tau, y) \in [0, T] \times \mathbb{R}^N \\ y_{N+1}, z_{N+1} \in \mathbb{R}}} \left(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta \psi(Y) - \frac{\delta}{T - \tau} \right),$$

where ψ is defined as the function ψ_2 in the proof of Proposition 4.7, is positive. For $j > 0$ let

$$M_j = \max_{\substack{\tau, s \in [0, T], y, z \in \mathbb{R}^N \\ y_{N+1}, z_{N+1} \in \mathbb{R}}} \bigg(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(s, z, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta \psi(Y) \\ - \frac{\delta}{T - \tau} - j|\tau - s|^2 - j|y - z|^2 \bigg),$$

and let $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \in ([0, T] \times \mathbb{R}^{N+1})^2$ be a point where M_j is attained. Classical arguments show that $M_j \to M$, $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \to (\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{\tau}, \overline{y}, \overline{z}_{N+1})$ as $j \to +\infty$, where $(\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{z}_{N+1})$ is a point where M is attained.

Remark that $0 < \overline{\tau} < T$, moreover, since $\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) > \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})$ and $\widehat{U}$ is non-decreasing in y_{N+1}, it is

$$\overline{y}_{N+1} > \overline{z}_{N+1}. \tag{6.7}$$

In particular $y_{N+1}^j \neq z_{N+1}^j$ and $0 < s_j$, $\tau_j < T$ for j large enough. Hence, for $r > 0$, we obtain the following viscosity inequalities

$$
\frac{\delta}{(T - \tau_j)^2} + j(t_j - s_j)
$$

$$
\leqslant L + C_N jr + \beta \mathcal{I}_1^{1,r}\big[\psi\big(\cdot, y_{N+1}^j\big), y^j\big] + \mathcal{I}_1^{2,r}\big[\widehat{U}\big(\tau^j, \cdot, y_{N+1}^j\big), y^j\big]
$$

$$
- W'\big(\widehat{U}\big(\tau^j, y^j, y_{N+1}^j\big) + p \cdot y^j\big) + \sigma\big(\tau^j, y^j\big) + \eta\Big[a_0 + \inf_{Y'}\big(\widehat{U}\big(\tau_j, Y'\big) - y_{N+1}'\big)
$$

$$
- \big(\widehat{U}\big(\tau^j, y^j, y_{N+1}^j\big) - y_{N+1}^j\big)\Big]\bigg(K \frac{y_{N+1}^j - z_{N+1}^j}{|y_{N+1}^j - z_{N+1}^j|} + \beta \partial_{y_{N+1}} \psi\big(y^j, y_{N+1}^j\big)\bigg), \tag{6.8}
$$

and

$$
j(t_j - s_j) \geqslant L - C_N jr + \mathcal{I}_1^{2,r}\big[\widehat{U}\big(s^j, \cdot, z_{N+1}^j\big), z^j\big] - W'\big(\widehat{U}\big(s^j, z^j, z_{N+1}^j\big) + p \cdot z^j\big) + \sigma\big(s^j, z^j\big)
$$

$$
+ \eta\Big[a_0 + \inf_{Y'}\big(\widehat{U}\big(s_j, Y'\big) - y_{N+1}'\big) - \big(\widehat{U}\big(s^j, z^j, z_{N+1}^j\big) - z_{N+1}^j\big)\Big]K \frac{y_{N+1}^j - z_{N+1}^j}{|y_{N+1}^j - z_{N+1}^j|}, \tag{6.9}
$$

where C_N is a constant depending on N. Since $(\tau^j, y^j, y_{N+1}^j, s^j, z^j, z_{N+1}^j)$ is a maximum point, we have

$$
\widehat{U}\big(\tau^j, y^j + x, y_{N+1}^j\big) - \widehat{U}\big(\tau^j, y^j, y_{N+1}^j\big) \leqslant \widehat{U}\big(s^j, z^j + x, z_{N+1}^j\big) - \widehat{U}\big(s^j, z^j, z_{N+1}^j\big)
$$

$$
+ \beta\big[\psi\big(y^j + x, y_{N+1}^j\big) - \psi\big(y^j, y_{N+1}^j\big)\big]
$$

for any $x \in \mathbb{R}^N$, which implies that for any $r > 0$

$$
\mathcal{I}_1^{2,r}\big[\widehat{U}\big(\tau^j, \cdot, y_{N+1}^j\big), y^j\big] \leqslant \mathcal{I}_1^{2,r}\big[\widehat{U}\big(s^j, \cdot, z_{N+1}^j\big), z^j\big] + \beta \mathcal{I}_1^{2,r}\big[\psi\big(\cdot, y_{N+1}^j\big), y^j\big].
$$

Hence, subtracting (6.8) with (6.9), sending $r \to 0^+$ and then $j \to +\infty$, we get

$$
\frac{\delta}{(T - \overline{\tau})^2} \leqslant \beta \mathcal{I}_1\big[\psi\big(\cdot, \overline{y}_{N+1}\big), \overline{y}\big] + W'\big(\widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) + p \cdot \overline{y}\big) - W'\big(\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) + p \cdot \overline{y}\big)
$$

$$
- \eta\big[\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) - (\overline{y}_{N+1} - \overline{z}_{N+1})\big]K \frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{|\overline{y}_{N+1} - \overline{z}_{N+1}|}
$$

$$
+ \beta \partial_{y_{N+1}} \psi(\overline{y}, \overline{y}_{N+1})\eta\Big[a_0 + \inf_{Y'}\big(\widehat{U}(\overline{\tau}, Y') - y_{N+1}'\big) - \big(\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \overline{y}_{N+1}\big)\Big]
$$

$$
\leqslant \|W''\|_\infty |\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})|
$$

$$
- K\eta\big[\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) - (\overline{y}_{N+1} - \overline{z}_{N+1})\big]\frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{|\overline{y}_{N+1} - \overline{z}_{N+1}|} + \beta C.
$$

Then, using (6.7) and that $K|\overline{y}_{N+1} - \overline{z}_{N+1}| < \widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})$, for β small enough, we finally obtain

$$
\big(\|W''\|_\infty + \eta - \eta K\big)\big(\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})\big) \geqslant 0,
$$

which is a contradiction for $K > 1 + \frac{\|W''\|_\infty}{\eta}$. $\square$

6.2. Ergodicity

Proposition 6.4 *(Ergodic properties). There exists a unique* $\lambda_\eta = \lambda_\eta(p, L)$ *such that the viscosity solution* $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ *of* (6.4) *with* $\eta \geqslant 0$, *satisfies*:

$$\left| U_\eta(\tau, Y) - \lambda_\eta \tau \right| \leqslant C_3 \quad \text{for all } \tau > 0, \ Y \in \mathbb{R}^{N+1}, \tag{6.10}$$

with C_3 *independent of* η. *Moreover*

$$L - \left\| W' \right\|_\infty - \| \sigma \|_\infty + \eta a_0 \leqslant \lambda_\eta \leqslant L + \left\| W' \right\|_\infty + \| \sigma \|_\infty + \eta a_0. \tag{6.11}$$

Proof. For simplicity of notations, in what follows we denote $U = U_\eta$ and $\lambda = \lambda_\eta$.

To prove the proposition we follow the proof of the analogue result in [24]. We proceed in three steps.

Step 1. *Existence.* The functions $W^+(\tau, Y) = C^+ \tau$ and $W^-(\tau, Y) = C^- \tau$, where

$$C^\pm = L \pm \left\| W' \right\|_\infty \pm \| \sigma \|_\infty + \eta a_0,$$

are respectively sub and supersolution of (6.4). Then the existence of a unique solution of (6.4) follows from Perron's method.

Step 2. *Control of the oscillations w.r.t. space.* We want to prove that there exists $C_1 > 0$ such that

$$\left| U(\tau, Y) - U(\tau, Z) \right| \leqslant C_1 \quad \text{for all } \tau \geqslant 0, \ Y, Z \in \mathbb{R}^{N+1}. \tag{6.12}$$

Step 2.1. For a given $k \in \mathbb{Z}^{N+1}$, we set $P \cdot k = l + \alpha$, with $l \in \mathbb{Z}$ and $\alpha \in [0, 1)$. The function $\widetilde{U}(\tau, Y) = U(\tau, Y + k) + \alpha$ is still a solution of (6.4), with $\widetilde{U}(0, Y) = \alpha$ Moreover

$$U(0, Y) = 0 \leqslant \widetilde{U}(0, Y) = \alpha \leqslant 1 = U(0, Y) + 1.$$

Then from the comparison principle for (6.4) and invariance by integer translations we deduce for all $\tau \geqslant 0$:

$$\left| U(\tau, Y + k) - U(\tau, Y) \right| \leqslant 1. \tag{6.13}$$

Step 2.2. We proceed as in [24] by considering the functions

$$M(\tau) := \sup_{Y \in \mathbb{R}^{N+1}} U(\tau, Y), \qquad m(\tau) := \inf_{Y \in \mathbb{R}^{N+1}} U(\tau, Y),$$

$$q(\tau) := M(\tau) - m(\tau) = \operatorname{osc} U(\tau, \cdot).$$

Let us assume that the extrema defining these functions are attained: $M(\tau) = U(\tau, Y^\tau)$, $m(\tau) = U(\tau, Z^\tau)$.

