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ORIGINAL ARTICLE

The Rough Intuitionistic Fuzzy Ideals of Intuitionistic Fuzzy Subrings in a Commutative Ring

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Abstract The aim of this paper is to give some definitions of rough intuitionistic fuzzy ideal, rough intuitionistic fuzzy radical, rough prime (primary) intuitionistic fuzzy ideal and rough semiprime intuitionistic fuzzy ideal of an intuitionistic fuzzy subring, and also to give some properties of such ideals. Moreover, we give their nature under homomorphism.

Keywords Intuitionistic fuzzy set · Upper and lower approximations · Rough intuitionistic fuzzy subring · Rough intuitionistic fuzzy radical · Rough (prime, primary and semiprime) intuitionistic fuzzy ideal

1. Introduction

The notion of rough sets was introduced by Pawlak [1] and is motivated by practical needs in classification and concept formation under insufficient and incomplete information. There are mainly two generalized methods, for the development of this theory, the constructive method and the algebraic method. By adopting these two methods the rough set theory has been combined with other mathematical theories

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such as model logic [2], lattice [3, 4], group theory [5], ring theory [6-8], fuzzy sets [9-11] and intuitionistic fuzzy sets (IF-sets for short) [12]. Among these research aspects, more efforts have been focused on the approximations of fuzzy sets and IF-sets.

After the introduction of fuzzy sets by Zadeh [13], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets, introduced by Atanassov [14-16] is one among them. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give the degree of membership and a degree of non-membership. Furthermore the sum of two degrees is not greater than 1.

In this paper, we study the rough intuitionistic fuzzy ideals (IF-ideals for short) of intuitionistic fuzzy subrings (IF-subrings for short) in a commutative ring in details.

The remainder of this paper is organized as follows: We review some results and definitions about rough sets and IF-sets in Sections 2 and 3. In Section 4, the concept of θ-lower and θ-upper approximations of IF-sets in a ring is introduced and its basic properties are discussed. The relations of rough IF-sets under the intuitionistic intrinsic product, intuitionistic sum and intuitionistic product are investigated respectively. A new definition of rough IF-ideal of IF-subrings is proposed in Section 5. In Section 6, we introduce the concept of rough IF-radicals of rough IF-ideals in IF-subrings. The definition of rough prime (primary) and semiprime IF-ideals, and their nature under homomorphism is elaborated in Section 7. Finally, a conclusion is stated at the end.

2. Basic Results on Rings, Congruence Relations and Rough Sets

In this section, some definitions and results of congruence relations and rough sets are discussed. For more details see [8].

A ring R is an algebraic structure $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations '+' and '·' (called addition and multiplication) such that $(R, +)$ is an abelian group, $(R, \cdot)$ is a semi-group and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in \mathrm{R}$.

If the multiplication is commutative, then R is said to be a commutative ring and R is said to have an identity say 1 if $a \cdot 1 = 1 \cdot a = a$, for all $a \in \mathrm{R}$. For the sake of simplicity, we shall write $ab = a \cdot b$ $(a, b \in \mathrm{R})$.

Let θ be an equivalence relation on a ring R. For $x \in \mathrm{R}$, the equivalence class (or coset) of x modulo θ is the set $[x]_\theta = \{y \in \mathrm{R} \mid (x, y) \in \theta\}$. In the rest of the paper R is a commutative ring with identity.

Definition 2.1 [8] *Let θ be an equivalence relation on R. Then θ is called a full congruence relation if $(a, b) \in \theta$ implies $(a + x, b + x) \in \theta$ and $(ax, bx) \in \theta$ for all $x \in R$.*

Theorem 2.1 [8] *Let θ be an equivalence relation on R. Then $(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a + c, b + d) \in \theta$, $(ac, bd) \in \theta$ and $(-a, -b) \in \theta$ for all $a, b, c, d \in R$.*

Lemma 2.1 [8] *Let θ be an equivalence relation on R. If $a, b \in R$, then*

(i) $[a]_\theta + [b]_\theta = [a + b]_\theta$.

(ii) $[-a]_\theta = -[a]_\theta$.

(iii) $\{xy \mid x \in [a]_\theta, y \in [b]_\theta\} \subseteq [ab]_\theta$.

If X and Y are non-empty subsets of R, let XY denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \in N, a_i \in X, b_i \in Y\}$.

Definition 2.2 [8] *A full congruence relation θ on R is called complete if $[ab]_\theta = \{xy \mid x \in [a]_\theta, y \in [b]_\theta\}$ for all $a, b \in R$.*

Definition 2.3 [8] *Let θ be a full congruence relation on R and X a subset of R. Then the sets*

$$\underline{\theta}(X) = \{x \in R \mid [x]_\theta \subseteq X\} \text{ and } \overline{\theta}(X) = \{x \in R \mid [x]_\theta \cap X\} \neq \emptyset$$

are called, respectively, the θ-lower and θ-upper approximations of the set X. $\theta(X) = (\underline{\theta}(X), \overline{\theta}(X))$ is called a rough set with respect to θ if $\underline{\theta}(X) \neq \overline{\theta}(X)$.

The following proposition is well known and easily seen.

Proposition 2.1 *For every subsets $X, Y \subseteq R$, we have*

(i) $\underline{\theta}(X) \subseteq X \subseteq \overline{\theta}(X)$.

(ii) *If $X \subseteq Y$, then $\underline{\theta}(X) \subseteq \underline{\theta}(Y)$ and $\overline{\theta}(X) \subseteq \overline{\theta}(Y)$.*

(iii) $\underline{\theta}(X \cap Y) = \underline{\theta}(X) \cap \underline{\theta}(Y)$ *and* $\underline{\theta}(X \cup Y) \supseteq \underline{\theta}(X) \cup \underline{\theta}(Y)$.

(iv) $\overline{\theta}(X \cup Y) = \overline{\theta}(X) \cup \overline{\theta}(Y)$ *and* $\overline{\theta}(X \cup Y) \subseteq \overline{\theta}(X) \cup \overline{\theta}(Y)$.

(v) $\underline{\theta}(\underline{\theta}(X)) = \underline{\theta}(X)$ *and* $\overline{\theta}(\overline{\theta}(X)) = \overline{\theta}(X)$.

A non-empty subset I of a ring R is an ideal if and only if for $a, b \in R$ and $r \in R$:

(i) $a, b \in I \Rightarrow a - b \in I$,

(ii) $a \in I, r \in R \Rightarrow ra \in I$.

Theorem 2.2 [8] *Let θ be a full congruence relation on R. If I is an ideal of R, then $\overline{\theta}(I)$ is an ideal of R.*

Theorem 2.3 [8] *Let θ be a full congruence relation on R and I be an ideal of R. If $\underline{\theta}(I)$ is a non-empty set, then it is equal to R.*

An ideal $P(\neq R)$ is said to be prime in a ring R, if for all $a, b \in R$, the condition $ab \in P$ implies that either $a \in P$ or $b \in P$.

An ideal $Q(\neq R)$ in a ring R is primary if for any $a, b \in R$: $a, b \in Q$ and $a \notin Q \Rightarrow b^n \in Q$ for some positive integer n.

Theorem 2.4 [8] *Let θ be a complete congruence relation on R and P be a prime (primary) ideal of R such that $\overline{\theta}(P) \neq R$. Then $\overline{\theta}(P)$ is a prime (primary) ideal of R.*

Theorem 2.5 [8] *Let θ be a full congruence relation on R and P be a prime (primary) ideal of R. If $\underline{\theta}(P)$ is a non-empty set, then $\underline{\theta}(P)$ is a prime (primary) ideal of R.*

An ideal S of R is said to be semiprime if, whenever $a^n \in S$ for some positive integer n, then $a \in S$.

Theorem 2.6 *Let θ be a complete congruence relation on R and P be a semiprime ideal of R such that $\overline{\theta}(P) \neq R$. Then $\overline{\theta}(P)$ is a semiprime ideal of R.*

Proof Since P is an ideal of R. Then by Theorem 2.2, we know that $\overline{\theta}(P)$ is an ideal of R. Now let $a^n \in \overline{\theta}(P)$ for some $a \in R$ and for some positive integer n, then $[a^n]_\theta \cap P \neq \emptyset$. Since θ is complete, so $\{u^n \mid u \in [a]_\theta\} \cap P \neq \emptyset$, and so there exists $u \in [a]_\theta$ such that $u^n \in P$, for some positive integer n. Since P is a semiprime ideal, we have $u \in P$. So $[a]_\theta \cap P \neq \emptyset$. Hence $a \in \overline{\theta}(P)$. Therefore $\overline{\theta}(P)$ is a semiprime ideal of R.