It is easy to see that $M(\tau)$ and $m(\tau)$ satisfy in the viscosity sense

$$\partial_\tau M \leqslant L + \mathcal{I}_1^2 \left[U\left(\tau, \cdot, y^\tau_{N+1}\right), y^\tau \right] - W'\left(M + P \cdot Y^\tau\right) + \sigma\left(\tau, y^\tau\right) + \eta \left[a_0 + m(\tau) - M(\tau)\right],$$

$$\partial_\tau m \geqslant L + \mathcal{I}_1^2 \left[U\left(\tau, \cdot, z^\tau_{N+1}\right), z^\tau \right] - W'\left(m + P \cdot Z^\tau\right) + \sigma\left(\tau, z^\tau\right) + \eta a_0.$$

Then q satisfies in the viscosity sense

$$
\begin{aligned}
\partial_\tau q \leqslant{} & \mathcal{I}_1^2\big[U\big(\tau,\cdot,y_{N+1}^\tau\big),y^\tau\big] - \mathcal{I}_1^2\big[U\big(\tau,\cdot,z_{N+1}^\tau\big),z^\tau\big] - W'\big(M + P\cdot Y^\tau\big) \\
& + W'\big(m + P\cdot Z^\tau\big) + \sigma\big(\tau,y^\tau\big) - \sigma\big(\tau,z^\tau\big) \\
\leqslant{} & \mathcal{I}_1^2\big[U\big(\tau,\cdot,y_{N+1}^\tau\big),y^\tau\big] - \mathcal{I}_1^2\big[U\big(\tau,\cdot,z_{N+1}^\tau\big),z^\tau\big] + 2\big\|W'\big\|_\infty + 2\|\sigma\|_\infty.
\end{aligned}
$$

Let us estimate the quantity $\mathcal{L}(\tau) := \mathcal{I}_1^2[U(\tau,\cdot,y_{N+1}^\tau),y^\tau] - \mathcal{I}_1^2[U(\tau,\cdot,z_{N+1}^\tau),z^\tau]$ from above by a function of q. Let us define $k^\tau \in \mathbb{Z}^{N+1}$ such that $Y^\tau - (Z^\tau + k^\tau) \in [0,1)^{N+1}$ and let $\widetilde{Z}^\tau := Z^\tau + k^\tau$. Using successively (6.13) and the first inequality in (6.5), we obtain:

$$
\begin{aligned}
\mathcal{L}(\tau) \leqslant{} & \int\limits_{|z|>1} \big(U\big(\tau,y^\tau+z,y_{N+1}^\tau\big) - U\big(\tau,Y^\tau\big)\big)\mu(dz) \\
& - \int\limits_{|z|>1} \big(U\big(\tau,\widetilde{Z}^\tau+z,\widetilde{Z}_{N+1}^\tau\big) - U\big(\tau,Z^\tau\big)\big)\mu(dz) + \overline{\mu} \\
\leqslant{} & \int\limits_{|z|>1} \big(U\big(\tau,y^\tau+z,y_{N+1}^\tau\big) - U\big(\tau,Y^\tau\big)\big)\mu(dz) \\
& - \int\limits_{|z|>1} \big(U\big(\tau,\widetilde{Z}^\tau+z,y_{N+1}^\tau\big) - U\big(\tau,Z^\tau\big)\big)\mu(dz) + 2\overline{\mu},
\end{aligned}
$$

where $\overline{\mu} = \|\mu_0\|_{L^1(\mathbb{R}^N\setminus B_1(0))}$. Now, let us introduce $c^\tau = \frac{y^\tau+\widetilde{Z}^\tau}{2}$ and $\delta^\tau = \frac{y^\tau-\widetilde{Z}^\tau}{2} \in [0,\frac{1}{2})^N$ so that $y^\tau = c^\tau + \delta^\tau$ and $\widetilde{Z}^\tau = c^\tau - \delta^\tau$. Hence

$$
\begin{aligned}
\mathcal{L}(\tau) \leqslant{} & 2\overline{\mu} + \int\limits_{|z|>1} \big(U\big(\tau,c^\tau+z+\delta^\tau,y_{N+1}^\tau\big) - U\big(\tau,Y^\tau\big)\big)\mu(dz) \\
& - \int\limits_{|z|>1} \big(U\big(\tau,c^\tau+z-\delta^\tau,y_{N+1}^\tau\big) - U\big(\tau,Z^\tau\big)\big)\mu(dz) \\
\leqslant{} & 2\overline{\mu} + \int\limits_{|z-\delta^\tau|>1} \big(U\big(\tau,c^\tau+z,y_{N+1}^\tau\big) - U\big(\tau,Y^\tau\big)\big)\mu_0\big(z-\delta^\tau\big)\,dz \\
& - \int\limits_{|z+\delta^\tau|>1} \big(U\big(\tau,c^\tau+z,y_{N+1}^\tau\big) - U\big(\tau,Z^\tau\big)\big)\mu_0\big(z+\delta^\tau\big)\,dz \\
\leqslant{} & 2\overline{\mu} - \int\limits_{\{|z-\delta^\tau|>1\}\cap\{|z+\delta^\tau|>1\}} \big(U\big(\tau,Y^\tau\big) - U\big(\tau,Z^\tau\big)\big)\min\big\{\mu_0\big(z-\delta^\tau\big),\mu_0\big(z+\delta^\tau\big)\big\}\,dz \\
\leqslant{} & 2\overline{\mu} - c_0 q(\tau)
\end{aligned}
$$

where $c_0 > 0$. We conclude that q satisfies in the viscosity sense

$$
\partial_\tau q(\tau) \leqslant 2\big\|W'\big\|_\infty + 2\|\sigma\|_\infty + 2\overline{\mu} - c_0 q(\tau),
$$

with $q(0) = 0$, from which we obtain (6.12).

If the extrema are not attained, it suffices to consider for $\beta > 0$, $M_\beta(\tau) := \sup_{Y \in \mathbb{R}^{N+1}} (U(\tau, Y) - \beta \psi(Y))$, $m_\beta(\tau) := \inf_{Y \in \mathbb{R}^{N+1}} (U(\tau, Y) + \beta \psi(Y))$, and $q_\beta(\tau) := M_\beta(\tau) - m_\beta(\tau)$, where ψ is defined as the function ψ_2 in the proof of Proposition 4.7. By the properties of ψ, $M_\beta(\tau)$ and $m_\beta(\tau)$ are attained. Then, the previous argument shows that

$$q_\beta \leqslant C_1 + C\beta,$$

and passing to the limit as $\beta \to 0^+$ we get (6.12).

Step 3. *Control of the oscillations in time.* We follow [24] by introducing the two quantities:

$$\lambda^+(T) := \sup_{\tau \geqslant 0} \frac{U(\tau + T, 0) - U(\tau, 0)}{T} \quad \text{and} \quad \lambda^-(T) := \inf_{\tau \geqslant 0} \frac{U(\tau + T, 0) - U(\tau, 0)}{T},$$

and proving that they have a common limit as $T \to +\infty$. First let us estimate $\lambda^+(T)$ from above. The function $U^+(t, Y) := U(\tau, 0) + C_1 + C^+ t$, is a supersolution of (6.4) if $C^+ = L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0$. Since $U^+(0, Y) \geqslant U(\tau, Y)$ if C_1 is as in (6.12), by the comparison principle for (6.4) in the time interval $[\tau, \tau + \tau_0]$, for any $\tau_0 > 0$ and $t \in [0, \tau_0]$ we get

$$U(\tau + t, Y) \leqslant U(\tau, 0) + C_1 + C^+ t. \tag{6.14}$$

Similarly

$$U(\tau + t, Y) \geqslant U(\tau, 0) - C_1 + C^- t, \tag{6.15}$$

where $C^- = L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0$. We then obtain for $\tau_0 = t = T$ and $y = 0$:

$$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 - \frac{C_1}{T} \leqslant \lambda^-(T) \leqslant \lambda^+(T) \leqslant L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0 + \frac{C_1}{T}. \tag{6.16}$$

By definition of $\lambda^\pm(T)$, for any $\delta > 0$, there exists $\tau^\pm \geqslant 0$ such that

$$\left| \lambda^\pm(T) - \frac{U(\tau^\pm + T, 0) - U(\tau^\pm, 0)}{T} \right| \leqslant \delta.$$

Let us consider $\alpha, \beta \in [0, 1)$ such that $\tau^+ - \tau^- - \beta = k \in \mathbb{Z}$, and $U(\tau^+, 0) - U(\tau^+ - k, 0) + \alpha \in \mathbb{Z}$. From (6.12) we have

$$U(\tau^+, Y) \leqslant U(\tau^+, 0) + C_1 \leqslant U(\tau^+ - k, Y) + 2C_1 + (U(\tau^+, 0) - U(\tau^+ - k, 0))$$

$$\leqslant U(\tau^+ - k, Y) + 2\lceil C_1 \rceil + (U(\tau^+, 0) - U(\tau^+ - k, 0) + \alpha).$$

Since $\sigma(\cdot, y)$ and $W'(\cdot)$ are $\mathbb{Z}$-periodic, the comparison principle for (6.4) on the time interval $[\tau^+, \tau^+ + T]$ implies that:

$$U(\tau^+ + T, Y) \leqslant U(\tau^+ - k + T, Y) + 2\lceil C_1 \rceil + U(\tau^+, 0) - U(\tau^+ - k, 0) + 1.$$

Choosing $Y = 0$ in the previous inequality we get

$$U(\tau^+ + T, 0) - U(\tau^+, 0) \leqslant U(\tau^+ - k + T, 0) - U(\tau^+ - k, 0) + 2\lceil C_1 \rceil + 1$$
$$= U(\tau^- + \beta + T, 0) - U(\tau^- + \beta, 0) + 2\lceil C_1 \rceil + 1,$$

and setting $t = \beta$ and $\tau = \tau^- + T$ in (6.14) and $\tau = \tau^-$ in (6.15) we finally obtain:

$$T\lambda^+(T) \leqslant T\lambda^-(T) + 4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty + 2\delta T.$$

Since this is true for any $\delta > 0$, we conclude that:

$$\left| \lambda^+(T) - \lambda^-(T) \right| \leqslant \frac{4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty}{T}.$$

Now arguing as in [23] and [24], we conclude that there exist $\lim_{T \to +\infty} \lambda^\pm(T) =: \lambda$ and

$$\left| \lambda^\pm(T) - \lambda \right| \leqslant \frac{4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty}{T},$$

which implies that

$$\left| U(T, 0) - \lambda T \right| \leqslant 4\lceil C_1 \rceil + 1 + 2\|W'\|_\infty + 2\|\sigma\|_\infty,$$

and then, using (6.12) we get (6.10). The uniqueness of λ follows from (6.10). Finally, (6.11) is obtained from (6.16) as $T \to +\infty$. $\quad\square$

6.3. Proof of Theorem 1.1

Let us consider the viscosity solution of (6.4) for $\eta = 0$. By Proposition 6.4 we know that there exists a unique λ such that $U(\tau, Y)/\tau$ converges to λ as τ goes to $+\infty$ for any $Y \in \mathbb{R}^{N+1}$. Moreover, by Proposition 4.6, $U(\tau, y, 0)$ is viscosity solution of (1.6). Hence, the theorem follows immediately from the uniqueness of the viscosity solution of (1.6).

6.4. Proof of Proposition 6.1

Step 1. *Definition of $W_\eta^\pm$.* Let us denote by U_η^+ the solution of (6.4) with $a_0 = C_1$, where C_1 is defined as in (6.12), and by U_η^- the solution of (6.4) with $a_0 = 0$. Let $\lambda_\eta^+ = \lim_{\tau \to +\infty} \frac{U_\eta^+(\tau, Y)}{\tau}$ and $\lambda_\eta^- = \lim_{\tau \to +\infty} \frac{U_\eta^-(\tau, Y)}{\tau}$; the existence of λ_η^+ and λ_η^- is guaranteed by Proposition 6.4. Now, we set

$$W_\eta^+(\tau, Y) := U_\eta^+(\tau, Y) - \lambda_\eta^+ \tau$$

and

$$W_\eta^-(\tau, Y) := U_\eta^-(\tau, Y) - \lambda_\eta^- \tau.$$

Step 2. *Limits of $\lambda_\eta^\pm$.* By stability (see e.g. [7]), for $\eta \to 0^+$ the sequence $(U_\eta^+)_\eta$ converges to U solution of (6.4) with $\eta = 0$. Moreover by (6.11) the sequence $(\lambda_\eta^+)_\eta$ is bounded. Take a subsequence $\eta_n \to 0$ as $n \to +\infty$ such that $\lambda_{\eta_n}^+ \to \lambda_\infty$ as $n \to +\infty$. We want to show that $\lambda_\infty = \lambda$, where $\lambda = \lim_{\tau \to +\infty} \frac{U(\tau, Y)}{\tau}$. By the proof of Theorem 1.1, we know that λ is the same quantity defined in Theorem 1.1. Using (6.10), we get

$$|\lambda - \lambda_\infty| \leqslant \left|\lambda - \frac{U(\tau,0)}{\tau}\right| + \left|\frac{U(\tau,0)}{\tau} - \frac{U_{\eta_n}^+(\tau,0)}{\tau}\right| + \left|\frac{U_{\eta_n}^+(\tau,0)}{\tau} - \lambda_{\eta_n}^+\right| + \left|\lambda_{\eta_n}^+ - \lambda_\infty\right|$$

$$\leqslant \left|\lambda - \frac{U(\tau,0)}{\tau}\right| + \left|\frac{U(\tau,0)}{\tau} - \frac{U_{\eta_n}^+(\tau,0)}{\tau}\right| + \frac{C_3}{\tau} + \left|\lambda_{\eta_n}^+ - \lambda_\infty\right|$$

where C_3 does not depend on n. Then, passing to the limit first as $n \to +\infty$ and then as $\tau \to +\infty$, we obtain that $\lambda = \lambda_\infty$. This implies that $\lambda_\eta^+ \to \lambda$ as $\eta \to 0$.

The same argument shows that $\lambda_\eta^- \to \lambda$ as $\eta \to 0$.

Step 3. W_η^+ and W_η^- are respectively sub and supersolutions. Since by (6.12), $C_0 + \inf_{Y'} U_\eta^+(\tau, Y') - U_\eta^+(\tau, Y) \geqslant 0$, W_η^+ is supersolution of (3.6) with $\lambda = \lambda_\eta^+$. Moreover, by (6.10), W_η^+ is bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ uniformly w.r.t. η: $|W_\eta^+(\tau, Y)| \leqslant C_3$ for all $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$.

Step 4. Regularity properties of $W_\eta^\pm$. By (6.5), W_η^+ is Lipschitz continuous w.r.t. y_{N+1} and $-1 \leqslant \partial_{y_{N+1}} W_\eta^+ \leqslant \frac{\|W''\|_\infty}{\eta}$. This implies that W_η^+ is also a viscosity subsolution of

$$\begin{cases} \lambda_\eta^+ + \partial_\tau V = L + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] - W'\big(V + \lambda_\eta^+ \tau + P \cdot Y\big) \\ \qquad\qquad + \sigma(\tau, y) + C_1\big(\|W''\|_\infty + \eta\big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \qquad (6.17) \\ V(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases}$$

By Proposition 4.6, W_η^+ is supersolution of (3.6) and subsolution of (6.17) in $\mathbb{R}^+ \times \mathbb{R}^N$ for any $y_{N+1} \in \mathbb{R}$. Then by Proposition 4.7, W_η^+ is of class C^α w.r.t. y uniformly in y_{N+1} and η, for any $0 < \alpha < 1$.

Similar arguments show that W_η^- is subsolution of (3.6) with $\lambda = \lambda_\eta^-$, is bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, Lipschitz continuous w.r.t. y_{N+1} with $-1 \leqslant \partial_{y_{N+1}} W_\eta^+ \leqslant \frac{\|W''\|_\infty}{\eta}$ and Hölder continuous w.r.t. y. This concludes the proof of Proposition 6.1. $\quad\square$

6.5. Proof of Proposition 5.4

The continuity of $\overline{H}(p, L)$ follows from stability of viscosity solutions of (1.6) (see e.g. [7]) and from (6.10). Indeed, let (p_n, L_n) be a sequence converging to (p_0, L_0) as $n \to +\infty$ and set $\lambda_n = \lambda(p_n, L_n)$, $n \geqslant 0$. By (6.10), we have for any $\tau > 0$

$$\left|\lambda_n - \frac{w_n(\tau, y)}{\tau}\right| \leqslant \frac{C_3}{\tau}.$$

Stability of viscosity solutions of (1.6) implies that w_n converges locally uniformly in (τ, y) to a function w_0 which is a solution of (1.6) with $(p, L) = (p_0, L_0)$. This implies that $\limsup_{n \to +\infty} |\lambda_n - \lambda_0| \leqslant \frac{2C_3}{\tau}$ for any $\tau > 0$. Hence, we conclude that $\lim_{n \to +\infty} \lambda_n = \lambda_0$.

Property (i) is an immediate consequence of (6.11).

The monotonicity in L of $\overline{H}(p, L)$ comes from the comparison principle.