Theorem 2.7 *Let θ be a full congruence relation on R and P be a semiprime ideal of R. If $\underline{\theta}(P)$ is a non-empty set, then $\underline{\theta}(P)$ is a semiprime ideal of R.*

Proof Analogous to the proof of Theorem 2.6.

Theorem 2.8 [8] *Let f be an epimorphism (an onto homomorphism) of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Then*

(i) $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (f(a), f(b)) \in \theta_2\}$ *is a full congruence relation on R_1.*

(ii) *If θ_2 is complete and f is one-to-one, then θ_1 is complete.*

(iii) $f(\overline{\theta}_1(X)) = \overline{\theta}_2(f(X))$.

(iv) $f(\underline{\theta}_1(X)) \subseteq \underline{\theta}_2(f(X))$. *If f is one-to-one, then $f(\underline{\theta}_1(X)) = \underline{\theta}_2(f(X))$.*

Theorem 2.9 [8] *Let f be an epimorphism (an onto homomorphism) of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Let P be a subset of R_1. If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (f(a), f(b)) \in \theta_2\}$, then*

(i) *$\overline{\theta}_1(P)$ is an ideal of R_1 if and only if $\overline{\theta}_2(f(P))$ is an ideal of R_2.*

(ii) *If θ_2 is complete, $\overline{\theta}_1(P)$ is a prime ideal of R_1 if and only if $\overline{\theta}_2(f(P))$ is a prime ideal of R_2.*

(iii) *If θ_2 is complete, $\overline{\theta}_1(P)$ is a primary ideal of R_1 if and only if $\overline{\theta}_2(f(P))$ is a primary ideal of R_2.*

For semiprime ideal, we have the following result:

(iv) *If θ_2 is complete, $\overline{\theta}_1(P)$ is a semiprime ideal of R_1 if and only if $\overline{\theta}_2(f(P))$ is a semiprime ideal of R_2.*

Proof (iv) Assume that $\overline{\theta}_1(P)$ is a semiprime ideal of R_1. Suppose that for some $x \in R_2$ and some positive integer n, such that $x^n \in \overline{\theta}_2(f(P))$. Then there exists $a \in R_1$ such that $f(a) = x$. Thus $[f(a^n)]_{\theta_2} \cap f(P) \neq \emptyset$. Since θ_2 is complete, so $\{f(u^n) \mid f(u) \in [f(a)]_{\theta_2}\} \cap f(P) \neq \emptyset$, and so there exists $f(u) \in [f(a)]_{\theta_2}$ such that $f(u^n) \in f(P)$ for some positive integer n. Then we have $u \in [a]_{\theta_1}$, and there exists $c \in P$ such that $f(c) = f(u^n)$. Hence $u^n \in [a^n]_{\theta_1}$ and $c \in [u^n]_{\theta_1}$. So $c \in [a^n]_{\theta_1}$.

Thus $[a^n]_{\theta_1} \cap P \neq \emptyset$ which implies that $a^n \in \overline{\theta_1}(P)$. Since $\overline{\theta}_1(P)$ is a semiprime ideal of R_1, we have $a \in \overline{\theta}_1(P)$. From this and Theorem 2.8, we obtain that $x = f(a) \in f(\overline{\theta}_1(P)) = \overline{\theta}_2(f(P))$. This means that $\overline{\theta}_2(f(P))$ is a semiprime ideal of R_2. Conversely, assume that $\overline{\theta}_2(f(P))$ is a semiprime ideal of R_2. Suppose that for some $x \in R_1$ and some positive integer n, such that $x \in \overline{\theta}_1(P)$. Then by Thorem 2.8, we obtain that $f(x^n) \in f(\overline{\theta}_1(P)) = \overline{\theta}_2(f(P))$. Since $\overline{\theta}_2(f(P))$ is a semiprime ideal, we have $f(x) \in f(\overline{\theta}_1(P))$. If $f(x) \in f(\overline{\theta}_1(P))$, then there exists $a \in \overline{\theta}_1(P)$ such that $f(x) = f(a)$. Thus $[a]_{\theta_1} \cap P \neq \emptyset$ and $x \in [a]_{\theta_1}$. So $[x]_{\theta_1} \cap P \neq \emptyset$ which implies that $x \in \overline{\theta}_1(P)$. This means that $\overline{\theta}_1(P)$ is a semiprime ideal of R_1.

Theorem 2.10 [8] *Let f be an isomorphism of a ring R_1 to a ring R_2 and let θ_2 be a complete congruence relation on R_2. Let P be a subset of R_1. If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (f(a), f(b)) \in \theta_2\}$, then*

(i) *$\underline{\theta}_1(P)$ is an ideal of R_1 if and only if $\underline{\theta}_2(f(P))$ is an ideal of R_2.*

(ii) *$\underline{\theta}_1(P)$ is a prime ideal of R_1 if and only if $\underline{\theta}_2(f(P))$ is a prime ideal of R_2.*

(iii) *$\underline{\theta}_1(P)$ is a primary ideal of R_1 if and only if $\underline{\theta}_2(f(P))$ is a primary ideal of R_2.*

For semiprime ideal we have the following result:

(iv) *$\underline{\theta}_1(P)$ is a semiprime ideal of R_1 if and only if $\underline{\theta}_2(f(P))$ is a semiprime ideal of R_2.*

Proof We prove only part (iv). By Theorem 2.8, we have $f(\underline{\theta}_1(X)) = \underline{\theta}_2(f(X))$. Now, remaining part is analogous by Theorem 2.9.

3. Basic Results on Intuitionistic Fuzzy Sets

A fuzzy set of R is a function $\mu : R \longrightarrow [0, 1]$. Let μ be a fuzzy set of R. For $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in R \mid \mu(x) \geq \alpha\}$ is called an α-level cut set of μ.

Let μ and λ be two fuzzy sets on X. Then μ is called a fuzzy subset of λ if $\mu(x) \leq \lambda(x)$ for all $x \in X$ and it is written as $\mu \subseteq \lambda$.

A fuzzy set μ in a ring R is called a fuzzy subring of R [17] if it satisfies:

(i) $(\forall x, y \in R)$ $(\mu(x - y) \geq \mu(x) \wedge \mu(y))$ and

(ii) $(\forall x, y \in R)$ $(\mu(xy) \geq \mu(x) \wedge \mu(y))$.

A fuzzy set μ in a ring R is called a fuzzy ideal of R [17] if it satisfies:

(i) $(\forall x, y \in R)$ $(\mu(x - y) \geq \mu(x) \wedge \mu(y))$ and

(ii) $(\forall x, y \in R)$ $(\mu(xy) \geq \mu(x))$.

An intuitionistic fuzzy set (IF-set for short) A in a ring R is an object having the form

$$A = \{< x, \mu_A(x), \nu_A(x) >\mid x \in R\},$$

where the functions $\mu_A : R \longrightarrow [0,1]$ and $\nu_A : R \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in R$ for the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in R$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IF-set $A = \{< x, \mu_A(x), \nu_A(x) >| x \in R\}$. Let $IFS(R)$ be the set of all IF-sets of R.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. Then

(i) $A \subseteq B \Longleftrightarrow (\forall x \in X)\ (\mu_A(x) \le \mu_B(x), \nu_A(x) \ge \nu_B(x))$.

(ii) $A = B \Longleftrightarrow A \subseteq B$ and $B \subseteq A$.

(iii) $A \cup B = \{\mu_A \vee \mu_B, \nu_A \wedge \nu_B\}$.

(iv) $A \cap B = \{\mu_A \wedge \mu_B, \nu_A \vee \nu_B\}$.

(v) $0_\sim = (0,1)$ and $1_\sim = (1,0)$.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. Then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy subset (IF-subset for short) of $B = (\mu_B, \nu_B)$ if $A \subseteq B$.