Let us show (iii). Let v be the solution of (1.5) and $\lambda = \lambda(p, L)$. Set $\tilde{v}(\tau, y) := v(\tau, -y)$. Remark that $\mathcal{I}_1[\tilde{v}(\tau, \cdot), y] = \mathcal{I}_1[v(\tau, \cdot), -y]$. If $\sigma(\tau, \cdot)$ is even then $\tilde{v}$ satisfies

$$\begin{cases} \lambda + \partial_\tau \tilde{v} = \mathcal{I}_1\big[\tilde{v}(\tau, \cdot), y\big] + L - W'(\tilde{v} + \lambda t - p \cdot y) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \tilde{v}(0, y) = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

By the uniqueness of λ we deduce that $\lambda(L, p) = \lambda(L, -p)$, i.e. (iii). $\quad\square$

Finally let us turn to (iv). Define $\widetilde{v}(\tau, y) := -v(\tau, -y)$. If $W'(\cdot)$ and $\sigma(\tau, \cdot)$ are odd functions, $\widetilde{v}$ satisfies

$$\begin{cases} -\lambda + \partial_\tau \widetilde{v} = \mathcal{I}_1[\widetilde{v}(\tau, \cdot), y] - L - W'(\widetilde{v} - \lambda t + p \cdot y) + \sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \widetilde{v}(0, y) = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

As before, we conclude that $\lambda(-L, p) = -\lambda(L, p)$, i.e. (iv). $\quad\square$

7. Smooth approximate correctors

In this section, we prove the existence of approximate correctors that are smooth w.r.t. y_{N+1}, namely Proposition 3.1. We first need the following lemma:

Lemma 7.1. *Let $u_1, u_2 \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ be viscosity subsolutions (resp., supersolutions) of (3.6) in $\mathbb{R}^+ \times \mathbb{R}^N$, then $u_1 + u_2$ is viscosity subsolution (resp., supersolution) of*

$$\begin{cases} 2\lambda + \partial_\tau v = 2L + \mathcal{I}_1[v] - W'(u_1 + P \cdot Y + \lambda\tau) \\ \qquad\qquad - W'(u_2 + P \cdot Y + \lambda\tau) + 2\sigma(\tau, y) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ v(0, y) = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

For the proof see Lemma 5.8 in [8].

Next, let us consider a positive smooth function $\rho : \mathbb{R} \to \mathbb{R}$, with support in $B_1(0)$ and mass 1. We define a sequence of mollifiers $(\rho_\delta)_\delta$ by $\rho_\delta(s) = \frac{1}{\delta}\rho(\frac{s}{\delta})$, $s \in \mathbb{R}$. Let W_η^+ (resp. W_η^-) be the Lipschitz supersolution (resp. subsolution) of (3.6) with $\lambda = \lambda_\eta^+$ (resp. $\lambda = \lambda_\eta^-$), whose existence is guaranteed by Proposition 6.1. We define

$$V_{\eta,\delta}^\pm(t, y, y_{N+1}) := W_\eta^\pm(t, y, \cdot) \star \rho_\delta(\cdot) = \int_\mathbb{R} W_\eta^\pm(t, y, z)\rho_\delta(y_{N+1} - z)\,dz. \qquad (7.1)$$

Lemma 7.2. *The functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$ are respectively super and subsolution of*

$$\begin{cases} \lambda_\eta^\pm + \partial_\tau V_{\eta,\delta}^\pm = L + \mathcal{I}_1\big[V_{\eta,\delta}^\pm(\tau, \cdot, y_{N+1})\big] \\ \qquad\qquad + \sigma(\tau, y) - \int_\mathbb{R} W'\big(W_\eta^\pm(\tau, y, z) \\ \qquad\qquad + p \cdot y + z + \lambda_\eta^\pm \tau\big)\rho_\delta(y_{N+1} - z)\,dz & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V_\eta^\pm(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \qquad (7.2)$$

Proof. We prove the lemma for supersolutions. Let $Q_h^e = e + [-h/2, h/2)$, $\overline{\rho}_\delta(e, h) = \int_{Q_h^e} \rho_\delta(y)\,dy$ and

$$I_h(\tau, y, y_{N+1}) = \sum_{e \in h\mathbb{Z}} W_\eta^+(\tau, y, y_{N+1} - e)\overline{\rho}_\delta(e, h).$$

The function I_h is a discretization of the convolution integral and by classical results, converges uniformly to $V_{\eta,\delta}^+$ as $h \to 0$. By Proposition 4.6, W_η^+ is a viscosity supersolution of (3.6) also in $\mathbb{R}^+ \times \mathbb{R}^N$. Then, by Lemma 7.1, for any $y_{N+1} \in \mathbb{R}$, $I_h(\tau, y, y_{N+1})$ is a supersolution of

$$
\begin{cases}
\lambda_\eta^+ + \partial_\tau V = L + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] + \sigma(\tau, y) \displaystyle\sum_{e \in h\mathbb{Z}} \overline{\rho}_\delta(e, h) \\
\qquad - \displaystyle\sum_{e \in h\mathbb{Z}} W'\big(W_\eta^+(\tau, y, y_{N+1} - e) \\
\qquad + p \cdot y + (y_{N+1} - e) + \lambda_\eta^+ \tau\big)\overline{\rho}_\delta(e, h) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\
V(0, y) = 0 & \text{on } \mathbb{R}^N.
\end{cases}
$$

Using the stability result for viscosity solution of non-local equations, see [7], we conclude that $V_{\eta,\delta}^+$ is supersolution of (7.2) in $\mathbb{R}^+ \times \mathbb{R}^N$ and hence also in $\mathbb{R}^+ \times \mathbb{R}^{N+1}$. $\square$

7.1. Proof of Proposition 3.1

We first show that the functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$, defined in (7.1), are respectively super and subsolution of

$$
\begin{cases}
\lambda_\eta^\pm + \partial_\tau V_{\eta,\delta}^\pm = L + \mathcal{I}_1\big[V_{\eta,\delta}^\pm(\tau, \cdot, y_{N+1})\big] - W'\big(V_{\eta,\delta}^\pm + P \cdot Y + \lambda_\eta^\pm \tau\big) \\
\qquad\qquad + \sigma(\tau, y) \mp C_{\eta,\delta} & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \quad (7.3) \\
V_\eta^\pm(0, Y) = 0 & \text{on } \mathbb{R}^{N+1},
\end{cases}
$$

where $C_{\eta,\delta} = \|W''\|_\infty(2\delta \|W''\|_\infty/\eta + \delta)$. Using (6.2) and the properties of the mollifiers, we get

$$
\left| W'\big(V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) + p \cdot y + y_{N+1} + \lambda_\eta^\pm \tau\big) \right.
$$

$$
\left. - \int_{\mathbb{R}} W'\big(W_\eta^\pm(\tau, y, z) + p \cdot y + z + \lambda_\eta^\pm \tau\big)\rho_\delta(y_{N+1} - z)\, dz \right|
$$

$$
\leqslant \int_{\mathbb{R}} \left| W'\big(V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) + p \cdot y + y_{N+1} + \lambda_\eta^\pm \tau\big) \right.
$$

$$
\left. - W'\big(W_\eta^\pm(\tau, y, z) + p \cdot y + z + \lambda_\eta^\pm \tau\big) \right| \rho_\delta(y_{N+1} - z)\, dz
$$

$$
\leqslant \|W''\|_\infty \int_{\mathbb{R}} \big[|V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) - W_\eta^\pm(\tau, y, z)| + |y_{N+1} - z|\big]\rho_\delta(y_{N+1} - z)\, dz
$$

$$
\leqslant \|W''\|_\infty \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |W_\eta^\pm(\tau, y, r) - W_\eta^\pm(\tau, y, z)|\rho_\delta(y_{N+1} - r)\, dr + |y_{N+1} - z|\right]\rho_\delta(y_{N+1} - z)\, dz
$$

$$
\leqslant \|W''\|_\infty \int_{\mathbb{R}} \left[\int_{|y_{N+1} - r| \leqslant \delta} \frac{\|W''\|_\infty}{\eta}|r - z|\rho_\delta(y_{N+1} - r)\, dr + |y_{N+1} - z|\right]\rho_\delta(y_{N+1} - z)\, dz
$$

$$
\leqslant \|W''\|_\infty \int_{|y_{N+1} - z| \leqslant \delta} \left[\frac{\|W''\|_\infty}{\eta}\big(|y_{N+1} - z| + \delta\big) + |y_{N+1} - z|\right]\rho_\delta(y_{N+1} - z)\, dz
$$

$$
\leqslant \|W''\|_\infty \left(2\delta \frac{\|W''\|_\infty}{\eta} + \delta\right).
$$