Let f be a mapping from a ring R_1 to a ring R_2. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in R_1 and R_2 respectively. Then the pre-image of $B = (\mu_B, \nu_B)$ under f is defined to be an IF-set $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$, where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$ for all $x \in R_1$, and the image of $A = (\mu_A, \nu_A)$ under f is defined to be an IF-set $f(A) = (\mu_{f(A)}, \nu_{f(A)})$, where

$$\mu_{f(A)}(y) = \begin{cases} \bigvee_{y=f(x)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{f(A)}(y) = \begin{cases} \bigwedge_{y=f(x)} \nu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in R_2$.

Definition 3.1 [18] *An IF-set $A = (\mu_A, \nu_A)$ in a ring R is called an IF-subring of R if it satisfies the following conditions:*

(i) $(\forall x, y \in R)\ (\mu_A(x - y) \ge \mu_A(x) \wedge \mu_A(y))$.

(ii) $(\forall x, y \in R)\ (\mu_A(xy) \ge \mu_A(x) \wedge \mu_A(y))$.

(iii) $(\forall x, y \in R)\ (\nu_A(x - y) \le \nu_A(x) \vee \nu_A(y))$.

(iv) $(\forall x, y \in R)\ (\nu_A(xy) \le \nu_A(x) \vee \nu_A(y))$.

Definition 3.2 [18] *An IF-set $A = (\mu_A, \nu_A)$ in a ring R is called an IF-ideal of R if it satisfies the following conditions:*

(i) $(\forall x, y \in R)\ (\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y))$.

(ii) $(\forall x, y \in R)\ (\mu_A(xy) \geq \mu_A(y))$.

(iii) $(\forall x, y \in R)\ (\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y))$.

(iv) $(\forall x, y \in R)\ (\nu_A(xy) \leq \nu_A(y))$.

Definition 3.3 [15, 16] *Let $A = (\mu_A, \nu_A)$ be an IF-set in a ring R and $\alpha,\ \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. The $(\alpha,\ \beta)$-level cut $A^{(\alpha,\ \beta)}$ will be $\{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$. Analogously, the $(\alpha,\ \beta)$-level strong cut $A_s^{(\alpha,\ \beta)}$ will be $\{x \in R \mid \mu_A(x) > \alpha, \nu_A(x) < \beta\}$ and finally $A^{(0,\ 0)} = \{x \in R \mid \mu_A(x) = 0,\ \nu_A(x) = 0\}$.*

Theorem 3.1 *Let $A = (\mu_A, \nu_A)$ be an IF-set of R. Then $A = (\mu_A, \nu_A)$ is an IF-subring of R if and only if $A^{(\alpha,\ \beta)}$ and $A_s^{(\alpha,\ \beta)}$ are, if they are non-empty, subrings of R for $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$.*

Proof See Theorem 5.6 [19].

It is easy to prove that Theorem 3.1 is also valid if we replace IF-subring by IF-ideal.

4. Rough IF-subrings and IF-ideals

In this section, let θ be a full congruence relation on a ring R, and $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets of R and N denote the set of all positive integers.

Definition 4.1 *Let θ be a full congruence relation on R, and $A = (\mu_A, \nu_A)$ be an IF-set of R. Then we define the fuzzy sets $\underline{\theta}(\mu_A)$, $\overline{\theta}(\mu_A)$, $\underline{\theta}(\nu_A)$, $\overline{\theta}(\nu_A)$ as follows:*

$$\underline{\theta}(\mu_A)(x) = \bigwedge_{a \in [x]_\theta} \mu_A(a) \ \ and\ \ \overline{\theta}(\mu_A)(x) = \bigvee_{a \in [x]_\theta} \mu_A(a),$$

$$\underline{\theta}(\nu_A)(x) = \bigvee_{a \in [x]_\theta} \nu_A(a) \ \ and\ \ \overline{\theta}(\nu_A)(x) = \bigwedge_{a \in [x]_\theta} \nu_A(a).$$

The fuzzy sets $\underline{\theta}(\mu_A)$ and $\overline{\theta}(\mu_A)$ are called, respectively, the θ-lower and θ-upper approximations of the fuzzy set μ_A; and the fuzzy sets $\underline{\theta}(\nu_A)$ and $\overline{\theta}(\nu_A)$ are called, respectively, the θ-lower and θ-upper approximations of the fuzzy set ν_A; and the IF-sets $\underline{\theta}(A) = (\underline{\theta}(\mu_A),\ \underline{\theta}(\nu_A))$ and $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$ are called, respectively, the θ-lower and θ-upper approximations of the IF-set $A = (\mu_A, \nu_A)$.

$\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ is called a rough IF-set with respect to θ if $\underline{\theta}(A) \neq \overline{\theta}(A)$, that is, $\underline{\theta}(\mu_A) \neq \overline{\theta}(\mu_A)$ and $\underline{\theta}(\nu_A) \neq \overline{\theta}(\nu_A)$.

Definition 4.2 *Let $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ and $\theta(B) = (\underline{\theta}(B),\ \overline{\theta}(B))$ be two rough IF-sets of R. Then*

(i) $\theta(A) \subseteq \theta(B)$ *if and only if* $\underline{\theta}(A) \subseteq \underline{\theta}(B), \overline{\theta}(A) \subseteq \overline{\theta}(B)$.

(ii) $\theta(A) = \theta(B) \Longleftrightarrow \theta(A) \subseteq \theta(B)$ and $\theta(B) \subseteq \theta(A)$.

(iii) $\theta(A) \cup \theta(B) = (\underline{\theta}(A) \cup \underline{\theta}(B),\ \overline{\theta}(A) \cup \overline{\theta}(B))$.

(iv) $\theta(A) \cap \theta(B) = (\underline{\theta}(A) \cap \underline{\theta}(B),\ \overline{\theta}(A) \cap \overline{\theta}(B))$.

Let $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ and $\theta(B) = (\underline{\theta}(B),\ \overline{\theta}(B))$ be two rough IF-sets of R. Then $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ is called a rough intuitionistic fuzzy subset (rough IF-subset for short) of $\theta(B) = (\underline{\theta}(B),\ \overline{\theta}(B))$ if $\theta(A) \subseteq \theta(B)$.

Proposition 4.1 *Let θ be a full congruence relation on R, and $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets of R. Then*

(i) $\underline{\theta}(A) \subseteq A \subseteq \overline{\theta}(A)$.

(ii) *If $A \subseteq B$ then $\underline{\theta}(A) \subseteq \underline{\theta}(B)$ and $\overline{\theta}(A) \subseteq \overline{\theta}(B)$.*

(iii) $\underline{\theta}(A \cap B) = \underline{\theta}(A) \cap \underline{\theta}(B)$ *and* $\underline{\theta}(A \cup B) = \underline{\theta}(A) \cup \underline{\theta}(B)$.

(iv) $\overline{\theta}(A \cap B) = \overline{\theta}(A) \cap \overline{\theta}(B)$ *and* $\overline{\theta}(A \cup B) = \overline{\theta}(A) \cup \overline{\theta}(B)$.

(v) $\underline{\theta}(\underline{\theta}(A)) = \underline{\theta}(A)$ *and* $\overline{\theta}(\overline{\theta}(A)) = \overline{\theta}(A)$.

(vi) $\underline{\theta}(\overline{\theta}(A)) = \overline{\theta}(A)$ *and* $\overline{\theta}(\underline{\theta}(A)) = \underline{\theta}(A)$.

(vii) $\underline{\theta}(\mu_A)(x) = \underline{\theta}(\mu_A)(a)$ *and* $\underline{\theta}(\nu_A)(x) = \underline{\theta}(\nu_A)(a)$, *for all $x \in R$ and $a \in [x]_\theta$.*

(viii) $\overline{\theta}(\mu_A)(x) = \overline{\theta}(\mu_A)(a)$ *and* $\overline{\theta}(\nu_A)(x) = \overline{\theta}(\nu_A)(a)$, *for all $x \in R$ and $a \in [x]_\theta$.*

Proof Straightforward.

Proposition 4.2 *Let θ be a full congruence relation on R, $A = (\mu_A, \nu_A)$ be an IF-set of R and $X \subseteq R$. Then*

(i) $\underline{\theta}(\mu_A)\ (\overline{\theta}(X)) = \underline{\theta}(\mu_A)(X)$ *and* $\underline{\theta}(\nu_A)\ (\overline{\theta}(X)) = \underline{\theta}(\nu_A)(X)$.

(ii) $\overline{\theta}(\mu_A)\ (\overline{\theta}(X)) = \overline{\theta}(\mu_A)(X)$ *and* $\overline{\theta}(\nu_A)\ (\overline{\theta}(X)) = \overline{\theta}(\nu_A)(X)$.

(iii) $\underline{\theta}(\mu_A)\ (\underline{\theta}(X)) \subseteq \underline{\theta}(\mu_A)(X)$ *and* $\underline{\theta}(\nu_A)\ (\underline{\theta}(X)) \subseteq \underline{\theta}(\nu_A)(X)$.

(iv) $\overline{\theta}(\mu_A)\ (\underline{\theta}(X)) \subseteq \overline{\theta}(\mu_A)(X)$ *and* $\overline{\theta}(\nu_A)\ (\underline{\theta}(X)) \subseteq \overline{\theta}(\nu_A)(X)$,

where for every $Y \subseteq R$, $\underline{\theta}(\mu_A)(Y) = \{\underline{\theta}(\mu_A)(y) \mid y \in Y\}$, $\underline{\theta}(\nu_A)(Y) = \{\underline{\theta}(\nu_A)(y) \mid y \in Y\}$, $\overline{\theta}(\mu_A)(Y) = \{\overline{\theta}(\mu_A)(y) \mid y \in Y\}$, $\overline{\theta}(\nu_A)(Y) = \{\overline{\theta}(\nu_A)(y) \mid y \in Y\}$.

Proof Follows from Propositions 4.1 and 2.1.

Theorem 4.1 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF -subring (IF-ideal) of R, then so is $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$.*

Proof (i) For any $x, y \in R$, we have

$$\begin{aligned}
\overline{\theta}(\mu_A)(x-y) &= \bigvee_{z \in [x-y]_\theta} \mu_A(z) \\
&\geq \bigvee_{a \in [x]_\theta,\ b \in [y]_\theta} \Big(\mu_A(a) \wedge \mu_A(b)\Big) \\
&= \Big(\bigvee_{a \in [x]_\theta} \mu_A(a)\Big) \wedge \Big(\bigvee_{b \in [y]_\theta} \mu_A(b)\Big) \\
&= \overline{\theta}(\mu_A)(x) \wedge \overline{\theta}(\mu_A)(y).
\end{aligned}$$

(ii) For any $x, y \in R$, we have

$$\begin{aligned}\overline{\theta}(\mu_A)(xy) &= \bigvee_{z\in[xy]_\theta} \mu_A(z)\\ &\geq \bigvee_{a\in[x]_\theta,\ b\in[y]_\theta} \Big(\mu_A(a) \wedge \mu_A(b)\Big)\\ &= \Big(\bigvee_{a\in[x]_\theta} \mu_A(a)\Big) \wedge \Big(\bigvee_{b\in[y]_\theta} \mu_A(b)\Big)\\ &= \overline{\theta}(\mu_A)(x) \wedge \overline{\theta}(\mu_A)(y).\end{aligned}$$

(iii) For any $x, y \in R$, we have

$$\begin{aligned}\overline{\theta}(\nu_A)(x-y) &= \bigwedge_{z\in[x-y]_\theta} \nu_A(z)\\ &\leq \bigwedge_{a\in[x]_\theta,\ b\in[y]_\theta} \Big(\nu_A(a) \vee \nu_A(b)\Big)\\ &= \Big(\bigwedge_{a\in[x]_\theta} \nu_A(a)\Big) \vee \Big(\bigwedge_{b\in[y]_\theta} \nu_A(b)\Big)\\ &= \overline{\theta}(\nu_A)(x) \vee \overline{\theta}(\nu_A)(y).\end{aligned}$$

(iv) For any $x, y \in R$, we have

$$\begin{aligned}\overline{\theta}(\nu_A)(xy) &= \bigwedge_{z\in[xy]_\theta} \nu_A(z)\\ &\leq \bigwedge_{a\in[x]_\theta,\ b\in[y]_\theta} \Big(\nu_A(a) \vee \nu_A(b)\Big)\\ &= \Big(\bigwedge_{a\in[x]_\theta} \nu_A(a)\Big) \vee \Big(\bigwedge_{b\in[y]_\theta} \nu_A(b)\Big)\\ &= \overline{\theta}(\nu_A)(x) \vee \overline{\theta}(\nu_A)(y).\end{aligned}$$

At last,

$$\overline{\theta}(\mu_A)(xy) = \bigvee_{z\in[xy]_\theta} \mu_A(z) \geq \bigvee_{b\in[y]_\theta} \mu_A(xb) \geq \bigvee_{b\in[y]_\theta} \mu_A(b) = \overline{\theta}(\mu_A)(y)$$

and

$$\overline{\theta}(\nu_A)(xy) = \bigwedge_{z\in[xy]_\theta} \nu_A(z) = \bigwedge_{b\in[y]_\theta} \nu_A(xb) \leq \bigwedge_{b\in[y]_\theta} \nu_A(b) = \overline{\theta}(\nu_A)(y).$$

Theorem 4.2 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-subring (IF-ideal) of R, then so is $\underline{\theta}(A) = (\underline{\theta}(\mu_A),\ \underline{\theta}(\nu_A))$.*

Proof Analogous to the proof of Theorem 4.1.

Let $A = (\mu_A, \nu_A)$ be an IF-set of R, and $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ a rough IF-set. If $\underline{\theta}(A) = (\underline{\theta}(\mu_A),\ \underline{\theta}(\nu_A))$ and $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$ are IF-subrings (IF-ideals) of R, then we call $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ a rough IF-subring (IF-ideal). In view of this we have the following corollary:

Corollary 4.1 *If $A = (\mu_A, \nu_A)$ is an IF-subring (IF-ideal) of R, then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is a rough IF-subring (rough IF-ideal) of R.*

Theorem 4.3 *Let $A = (\mu_A, \nu_A)$ be an IF-set of R. Then $A = (\mu_A, \nu_A)$ is an IF-ideal of R if and only if $A^{(\alpha,\beta)}$ and $A_s^{(\alpha,\beta)}$ are, if they are non-empty, ideals of R for $(\alpha, \beta) \leq (\mu_A(0), \nu_A(0))$.*

Proof The proof is easy and hence omitted.

Theorem 4.4 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-set of R and $\alpha, \beta \in [0, 1]$, then*

$$(\underline{\theta}(A))^{(\alpha,\beta)} = \underline{\theta}(A^{(\alpha,\beta)}) \text{ and } (\underline{\theta}(A))_s^{(\alpha,\beta)} = \underline{\theta}(A_s^{(\alpha,\beta)}).$$

Proof See Theorem 3.7 [8].

Definition 4.3 [19] *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. The intuitionistic intrinsic product of $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ is defined to be the IF-set $A * B = (\mu_{A*B}, \nu_{A*B})$ in R given by*

$$\mu_{A*B}(x) = \bigvee_{\sum_{i=1}^{k} a_i b_i = x} \left\{ \bigwedge_{1\leq i\leq k} \Big(\mu_A(a_i) \wedge \mu_B(b_i)\Big) \mid k \in N \right\},$$

$$\nu_{A*B}(x) = \bigwedge_{\sum_{i=1}^{k} a_i b_i = x} \left\{ \bigvee_{1\leq i\leq k} \Big(\nu_A(a_i) \vee \nu_B(b_i)\Big) \mid k \in N \right\},$$

if we can express $x = \sum_{i=1}^{k} a_i b_i$ for some $a_i b_i \in R$, where each $a_i b_i \neq 0$ and $k \in N$. Otherwise, we define $A * B = 0_\sim$, that is, $\mu_{A*B}(x) = 0$ and $\nu_{A*B}(x) = 1$.

Theorem 4.5 *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. Then*

(i) $\overline{\theta}(\mu_{A*B}) = \overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)$; (ii) $\underline{\theta}(\mu_{A*B}) = \underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)$;

(iii) $\overline{\theta}(\nu_{A*B}) = \overline{\theta}(\nu_A) * \overline{\theta}(\nu_B)$; (iv) $\underline{\theta}(\nu_{A*B}) = \underline{\theta}(\nu_A) * \underline{\theta}(\nu_B)$.