From this estimate and Lemma 7.2, we deduce that $V_{\eta,\delta}^{+}$ and $V_{\eta,\delta}^{-}$ are respectively super and subsolution of (7.3). Now, we choose $\delta = \delta(\eta)$ such that $\|W''\|_{\infty}(2\delta\|W''\|_{\infty}/\eta + \delta) = o_{\eta}(1)$ as $\eta \to 0$ and define

$$V_{\eta}^{\pm}(\tau, Y) := V_{\eta,\delta(\eta)}^{\pm}(\tau, Y).$$

Then the functions $V_{\eta}^{\pm}$ are the desired super and subcorrectors. Indeed, we have already shown that they are super and subsolution of (3.7) with λ_{η}^{+} and λ_{η}^{-} satisfying (3.8). Properties (i) and (ii) of Proposition 5.4 can be shown as in the proof of the proposition. Finally, (3.9)–(3.11) easily follow from (6.1)–(6.3) and the properties of the mollifiers. $\quad\square$

Appendix A

Proof of Proposition 4.7. *Heuristic arguments.* Before entering in the proof, let us start with a heuristic explanation. Indeed, replacing $\partial_t u$ by u, we should get a similar result for a stationary solution of

$$\mathcal{I}_1[u] + g_2 \leqslant u \leqslant \mathcal{I}_1[u] + g_1.$$

At a point (x, y), with $x \neq y$, of supremum of

$$u(x) - u(y) - K|x - y|^{\alpha}$$

we have for $r > 0$

$$\begin{cases} u(x) \leqslant g_1 + K\mathcal{I}_1^{1,r}\big[|\cdot - y|^{\alpha}, x\big] + \mathcal{I}_1^{2,r}[u, x], \\ u(y) \geqslant g_2 - K\mathcal{I}_1^{1,r}\big[|x - \cdot|^{\alpha}, y\big] + \mathcal{I}_1^{2,r}[u, y]. \end{cases}$$

Setting $e = \frac{x-y}{|x-y|}$, $\varphi_{\alpha}(z) = |z|^{\alpha}$ and using the homogeneity of the functions, we get for $r = \sigma|x - y|$

$$\mathcal{I}_1^{1,r}\big[|\cdot - y|^{\alpha}, x\big] = -|x - y|^{\alpha-1}c_{\alpha}^{\sigma} = \mathcal{I}_1^{1,r}\big[|x - \cdot|^{\alpha}, y\big] \quad \text{with} \quad -c_{\alpha}^{\sigma} = \mathcal{I}_1^{1,\sigma}[\varphi_{\alpha}, e].$$

Therefore we get

$$u(x) - u(y) - K|x - y|^{\alpha} \leqslant g_1 - g_2 - K|x - y|^{\alpha} - 2K|x - y|^{\alpha-1}c_{\alpha}^{\sigma} + \mathcal{I}_1^{2,r}[u, x] - \mathcal{I}_1^{2,r}[u, y].$$

By the maximal property of (x, y), for any $z \in \mathbb{R}^N$ we have

$$u(x + z) - u(y + z) \leqslant u(x) - u(y)$$

which implies that

$$\mathcal{I}_1^{2,r}[u, x] - \mathcal{I}_1^{2,r}[u, y] \leqslant 0.$$

We conclude that

$$u(x) - u(y) - K|x - y|^{\alpha} \leqslant g_1 - g_2 - K|x - y|^{\alpha} - 2K|x - y|^{\alpha-1}c_{\alpha}^{\sigma}.$$

We can show that $c_{\alpha}^{\sigma} > 0$, for σ small enough and then an optimization on $|x - y|$ shows that for K large enough, the right hand side is negative. This shows the Hölder estimate.

It turns out that the condition $c_{\alpha}^{\sigma} > 0$ is not satisfied for large values of σ.

Rigorous proof. We use standard techniques from the theory of regularity of viscosity solutions of uniformly elliptic second-order local operators, see [28], adapted to our context.

We argue by contradiction, assuming that u does not belong to $C_x^\alpha(\mathbb{R}^+ \times \mathbb{R}^N)$. Let $u^{\epsilon,\epsilon'}$ and $u_{\epsilon,\epsilon'}$ be respectively the double-parameters sup and inf convolution of u in $\mathbb{R}^+ \times \mathbb{R}^N$, i.e.

$$u^{\epsilon,\epsilon'}(t,x) = \sup_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N} \left(u(s,y) - \frac{1}{2\epsilon}|x-y|^2 - \frac{1}{2\epsilon'}(t-s)^2 \right),$$

$$u_{\epsilon,\epsilon'}(t,x) = \inf_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N} \left(u(s,y) + \frac{1}{2\epsilon}|x-y|^2 + \frac{1}{2\epsilon'}(t-s)^2 \right).$$

Then $u^{\epsilon,\epsilon'}$ is semiconvex and is a subsolution of

$$\partial_t u^{\epsilon,\epsilon'} = \mathcal{I}_1\big[u^{\epsilon,\epsilon'}(t,\cdot)\big] + g_1 \quad \text{in } (t_{\epsilon'}, +\infty) \times \mathbb{R}^N$$

and $u_{\epsilon,\epsilon'}$ is semiconcave and is a supersolution of

$$\partial_t u_{\epsilon,\epsilon'} = \mathcal{I}_1\big[u_{\epsilon,\epsilon'}(t,\cdot)\big] + g_2 \quad \text{in } (t_{\epsilon'}, +\infty) \times \mathbb{R}^N,$$

where $t_{\epsilon'} \to 0$ as $\epsilon' \to 0$, see e.g. Proposition III.2 in [5].

Since u is not Hölder continuous in x, there exists $\alpha \in (0,1)$ such that for any $K > 0$ and $\epsilon, \epsilon' > 0$

$$\sup_{(t,x_1,x_2)\in\mathbb{R}^+\times\mathbb{R}^{2N}} u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(t,x_2) - K|x_1-x_2|^\alpha$$

$$\geqslant \sup_{(t,x_1,x_2)\in\mathbb{R}^+\times\mathbb{R}^{2N}} u(t,x_1) - u(t,x_2) - K|x_1-x_2|^\alpha$$

$$> 0.$$

In order to make the supremum attained at some point, let us introduce smooth positive functions $\psi_1(t)$ and $\psi_2(x)$ with bounded first and second derivatives such that $\psi_1(t) \to +\infty$ as $t \to +\infty$, $\psi_2(x) \to +\infty$ as $|x| \to +\infty$ and there exists $K_0 > 0$ such that $|\psi_2(x)| \leqslant K_0(1 + \sqrt{|x|})$. The last assumption on ψ_2 assures that $\mathcal{I}_1^2[\psi_2]$ is finite at any point. Then, for any $K > 0$ and $\epsilon, \epsilon' > 0$ and $\beta > 0$ small enough, the supremum on $\mathbb{R}^+ \times \mathbb{R}^{2N}$ of the function

$$u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(t,x_2) - \phi(t,x_1,x_2), \tag{A.1}$$

where

$$\phi(t,x_1,x_2) = K|x_1-x_2|^\alpha + \beta\psi_1(t) + \beta\psi_2(x_1),$$

is positive and is attained at some point $(\bar{t}, \bar{x}_1, \bar{x}_2) \in [0, +\infty) \times \mathbb{R}^{2N}$. For ϵ, ϵ' small enough, $\bar{x}_1 \neq \bar{x}_2$. Moreover, since $u^{\epsilon,\epsilon'}(0,x) = u_{\epsilon,\epsilon'}(0,x) = 0$ for any $x \in \mathbb{R}^N$, it turns out that actually $\bar{t} > t_{\epsilon'}$. Remark that

$$|\bar{x}_1 - \bar{x}_2| \leqslant \left(\frac{2 \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^N} |u(t,x)|}{K} \right)^{\frac{1}{\alpha}}. \tag{A.2}$$

The function (A.1) is semiconvex, hence, by Aleksandrov's Theorem, twice differentiable almost everywhere. Let us now introduce a perturbation of it, for which we can choose maximum points of twice differentiability. First we transform $(\bar{t}, \bar{x}_1, \bar{x}_2)$ into a strict maximum point. In order to do that,

we consider a smooth function $h : \mathbb{R}^+ \to \mathbb{R}$, with compact support, such that $h(0) = 0$ and $h(s) > 0$ for $0 < s < 1$ and we set $\theta(t, x_1, x_2) = h((t - \bar{t})^2) + h(|x_1 - \bar{x}_1|^2) + h(|x_2 - \bar{x}_2|^2)$. Clearly $(\bar{t}, \bar{x}_1, \bar{x}_2)$ is a strict maximum point of $u^{\epsilon, \epsilon'}(t, x_1) - u_{\epsilon, \epsilon'}(t, x_2) - \phi(t, x_1, x_2) - \theta(t, x_1, x_2)$. Next we consider a smooth function $\chi : \mathbb{R}^N \to \mathbb{R}$ such that $\chi(x) = 1$ if $|x| \leqslant 1/2$ and $\chi(x) = 0$ for $|x| \geqslant 1$.

By Jensen's Lemma, see e.g. Lemma A.3 of [9], for every small and positive δ there exist $s^\delta \in \mathbb{R}$, $q_1^\delta, q_2^\delta \in \mathbb{R}^N$ with $|s^\delta|, |q_1^\delta|, |q_2^\delta| \leqslant \delta$ such that the function

$$\Phi(t, x_1, x_2) = u^{\epsilon, \epsilon'}(t, x_1) - u_{\epsilon, \epsilon'}(t, x_2) - K|x_1 - x_2|^\alpha - \varphi_1(t, x_1) - \varphi_2(x_2), \tag{A.3}$$

where

$$\varphi_1(t, x_1) = \beta \psi_1(t) + \beta \psi_2(x_1) + h((t - \bar{t})^2) + h(|x_1 - \bar{x}_1|^2) + s^\delta t + \chi(x_1 - \bar{x}_1) q_1^\delta \cdot x_1,$$
$$\varphi_2(x_2) = h(|x_2 - \bar{x}_2|^2) + \chi(x_2 - \bar{x}_2) q_2^\delta \cdot x_2,$$

has a maximum at $(t^\delta, x_1^\delta, x_2^\delta)$, with

$$|t_\delta - \bar{t}|, |x_1^\delta - \bar{x}_1|, |x_2^\delta - \bar{x}_2| \leqslant \delta \tag{A.4}$$

and $u^{\epsilon, \epsilon'}(t, x_1) - u_{\epsilon, \epsilon'}(t, x_2)$ is twice differentiable at $(t^\delta, x_1^\delta, x_2^\delta)$. In particular $u^{\epsilon, \epsilon'}$ is twice differentiable w.r.t. x_1 at (t^δ, x_1^δ) and $u_{\epsilon, \epsilon'}$ is twice differentiable w.r.t. x_2 at (t^δ, x_2^δ). The function χ has been introduced to make $\mathcal{I}_1^2[\varphi_1]$ and $\mathcal{I}_1^2[\varphi_2]$ finite.

For δ small enough, we can assume $x_1^\delta \neq x_2^\delta$ and this will allow us to compute the derivatives of (A.3). Since $(t^\delta, x_1^\delta, x_2^\delta)$ is a maximum point, we have

$$\nabla_{x_1} u^{\epsilon, \epsilon'}(t^\delta, x_1^\delta) = \nabla_{x_1} \varphi_1(t^\delta, x_1^\delta) + \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta),$$
$$\nabla_{x_2} u_{\epsilon, \epsilon'}(t^\delta, x_2^\delta) = -\nabla_{x_2} \varphi_2(x_2^\delta) + \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta). \tag{A.5}$$

Moreover the inequalities

$$\Phi(t^\delta, x_1^\delta + z, x_2^\delta) \leqslant \Phi(t^\delta, x_1^\delta, x_2^\delta),$$
$$\Phi(t^\delta, x_1^\delta, x_2^\delta + z) \leqslant \Phi(t^\delta, x_1^\delta, x_2^\delta),$$
$$\Phi(t^\delta, x_1^\delta + z, x_2^\delta + z) \leqslant \Phi(t^\delta, x_1^\delta, x_2^\delta),$$

for any $z \in \mathbb{R}^N$, with together (A.5), give respectively:

$$u^{\epsilon, \epsilon'}(t^\delta, x_1^\delta + z) - u^{\epsilon, \epsilon'}(t^\delta, x_1^\delta) - \nabla_{x_1} u^{\epsilon, \epsilon'}(t^\delta, x_1^\delta) \cdot z$$
$$\leqslant \varphi_1(t^\delta, x_1^\delta + z) - \varphi_1(t^\delta, x_1^\delta) - \nabla_{x_1} \varphi_1(t^\delta, x_1^\delta) \cdot z$$
$$+ K|x_1^\delta + z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha - \alpha K|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta) \cdot z, \tag{A.6}$$

$$-(u_{\epsilon, \epsilon'}(t^\delta, x_2^\delta + z) - u_{\epsilon, \epsilon'}(t^\delta, x_2^\delta) - \nabla_{x_2} u_{\epsilon, \epsilon'}(t^\delta, x_2^\delta) \cdot z)$$
$$\leqslant \varphi_2(x_2^\delta + z) - \varphi_2(x_2^\delta) - \nabla_{x_2} \varphi_2(x_2^\delta) \cdot z$$
$$+ K|x_1^\delta - z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha + \alpha K|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta) \cdot z, \tag{A.7}$$

and for any $r > 0$

$$u^{\epsilon,\epsilon'}\big(t^\delta, x_1^\delta + z\big) - u^{\epsilon,\epsilon'}\big(t^\delta, x_1^\delta\big) - \nabla_{x_1} u^{\epsilon,\epsilon'}\big(t^\delta, x_1^\delta\big) \cdot z \mathbf{1}_{B_r}(z)$$

$$\leqslant u_{\epsilon,\epsilon'}\big(t^\delta, x_2^\delta + z\big) - u_{\epsilon,\epsilon'}\big(t^\delta, x_2^\delta\big) - \nabla_{x_2} u_{\epsilon,\epsilon'}\big(t^\delta, x_2^\delta\big) \cdot z \mathbf{1}_{B_r}(z)$$

$$+ \varphi_1\big(t^\delta, x_1^\delta + z\big) - \varphi_1\big(t^\delta, x_1^\delta\big) - \nabla_{x_1}\varphi_1\big(t^\delta, x_1^\delta\big) \cdot z \mathbf{1}_{B_r}(z)$$

$$+ \varphi_2\big(x_2^\delta + z\big) - \varphi_2\big(x_2^\delta\big) - \nabla_{x_2}\varphi_2\big(x_2^\delta\big) \cdot z \mathbf{1}_{B_r}(z), \tag{A.8}$$

where $B_r = B_r(0)$. The last inequality in particular implies that

$$\mathcal{I}_1^{2,r}\big[u^{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_1^\delta\big] \leqslant \mathcal{I}_1^{2,r}\big[u_{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_2^\delta\big] + \mathcal{I}_1^{2,r}\big[\varphi_1\big(t^\delta, \cdot\big), x_1^\delta\big] + \mathcal{I}_1^{2,r}\big[\varphi_2, x_2^\delta\big]. \tag{A.9}$$

Next, in order to test, we need to double the time variables. Hence, for $j > 0$, let us consider the maximum point (t^j, x_1^j, s^j, x_2^j) of the function

$$u^{\epsilon,\epsilon'}(t, x_1) - u_{\epsilon,\epsilon'}(s, x_2) - \Psi(t, x_1, x_2) - \frac{j}{2}|t - s|^2,$$

where

$$\Psi(t, x_1, x_2) = K|x_1 - x_2|^\alpha + \varphi_1(t, x_1) + \varphi_2(x_2) + \big|t - t^\delta\big|^2 + \big|x_1 - x_1^\delta\big|^2 + \big|x_2 - x_2^\delta\big|^2,$$

on $Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_1^\delta) \times Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_2^\delta)$, for $\overline{\rho} > 0$ sufficiently small. Standard arguments show that $(t^j, x_1^j, s^j, x_2^j) \to (t^\delta, x_1^\delta, t^\delta, x_2^\delta)$ as $j \to +\infty$. Hence for j large enough there exists $\rho > 0$ such that $Q_{\rho,\rho}(t^j, x_1^j) \times Q_{\rho,\rho}(s^j, x_2^j) \subset Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_1^\delta) \times Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_2^\delta)$ and $x_1^j \neq x_2^j$. Testing, we get

$$j\big(t^j - s^j\big) + 2\big(t^j - t^\delta\big) + \partial_t\varphi_1\big(t^j, x_1^j\big) \leqslant \mathcal{I}_1^{1,\rho}\big[\Psi\big(t^j, \cdot, x_2^j\big), x_1^j\big] + \mathcal{I}_1^{2,\rho}\big[u^{\epsilon,\epsilon'}\big(t^j, \cdot\big), x_1^j\big] + g_1,$$

$$j\big(t^j - s^j\big) \geqslant -\mathcal{I}_1^{1,\rho}\big[\Psi\big(t^j, x_1^j, \cdot\big), x_2^j\big] + \mathcal{I}_1^{2,\rho}\big[u_{\epsilon,\epsilon'}\big(s^j, \cdot\big), x_2^j\big] + g_2.$$

Subtracting the two last inequalities, and then letting $j \to +\infty$, we have

$$\partial_t\varphi_1\big(t^\delta, x_1^\delta\big) \leqslant \mathcal{I}_1^{1,\rho}\big[\Psi\big(t^\delta, \cdot, x_2^\delta\big), x_1^\delta\big] + \mathcal{I}_1^{1,\rho}\big[\Psi\big(t^\delta, x_1^\delta, \cdot\big), x_2^\delta\big]$$

$$+ \mathcal{I}_1^{2,\rho}\big[u^{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_1^\delta\big] - \mathcal{I}_1^{2,\rho}\big[u_{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_2^\delta\big] + g_1 - g_2.$$