Proof (i) For any $y \in R$, we have

$$\begin{aligned}
\overline{\theta}(\mu_{A*B})(y) &= \bigvee_{x\in[y]_\theta} \mu_{A*B}(x) \\
&= \bigvee_{x\in[y]_\theta} \left\{ \bigvee_{\sum_{i=1}^{k} a_i b_i = x} \left\{ \bigwedge_{1\leq i\leq k} \Big(\mu_A(a_i) \wedge \mu_B(b_i)\Big) \mid k \in N \right\} \right\} \\
&\geq \bigvee_{\sum_{i=1}^{m} c_i d_i = y} \left\{ \bigwedge_{1\leq i\leq m} \left\{ \Big(\bigvee_{a_i\in[c_i]_\theta} \mu_A(a_i) \Big) \wedge \Big(\bigvee_{b_i\in[d_i]_\theta} \mu_A(b_i) \Big) \right\} \mid m \in N \right\} \\
&= \bigvee_{\sum_{i=1}^{m} c_i d_i = y} \left\{ \bigwedge_{1\leq i\leq m} \left\{ \Big(\overline{\theta}(\mu_A(c_i))\Big) \wedge \Big(\overline{\theta}(\mu_B(d_i))\Big) \right\} \mid m \in N \right\} \\
&= \Big(\overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)\Big)(y).
\end{aligned}$$

Therefore $\overline{\theta}(\mu_{A*B}) \supseteq \overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)$. On the other hand, let $x \in R$ and let $x = \sum_{i=1}^{k} a_i b_i$, where $a_i b_i \neq 0$ in R. Since $\mu_A(a_i) \wedge \mu_B(b_i) \leq \overline{\theta}(\mu_A)(a_i) \wedge \overline{\theta}(\mu_B)(b_i)$ for $1 \leq i \leq k$. Thus

$$\bigwedge_{1\leq i\leq k} \Big(\mu_A(a_i) \wedge \mu_B(b_i)\Big) \leq \bigwedge_{1\leq i\leq k} \Big(\overline{\theta}(\mu_A)(a_i) \wedge \overline{\theta}(\mu_B)(b_i)\Big).$$

Therefore $\mu_{A*B}(x) \leq \Big(\overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)\Big)(x)$. By Proposition 4.1, we have

$$\overline{\theta}(\mu_{A*B})(x) \leq \Big(\overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)\Big)(x),$$

that is,

$$\overline{\theta}(\mu_{A*B}) \subseteq \overline{\theta}(\mu_A) * \overline{\theta}(\mu_B).$$

Hence $\overline{\theta}(\mu_{A*B}) = \overline{\theta}(\mu_A) * \overline{\theta}(\mu_B)$.

(ii) For any $y \in R$, we have

$$\begin{aligned}
\underline{\theta}(\mu_{A*B})(y) &= \bigwedge_{x\in[y]_\theta} \mu_{A*B}(x) \\
&= \bigwedge_{x\in[y]_\theta} \Big\{ \bigvee_{\sum_{i=1}^{k} a_i b_i = x} \Big\{ \bigwedge_{1\leq i\leq k} \Big(\mu_A(a_i) \wedge \mu_B(b_i)\Big) \mid k \in N\Big\}\Big\} \\
&\leq \bigvee_{\sum_{i=1}^{m} c_i d_i = y} \Big\{ \bigwedge_{1\leq i\leq m} \Big\{\Big(\bigwedge_{a_i\in[c_i]_\theta} \mu_A(a_i)\Big) \wedge \Big(\bigwedge_{b_i\in[d_i]_\theta} \mu_A(b_i)\Big)\Big\} \mid m \in N\Big\} \\
&= \bigvee_{\sum_{i=1}^{m} c_i d_i = y} \Big\{ \bigwedge_{1\leq i\leq m} \Big\{\Big(\underline{\theta}(\mu_A(c_i))\Big) \wedge \Big(\underline{\theta}(\mu_B(d_i))\Big)\Big\} \mid m \in N\Big\} \\
&= \Big(\underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)\Big)(y).
\end{aligned}$$

Therefore $\underline{\theta}(\mu_{A*B}) \subseteq \underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)$. On the other hand, let $x \in R$ and let $x = \sum_{i=1}^{k} a_i b_i$, where $a_i b_i \neq 0$ in R. Since $\mu_A(a_i) \wedge \mu_B(b_i) \geq \underline{\theta}(\mu_A)(a_i) \wedge \underline{\theta}(\mu_B)(b_i)$ for $1 \leq i \leq k$. Thus

$$\bigwedge_{1\leq i\leq k} \Big(\mu_A(a_i) \wedge \mu_B(b_i)\Big) \geq \bigwedge_{1\leq i\leq k} \Big(\underline{\theta}(\mu_A)(a_i) \wedge \underline{\theta}(\mu_B)(b_i)\Big).$$

Therefore $\mu_{A*B}(x) \geq \Big(\underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)\Big)(x)$. By Proposition 4.1, we have

$$\underline{\theta}(\mu_{A*B})(x) \geq \Big(\underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)\Big)(x),$$

that is,

$$\underline{\theta}(\mu_{A*B}) \supseteq \underline{\theta}(\mu_A) * \underline{\theta}(\mu_B).$$

Hence $\underline{\theta}(\mu_{A*B}) = \underline{\theta}(\mu_A) * \underline{\theta}(\mu_B)$. The proofs of part (iii) and (iv) are omitted as they are easy.

Definition 4.4 [19] *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. The intuitionistic sum of $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ is defined to be the IF-set $A + B = (\mu_{A+B}, \nu_{A+B})$ in R given by*

$$\mu_{A+B}(x) = \begin{cases} \bigvee_{a+b=x} \Big(\mu_A(a) \wedge \mu_B(b)\Big), & \text{if } x = a + b, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{A+B}(x) = \begin{cases} \bigwedge_{a+b=x} \Big(\nu_A(a) \vee \nu_B(b)\Big), & \text{if } x = a + b, \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 4.6 *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. Then*

(i) $\overline{\theta}(\mu_{A+B}) = \overline{\theta}(\mu_A) + \overline{\theta}(\mu_B)$; (ii) $\underline{\theta}(\mu_{A+B}) = \underline{\theta}(\mu_A) + \underline{\theta}(\mu_B)$;
(iii) $\overline{\theta}(\nu_{A+B}) = \overline{\theta}(\nu_A) + \overline{\theta}(\nu_B)$; (iv) $\underline{\theta}(\nu_{A+B}) = \underline{\theta}(\nu_A) + \underline{\theta}(\nu_B)$.

The proofs are easy and so left to the reader.

Definition 4.5 *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. The intuitionistic product of $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ is defined to be the IF-set $A \bullet B = (\mu_{A\bullet B}, \nu_{A\bullet B})$ in R given by*

$$\mu_{A\bullet B}(x) = \begin{cases} \bigvee_{ab=x} \Big(\mu_A(a) \wedge \mu_B(b)\Big), & \text{if } x = ab, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{A\bullet B}(x) = \begin{cases} \bigwedge_{ab=x} \Big(\nu_A(a) \vee \nu_B(b)\Big), & \text{if } x = ab, \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 4.7 *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF-sets in a ring R. Then*

(i) $\overline{\theta}(\mu_{A\bullet B}) = \overline{\theta}(\mu_A) \bullet \overline{\theta}(\mu_B)$; (ii) $\underline{\theta}(\mu_{A\bullet B}) = \underline{\theta}(\mu_A) \bullet \underline{\theta}(\mu_B)$;
(iii) $\overline{\theta}(\nu_{A\bullet B}) = \overline{\theta}(\nu_A) \bullet \overline{\theta}(\nu_B)$; (iv) $\underline{\theta}(\nu_{A\bullet B}) = \underline{\theta}(\nu_A) \bullet \underline{\theta}(\nu_B)$.

The proofs are easy and so left to the reader.

Let θ be a full congruence relation on R. $A = (\mu_A, \nu_A) \in IFS(R)$ is called a fixed point of θ-lower approximation, if $\underline{\theta}(A) = A$, i.e., $\underline{\theta}(\mu_A) = \mu_A$, $\underline{\theta}(\nu_A) = \nu_A$ and we put $Fix\ (\underline{\theta}) = \{ A \in IFS(R) \mid \underline{\theta}(A) = A\}$, i.e., $Fix\ (\underline{\theta}) = \{ A = (\mu_A, \nu_A) \in IFS(R) \mid \underline{\theta}(\mu_A) = \mu_A, \underline{\theta}(\nu_A) = \nu_A\}$.