Since $u^{\epsilon,\epsilon'}(t^\delta, \cdot)$ and $u_{\epsilon,\epsilon'}(t^\delta, \cdot)$ are twice differentiable respectively at $x_1 = x_1^\delta$ and $x_2 = x_2^\delta$, we can pass to the limit as $\rho \to 0^+$ and obtain

$$\partial_t\varphi_1\big(t^\delta, x_1^\delta\big) \leqslant \mathcal{I}_1\big[u^{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_1^\delta\big] - \mathcal{I}_1\big[u_{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_2^\delta\big] + g_1 - g_2.$$

Using (A.9), we finally get

$$\partial_t\varphi_1\big(t^\delta, x_1^\delta\big) \leqslant \mathcal{I}_1^{1,r}\big[u^{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_1^\delta\big] - \mathcal{I}_1^{1,r}\big[u_{\epsilon,\epsilon'}\big(t^\delta, \cdot\big), x_2^\delta\big]$$

$$+ \mathcal{I}_1^{2,r}\big[\varphi_1\big(t^\delta, \cdot\big), x_1^\delta\big] + \mathcal{I}_1^{2,r}\big[\varphi_2, x_2^\delta\big] + g_1 - g_2. \tag{A.10}$$

Next, let us estimate the term $\mathcal{I}_1^{1,r}[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] - \mathcal{I}_1^{1,r}[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta]$ and show that it contains a main negative part. For $0 < \nu_0 < 1$, let us denote

$$A_r := \big\{z \in B_r(0),\ \big|z \cdot \big(x_1^\delta - x_2^\delta\big)\big| \geqslant \nu_0 |z| \big|x_1^\delta - x_2^\delta\big|\big\}.$$

Then

$$
\mathcal{I}_1^{1,r}\big[u^{\epsilon,\epsilon'}\big(t^\delta,\cdot\big),x_1^\delta\big] - \mathcal{I}_1^{1,r}\big[u_{\epsilon,\epsilon'}\big(t^\delta,\cdot\big),x_2^\delta\big]
$$

$$
= \int\limits_{A_r} \big[u^{\epsilon,\epsilon'}\big(t^\delta,x_1^\delta + z\big) - u^{\epsilon,\epsilon'}\big(t^\delta,x_1^\delta\big) - \nabla_{x_1} u^{\epsilon,\epsilon'}\big(t^\delta,x_1^\delta\big)\cdot z
$$

$$
- \big(u_{\epsilon,\epsilon'}\big(t^\delta,x_2^\delta + z\big) - u_{\epsilon,\epsilon'}\big(t^\delta,x_2^\delta\big) - \nabla_{x_2} u_{\epsilon,\epsilon'}\big(t^\delta,x_2^\delta\big)\cdot z\big)\big]\mu(dz)
$$

$$
+ \int\limits_{B_r\backslash A_r} [\ldots]\mu(dz)
$$

$$
= T_1 + T_2.
$$

From (A.8) we have

$$
T_2 \leqslant C.
$$

Here and henceforth C denotes various positive constants independent of the parameters. Let us estimate T_1. Using (A.6) and (A.7), and successively making the change of variable $z \to -z$, we get the following estimate of T_1:

$$
T_1 \leqslant \int\limits_{A_r} \big[K\big|x_1^\delta + z - x_2^\delta\big|^\alpha - K\big|x_1^\delta - x_2^\delta\big|^\alpha - \alpha K\big|x_1^\delta - x_2^\delta\big|^{\alpha-2}\big(x_1^\delta - x_2^\delta\big)\cdot z\big]\mu(dz) + C
$$

$$
+ \int\limits_{A_r} \big[K\big|x_1^\delta - z - x_2^\delta\big|^\alpha - K\big|x_1^\delta - x_2^\delta\big|^\alpha + \alpha K\big|x_1^\delta - x_2^\delta\big|^{\alpha-2}\big(x_1^\delta - x_2^\delta\big)\cdot z\big]\mu(dz)
$$

$$
= 2\int\limits_{A_r} \big[K\big|x_1^\delta + z - x_2^\delta\big|^\alpha - K\big|x_1^\delta - x_2^\delta\big|^\alpha - \alpha K\big|x_1^\delta - x_2^\delta\big|^{\alpha-2}\big(x_1^\delta - x_2^\delta\big)\cdot z\big]\mu(dz) + C
$$

$$
\leqslant \alpha K \int\limits_{A_r} \sup_{|t|\leqslant 1}\big\{\big|x_1^\delta - x_2^\delta + tz\big|^{\alpha-4}\big(\big|x_1^\delta - x_2^\delta + tz\big|^2|z|^2 - (2-\alpha)\big[\big(x_1^\delta - x_2^\delta + tz\big)\cdot z\big]^2\big)\big\}\mu(dz) + C.
$$

Let us fix $r = \sigma|x_1^\delta - x_2^\delta|$, $\sigma > 0$, then for $z \in A_r$

$$
\big|x_1^\delta - x_2^\delta + tz\big| \leqslant (1+\sigma)\big|x_1^\delta - x_2^\delta\big|,
$$

$$
\big|\big(x_1^\delta - x_2^\delta + tz\big)\cdot z\big| \geqslant \big|\big(x_1^\delta - x_2^\delta\big)\cdot z\big| - |z|^2 \geqslant (\nu_0 - \sigma)\big|x_1^\delta - x_2^\delta\big||z|.
$$

Let us choose $0 < \sigma < \nu_0 < 1$ such that

$$
C_0 := -(1+\sigma)^2 + (2-\alpha)(\nu_0 - \sigma)^2 > 0,
$$

then

$$
T_1 \leqslant -CC_0 K\big|x_1^\delta - x_2^\delta\big|^{\alpha-2}\int\limits_{A_r} |z|^2\mu(dz) + C.
$$

By homogeneity

$$\int\limits_{A_r} |z|^2 \mu(dz) = Cr.$$

Then, we conclude

$$T_1 \leqslant -CC_0 K \big|x_1^\delta - x_2^\delta\big|^{\alpha-2} r + C \leqslant -CC_0 K \big|x_1^\delta - x_2^\delta\big|^{\alpha-1} + C,$$

and from (A.10)

$$CC_0 K \big|x_1^\delta - x_2^\delta\big|^{\alpha-1} \leqslant -\partial_t \varphi_1\big(t^\delta, x_1^\delta\big) + g_1 - g_2 + C$$
$$+ \mathcal{I}_1^{2,r}\big[\varphi_1(t^\delta, \cdot), x_1^\delta\big] + \mathcal{I}_1^{2,r}\big[\varphi_2, x_2^\delta\big]$$
$$\leqslant g_1 - g_2 + C.$$

Letting δ go to 0, from the previous inequalities and (A.4) we finally obtain

$$K|\bar{x}_1 - \bar{x}_2|^{\alpha-1} \leqslant C,$$

where C is independent of K. This is a contradiction for K large enough, because of (A.2), hence $u \in C_x^\alpha(\mathbb{R}^+ \times \mathbb{R}^N)$. $\quad\square$

Proof of Proposition 6.2. Let us define the functions $V_1(\tau, Y) := e^{-k\tau} U_1(\tau, Y)$ and $V_2(\tau, Y) := e^{-k\tau} U_2(\tau, Y)$, where $k := \|W''\|_\infty + 1$. It is easy to see that V_1 and V_2 are respectively sub and supersolution of

$$\begin{cases} \partial_\tau V = Le^{-k\tau} + \mathcal{I}_1\big[V(\tau, \cdot, y_{N+1})\big] + g(\tau, Y, V) \\ \qquad + \eta\Big[a_0 + e^{k\tau}\Big(\inf\limits_{Y'} V(\tau, Y') - V(\tau, Y)\Big)\Big]\big|\partial_{y_{N+1}} V + e^{-k\tau}\big| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \quad (A.11)$$

where $g(\tau, Y, V) = -e^{-k\tau} W'(e^{k\tau} V + P \cdot Y) - kV + e^{-k\tau}\sigma(\tau, y)$. Remark that, by the choice of k,

$$g(\tau, Y, V_1) - g(\tau, Z, V_2) \leqslant -(V_1 - V_2) + e^{-k\tau}\big(\|W''\|_\infty |P| + \|\sigma'\|_\infty\big)|Y - Z|. \quad (A.12)$$

To prove the comparison between U_1 and U_2, it suffices to show that $V_1(\tau, Y) \leqslant V_2(\tau, Y)$ for all $(\tau, Y) \in (0, T) \times \mathbb{R}^{N+1}$ and for any $T > 0$.