Proposition 4.3 *Let θ_1 and θ_2 be two full congruence relations on a ring R. Then the following statements are equivalent:*

(i) *For each $A = (\mu_A, \nu_A) \in IFS(R)$, $\underline{\theta}_1(A) \subseteq \underline{\theta}_2(A)$.*

(ii) $Fix\ (\underline{\theta}_1) \subseteq Fix\ (\underline{\theta}_2)$.

Proof (i) $\Longrightarrow$ (ii)

Let $A = (\mu_A, \nu_A) \in IFS(R)$ and $\underline{\theta}_1(A) = A$. Then $A = \underline{\theta}_1(A) \subseteq \underline{\theta}_2(A) \subseteq A$, so it follows that $\underline{\theta}_2(A) = A$.

(ii) $\Longrightarrow$ (i)

Let $A = (\mu_A, \nu_A) \in IFS(R)$. Since by Proposition 4.1, $\underline{\theta}_1(A) \in Fix\ (\underline{\theta}_1)$, we conclude that $\underline{\theta}_1(A) \in Fix\ (\underline{\theta}_2)$. Thus, $\underline{\theta}_1(A) = \underline{\theta}_2(\underline{\theta}_1(A)) \subseteq \underline{\theta}_2(A)$.

Let θ be a full congruence relation on R. $A = (\mu_A, \nu_A) \in IFS(R)$ is called a fixed point of θ-upper approximation, if $\overline{\theta}(A) = A$, i.e., $\overline{\theta}(\mu_A) = \mu_A$, $\overline{\theta}(\nu_A) = \nu_A$ and we put $Fix\ (\overline{\theta}) = \{ A \in IFS(R) \mid \overline{\theta}(A) = A\}$, i.e., $Fix\ (\overline{\theta}) = \{ A = (\mu_A, \nu_A) \in IFS(R) \mid \overline{\theta}(\mu_A) = \mu_A, \overline{\theta}(\nu_A) = \nu_A\}$.

Proposition 4.4 *Let θ_1 and θ_2 be two full congruence relations on a ring R. Then the following statements are equivalent:*

(i) *For each $A = (\mu_A, \nu_A) \in IFS(R)$, $\overline{\theta}_1(A) \subseteq \overline{\theta}_2(A)$.*

(ii) $Fix\ (\overline{\theta}_2) \subseteq Fix\ (\overline{\theta}_1)$.

Proof (i) $\Longrightarrow$ (ii)

Let $A = (\mu_A, \nu_A) \in IFS(R)$ and $\overline{\theta}_2(A) = A$. Then $A \subseteq \overline{\theta}_1(A) \subseteq \overline{\theta}_2(A) = A$, so it follows that $\underline{\theta}_1(A) = A$.

(ii) $\Longrightarrow$ (i)

Let $A = (\mu_A, \nu_A) \in IFS(R)$. Since $\overline{\theta}_2(A) \in Fix\ (\overline{\theta}_2)$, we have $\overline{\theta}_2(A) \in Fix\ (\overline{\theta}_1)$. Thus, $\overline{\theta}_1(A) \subseteq \overline{\theta}_1(\overline{\theta}_2(A)) = \overline{\theta}_2(A)$.

5. Rough IF-ideals of IF-subrings

In this section, we study lower and upper approximations of IF-ideals of IF-subrings. In the rest of the paper $B = (\mu_B, \nu_B)$ is always IF-subring of a commutative ring R.

Definition 5.1 *Let $A = (\mu_A, \nu_A)$ be an IF-set of R and $B = (\mu_B, \nu_B)$ be an IF-subring of R such that $A \subseteq B$. Then the IF-set $A = (\mu_A, \nu_A)$ is called an IF-ideal of $B = (\mu_B, \nu_B)$ if it satisfies the following conditions:*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ *for every $x, y \in R$.*

(ii) $\mu_A(xy) \geq \mu_B(x) \wedge \mu_A(y)$ *for every $x, y \in R$.*

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$ *for every $x, y \in R$.*

(iv) $\nu_A(xy) \leq \nu_B(x) \vee \nu_A(y)$ *for every* $x, y \in R$.

If $A = (\mu_A, \nu_A)$ and $C = (\mu_C, \nu_C)$ are IF-deals of $B = (\mu_B, \nu_B)$, then so are $A * C = (\mu_{A*C}, \nu_{A*C})$ and $A \bullet C = (\mu_{A\bullet C}, \nu_{A\bullet C})$. (For the proof see Theorems 4.3 and 4.8 [19]).

Theorem 5.1 *Let $B = (\mu_B, \nu_B)$ be an IF-subring of R. Then $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$ if and only if $A^{(\alpha,\beta)}$ and $A_s^{(\alpha,\beta)}$ are, if they are non-empty, ideals of $B^{(\alpha,\beta)}$ and $B_s^{(\alpha,\beta)}$ for $(\alpha, \beta) \leq (\mu_A(0), \nu_A(0))$.*

Proof Suppose that $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, and $(\alpha, \beta) \leq (\mu_A(0), \nu_A(0))$. Let $x, y \in A^{(\alpha,\beta)}$ and $r \in B^{(\alpha,\beta)}$. Then $\mu_A(x) \geq \alpha$, $\nu_A(x) \leq \beta$, $\mu_A(y) \geq \alpha$, $\nu_A(y) \leq \beta$, $\mu_B(r) \geq \alpha$ and $\nu_B(r) \leq \beta$. Hence

$$\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha,$$
$$\mu_A(rx) \geq \mu_B(r) \wedge \mu_A(x) \geq \alpha,$$
$$\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y) \leq \beta,$$
$$\nu_A(rx) \leq \nu_B(r) \vee \nu_A(x) \leq \beta,$$

and therefore $x - y \in A^{(\alpha,\beta)}$ and $rx \in A^{(\alpha,\beta)}$ for all $x, y \in A^{(\alpha,\beta)}$, $r \in B^{(\alpha,\beta)}$. The converse part is proved by contradiction method and is left to the reader.

Theorem 5.2 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, then $\overline{\theta}(A) = (\overline{\theta}(\mu_A), \overline{\theta}(\nu_A))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Since $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, by Theorem 5.1, we know that $A^{(\alpha,\beta)}$ is, if it is non-empty, an ideal of $B^{(\alpha,\beta)}$. By Theorem 2.2, we obtain that $\overline{\theta}(A^{(\alpha,\beta)})$ is an ideal of $B^{(\alpha,\beta)}$. From this and Theorem 4.4, we know that $(\overline{\theta}(A))^{(\alpha,\beta)}$ is an ideal of $B^{(\alpha,\beta)}$. Now by Theorem 5.1, we conclude that $\overline{\theta}(A) = (\overline{\theta}(\mu_A), \overline{\theta}(\nu_A))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.

Theorem 5.3 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, then $\underline{\theta}(A) = (\underline{\theta}(\mu_A), \underline{\theta}(\nu_A))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Analogous to the proof of Theorem 5.1.

Let $A = (\mu_A, \nu_A)$ be an IF-subset of $B = (\mu_B, \nu_B)$ and $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ be a rough IF-subset. If $\underline{\theta}(A) = (\underline{\theta}(\mu_A), \underline{\theta}(\nu_A))$ and $\overline{\theta}(A) = (\overline{\theta}(\mu_A), \overline{\theta}(\nu_A))$ are IF-ideals of $B = (\mu_B, \nu_B)$, then we call $\theta_A = (\underline{\theta}(A), \overline{\theta}(A))$ a rough IF-ideal. In view of this we have the following corollary:

Corollary 5.1 *Let $A = (\mu_A, \nu_A)$ be an IF-ideal of $B = (\mu_B, \nu_B)$. Then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is a rough IF-deal of $B = (\mu_B, \nu_B)$.*

6. Rough IF-radical of Rough IF-ideals

Throughout this section, N denotes the set of all positive integers.

Definition 6.1 *Let $A = (\mu_A, \nu_A)$ be an IF-ideal of $B = (\mu_B, \nu_B)$. The intuitionistic fuzzy nil radical (IF-nil radical for short) of $A = (\mu_A, \nu_A)$ is defined to be an IF-set $R(A) = (\mu_{R(A)}, \nu_{R(A)})$ of $B = (\mu_B, \nu_B)$ defined by*

$$\mu_{R(A)}(x) = \Big(\bigvee_{n\in N} \mu_A(x^n)\Big) \wedge \mu_B(x)$$

and

$$\nu_{R(A)}(x) = \Big(\bigwedge_{n\in N} \nu_A(x^n)\Big) \vee \nu_B(x)$$

for all $x \in R$ and some $n \in N$.

Proposition 6.1 *For every IF-ideals $A = (\mu_A, \nu_A)$ and $C = (\mu_C, \nu_C)$ of $B = (\mu_B, \nu_B)$, we have*

(i) $A \subseteq R(A)$,

(ii) $A \subseteq C$ *implies* $R(A) \subseteq R(C)$ *and*

(iii) $R((R(A)) = R(A)$.

Proof See Propositions 4.6, 4.7 and 4.4 [20].

Theorem 6.1 *For any IF-ideals $A = (\mu_A, \nu_A)$ of $B = (\mu_B, \nu_B)$, then $R(A) = (\mu_{R(A)}, \nu_{R(A)})$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof See Theorem 4.3 [20].

Proposition 6.2 *For every IF-ideals $A = (\mu_A, \nu_A)$ and $C = (\mu_C, \nu_C)$ of $B = (\mu_B, \nu_B)$, we have*

$$R(A * C) = R(A \cap C) = R(A) \cap R(C)$$

and

$$R(A + C) \subseteq R(A) + R(C) = R(A + C).$$

Proof See Theorems 4.6 and 4.9 [19].

Theorem 6.2 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, then $\overline{\theta}(R(A)) = (\overline{\theta}(\mu_{R(A)}),\ \overline{\theta}(\nu_{R(A)}))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Straightforward from Theorems 6.1 and 5.2.

Theorem 6.3 *Let θ be a full congruence relation on R. If $A = (\mu_A, \nu_A)$ is an IF-ideal of $B = (\mu_B, \nu_B)$, then $\underline{\theta}(R(A)) = (\underline{\theta}(\mu_{R(A)}),\ \underline{\theta}(\nu_{R(A)}))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Straightforward from Theorems 6.1 and 5.3.

Let $A = (\mu_A, \nu_A)$ be an IF-subset of $B = (\mu_B, \nu_B)$ and $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ be a rough IF-set. If $\underline{\theta}(R(A)) = (\underline{\theta}(\mu_{R(A)}),\ \underline{\theta}(\nu_{R(A)}))$ and $\overline{\theta}(R(A)) = (\overline{\theta}(\mu_{R(A)}),\ \overline{\theta}(\nu_{R(A)}))$ are IF-ideals of $B = (\mu_B, \nu_B)$, then we call $\theta(R(A)) = (\underline{\theta}(R(A)),\ \overline{\theta}(R(A)))$ a rough IF-ideal. In view of this we have the following corollary:

Corollary 6.1 *Let $A = (\mu_A, \nu_A)$ be an IF-ideal of $B = (\mu_B, \nu_B)$. Then $\theta(R(A)) = (\underline{\theta}(R(A)),\ \overline{\theta}(R(A)))$ is a rough IF-deal of $B = (\mu_B, \nu_B)$.*

7. Rough Prime (Primary) and Semiprime IF-ideals

Throughout this section, N denotes the set of all positive integers and θ denote a complete congruence relation on R. For more details on fuzzy prime ideals see [21-23].

Definition 7.1 *A non-constant IF-ideal $A = (\mu_A, \nu_A)$ of $B = (\mu_B, \nu_B)$ is said to be prime IF-ideal of $B = (\mu_B, \nu_B)$ if it satisfies:*

$$\mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) \leq \mu_A(x) \vee \mu_A(y)$$

and

$$\nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) \geq \nu_A(x) \wedge \nu_A(y)$$

for all $x, y \in R$ and for all $n \in N$.

Definition 7.2 *A non-constant IF-ideal $A = (\mu_A, \nu_A)$ of $B = (\mu_B, \nu_B)$ is said to be primary IF-ideal of $B = (\mu_B, \nu_B)$ if it satisfies:*

$$\mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) \leq \mu_A(x) \vee \Big(\bigvee_{n \in N} \mu_A(y^n) \Big)$$

and

$$\nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) \geq \nu_A(x) \wedge \Big(\bigwedge_{n \in N} \nu_A(y^n) \Big)$$

for all $x, y \in R$ and for all $n \in N$.

Definition 7.3 *An IF-ideal $A = (\mu_A, \nu_A)$ of $B = (\mu_B, \nu_B)$ is said to be semiprime IF-ideal of $B = (\mu_B, \nu_B)$ if $R(A) = A$, that is,*

$$(\forall x \in R)\ (\mu_{R(A)}(x) = \mu_A(x) \text{ and } \nu_{R(A)}(x) = \nu_A(x)).$$

Theorem 7.1 *$A = (\mu_A, \nu_A)$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$ if and only if $A^{(\alpha,\, \beta)}$ is a prime ideal of $B^{(\alpha,\, \beta)}$ for $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$.*

Proof Suppose that $A = (\mu_A, \nu_A)$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$ and let $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$. By Theorem 5.1, $A^{(\alpha,\, \beta)}$ is a prime ideal of $B^{(\alpha,\, \beta)}$. Now, let $x, y \in B^{(\alpha,\, \beta)}$ with $xy \in A^{(\alpha,\, \beta)}$. Then $\mu_B(x) \geq \alpha$, $\nu_B(x) \leq \beta$, $\mu_B(y) \geq \alpha$, $\nu_B(y) \leq \beta$, $\mu_A(xy) \geq \alpha$, $\nu_A(xy) \leq \beta$. Hence

$$\mu_A(x) \vee \mu_A(y) \geq \mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) \geq \alpha$$

and

$$\nu_A(x) \wedge \nu_A(y) \leq \nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) \leq \beta.$$

Therefore, either $x \in A^{(\alpha,\, \beta)}$ or $y \in A^{(\alpha,\, \beta)}$. This shows that $A^{(\alpha,\, \beta)}$ is a prime ideal of $B^{(\alpha,\, \beta)}$. Conversely, suppose that $A^{(\alpha,\, \beta)}$ is a prime ideal of $B^{(\alpha,\, \beta)}$ for $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$. By Theorem 5.1, $A = (\mu_A, \nu_A)$ is a IF-ideal of $B = (\mu_B, \nu_B)$. The proof of the theorem will be complete if we show that,

$$\mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) \leq \mu_A(x) \vee \mu_A(y)$$

and

$$\nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) \geq \nu_A(x) \wedge \nu_A(x).$$

Suppose if possible,

$$\mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) > \mu_A(x) \vee \mu_A(y)$$

and

$$\nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) < \nu_A(x) \wedge \nu_A(y).$$

Since $A^{(\alpha,\,\beta)}$ is a prime ideal of $B^{(\alpha,\,\beta)}$, for x, y in $B^{(\alpha,\,\beta)}$ with $xy \in A^{(\alpha,\,\beta)}$, let us choose α and β such that

$$\mu_A(xy) \wedge \mu_B(x) \wedge \mu_B(y) \geq \alpha > \mu_A(x) \vee \mu_A(y)$$

and

$$\nu_A(xy) \vee \nu_B(x) \vee \nu_B(y) \leq \beta < \nu_A(x) \wedge \nu_A(y).$$

It follows that $xy \in A^{(\alpha,\,\beta)}$ implies $x \notin A^{(\alpha,\,\beta)}$ and $y \notin A^{(\alpha,\,\beta)}$. This is a contradiction because $xy \in A^{(\alpha,\,\beta)}$ implies either $x \in A^{(\alpha,\,\beta)}$ or $y \in A^{(\alpha,\,\beta)}$. Therefore our assumption is wrong and we conclude the theorem.