Suppose by contradiction that $M = \sup_{(\tau, Y) \in (0, T) \times \mathbb{R}^{N+1}} (V_1(\tau, Y) - V_2(\tau, Y)) > 0$ for some $T > 0$. Define for small $\nu_1, \nu_2, \beta, \delta > 0$ the function $\phi \in C^2((\mathbb{R}^+ \times \mathbb{R}^{N+1})^2)$ by

$$\phi(\tau, Y, s, Z) = \frac{1}{2\nu_1}|\tau - s|^2 + \frac{1}{2\nu_2}|Y - Z|^2 + \beta\psi(Y) + \frac{\delta}{T - \tau},$$

where ψ is defined as the function ψ_2 in the proof of Proposition 4.7. The supremum of $V_1(\tau, Y) - V_2(s, Z) - \phi(\tau, Y, s, Z)$ is attained at some point $(\bar{\tau}, \bar{Y}, \bar{s}, \bar{Z}) \in ((0, T) \times \mathbb{R}^{N+1})^2$. Standard arguments show that, because U_1 and U_2 are assumed bounded

$$(\overline{\tau}, \overline{Y}, \overline{s}, \overline{Z}) \to (\widehat{\tau}, \widehat{\tau}, \widehat{Y}, \widehat{Z}) \quad \text{as } \nu_1 \to 0,$$

$$V_1(\overline{\tau}, \overline{Y}) \to V_1(\widehat{\tau}, \widehat{Y}), \qquad V_2(\overline{s}, \overline{Z}) \to V_2(\widehat{\tau}, \widehat{Z}) \quad \text{as } \nu_1 \to 0,$$

where $(\widehat{\tau}, \widehat{Y}, \widehat{Z})$ is a maximum point of $V_1(\tau, Y) - V_2(\tau, Z) - \frac{1}{2\nu_2}|Y - Z|^2 - \beta\psi(Y) - \frac{\eta}{T-\tau}$. Moreover, it is easy to see that

$$\limsup_{\nu_1 \to 0} \inf_{Y'} V_1(\overline{\tau}, Y') \leqslant \inf_{Y'} V_1(\widehat{\tau}, Y'), \qquad \liminf_{\nu_1 \to 0} \inf_{Y'} V_2(\overline{s}, Y') \geqslant \inf_{Y'} V_2(\widehat{\tau}, Y').$$

Since V_1 and V_2 are respectively sub and supersolution of (A.11), for any $r > 0$ we have

$$\frac{\delta}{(T - \overline{\tau})^2} + \frac{\overline{\tau} - \overline{s}}{\nu_1}$$

$$\leqslant Le^{-k\overline{\tau}} + \frac{C_N r}{\nu_2} + \beta \mathcal{I}_1^{1,r}\big[\psi(\cdot, \overline{y}_{N+1}), \overline{y}\big] + \mathcal{I}_1^{2,r}\big[V_1(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] + g\big(\overline{\tau}, \overline{Y}, V_1(\overline{\tau}, \overline{Y})\big)$$

$$+ \eta\bigg[a_0 + e^{k\overline{\tau}}\Big(\inf_{Y'} V_1(\overline{\tau}, Y') - V_1(\overline{\tau}, \overline{Y})\Big)\bigg]\bigg|\frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{\nu_2} + \beta\partial_{y_{N+1}}\psi(\overline{Y}) + e^{-k\overline{\tau}}\bigg| \quad \text{(A.13)}$$

and

$$\frac{\overline{\tau} - \overline{s}}{\nu_1} \geqslant Le^{-k\overline{s}} - \frac{C_N r}{\nu_2} + \mathcal{I}_1^{2,r}\big[V_2(\overline{s}, \cdot, \overline{z}_{N+1}), \overline{z}\big] + g\big(\overline{s}, \overline{Z}, V_2(\overline{s}, \overline{Z})\big)$$

$$+ \eta\bigg[a_0 + e^{k\overline{s}}\Big(\inf_{Y'} V_2(\overline{s}, Y') - V_2(\overline{s}, \overline{Z})\Big)\bigg]\bigg|\frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{\nu_2} + e^{-k\overline{s}}\bigg|, \quad \text{(A.14)}$$

where C_N is a constant depending on the dimension N. Since $(\overline{\tau}, \overline{Y}, \overline{s}, \overline{Z})$ is a maximum point, we have

$$V_1(\overline{\tau}, \overline{y} + x, \overline{y}_{N+1}) - V_1(\overline{\tau}, \overline{Y}) \leqslant V_2(\overline{s}, \overline{z} + x, \overline{z}_{N+1}) - V_2(\overline{s}, \overline{Z}) + \beta\big[\psi(\overline{y} + x, \overline{y}_{N+1}) - \psi(\overline{Y})\big],$$

for any $x \in \mathbb{R}^N$, which implies that for any $r > 0$

$$\mathcal{I}_1^{2,r}\big[V_1(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}\big] \leqslant \mathcal{I}_1^{2,r}\big[V_2(\overline{s}, \cdot, \overline{z}_{N+1}), \overline{z}\big] + \beta\mathcal{I}_1^{2,r}\big[\psi(\cdot, \overline{y}_{N+1}), \overline{y}\big].$$

Then, subtracting (A.13) with (A.14) and letting $r \to 0^+$, we get

$$\frac{\delta}{(T - \overline{\tau})^2} \leqslant L\big(e^{-k\overline{\tau}} - e^{-k\overline{s}}\big) + \beta\mathcal{I}_1\big[\psi(\cdot, \overline{y}_{N+1}), \overline{y}\big] + g\big(\overline{\tau}, \overline{Y}, V_1(\overline{\tau}, \overline{Y})\big) - g\big(\overline{s}, \overline{Z}, V_2(\overline{s}, \overline{Z})\big)$$

$$+ \eta\bigg[a_0 + e^{k\overline{\tau}}\Big(\inf_{Y'} V_1(\overline{\tau}, Y') - V_1(\overline{\tau}, \overline{Y})\Big)\bigg]\bigg|\frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{\nu_2} + \beta\partial_{y_{N+1}}\psi(\overline{Y}) + e^{-k\overline{\tau}}\bigg|$$

$$- \eta\bigg[a_0 + e^{k\overline{s}}\Big(\inf_{Y'} V_2(\overline{s}, Y') - V_2(\overline{s}, \overline{Z})\Big)\bigg]\bigg|\frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{\nu_2} + e^{-k\overline{s}}\bigg|.$$

Next, letting $\nu_1 \to 0$ and using (A.12), we obtain

$$
\frac{\delta}{(T - \widehat{\tau})^2}
$$
$$
\leqslant \beta \mathcal{I}_1 \big[\psi(\cdot, \widehat{y}_{N+1}), \widehat{y} \big] - \big(V_1(\widehat{\tau}, \widehat{Y}) - V_2(\widehat{\tau}, \widehat{Z}) \big) + e^{-k\widehat{\tau}} \big(\|W''\|_\infty |P| + \|\sigma'\|_\infty \big) |\widehat{Y} - \widehat{Z}| + C\beta
$$
$$
+ \eta e^{k\widehat{\tau}} \Big[\inf_{Y'} V_1(\widehat{\tau}, Y') - \inf_{Y'} V_2(\widehat{\tau}, Y') - \big(V_1(\widehat{\tau}, \widehat{Y}) - V_2(\widehat{\tau}, \widehat{Z}) \big) \Big] \Big| \frac{\widehat{y}_{N+1} - \widehat{z}_{N+1}}{\nu_2} + e^{-k\widehat{\tau}} \Big|.
$$

$$(A.15)$$

It is easy to prove that

$$
\liminf_{(\beta, \delta) \to (0,0)} \big(V_1(\widehat{\tau}, \widehat{Y}) - V_2(\widehat{\tau}, \widehat{Z}) \big) \geqslant M \tag{A.16}
$$

and

$$
\frac{|\widehat{Y} - \widehat{Z}|^2}{\nu_2} \leqslant C,
$$

where C is independent of β and δ. Up to subsequence, $\widehat{\tau} \to \tau_0 \in [0, T]$ as $(\beta, \delta) \to (0, 0)$ and by (A.16), we have

$$
\limsup_{(\beta, \delta) \to (0,0)} \Big[\inf_{Y'} V_1(\widehat{\tau}, Y') - \inf_{Y'} V_2(\widehat{\tau}, Y') - \big(V_1(\widehat{\tau}, \widehat{Y}) - V_2(\widehat{\tau}, \widehat{Z}) \big) \Big]
$$
$$
\leqslant \inf_{Y'} V(\tau_0, Y') - \inf_{Y'} V_2(\tau_0, Y') - \sup_{Y'} \big(V_1(\tau_0, Y') - V_2(\tau_0, Y') \big)
$$
$$
\leqslant 0.
$$

Then, passing to the limit first as $(\beta, \delta) \to (0, 0)$ and then as $\nu_2 \to 0$ in (A.15) we finally get the contradiction:

$$
M \leqslant 0,
$$

and this concludes the proof of the comparison theorem. $\quad\square$

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