Similar to Theorem 7.1, we have the following two theorems:

Theorem 7.2 *$A = (\mu_A, \nu_A)$ is a primary IF-ideal of $B = (\mu_B, \nu_B)$ if and only if $A^{(\alpha,\,\beta)}$ is a primary ideal of $B^{(\alpha,\,\beta)}$ for $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$.*

Theorem 7.3 *$A = (\mu_A, \nu_A)$ is a smiprime IF-ideal of $B = (\mu_B, \nu_B)$ if and only if $A^{(\alpha,\,\beta)}$ is a semiprime ideal of $B^{(\alpha,\,\beta)}$ for $(\alpha,\ \beta) \leq (\mu_A(0), \nu_A(0))$.*

Theorem 7.4 *If $A = (\mu_A, \nu_A)$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$, then $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$ is an IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Since $A = (\mu_A, \nu_A)$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$, by Theorem 7.1, we conclude that $A^{(\alpha,\,\beta)}$ is, if it is non-empty, a prime ideal of $B^{(\alpha,\,\beta)}$. Then by Theorem 2.4, we obtain that $\overline{\theta}(A^{(\alpha,\,\beta)})$ is a prime ideal of $B^{(\alpha,\,\beta)}$. From this and Theorem 4.4, we know that $(\overline{\theta}(A))^{(\alpha,\,\beta)}$ is a prime ideal of $B^{(\alpha,\,\beta)}$. Using Theorem 5.1, we obtain that $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$.

Theorem 7.5 *If $A = (\mu_A, \nu_A)$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$, then $\underline{\theta}(A) = (\underline{\theta}(\mu_A),\ \underline{\theta}(\nu_A))$ is a prime IF-ideal of $B = (\mu_B, \nu_B)$.*

Proof Analogous to the proof of Theorem 7.4.

It is easy to prove that Theorems 7.4 and 7.5 are also valid if we replace prime IF-ideal by primary IF-ideal and semiprime IF-ideal.

Let $A = (\mu_A, \nu_A)$ be an IF-subset of $B = (\mu_B, \nu_B)$ and $\theta(A) = (\underline{\theta}(A),\ \overline{\theta}(A))$ be a rough IF-set. If $\underline{\theta}(A) = (\underline{\theta}(\mu_A),\ \underline{\theta}(\nu_A))$ and $\overline{\theta}(A) = (\overline{\theta}(\mu_A),\ \overline{\theta}(\nu_A))$ are prime (primary, semiprime) IF-ideals of $B = (\mu_B, \nu_B)$, then we call $\theta_A = (\underline{\theta}(A), \overline{\theta}(A))$ a rough prime (primary, semiprime) IF-ideal. In view of this we have the following corollary:

Corollary 7.1 *Let $A = (\mu_A, \nu_A)$ be an IF-ideal of $B = (\mu_B, \nu_B)$. Then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ a rough prime (primary, semiprime) IF-deal of $B = (\mu_B, \nu_B)$.*

Theorem 7.6 *Let f be an epimorphism (an onto homomorphism) of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Let $B_1 = (\mu_{B_1}, \nu_{B_1})$ and $B_2 = (\mu_{B_2}, \nu_{B_2})$ be IF-subrings of R_1 and R_2, respectively, and $A = (\mu_A, \nu_A)$ be an IF-subset of $B_1 = (\mu_{B_1}, \nu_{B_1})$. If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (f(a), f(b)) \in \theta_2\}$, then*

(i) $\overline{\theta}_1(A) = (\overline{\theta}_1(\mu_A), \overline{\theta}_1(\nu_A))$ *is an IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\overline{\theta}_2(f(A)) = (\overline{\theta}_2(\mu_{f(A)}), \overline{\theta}_2(\nu_{f(A)}))$ *is an IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(ii) $\overline{\theta}_1(A) = (\overline{\theta}_1(\mu_A), \overline{\theta}_1(\nu_A))$ *is a prime IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\overline{\theta}_2(f(A)) = (\overline{\theta}_2(\mu_{f(A)}), \overline{\theta}_2(\nu_{f(A)}))$ *is a prime IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(iii) $\overline{\theta}_1(A) = (\overline{\theta}_1(\mu_A), \overline{\theta}_1(\nu_A))$ *is a primary IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\overline{\theta}_2(f(A)) = (\overline{\theta}_2(\mu_{f(A)}), \overline{\theta}_2(\nu_{f(A)}))$ *is a primary IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(iv) $\overline{\theta}_1(A) = (\overline{\theta}_1(\mu_A), \overline{\theta}_1(\nu_A))$ *is a semiprime IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\overline{\theta}_2(f(A)) = (\overline{\theta}_2(\mu_{f(A)}), \overline{\theta}_2(\nu_{f(A)}))$ *is a semiprime IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

Moreover if f is one to one, then we have

(v) $\underline{\theta}_1(A) = (\underline{\theta}_1(\mu_A), \underline{\theta}_1(\nu_A))$ *is an IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\underline{\theta}_2(f(A)) = (\underline{\theta}_2(\mu_{f(A)}), \underline{\theta}_2(\nu_{f(A)}))$ *is an IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(vii) $\underline{\theta}_1(A) = (\underline{\theta}_1(\mu_A), \underline{\theta}_1(\nu_A))$ *is a prime IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\underline{\theta}_2(f(A)) = (\underline{\theta}_2(\mu_{f(A)}), \underline{\theta}_2(\nu_{f(A)}))$ *is a prime IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(vii) $\underline{\theta}_1(A) = (\underline{\theta}_1(\mu_A), \underline{\theta}_1(\nu_A))$ *is a primary IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\underline{\theta}_2(f(A)) = (\underline{\theta}_2(\mu_{f(A)}), \underline{\theta}_2(\nu_{f(A)}))$ *is a primary IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

(viii) $\underline{\theta}_1(A) = (\underline{\theta}_1(\mu_A), \underline{\theta}_1(\nu_A))$ *is a semiprime IF-ideal of* $B_1 = (\mu_{B_1}, \nu_{B_1})$ *if and only if* $\underline{\theta}_2(f(A)) = (\underline{\theta}_2(\mu_{f(A)}), \underline{\theta}_2(\nu_{f(A)}))$ *is a semiprime IF-ideal of* $B_2 = (\mu_{B_2}, \nu_{B_2})$.

Proof (i) By Theorem 5.1, we conclude that $\overline{\theta}_1(A) = (\overline{\theta}_1(\mu_A), \overline{\theta}_1(\nu_A))$ is an IF-ideal of $B_1 = (\mu_{B_1}, \nu_{B_1})$ if and only if $(\overline{\theta}_1(A))_s^{(\alpha, \beta)}$ are, if they are non-empty, an ideal of $B_{1_s}^{(\alpha, \beta)}$ for $\alpha, \beta \in [0, 1]$. Using Theorem 4.4, we have $(\overline{\theta}_1(A))_s^{(\alpha, \beta)} = \overline{\theta}_1(A_s^{(\alpha, \beta)})$. By Theorem 2.9, we obtain that $\overline{\theta}_1(A_s^{(\alpha, \beta)})$ is an ideal of $B_{1_s}^{(\alpha, \beta)}$ if and only if $\overline{\theta}_2(f(A_s^{(\alpha, \beta)}))$ is an ideal of $B_{2_s}^{(\alpha, \beta)}$. It is clear that $f(A_s^{(\alpha, \beta)}) = (f(A))_s^{(\alpha, \beta)}$. From this and Theorem 5.1, we have

$$\overline{\theta}_2(f(A_s^{(\alpha, \beta)})) = \overline{\theta}_2((f(A))_s^{(\alpha, \beta)}) = \overline{\theta}_2(f(A))_s^{(\alpha, \beta)}.$$

By Theorem 5.1, we obtain $\overline{\theta}_2(f(A))_s^{(\alpha, \beta)}$ is an ideal of $B_{2_s}^{(\alpha, \beta)}$ for every $\alpha, \beta \in [0, 1]$ if and only if $\overline{\theta}_2(f(A)) = (\overline{\theta}_2(\mu_{f(A)}), \overline{\theta}_2(\nu_{f(A)}))$ is an IF-ideal of $B_2 = (\mu_{B_2}, \nu_{B_2})$.

The proof of other parts is similar.

8. Conclusion

The study of properties of rough sets on a ring is a meaningful research topic for rough set theory.The existing research of rough sets on a ring is mainly concerned with crisp sets. In this paper we concentrate our study on algebraic properties of rough IF-sets with respect to a ring. Here we substitute a universal set by a ring and we study on the algebraic structure of rough IF-ideals of IF-subrings in details.

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